

Abstract : Unsupervised iterative reconstruction algorithms based on a Bayesian approach for piecewise constant images are presented. Such images can be expressed via a sparse representation and the reconstruction problem can be addressed using sparsity enforcing priors. We focus on sparsity enforcing priors expressed as **Normal variance mixture**, considering three mixing distributions : Inverse Gamma distribution, corresponding to Student-t prior, general inverse Gaussian distribution with the real parameter fixed, corresponding to Normal-inverse Gaussian prior and Gamma distribution corresponding to Variance-Gamma prior. We present and discuss the corresponding iterative algorithms considering the Joint Maximum A Posteriori estimation showing simulations results for 3D X-ray Computed Tomography.

Data Typical discretized linear forward model for image reconstruction :

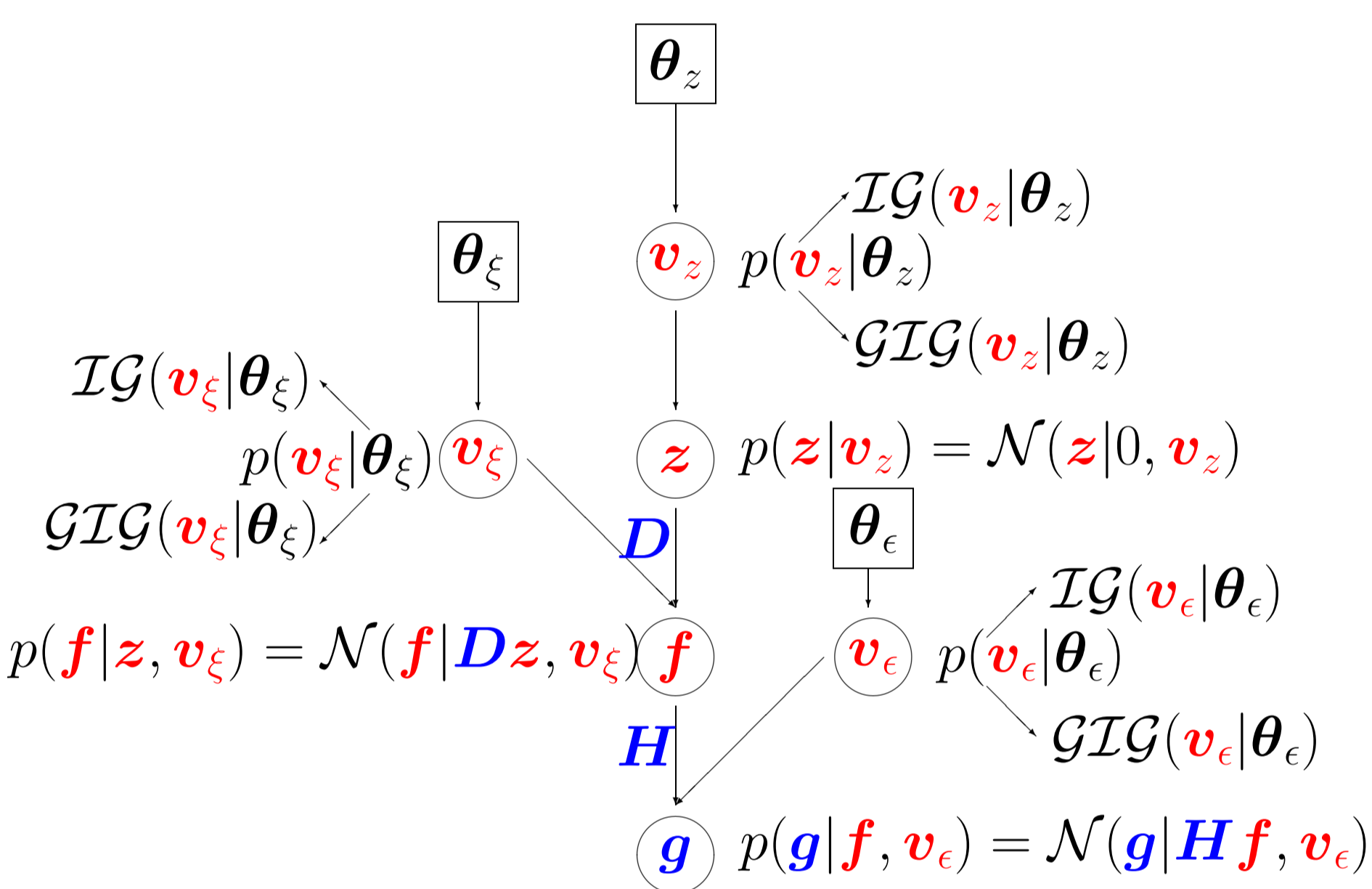
$$g = Hf + \epsilon, \quad (1)$$

Piecewise continuous images f can be expressed as a transformation applied on a sparse structure z accounting for the uncertainties,

