Longitudinal vibrations of a thin rod interacting with an immobile limiter

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Abstract—A solution of the problem of longitudinal vibrations of an elastic rod interacting with a one-sided limiter is presented. The methods of frequency–time analysis are used to obtain exact solutions that describe free and forced vibrations. Frequency characteristics of the system are determined. The obtained exact solution is compared to the approximate solutions found earlier.

Problems of collisions of a rod end with immobile barriers are frequently met in calculations of different elements in a machine structure, including the analysis of the dynamics of ultrasonic production machines [1, 2]. The main dynamic effects observed in such systems are considered in [3, 4]. The calculation methods based on the concept of equivalent linearization that were used in those studies enabled their authors to perform an initial theoretical study of the system and derive relations that are required for production of actual machines. However, some issues related to the laws of motion of such systems remained unexplored. A number of these issues will be considered in this study.

Using the methods of the theory of vibration–impact systems with distributed impact elements [5–8], we consider the problem of the interaction of a rod made of viscoelastic material that vibrates longitudinally under the effect of external forces.

1. We assume that the methods for measuring independent variables x and t that determine the sought dislocation of the rod cross sections $u(x, t)$ are selected in such a way that the linear density of the rod and the product ES (E and S are the elasticity modulus of the rod material and the area of a rod cross section, respectively) are equal to one. Then, for the selected methods of measurement, the sound velocity in the rod material also proves to be unity. Without loss of generality, the coordinates of the ends of the rod in a static state may be set equal to $x = 0$ and $x = 1/2$. We represent the model under study as an equation of motion and boundary conditions (Fig. 1)

$$\Box u - \Phi(u) = \varepsilon H(t, x, u_t, u_x, \ldots), \quad u(0, t) = 0, \quad u_x(1/2, t) = 0, \tag{1}$$

where $u = u(x, t)$ is the dislocation of the cross section x of the rod at the moment of time t, $\Box u \equiv u_{tt} - u_{xx}$ is the d'Alembertian, ε is a small parameter, and H is a nonlinear T-periodic function of time that depends on independent variables and higher partial derivatives of the displacement. We assume that the smoothness of the function H ensures the existence and uniqueness of a solution at least in the sense of generalized functions [5]. If an ultrasonic production machine is calculated, the function H may simulate a tool connected to the end of the rod and the treated medium.

The function $\Phi(u)$ in Eq. (1) describes the interaction of the free end of the rod with an immobile limiter (for example, the treated surface). Let the interaction take place at $u(1/2, t) = \Delta$. If $\Delta > 0$, we consider a system with a gap; if $\Delta < 0$, a system with interference. Therefore, for all values of t,

$$u\left(\frac{1}{2}, t\right) \le \Delta. \tag{2}$$

When the rod comes into contact with the limiter, during some time, its end section $x = 1/2$ is at rest while contacting the limiter. In this case, inequality (2) becomes an equality. If t_k is the moment when the kth interaction begins and θ_k is the moment when it ends, the equality in (2) is realized for $\in [t_k, \theta_k]$, so that

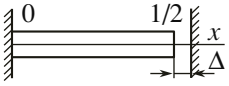

Fig. 1.

$$\Phi[u(^1/_2, t)] = \sum_k R_k(t)\delta(x - {}^1/_2)[\eta(t - t_k) - \eta(t - \theta_k)], \tag{3}$$

where $k = 1, 2, \ldots$ are integers; $\delta(x)$ and $\eta(t)$ are the Dirac and Heaviside functions; and

$$R_k(t) = u_x(^1/_2, t) \le 0, \quad t \in [t_k, \theta_k] \tag{4}$$

is the reaction force of the limiter.

The presence of dwells of the contacting cross section of the rod results in a situation that seems paradoxical for the standard impact theory, which assumes that the striker bounces instantaneously. The energy proves not to be lost during the impact; nevertheless, the impact proves to be similar to an absolutely inelastic impact.

If (2) has the form of a strict inequality, relation (4) becomes an equality and the contact ends. Therefore, the effect of a limiter is equivalent to the occurrence of regular pressing of the rod end to the limiter.

