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# Stubborn ISS Redesign for Nonlinear High-Gain Observers

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**Abstract:** We propose a stubborn design paradigm for nonlinear high-gain observers, resulting in enforcing a saturation nonlinearity with variable limits on the output injection term. We analyze the input-to-state stability properties of the estimation error dynamics, and we show that the stubborn extension does not lead to a worse bound than the original high-gain design in terms of both peaking and sensitivity to disturbances and measurement noise. We illustrate by simulation that the proposed “stubborn” modification is actually effective at improving the transient response of the observer and the rejection of measurement outliers.

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## 1. INTRODUCTION

The high-gain observer is the most popular tool for estimation in the area of nonlinear control. Such an observer was proposed by Bornard and Hammouri (1991) (see also Gauthier and Kupka (2001)), and it is based on the idea of dominating the effect of uncertainties or nonlinear terms in the dynamics of the estimation error by using a sufficiently large gain. The main drawback of this approach is the occurrence of a strong peaking in the transient, which may cause the destabilization of the control loop if the high-gain observer is used in cascade with a feedback regulator (Khalil and Praly, 2014). Another consequence of the adoption of a large gain is the increased sensitivity to measurement noise. Concerning these issues, Astolfi and Marconi (2015) have recently proposed a “low-power” evolution of the high-gain observer. Here, we will investigate a new high-gain observer for nonlinear systems with the goal of reducing the effect of the measurement noise, especially in case of outliers, by taking advantage of the construction presented in (Alessandri and Zaccarian, 2015, 2016).

In the research area of high-gain observers, various gain adaptation methods have been proposed to account for the presence of disturbances acting on the system. For example, a switching-gain tuning is proposed by Ahrens and Khalil (2009); in (Boizot et al., 2010; Oueder et al., 2012) moving-horizon schemes are suggested to set the gain; nonlinear adaptation laws are described in (Ibrir, 2009) and (Sanfelice and Praly, 2011); resetting rules have been investigated by Prieur et al. (2012).

The stability of the estimation error of observers for systems without noise is treated by using Lyapunov functions.

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However, input-to-state stability (ISS) turns out to be really helpful for this task, because the effects of both system disturbances and measurement noise are easily taken into account, together with stable behaviors in the absence of external signals. As a consequence, the use of ISS analysis has been considered in various works on estimation for nonlinear systems (see, e.g., Arcaç and Kokotovic (2001); Chaves and Sontag (2002); Karafyllis and Kravaris (2009); Alessandri et al. (2010); Postoyan and Nesic (2012); Alessandri (2013)). Concerning observer design for nonlinear systems, the connection between ISS and passivity with respect to measurement noise has been addressed by Shim et al. (2003).

In this paper, we will present a new high-gain observer structure as compared with the standard high-gain observer reported in the literature. The novelty consists in using a saturated output error, which allows to reduce the effect of the measurement noise thanks to the “stubborn” adaptation of the saturation threshold according to the approach proposed by Alessandri and Zaccarian (2016). Such an adaptation is called “stubborn” because in light of a persistent zero output error, the observer shrinks to zero the saturation threshold regulating the trimming action (performed by the saturation nonlinearity) on the output injection term, thereby making the observer increasingly “stubborn” about its current estimate. Thus, the role played by the saturation is different from the one addressed in (Astolfi et al., 2016b), where the use of a nested-saturation low-power high-gain observer is analyzed as to stability and sensitivity to measurement noise. For the proposed stubborn high-gain observer, here we will investigate the ISS stability first only with the presence of a system disturbance and later with in addition a measurement noise. It is suitable to verify ISS when dealing

with estimation for nonlinear systems and especially with measurements affected by impulsive disturbances since ISS may not hold even if the estimation error is asymptotically stable in the absence of noises (see (Shim et al., 2003, Section 5, p. 890)).

The paper is organized as follows. In Section 2, we present the system framework and the essential results about standard high-gain observers. The proposed observer and its ISS property on the estimation error are described in Sections 3 and 4. A simulation comparison between the standard high-gain observer and our stubborn high-gain observer is shown in Section 5. Some conclusions are drawn in Section 6.