$$f = Dz + \xi. \quad (2)$$

inversion based on a Bayesian approach, building an hierarchical prior model accounting for the sparse structure of z , using sparsity enforcing priors.

Hierarchical prior model



Student-t prior expressed as a Normal variance mixture, with the mixing distribution an IG distribution. The Student-t Prior Model (StPM), considers a zero-mean \mathcal{N} distribution for $z_j|v_{z_j}$ and an IG distribution for the variance $v_{z_j}|\alpha_z, \beta_z$, with the corresponding shape and scale parameters, α_z and β_z :

$$\text{StPM} : \begin{cases} p(z_j|0, v_{z_j}) = \mathcal{N}(z_j|0, v_{z_j}) \\ p(v_{z_j}|\alpha_z, \beta_z) = IG(v_{z_j}|\alpha_z, \beta_z) \end{cases} \quad (3)$$

The marginal of the joint probability distribution $p(z_j|\alpha_z, \beta_z)$ is a two parameters Student-t distribution with the probability density :

$$p(z_j|\alpha_z, \beta_z) = \frac{\Gamma(\alpha_z + \frac{1}{2})}{\sqrt{2\pi\beta_z}\Gamma(\alpha_z)} \left(1 + \frac{z_j^2}{2\beta_z}\right)^{-(\alpha_z + \frac{1}{2})} \quad (4)$$

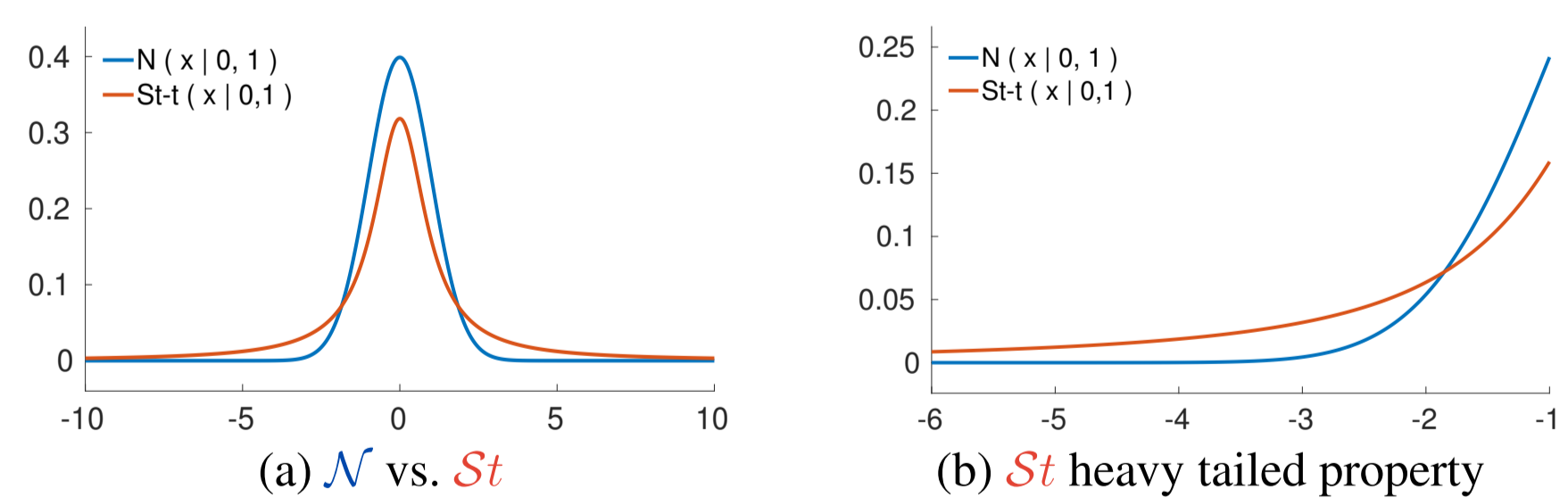


FIGURE 1 – Comparison between \mathcal{N} and St distribution.