Equation (1) is the nonlinear Klein–Gordon equation. Assuming that the process is periodic, we denote as T the period of the stationary standing wave that appears in the case of nonlinear periodic vibrations. In accordance with the methods of frequency–time analysis of vibration impact processes, we represent the integral equation describing the process in the following form [8, 9]:

$$u(x, t) = \int_0^{T\,1/2}\int_0 \chi(x, z, t - s)\{\Phi[u(z, s)] + \varepsilon H(s, x, u_t, u_x, \ldots)\}dzds, \tag{5}$$

where

$$\chi(x, z, t) = 4\sum_{k=0}^{\infty} \frac{1}{2k+1}\sin[(2k+1)\pi x]\sin[(2k+1)\pi z]\chi_{2k+1}(t) \tag{6}$$

is the T-periodic Green's function of the rod with boundary conditions (1) and the function

$$\chi_{2k+1}(t) = T^{-1}\left\{[(2k+1)\pi]^{-2} + 2\sum_{n=0}^{\infty}[(2k+1)^2\pi^2 - (2n+1)^2\omega^2]^{-1}\cos(2k+1)\omega t\right\}$$

is the periodic Green's function of linear oscillators with the rod eigenfrequencies $\Omega_k = (2k + 1)\pi$ and $\omega = 2\pi T^{-1}$.

2. To implement general approaches to the approximate study of resonance regimes [8–10], we consider a degenerate conservative model. To this end, we set $\varepsilon = 0$ in (1). Initial conditions are represented in the form

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = 0, \tag{7}$$

For simplicity, our consideration is limited by initial conditions that ensure in the linear case standing waves whose shape is similar to the first linear eigenmode (Fig. 2a, curve *1*). We assume that the function $u_0(x)$ in (7) has a single maximum on the boundary of the segment [0, 1/2] at $x = 1/2$.

We consider an important example. Let $u_0(x) \equiv A_0(x) = -2E_0x$ (Fig. 2a, curve *2*). Obviously, the parameter $E_0 \ge 0$ determines in a unique way the total energy of the system. We denote the sought solution as $A(x, t)$.

In the linear case, the profiles of displacements (profiles of the standing wave) at any moment of time will have the shape of a trapezoid (Fig. 2b) with the exception of the moments of time $t = 1/2nT_0 = n$ ($n = 0$, 1, 2, …; $T_0 = 2$ is the period of the free vibrations of the rod) when the trapezoids degenerate into triangles. The free cross section of the rod moves with the constant velocity $\upsilon = 2D_0$, which changes its direction after every half-period when the rod deformation is maximal.

If the system contains a limiter, at the moments of time $t = t_k = t_1 + kT$ when the equality $A(0, t_k) = \Delta$ holds, the free cross section of the rod $x = 1/2$ reaches the limiter, instantaneously stops, and for some time remains in contact with the limiter. During the impact, the wave profile $A(x, t)$ has the shape of a tooth (Fig. 2c). In the utmost position, the part of the rod located to the left of the tooth tip is stretched and that to the left is compressed. The wave then moves in the opposite direction and the halt of the cross section ends at the

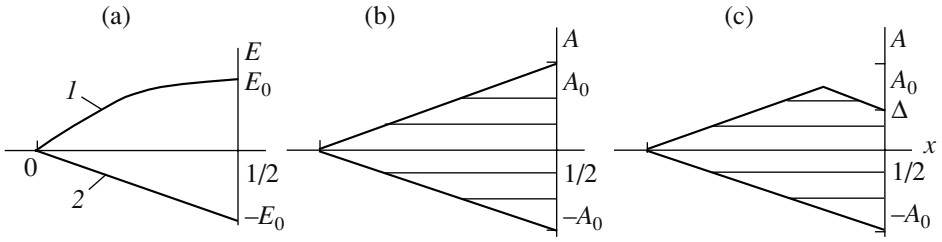

Fig. 2.

moment $\theta_k = \theta_1 + kT$. Under the natural assumption $E_0 > |\Delta|$, the moments of the start and end of the impact and the period of vibrations are determined by the following equalities:

$$t_1 = (E_0 + \Delta)/2E_0, \quad \theta_1 = 1, \quad T = (3E_0 + \Delta)/2E_0. \tag{8}$$