## 2. PROBLEM DESCRIPTION AND PRELIMINARIES

We consider systems of the form

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \dots, x_i, u(t)), \quad i = 1, \dots, n-1 \\ \dot{x}_n &= \varphi_n(x_1, \dots, x_n, u(t), d(t)), \\ y &= x_1 + \nu(t)\end{aligned}\quad (1)$$

where the state  $x := (x_1, \dots, x_n) \in \mathbb{R}^n$  evolves in a given compact subset  $X$  of  $\mathbb{R}^n$ , the input  $t \mapsto u(t)$  is any function assumed to be known evolving in a compact subset  $U$  of  $\mathbb{R}^m$ , and  $y \in \mathbb{R}$  is the measured output. We suppose that functions  $\varphi_i, i = 1, \dots, n$  are locally Lipschitz. The function  $t \mapsto d(t)$  is an unknown, bounded disturbance representing exogenous signals or model uncertainties. The function  $t \mapsto \nu(t)$  represents a measurement noise and it is assumed to be bounded for all  $t \geq 0$ . For any mapping  $t \mapsto s(t)$ , let  $|s|_\infty := \sup_{t \geq 0} |s(t)|$ , where  $|s(t)| := \sqrt{s(t)^\top s(t)}$ . For this class of systems we can design a high-gain observer as follows:

$$\begin{aligned}\dot{\hat{x}}_i &= \hat{x}_{i+1} + \hat{\varphi}_i(\hat{x}_1, \dots, \hat{x}_i, u(t)) + k_i \ell^i (y - \hat{x}_1), \\ & \quad i = 1, \dots, n-1\end{aligned}\quad (2)$$

$$\dot{\hat{x}}_n = \hat{\varphi}_n(\hat{x}_1, \dots, \hat{x}_n, u(t)) + \ell^n k_n (y - \hat{x}_1),$$

where  $\hat{x} := (\hat{x}_1, \dots, \hat{x}_n) \in \mathbb{R}^n$ ;  $k_1, \dots, k_n$  are coefficients to be chosen such that

$$s^n + k_1 s^{n-1} + \dots + k_{n-1} s + k_n$$

is a Hurwitz polynomial and where  $\ell \geq 1$  is a positive scalar usually denoted as ‘‘high-gain parameter.’’ Functions  $\hat{\varphi}_i$  are continuous, bounded outside  $X \times U$  and satisfy the following conditions:

$$\begin{aligned}\hat{\varphi}_i(x_1, \dots, x_i, u) &= \varphi_i(x_1, \dots, x_i, u), \quad i = 1, \dots, n-1, \\ \hat{\varphi}_n(x_1, \dots, x_n, u) &= \varphi_n(x_1, \dots, x_n, u, 0),\end{aligned}$$

for all  $(x, u) \in X \times U$  and

$$|\hat{\varphi}_i(\hat{x}_1, \dots, \hat{x}_i, u) - \varphi_i(x_1, \dots, x_n, u)| \leq L_i \sum_{j=1}^i |\hat{x}_j - x_j| \quad (3)$$

for  $i = 1, \dots, n-1$  and

$$|\hat{\varphi}_n(\hat{x}_1, \dots, \hat{x}_n, u) - \varphi_n(x_1, \dots, x_n, u, d)| \leq L_n \sum_{j=1}^n |\hat{x}_j - x_j| + R|d|_\infty \quad (4)$$

for all  $(x, u, \hat{x}) \in X \times U \times \mathbb{R}^n$  and  $t \mapsto d(t)$  with  $|d|_\infty$  bounded. Standard results show that if the high-gain parameter  $\ell$  is chosen large enough then observer (2) satisfies the following bound (see, for instance, Khalil and Praly (2014)):

$$\begin{aligned}|\hat{x}_i(t) - x_i(t)| &\leq a_1 \ell^{i-1} \exp(-a_2 \ell t) |\hat{x}(0) - x(0)| \\ &\quad + \frac{a_3}{\ell^{n-(i-1)}} |d|_\infty + a_4 \ell^{i-1} |\nu|_\infty\end{aligned}\quad (5)$$

for all  $t \geq 0$  and for some constants  $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$  independent of  $\ell$  and for any initial condition  $(x(0), \hat{x}(0)) \in X \times \mathbb{R}^n$ .

As pretty well-known and evident in (5), the growth of  $\ell$  makes the estimation error more robust to system noise but it increases also the sensitivity to measurement noise. This motivates the investigation of more effective ‘‘high-gain like’’ observers, as will be shown in the next sections.