Normal Inverse Gaussian expressed as a Normal variance mixture with the mixing distribution a GIG distribution with the real parameter fixed $p = -1/2$. The Normal-Inverse Gaussian Prior Model (NIGPM), considers zero-mean \mathcal{N} distributions for $z_j|v_{z_j}$, and generalized inverse Gaussian distributions for the variances $v_{z_j}|\gamma_z^2, \delta_z^2$, with the corresponding parameters γ_z^2, δ_z^2 and $p_z = -1/2$:

$$\text{NIGPM} : \begin{cases} p(z_j|0, v_{z_j}) = \mathcal{N}(z_j|0, v_{z_j}) \\ p(v_{z_j}|\gamma_z^2, \delta_z^2) = GIG(v_{z_j}|\gamma_z^2, \delta_z^2, p_z = -\frac{1}{2}) \end{cases} \quad (5)$$

The marginal of the joint probability distribution $p(z_j, |\gamma_z, \delta_z)$ is a NIG distribution with zero location and asymmetry β parameters :

$$p(z_j|\gamma_z, \delta_z) = \frac{\gamma_z \delta_z \mathcal{K}_1(\gamma_z \sqrt{\delta_z^2 + z_j^2})}{\pi \sqrt{\delta_z^2 + z_j^2}} \exp\{\gamma_z \delta_z z_j\}, \quad (6)$$

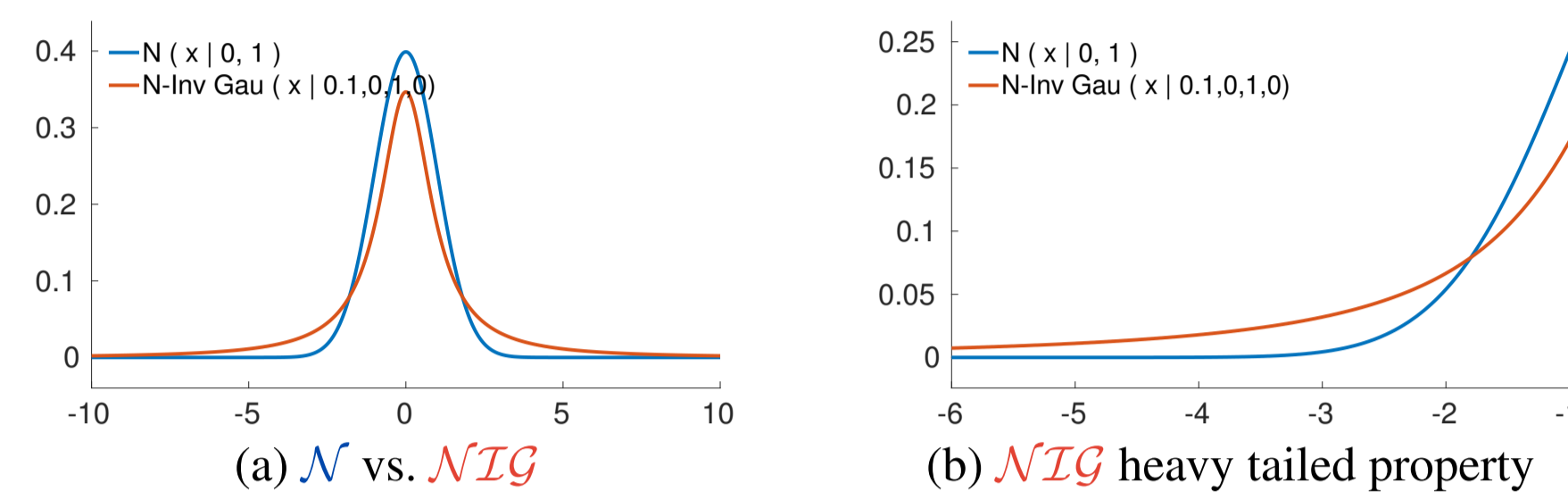


FIGURE 2 – Comparison between \mathcal{N} and NIG distribution.