Substituting (6) and (8) into (5), we obtain

$$A(E_0, x, t) = 2E_0 \sum_{k=1}^{\infty} D_k(t^*; \omega, t) \sin[(2k+1)\pi x], \tag{9}$$

where $t^* = (t_2 - t_1) = 1 - t_1$ is the duration of the impact and

$$D_k(t^*; \omega t) = t^*[T(2k+1)^2 \pi t_1^2]^{-1}$$

$$+ 2T^{-1} \sum_{n=0}^{\infty} [(2n+1)\omega]^{-2}[(2k+1)^2\pi^2 - (2n+1)^2\omega^2]^{-1} \cos n\omega t.$$

The solution A_0 may be used as a generating solution in studies of resonant processes. However, it was obtained for initial conditions of a special type and, at first glance, does not ensure the required generality. Using the results of [5, 6, 10], we will show that, if this solution was found for initial conditions that are similar in some sense to those that generate the first eigenmode of linear vibrations, it has properties suitable for the analysis of resonant standing waves.

3. We seek a solution of the nonlinear problem with initial conditions that have a single extremum at $x = 0$ in the form

$$u(x, t) = A_0[z(x, t), \tau(x, t)], \quad E_0 = 1, \tag{10}$$

where A_0 is the solution found in section 2 and z and τ are new variables.

Considering the conservative problem, we substitute (10) into (1) at $\varepsilon = 0$ and, after transformations, arrive at the expression

$$(\tau_t^2 - \tau_x^2)A_{0\tau\tau} - (z_x^2 - z_t^2)A_{0zz} + (z_{tt} - z_{xx})A_{0z} + (\tau_{tt} - \tau_{xx})A_{0\tau} - \Phi[A_0(z, \tau)] = 0,$$

where differentiation is understood in the sense of generalized functions.

Requiring that the transformed equation of motion retain its structure, we constrain the functions z and τ by setting

$$z(x, t) = g(x - t) + g(x + t), \quad \tau(x, t) = g(x - t) - g(x + t). \tag{11}$$

We obtain

$$2g'(x - t)g'(x + t)[A_{0\tau\tau} - A_{0zz}] - \Phi[A_0(z, \tau)] = 0. \tag{12}$$

By virtue of (3) and (4), $\Phi > 0$ and, as follows from formula (11), the introduced function g should be monotonic. Subsequent analysis will show that it should be monotonically increasing; i.e., $g' > 0$ on the entire numerical axis. We select this function using the form of the initial condition $u_0(x)$ set on the segment [0, 1/2]. Since $u_0(x) = A_0[z(x, 0), 0] = -2x = 4|g(x)| - 1$, by additionally defining the function $g(x)$ on the segment [1/2, 1] in an antisymmetric way, one can set

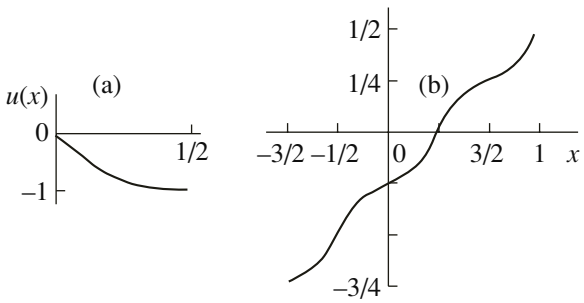

Fig. 3.

$$g(x) = \begin{cases} -{}^1\!/_4[1 + u_0(x)] & 0 \le x \le {}^1\!/_2 \\ 0 & x = {}^1\!/_2 \\ {}^1\!/_4[1 + u_0(x)] & {}^1\!/_2 \le x \le 1 \end{cases}.$$

As a result, $g(x)$ is an odd function that increases on the [–1/2, 1/2] segment. We can define it on the entire numerical axis using two equalities,

$$g(x + 1) = {}^1\!/_2 + g(x), \quad g(-x) = -{}^1\!/_2 - g(x). \tag{13}$$

Figure 3 shows plots of the possible initial configuration of the wave (Fig. 3a) and the function $g(x)$ that corresponds to this wave (Fig. 3b). Formulas (13) and (11) show that $z(x, t) \in [0, 1]$ and $z(x, t + 2) = z(x, t)$; i.e., the function z has a period coinciding with the period of free vibrations of the linear rod ($T_0 = 2$). We have also $\tau(x, t + 2) = \tau(x, t) + 2$.