## 3. STUBBORN REDESIGN FOR HIGH-GAIN OBSERVERS

In order to improve the performances of high-gain observer (2) in presence of impulsive measurement noise (outliers), we follow the *stubborn* construction proposed by Alessandri and Zaccarian (2016), namely we saturate the output injection term with a dynamical saturation level  $\sigma$  as follows:

$$\begin{aligned}\dot{\hat{x}}_i &= \hat{x}_{i+1} + \hat{\varphi}_i(\hat{x}_1, \dots, \hat{x}_i, u(t)) + k_i \ell^i \text{sat}_{\bar{\sigma}}(y - \hat{x}_1), \\ & \quad i = 1, \dots, n-1 \\ \dot{\hat{x}}_n &= \hat{\varphi}_n(\hat{x}_1, \dots, \hat{x}_n, u(t)) + k_n \ell^n \text{sat}_{\bar{\sigma}}(y - \hat{x}_1), \\ \dot{\sigma} &= -\ell \lambda \sigma + \ell (\lambda + \epsilon) (y - \hat{x}_1)^2, \quad \sigma \geq 0\end{aligned}\quad (6)$$

where  $\lambda > 0, \epsilon > 0, \sigma \in \mathbb{R}_{\geq 0}$  and  $\bar{\sigma} := \sqrt{\sigma}$ , which is well defined from  $\sigma \geq 0$ . We want to prove that observer (6) is still an exponentially convergent observer. In the rest of this section we will consider the case in which the measurement noise is not present, namely  $\nu(t) = 0$  for all  $t \geq 0$ . Comments about the behaviour of the observer (6) in presence of measurement noise are given in Section 4. The larger  $\lambda$ , the slower the reaction to the occurrence of an outlier in the measurements. In practice,  $\lambda$  should be taken to make  $\ell \lambda$  small enough to keep the threshold  $\bar{\sigma}$  small before the next outlier’s rise.

*Proposition 1.* Consider system (1) and observer (6) when there is no measurement noise, namely  $\nu(t) = 0$  for all  $t \geq 0$ . Let  $k_1, \dots, k_n$  be fixed such that the all the roots of

$$s^n + k_1 s^{n-1} + \dots + k_{n-1} s + k_n$$

have (strictly) negative real part, and let  $\lambda > 0$ . There exist  $\epsilon^* > 0$  and  $\ell^* \geq 1$  such that for each  $\epsilon > \epsilon^*$  there exist  $\alpha_i > 0, i = 1, 2, 3$ , and the following bound

$$\begin{aligned}|\hat{x}_i(t) - x_i(t)| &\leq \alpha_1 \ell^{i-1} \exp(-\alpha_2 \ell t) |\hat{x}(0) - x(0)| \\ &\quad + \frac{\alpha_3}{\ell^{n-(i-1)}} |d|_\infty\end{aligned}\quad (7)$$

holds for any  $\ell \geq \ell^*$ , for all  $i = 1, \dots, n$ , for any  $t \geq 0$  and for all initial conditions  $(x(0), \hat{x}(0), \sigma(0)) \in X \times \mathbb{R}^n \times \{0\}$ .

**Proof.** First of all, note that, by picking any initial condition  $\sigma(0) \geq 0$ , we get  $\sigma(t) \geq 0$  for all  $t \geq 0$  so that the observer dynamics generates complete solutions. Now consider the following (standard) change of coordinates

$$e_i := \frac{\hat{x}_i - x_i}{\ell^{i-1}}, \quad (8)$$

by which system (6) is transformed into

$$\begin{aligned} \dot{e} &= \ell(A - KC)e + \ell Kq + \Delta \\ \dot{\sigma} &= -\ell\lambda\sigma + \ell(\lambda + \epsilon)e^\top C^\top Ce, \quad \sigma \geq 0 \\ q &= Ce - \text{sat}_{\bar{\sigma}}(Ce) \end{aligned} \quad (9)$$

where we used the compact notation  $K := \text{col}(k_1, \dots, k_n)$ ,

$$A := \begin{pmatrix} 0_{(n-1) \times 1} & I_{(n-1) \times (n-1)} \\ 0 & 0_{1 \times (n-1)} \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0_{1 \times (n-1)} \end{pmatrix}$$

and  $\Delta := \text{col}(\Delta_1, \dots, \Delta_n)$  with

$$\Delta_i := \frac{1}{\ell^{i-1}} \left[ \hat{\varphi}_i(e_1 + x_1, \ell e_2 + x_2, \dots, \ell^{i-1}e_i + x_i, u) - \varphi_i(x_1, x_2, \dots, x_i, u) \right]$$

for  $i = 1, \dots, n-1$  and

$$\Delta_n := \frac{1}{\ell^{n-1}} \left[ \hat{\varphi}_n(e_1 + x_1, \dots, \ell^{n-1}e_n + x_n, u) - \varphi_n(x_1, \dots, x_n, u, d) \right].$$