Variance Gamma prior Gamma distribution considered as the mixing distribution for a Normal variance mixture, leads to the Variance-Gamma distribution as prior. The Variance-Gamma Prior Model (VGPM), Equation (7), considers zero-mean Normal distributions for $z_j|v_{z_j}$, and Gamma distributions with the corresponding shape and scale parameters α_z and β_z for the variances $v_{z_j}|\alpha_z, \beta_z$:

$$\text{VGPM} : \begin{cases} p(z_j|0, v_{z_j}) = \mathcal{N}(z_j|0, v_{z_j}) \\ p(v_{z_j}|\alpha_z, \beta_z) = \mathcal{G}(v_{z_j}|\alpha_z, \beta_z) \end{cases} \quad (7)$$

The marginal of the joint probability distribution $p(z_j|\alpha_z, \beta_z)$ is a \mathcal{VG} distribution with the zero location and asymmetry parameters :

$$\mathcal{VG}(x|\alpha, \beta) = \frac{\beta^{2\alpha} |x|^{\alpha - \frac{1}{2}} \mathcal{K}_{\alpha - \frac{1}{2}}(\alpha |x|)}{\sqrt{\pi} \Gamma(\alpha) (2\beta)^{\alpha - \frac{1}{2}}}. \quad (8)$$

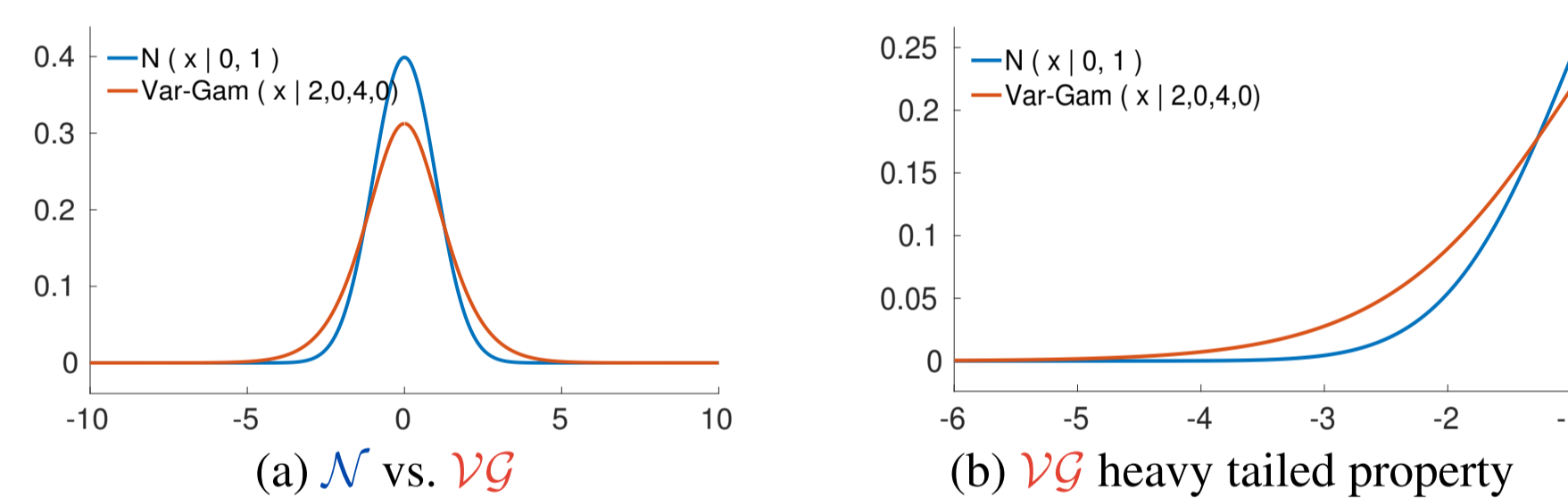


FIGURE 3 – Comparison between the \mathcal{N} and \mathcal{VG} distribution.

JMAP iterative algorithm Unknowns estimated by minimizing the criterion \mathcal{L} :

$$(\hat{f}, \hat{z}, \hat{v}_\xi, \hat{v}_e, \hat{v}_z) = \arg \min_{(f, z, v_\xi, v_e, v_z)} \mathcal{L}(f, z, v_\xi, v_e, v_z), \quad (9)$$

where the criterion \mathcal{L} is defined as :

$$\mathcal{L}(f, z, v_\xi, v_e, v_z) = -\ln p(f, z, v_\xi, v_e, v_z|g) \quad (10)$$

Alternate optimization of the criterion \mathcal{L} with respect to the each unknown :