We will show now that representation (10) corresponds to the initial problem. Since the limiter acts on the rod only when the rod contacts the limiter at the single point $x = 1/2$, the representation of the solution should satisfy the wave equation. As follows from (12), if $A_0(x, t)$ is a solution of the wave equation, $A_0[g(x - t) + g(x + t), g(x - t) - g(x + t)]$ is also a solution of the wave equation. In addition, inequality (2) should be satisfied, but since A_0 is a solution of the considered conservative problem and the function $g(x)$ is antisymmetric with respect to $x = 1/2$, $u(x, t) = A_0[0, 2g(t)] \le \Delta$ and this inequality is actually satisfied Third, inequality (4) should be satisfied. Since, for the wave A_0 during the entire dwell of the rod on the limiter $R_0 = A_{0x}(+0) = 2$, given the increase in the function g, in accordance with (10), we obtain $R_0 = 4g'(t) > 0$.

Thus, expression (10) actually determines the sought process. Therefore, using formula (9) for $E_0 = 1$ and $t_1 = 1/2(1 + \Delta)$, we obtain

$$u(x, t) = 8\sum_{k=1}^{\infty} D_{2k+1}[(\omega; \tau(x, t))]\sin\{(2k + 1)\pi[z(x, t) + {}^1\!/_2]\}. \tag{14}$$

The periods of the functions D_{2k+1} in τ are $T_\tau = (3 + \Delta)/2$. We also have $z(x, t + 2) = z(x, t)$ and $\tau(x, t + 2) = \tau(x, t) + 2$. Therefore, standing wave (14) may be periodic only provided the number $T_0 = 2$ and Δ are incommensurable. This is possible only if Δ is a rational number. If the gap is an irrational number, formula (14) determines a quasi-periodic process [3, 4, 10].

Therefore, $\{A_0(E_0, x, t)\}$ is the only family of periodic standing waves that is weakly sensitive to variations in the value of the gap (interference). The loss of the process periodicity when a rational value is replaced with an irrational value implies that periodic regimes of other types practically cannot appear. Therefore, it is natural to assume that the resonant periodic regime that is established in the initial system when $\varepsilon \ne 0$ is described in terms of the elements of the family $\{A_0(E_0, x, t)\}$.

4. We consider now frequency properties of regimes (14) $A_0(E_0, x, t)$. Taking into account that $T = 2\pi/\omega$ (where ω is the process frequency), we solve the third relation in (8) with respect to $E_0(\omega) > 0$. After transformations, we obtain

$$E_0(\omega) = \Delta\omega(4\pi - 3\omega)^{-1}. \tag{15}$$

Figure 4 shows plots of dependence (15) for different values of the gap. For systems with a gap ($\Delta > 0$), strict anisochronism takes place and only waves with frequencies $\pi < \omega < 4/3\pi$ may exist. If $\Delta = 0$, the frequency $\omega = 4/3\pi$ and the motion is isochronic. Finally, for a system with interference ($\Delta < 0$), weak anisochronism takes place and $4/3\pi < \omega < 2\pi$. The frequency $\omega = \pi$ corresponds to linear vibrations when the rod touches the limiter in the case of $\Delta > 0$ and $E_0 = \Delta$. The frequency $\omega = 2\pi$ corresponds to free vibrations of the rod pressed to the limiter for $\Delta < 0$ and $E_0 < |\Delta|$. The curves presented are energy analogues of the skeleton curves in the theory of nonlinear vibrations. If $\Delta > 0$, the skeleton curve is complemented by a vertical segment $\omega = \pi$ that corresponds to free vibrations of the free end of the rod within the gap for $E_0 < \Delta$.