Note that, using (3) and (4), it is not hard to see that

$$|\Delta| \leq L|e| + \ell^{-(n-1)}R\bar{d}, \quad (10)$$

with  $\bar{d} = |d|_\infty$  for some  $L > 0$  (independent of  $\ell$ ) and for all  $x \in X$ ,  $e \in \mathbb{R}^n$ . By using the time-rescaling

$$t \mapsto \tau := \ell t, \quad (11)$$

system (9) reads

$$\begin{aligned} e' &= (A - KC)e + Kq + \ell^{-1}\Delta \\ \sigma' &= -\lambda\sigma + (\lambda + \epsilon)e^\top C^\top Ce, \quad \sigma \geq 0 \\ q &= Ce - \text{sat}_{\bar{\sigma}}(Ce) \end{aligned} \quad (12)$$

where  $e' := de/d\tau$  and  $\sigma' := d\sigma/d\tau$ . Now consider the following Lyapunov function

$$V = e^\top Pe + \zeta\sigma + \eta \max\{e^\top C^\top Ce - \sigma, 0\}, \quad (13)$$

with  $P$  solution of

$$P(A - KC) + (A - KC)^\top P = -I,$$

and  $\eta > \zeta > 0$  parameters to be chosen. Such a Lyapunov function is locally Lipschitz and thus treatable by using Clarke's generalized gradient (Clarke, 1990) that, among other things, enjoys useful and well-established properties (see Clarke (1990, Theorem 2.5.1, p. 63 and corollary, p. 64) and also Teel and Praly (2000, p. 99)). Note that the following inequality

$$\lambda_{\min}(P)|e|^2 + \zeta\sigma \leq V \leq (\lambda_{\max}(P) + \eta)|e|^2 + \zeta\sigma \quad (14)$$

holds for any  $e, \sigma$ . The function  $V$  is positive definite with respect to the origin in the set  $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$  where the dynamics is constrained. Indeed, the first term is positive definite and the second term is non-negative. As mentioned above, function  $V$  is non-differentiable in the zero measure set where  $e^\top C^\top Ce - \sigma = 0$ . Below, we prove the results of the proposition by establishing decrease of  $V$  wherever it

is differentiable. In particular, we split the analysis in two cases.

Case 1:  $e^\top C^\top Ce < \sigma$ . In this case  $q = 0$  and the dynamics (12) reduces to

$$\begin{aligned} e' &= (A - KC)e + \ell^{-1}\Delta \\ \sigma' &= -\lambda\sigma + (\sigma + \epsilon)e^\top C^\top Ce, \quad \sigma \geq 0 \end{aligned} \quad (15)$$

and (13) specializes to  $V = e^\top Pe + \zeta\sigma$ . As a consequence its derivative is given by

$$\begin{aligned} \dot{V} &= 2e^\top P((A - KC)e + \ell^{-1}\Delta) \\ &\quad + \zeta(-\lambda\sigma + (\lambda + \epsilon)(Ce)^\top) \\ &\leq -|e|^2 + 2\ell^{-1}e^\top P\Delta - \zeta\lambda\sigma + \zeta(\lambda + \epsilon)|e|^2. \end{aligned}$$

Furthermore, by using inequality (10) and standard Young's inequality we can write

$$\begin{aligned} 2\ell^{-1}e^\top P\Delta &\leq 2\ell^{-1}|e||P|(L|e| + \ell^{-(n-1)}R\bar{d}) \\ &\leq (1 - a_0)|e|^2 + \frac{c_0}{\ell^{2n}}\bar{d}^2 \end{aligned}$$

with  $a_0$  and  $c_0$  defined as

$$a_0 := \frac{3}{4} - 2\ell^{-1}|P|L, \quad c_0 := 4(|P|R)^2. \quad (16)$$

As a consequence, we obtain

$$\dot{V} \leq -(1 - (1 - a_0) - \zeta(\lambda + \epsilon))|e|^2 - \zeta\lambda\sigma + \frac{c_0}{\ell^{2n}}\bar{d}^2,$$

and therefore

$$\dot{V} \leq -a_1|e|^2 - \zeta\lambda\sigma + \frac{c_0}{\ell^{2n}}\bar{d}^2 \quad (17)$$

with  $a_1 := a_0 - \zeta(\lambda + \epsilon)$ . It is easily seen that  $a_1 > 0$  by choosing  $\zeta$  small enough and  $\ell$  large enough. For example, let  $\zeta$  be any positive real number satisfying

$$\zeta < \frac{1}{8(\lambda + \epsilon)}. \quad (18)$$

Then, for any  $\ell \geq \ell^*$  with

$$\ell^* := \max\{4|P|L, 1\}, \quad (19)$$

we get  $a_0 > 1/4$  and  $a_1 > 1/8$ .

