Unsupervised sparsity enforcing iterative algorithms for 3D image reconstruction in X-ray Computed Tomography

Mirece DUMITRU, Nicolas GAC, Li WANG, Ali MOHAMMAD-DJAFARI

Laboratoire des Signaux et Systèmes (L2S, CNRS - CentraleSupelec - Université Paris Saclay)

CentraleSupelec, Plateau de Moulon, 91192, Gif-sur-Yvette, France

The 14th International Conference on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine

18-21 June 2017, Xi’an, China

Abstract : Unsupervised iterative reconstruction algorithms based on a Bayesian approach for piecewise constant images are presented. Such images can be expressed via a sparse representation and the reconstruction problem can be addressed using sparsity enforcing priors. We focus on sparsity enforcing priors expressed as Normal variance mixture, considering three mixing distributions: Inverse Gamma distribution, corresponding to Student-t prior, general inverse Gaussian distribution with the real parameter fixed, corresponding to Normal-inverse Gaussian prior and Gamma distribution corresponding to Variance-Gamma prior. We present and discuss the corresponding iterative algorithms considering the Joint Maximum A Posteriori estimation showing simulations results for 3D X-ray Computed Tomography.

Data: Typical discretized linear forward model for image reconstruction:

\[ g = H f + \epsilon, \]  

(1)

Piecewise continuous images can be expressed as a transformation applied on a sparse structure \( f \) accounting for the uncertainties,

\[ f = D z + \xi, \]  

(2)

inversion based on a Bayesian approach, building an hierarchical prior model accounting for the sparse structure of \( z \), using sparsity enforcing priors.

Hierarchical prior model

\[ \begin{align*}
\mathcal{G} \left( \nu | \theta \right) &= \left\{ \begin{array}{ll}
\mathcal{N} \left( 0, \frac{\nu}{\theta} \right) & \text{Student-t prior} \\
\mathcal{N} \left( 0, 1 \right) & \text{Normal Inverse Gaussian}
\end{array} \right. \\
\mathcal{G} \left( \nu, \alpha, \beta | \theta \right) &= \left\{ \begin{array}{ll}
\mathcal{N} \left( 0, \frac{\nu}{\theta} \right) & \text{Normal Gamma Prior} \\
\mathcal{N} \left( 0, 1 \right) & \text{Normal Inverse Gau}
\end{array} \right.
\end{align*} \]

Figure 1 – Comparison between \( \mathcal{N} \) and \( \mathcal{IG} \) distribution.

Student-t prior expressed as a Normal variance mixture, with the mixing distribution an \( \mathcal{IG} \) distribution. The Student-t Prior Model (SiPM), considers a zero-mean \( \mathcal{N} \) distribution for \( z|\nu \), and an \( \mathcal{IG} \) distribution for the variance \( \nu|\alpha, \beta \), with the corresponding shape and scale parameters, \( \alpha \) and \( \beta \):

\[ \text{SiPM : } \begin{align*}
\nu | \alpha, \beta &\sim \mathcal{IG}(\alpha, \beta) \\
\nu | z &\sim \mathcal{N}(z, \nu)
\end{align*} \]

(3)

Figure 2 – Comparison between \( \mathcal{N} \) and \( \mathcal{IG} \) distribution.

The marginal of the joint probability distribution \( p(z|\alpha, \beta) \) is a \( \mathcal{IG} \) distribution with the zero location and asymmetry parameters:

\[ \mathcal{IG}(\alpha|\beta) = \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\frac{\alpha}{2}) \sqrt{\pi \alpha}} \left( 1 + \frac{z^2}{\alpha} \right)^{-\left(\frac{\alpha}{2}\right)}. \]

(4)

Figure 3 – Comparison between the \( \mathcal{N} \) and \( \mathcal{IG} \) distribution.

Normal Inverse Gaussian expressed as a Normal variance mixture with the mixing distribution a \( \mathcal{IG} \) distribution with the real parameter fixed \( p = -1/2 \). The Normal-Inverse Gaussian (NIGPM), consists of zero-mean \( \mathcal{N} \) distributions for \( z|\nu \), and generalized inverse Gaussian distributions for the variances \( \nu|\gamma, \delta, p \), with the corresponding parameters \( \gamma, \delta \) and \( p = -1/2 \):

\[ \text{NIGPM : } \begin{align*}
\nu | \gamma, \delta, p &\sim \mathcal{IG}(\gamma, \delta) \\
\nu | z &\sim \mathcal{N}(z, \nu)
\end{align*} \]

(5)

The marginal of the joint probability distribution \( p(z|\gamma, \delta, p) \) is a \( \mathcal{IG} \) distribution with zero location and asymmetry parameters:

\[ \mathcal{IG}(\gamma|\delta) = \frac{\Gamma(\gamma + \frac{1}{2})}{\Gamma(\frac{\gamma}{2}) \sqrt{\pi \gamma}} \left( 1 + \frac{z^2}{\gamma} \right)^{-\left(\frac{\gamma}{2}\right)} \cos(\gamma \delta). \]

(6)


Simulations results

Figure 4 – Joint MAP iterative algorithm

\[ f = (H^T V_h H + \nu^H \nu)^{-1} (H^T V_h g + \nu^H D) \]

Update variances \( V_h, V, \nu \) for \( \mathcal{N}, \mathcal{IG}, \mathcal{IG} \) prior

Figure 5 – The Shepp Logan phantom (64^3) is used as the original image. 125 projections are simulated uniformly between \( 0^\circ \) and \( 180^\circ \). For the sparse representation of the image the multilevel Haar transform is used. Slice comparison between the original volume (top left) and MAP reconstructed volumes ST (top right), \( \mathcal{N} \) (bottom left), \( \mathcal{IG} \) (bottom right). SNR=3dB

Figure 6 – Comparison of the normalized mean squared error (NMSE) during iterations for the three prior models: ST, \( \mathcal{N} \) and \( \mathcal{IG} \).

\[ \text{JMAP iterative algorithm} \]

Unknowns estimated by minimizing the criterion \( \mathcal{L} \):

\[ \hat{f} = \arg \min_{f} \mathcal{L}(f, z, v_h, v) \]

(9)

Figure 7 – Influence of the prior hyperparameters : NMSE vs. iterations for ST (top), \( \mathcal{N} \) (middle) \( \mathcal{IG} \) (bottom) priors.

Conclusions Heavy tailed distributions expressed as Normal variance mixtures were considered in order to obtain analytical expressions for the unknowns of the model. The prior models and their corresponding JMAP iterative algorithms were developed and compared: the reconstruction accuracy is similar for the three prior models considered, but the rate of convergence is different and the sensibility to the prior hyperparameters is different. This results are encouraging. However we are now investigating methods to compute the posterior mean via Variational Bayesian Approximation (VBA). The structure of the algorithms is the same but the computations need the diagonal elements of the covariance matrices which are too costly for 3D applications.