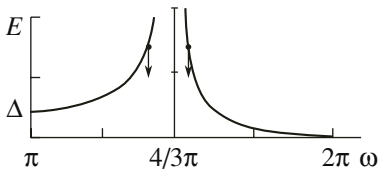

Fig. 4.

Bearing in mind the remark made in section 3, we will seek resonance regimes of motion described in terms of the family $\{A_0(E_0, x, t)\}$. We assume that $\Delta \neq 0$ and, for certainty, assume that in equation of motion (1) $\varepsilon H(t, x, u_t, u_x, \ldots) = \varepsilon[P(x, t) - 2bu_t]$.

Let the exciting force have the form

$$P(x, t) = P(x)\cos(\omega_1 t + \psi) = \sum_{m=1}^{\infty} P_m \sin m\pi x \sin \pi x \cos(\omega_1 t + \psi). \tag{16}$$

It is assumed that time is measured from the beginning of a contact between the rod and the limiter and the phase ψ is unknown. The obtained dependence $E_0(\omega)$ and its properties described above enable one to conclude that resonant regimes of the $u(x, t) \approx A_0(E_0, x, t) + \ldots$ type (where $E_1 = E_0(\omega) = \Delta\omega(4\pi - 3\omega)^{-1}$) may exist in the case of $\Delta > 0$ only in the frequency range $\pi < \omega < 4/3\pi$ and, in the case of $\Delta < 0$, in the range $4/3\pi < \omega < 2\pi$. Isochronic motions are not considered.

For an approximate description of resonant standing waves close (for small ε) to $\{A_0(E_0, x, t)\}$, we use the energy balance concept [4, 5, 8–10]. According to this concept, if a T_1-periodic process, for example, $u(x, t)$, is realized, which is supported by a T_1-periodic force (16), the work performed by the forces of excitation and dissipation must be balanced; i.e.,

$$\int_0^T P(x)\cos(\omega_1 t + \psi) u_t(x, t)\, dt = \int_0^T 2bu_t^2(x, t)\, dt.$$

To determine the unknown phase ψ, we substitute the required variables into Eq. (16) to obtain

$$\sin\psi \sum_{k=1}^{\infty} P_k \frac{\sin(\omega t_1/2)}{k^2(\pi^2 - \omega_1^2)} = 16bE_1(\pi\omega_1)^2 \sum_{m=1}^{\infty 1}\sum_{n=1}^{\infty 1} \frac{\sin^2(\omega t_1/2)}{(m^2\pi^2 - n^2\omega^2)^2}. \tag{17}$$

Here and in (14), $m = 2q + 1$. Denoting the right-hand side of (17) as $bE_1 Y(\omega_1)$ and considering the case $P(x, t) = P_1 \sin\pi x \cos(\omega_1 t + \psi)$ alone, we obtain two possible values of the phase and the condition for the existence of resonant regimes of motion ($|\sin\psi| \le 1$)

$$\psi_1 = \arcsin[bP_1^{-1}E_1 \left| Y(\omega_1)(\pi^2 - \omega_1^2)\sin^{-1}(\omega_1 t_1/2)\right|], \quad \psi_2 = \pi - \psi_1;$$
$$bE_1 \left| Y(\omega_1)(\pi^2 - \omega_1^2)\right| \le P_1 \left|\sin(\omega_1 t_1/2)\right|.$$

An analysis of the last inequality shows that, if $\omega_1 \to 4/3\pi \pm 0$ (the plus and minus signs correspond to the inequalities $\Delta > 0$ and $\Delta < 0$, respectively), $|Y(\omega_1)| \to \infty$. The dots in Fig. 4 shows the boundaries of the region of existence within which the equalities in the existence condition hold. Obviously, at these points, located in the vicinity of the "limiting frequency" $\omega = 4/3\pi$, vibrations stop, as is shown by the vertical arrows (Fig. 4). Obtained dependence (15) and Fig. 4 also show that the frequency pulling phenomenon, well-known in the standard theory of vibration impact systems [8–10], may be observed in the considered system as well.