Case 2:  $e^\top C^\top Ce \geq \sigma$ . In this case  $q \neq 0$  and  $V$  is computed as  $V = V_1 + V_2 + V_3$  with

$$V_1 = e^\top Pe, \quad V_2 = -(\eta - \zeta)\sigma, \quad V_3 = \eta e^\top C^\top Ce,$$

with derivatives along dynamics (12) given by

$$\begin{aligned} \dot{V}_1 &= 2e^\top P((A - KC)e + \ell^{-1}\Delta + Kq) \\ &\leq -a_0|e|^2 + 2e^\top PKq + \frac{c_0}{\ell^{2n}}\bar{d}^2 \end{aligned}$$

with  $a_0$  and  $c_0$  defined as in (16). For the other two parts of  $V$ , by recalling that  $\zeta > \eta$ , we obtain

$$\begin{aligned} \dot{V}_2 &= (\eta - \zeta)[\lambda\sigma - (\lambda + \epsilon)e^\top C^\top Ce] \\ &\leq -(\eta - \zeta)\epsilon e^\top C^\top Ce \\ &\leq -\frac{1}{2}(\eta - \zeta)\epsilon e^\top C^\top Ce - \frac{1}{2}(\eta - \zeta)\epsilon\sigma, \end{aligned}$$

and finally

$$\begin{aligned} \dot{V}_3 &= 2\eta e^\top C^\top Ce' \\ &= 2\eta e^\top C^\top [C(A - KC)e + \ell^{-1}\Delta_1 + k_1 q] \\ &\leq 2\eta|C(A - KC)||e|^2 + 2\eta(L_1\ell^{-1} + k_1)|Ce|^2 \end{aligned}$$

where we used the following inequalities derived from the sector properties of the saturations, from bound (3) with  $i = 1$ , and from  $CK = k_1$ :

$$|q| = |Ce - \text{sat}_{\bar{\sigma}}(Ce)| \leq |Ce| \quad (20a)$$

$$|\Delta_1| \leq L_1|Ce| \quad (20b)$$

for all  $(x, e) \in X \times \mathbb{R}^n$ . By combining the above derived bounds on  $\dot{V}_1$ ,  $\dot{V}_2$ , and  $\dot{V}_3$ , and by using

$$2e^\top PKq \leq \eta|e|^2 + \eta^{-1}|PK|^2|Ce|^2,$$

issued from (20), we finally get

$$\dot{V} = -a_2|e|^2 - a_3\sigma - (a_3 - a_4)|Ce|^2 + \frac{c_0}{\ell^{2n}}\bar{d}^2 \quad (21)$$

where

$$a_2 := a_0 - 2\eta(|C(A - KC)| + 1)$$

$$a_3 := \frac{1}{2}(\eta - \zeta)\epsilon$$

$$a_4 := \eta^{-1}|PK|^2 + 2\eta(L_1\ell^{-1} + k_1).$$

By imposing  $a_2 > 1/8$ ,  $a_3 \geq a_4$  and recalling that  $\ell \geq \ell^*$  with  $\ell^*$  defined in (19), we obtain that  $\eta$ ,  $\epsilon$ , and  $\zeta$  must be chosen in order to satisfy the following set of inequalities

$$\eta < \frac{1}{8(2|C(A - KC)| + 1)}, \quad (22)$$

$$\epsilon > \frac{4|PK|^2}{\eta^2} + 8(L_1 + k_1), \quad (23)$$

$$\zeta < \min \left\{ \frac{\eta}{2}, \frac{1}{8(\lambda + \epsilon)} \right\}. \quad (24)$$

*Combining the bounds.* By combining (17) and (21) with (22), (23), (24), we get that the bound

$$\dot{V} \leq -\frac{1}{8}(|e|^2 + 8\rho_0\sigma) + \frac{c_0}{\ell^{2n}}\bar{d}^2 \quad \text{for almost all } (e, \sigma) \quad (25)$$

holds for any  $\ell > \ell^*$ , with

$$\rho_0 < \min \left\{ \zeta\lambda, \frac{1}{2}(\eta - \zeta)\epsilon \right\}. \quad (26)$$

By using (14) we also get

$$\dot{V} \leq -\rho_1 V + \frac{c_0}{\ell^{2n}}\bar{d}^2 \quad \text{for almost all } (e, \sigma),$$

for some  $\rho_1 > 0$  independent of  $\ell$ . As a consequence, following the same derivations as in (Teel and Praly, 2000, pp. 99), we get along any solution

$$V(\tau) \leq \exp(-\rho_1\tau)V(0) + \frac{c_0}{\rho_1\ell^{2n}}\bar{d}^2.$$

Therefore, using again (14), recalling that  $\sigma(0) = 0$  and  $\tau = \ell t$ , we obtain

$$|e(t)| \leq \alpha_1 \exp(-\alpha_2 \ell t) |e(0)| + \frac{\alpha_3}{\ell^n} \bar{d}$$

with  $\alpha_2 = \rho_1/2$ , and for some  $\alpha_1 > 0$ ,  $\alpha_3 > 0$  independent of  $\ell$ . Finally bounds  $|\hat{x}_i - x_i| \leq \ell^{i-1}|e_i|$  and  $|e| \leq |x|$  can be used to get (7).  $\square$

#### 4. EFFECT OF THE MEASUREMENT NOISE

In this section we consider the effect of the measurement noise  $\nu$  on the stubborn high-gain observer (6). The proposed analysis follows standard Lyapunov approaches and it is able to capture the  $L^\infty$  gain between the measurement noise  $\nu$  and the error estimate. It is well known that this

analysis is in general too conservative. For instance, in (Astolfi et al. (2016a)) it has been shown that this type of analysis fails in capturing the low-pass filtering properties of the high-gain observer. Even if the recent tool proposed by Astolfi et al. (2016a) could be held to analyse the sensitivity properties of observer (6) with respect to high frequency measurement noise, this analysis would fail in catching the effects of impulsive disturbances. For this reason in this work we limit ourselves to study the  $L^\infty$  gain and in particular we want to show that the bounds for observer (6) are comparable with the bounds (5) one can find for standard high-gain observers (2), showing that the  $i$ -th error estimate  $\hat{x}_i - x_i$ , is proportional to  $\ell^{i-1}$ . More details about how the stubborn high-gain observer behaves in presence of measurement noise are shown in Section 5 through a simulation.

*Proposition 2.* Consider system (1) and observer (6) and let  $k_1, \dots, k_n, \ell^*, \lambda, \epsilon$  be fixed according to Proposition 1. Then there exists  $\alpha_i > 0$ ,  $i = 1, 2, 3, 4$ , such that for any  $\ell \geq \ell^*$  the following bound

$$|\hat{x}_i(t) - x_i(t)| \leq \alpha_1 \ell^{i-1} \exp(-\alpha_2 \ell t) |\hat{x}(0) - x(0)| + \frac{\alpha_3}{\ell^{n-(i-1)}} |d|_\infty + \alpha_4 \ell^{i-1} |\nu|_\infty \quad (27)$$

holds for all  $i = 1, \dots, n$ , for any  $t \geq 0$  and for all initial conditions  $(x(0), \hat{x}(0), \sigma(0)) \in X \times \mathbb{R}^n \times \{0\}$ .

**Proof.** In the sequel we will follow the main steps of the proof of Proposition 1. To begin with, consider the change of coordinates (8), by which system (6) reads as

$$\dot{e} = \ell(A - KC)e + \ell Kq + \Delta + \ell Kdq$$

$$\dot{\sigma} = -\ell\lambda\sigma + \ell(\lambda + \epsilon)(Ce)^2 + \ell(\lambda + \epsilon)[\nu^2 + 2\nu Ce]$$

$$q := Ce - \text{sat}_{\bar{\sigma}}(Ce)$$

$$dq := \text{sat}_{\bar{\sigma}}(Ce) - \text{sat}_{\bar{\sigma}}(Ce + \nu).$$

(28)

Then, by applying the time rescaling (11), we get

$$e' = (A - KC)e + Kq + \ell^{-1}\Delta + Kdq \quad (29)$$

$$\sigma' = -\lambda\sigma + (\lambda + \epsilon)(Ce)^2 + (\lambda + \epsilon)[\nu^2 + 2\nu Ce]$$

where as before we denoted  $e' = de/d\tau$ ,  $\sigma' = d\sigma/d\tau$ . By using the sector properties of the saturations we also have

$$|dq| = |\text{sat}_{\bar{\sigma}}(Ce) - \text{sat}_{\bar{\sigma}}(Ce + \nu)| \leq |\nu| \leq \bar{\nu} \quad (30)$$

for all  $t \geq 0$  with  $\bar{\nu} := |\nu|_\infty$ . Finally, consider the Lyapunov function  $V$  introduced in (13). Recall that the bound (25) holds as long as  $\ell \geq \ell^*$  and  $dq = 0$ . Moreover, it is not hard to see that the term  $dq$  is introducing additional terms in the derivative of  $V$ . As in the proof of Proposition 1, we split the analysis in two parts.