It should be noted that, if $\Delta > 0$ and $\omega_1 \to \pi$, the time of interaction between the rod and the limiter $t_1 \to 0$; the rod linearly vibrates within the gap and does not touch the limiter. If $\Delta < 0$ and $\omega_1 \to 2\pi$ (the vibration period $T_1 = 1$), the interaction time $t_1 \to T_1 = 1$ and the rod performs the same vibrations remaining in contact with the limiter as a rod with both ends fi ed.

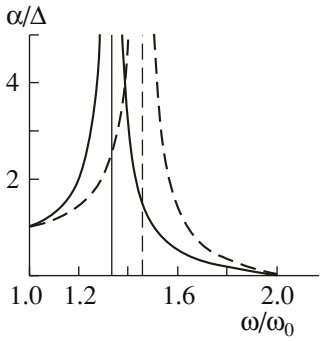

Fig. 5.

It should also be noted that, if in the case of $\Delta < 0$ a rod that was initially at rest is set with interference so that $u_0(x) = 2\Delta x$, to obtain resonant vibrations of the considered type, the rod should first be moved away from the limiter; i.e., the standing wave should be forcibly launched [2, 6, 10, 11]. It is easy to show that, as in standard vibration impact systems [6, 8–10], the amplitude pulling of resonant vibrations is possible in the considered system. The described nonlinear effects were observed many times in natural and numerical experiments [2, 6, 10, 11].

Therefore, the discovered regimes and the regimes of trapezoidal standing waves in systems with distributed impact elements [5–7, 10, 11] make it possible to establish an analogy between the fairly complex system under consideration and an elementary vibration impact system of the "impact oscillator" type [7–10].

5. It should be noted in conclusion that the obtained solution of the nonlinear problem is the exact solution. Therefore, it is possible to compare the results yielded by the exact solution and the approximate solution obtained in [1]. We make such a comparison by comparing the frequency characteristics of free vibrations. For convenience, we present the results that follow from the obtained solution in the dimensional form.

Let the rod length be l. The period of free vibrations of the rod with one end free and the other fi ed is then $T_0 = 4l/c$, and its first eigenfrequency $\omega_0 = 2\pi/T_0 = \pi c/2l$, where c is the velocity of sound in the material of the rod. By analogy with Fig. 3a, we assume that the rod is preliminarily compressed and the initial displacement of the free end is $A > |\Delta|$. Then the amplitude of the vibrations of the rod impacting the limiter is

$$a = (A + \Delta)/2, \tag{18}$$

the period of vibrations $T = (3 + \Delta/A)T_0/4$, and the frequency of vibrations

$$\omega = 4\omega_0/(3 + \Delta/A). \tag{19}$$

The condition $A > |\Delta|$ yields that the frequency of vibrations with impacts is $\omega > \omega_0$.

From (18) and (19), we obtain the following dependence of the frequency on the amplitude of vibrations:

$$\omega = \omega_0 \frac{4 - 2\Delta/a}{3 - \Delta/a}. \tag{20}$$

The function inverse to (20) determines the skeleton curve,

$$a = \Delta \frac{2 - \omega/\omega_0}{4 - 3\omega/\omega_0}. \tag{21}$$

Using (20) and (21), one can easily find the area of existence of the skeleton curves that coincide with those obtained in section 4 (Fig. 4). The skeleton curves plotted according to formula (21) are shown in Fig. 5 by solid lines.

When the same problem was solved in [1] by the harmonic linearization method, the following dependence was obtained for the skeleton curves:

$$a = 4\Delta\left(\pi \frac{\omega}{\omega_0} \cot \frac{\pi\omega}{2\omega_0} + 4\right)^{-1}. \tag{22}$$

The skeleton curves plotted according to formula (22) are shown in Fig. 5 by broken lines. The asymptotes of the skeleton curves are found as the solution of the equation $\pi(\omega/\omega_0)\cot(\pi\omega/2\omega_0) + 4 = 0$, which yields the following frequency of isochronic vibrations for $\Delta = 0$: $\omega = 1.458$.

A comparison of the curves shows that the exact and approximate solutions are in good qualitative agreement.

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