*Case 1:  $e^\top C^\top Ce < \sigma$ .* In this case  $q = 0$  and the dynamics (29) reduces to

$$e' = (A - KC)e + \ell^{-1}\Delta + Kdq$$

$$\sigma' = -\lambda\sigma + (\sigma + \epsilon)e^\top C^\top Ce + (\lambda + \epsilon)[\nu^2 + 2\nu Ce]. \quad (31)$$

Note that the second term of (13) is zero. As a consequence, by using the bound (30) and the inequality (25),

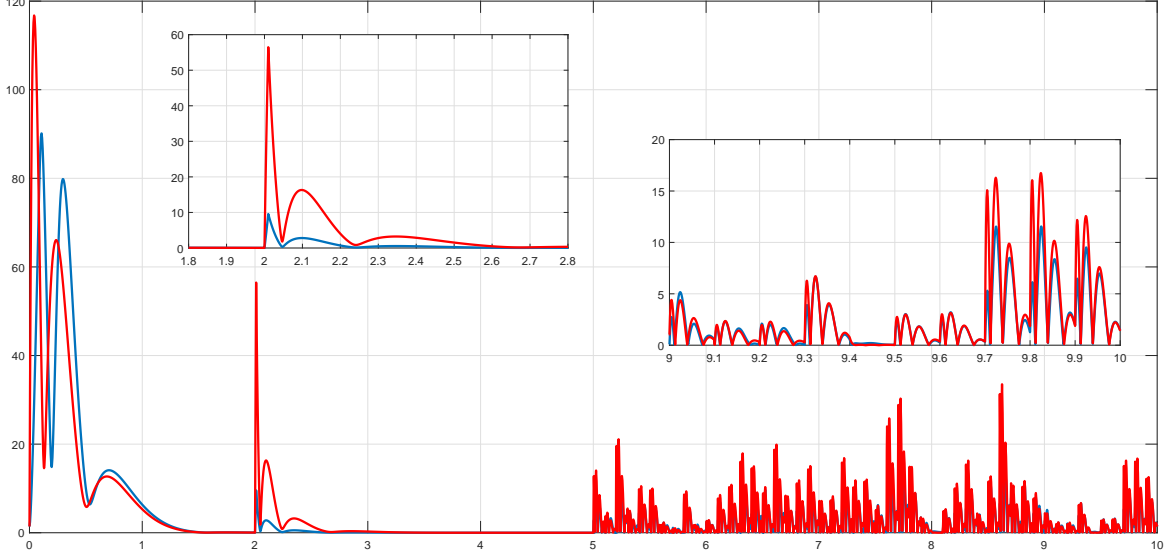


Fig. 1. Behaviour of the norm of the error dynamics  $|\hat{x}(t) - x(t)|$  of the standard high-gain observer (red line) and of the stubborn high-gain observer (blue line)

we get that the derivative of  $V$  along the solutions of (31) is given by

$$\begin{aligned} \dot{V} &\leq -\frac{1}{8}|e|^2 - \rho_0\sigma + \frac{c_0}{\ell^{2n}}\bar{d}_0^2 + 2e^\top PKdq \\ &\quad + \zeta(\lambda + \epsilon)[\nu^2 + 2\nu Ce] \\ &\leq -\frac{1}{8}|e|^2 - \rho_0\sigma + \frac{c_0}{\ell^{2n}}\bar{d}_0^2 + 2b_1|e||\nu| + b_2|\nu|^2 \\ &\leq -\frac{1}{16}|e|^2 - \rho_0\sigma + \frac{c_0}{\ell^{2n}}\bar{d}_0^2 + b_3\bar{\nu}^2 \end{aligned}$$

with

$$\begin{aligned} b_1 &:= |P||K| + \zeta(\lambda + \epsilon), \quad b_2 := \zeta(\lambda + \epsilon), \\ b_3 &:= 16b_1^2 + b_2, \end{aligned}$$

and a choice of  $\rho_0$  like in (26).

*Case 2:  $e^\top C^\top Ce > \sigma$ .* In this case  $q \neq 0$  and  $V$  is given by  $V = V_1 + V_2 + V_3$  as in the proof of Proposition 1. By using the bound (30) and the inequality (25), we get that the derivative of  $V$  along the solutions of (29) is given by

$$\begin{aligned} \dot{V} &\leq -\frac{1}{8}|e|^2 - \rho_0\sigma + \frac{c_0}{\ell^{2n}}\bar{d}_0^2 + 2e^\top PKdq \\ &\quad - (\eta - \zeta)(\lambda + \epsilon)[\nu^2 + 2\nu Ce] + 2\eta e^\top C^\top CKdq \\ &\leq -\frac{1}{8}|e|^2 - \rho_0\sigma + \frac{c_0}{\ell^{2n}}\bar{d}_0^2 + 2b_4|e||\nu| + b_5|\nu|^2 \\ &\leq -\frac{1}{16}|e|^2 - \rho_0\sigma + \frac{c_0}{\ell^{2n}}\bar{d}_0^2 + b_6\bar{\nu}^2 \end{aligned}$$

with

$$\begin{aligned} b_4 &:= |P||K| + (\eta - \zeta)(\lambda + \epsilon) + \eta k_1, \\ b_5 &:= (\eta - \zeta)(\lambda + \epsilon), \\ b_6 &:= 16b_4^2 + b_5. \end{aligned}$$

*Combining the bounds.* Let  $b_0 := \max\{b_3, b_6\}$ . Using the two previous inequalities on  $\dot{V}$ , we obtain that the following inequality

$$\dot{V} \leq -\frac{1}{16}|e|^2 - \rho_0\sigma + \frac{c_0}{\ell^{2n}}\bar{d}_0^2 + b_0\bar{\nu}^2$$

holds for almost all  $(e, \sigma)$ . By following the same arguments used in the proof of Proposition 1, we also get

$$V(\tau) \leq \exp(-\rho_2\tau)V(0) + \frac{c_0}{\rho_1\ell^{2n}}\bar{d}^2 + \frac{b_0}{\rho_1}\bar{\nu}^2$$

for some  $\rho_2 > 0$  independent of  $\ell$ . It is not hard to see that we have also

$$|e(t)| \leq \alpha_1 \exp(-\alpha_2 t) |e(0)| + \frac{\alpha_3}{\ell^n} \bar{d} + \alpha_4 \bar{\nu},$$

for some  $\alpha_i > 0$ ,  $i = 1, 2, 3, 4$  independent of  $\ell$ . Finally, the bounds  $|\hat{x}_i - x_i| \leq \ell^{i-1} |e_i|$  and  $|e| \leq |x|$  can be used to get (27).  $\square$

## 5. SIMULATION RESULTS

In the simulation we consider a forced Duffing oscillator described by

$$\ddot{w} = \alpha w^3 - \beta w + \sin(\omega t + \phi),$$

or alternatively, in the state space representation, by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 - \alpha x_1^3 - \beta x_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\omega x_3 \\ y &= x_1 + \nu. \end{aligned} \tag{32}$$

In the simulations, the parameters have been chosen as

$$\alpha = 0.5, \quad \beta = 1.3, \quad \omega = 3,$$

and the initial conditions  $x(0) = (1, 0, 0, -1)^\top$ . The parameters of the standard high-gain observer (2) have been selected as follows:

$$k_1 = 10, \quad k_2 = 35, \quad k_3 = 50, \quad k_4 = 24, \quad \ell = 4.$$

The same choice is made for the stubborn high-gain observer (6) where we have chosen  $\lambda = 0.1$  and  $\epsilon = 0.05$ . In both observers the initial conditions coincide with the origin.

The measurement noise  $\nu(t)$  is zero until  $t = 2$ . Then at  $t = 2$  an impulsive disturbance of amplitude 1 occurs (thus

simulating an *outlier*). Finally, from time  $t = 5$  we inject a high-frequency measurement noise simulated by filtering some white-noise with a high-pass filter.

Figure 1 shows the evolution of the norm of the estimation error  $|\hat{x}(t) - x(t)|$  for both observers. It is worth noticing the remarkable feature that, while the transients are slightly slowed down by the effect of the dynamics of the saturations, the stubborn observer has better properties in terms of rejection of outliers and in terms of peaking phenomenon. The responses to high-frequency measurement noise are comparable and do not show any significant improvement/deterioration.

## 6. CONCLUSIONS

The stubborn design paradigm recently proposed by Alessandri and Zaccarian (2016) in the context of linear Luenberger observers to handle measurement outliers has been extended to the case of nonlinear high-gain observers. The special structure of the high-gain dynamics allows for a natural selection of the “stubborn” parameters. Moreover, we show that the ISS properties of the estimation error dynamics from disturbances and measurement noise are not deteriorated as compared with the classical high-gain observer case. Simulation results show desirable features of the proposed construction in terms of rejection to outliers and peaking reduction.

Finally, we remark that the proposed technique can be applied without loss of generality also to other classes of nonlinear observers, such as the ones proposed by Gauthier and Kupka (2001) for systems in non-strict feedback form (see pp. 95, *Luenberger style observers*) or the novel class of *low-power high-gain observers* recently introduced by Astolfi and Marconi (2015).

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