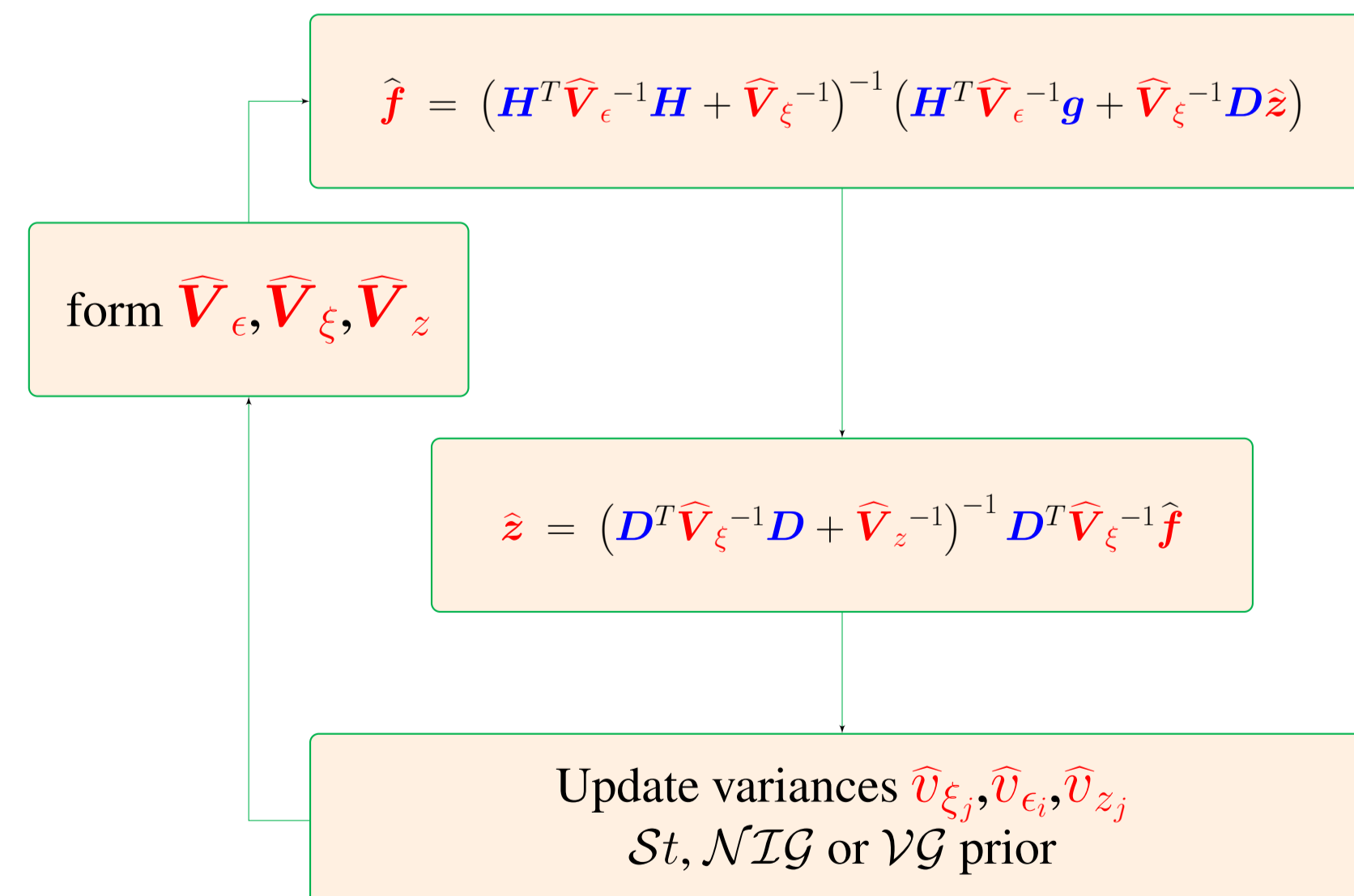


FIGURE 4 – Joint MAP iterative algorithm

Perspectives Mircea Dumitru, Li Wang, Nicolas Gac, Ali Mohammad-Djafari (2017). Performance Comparison of Bayesian Iterative Algorithms for Three Classes of Sparsity Enforcing Priors with Application in Computed Tomography. In 2017 IEEE International Conference on Image Processing in Beijing, China.

Simulations results

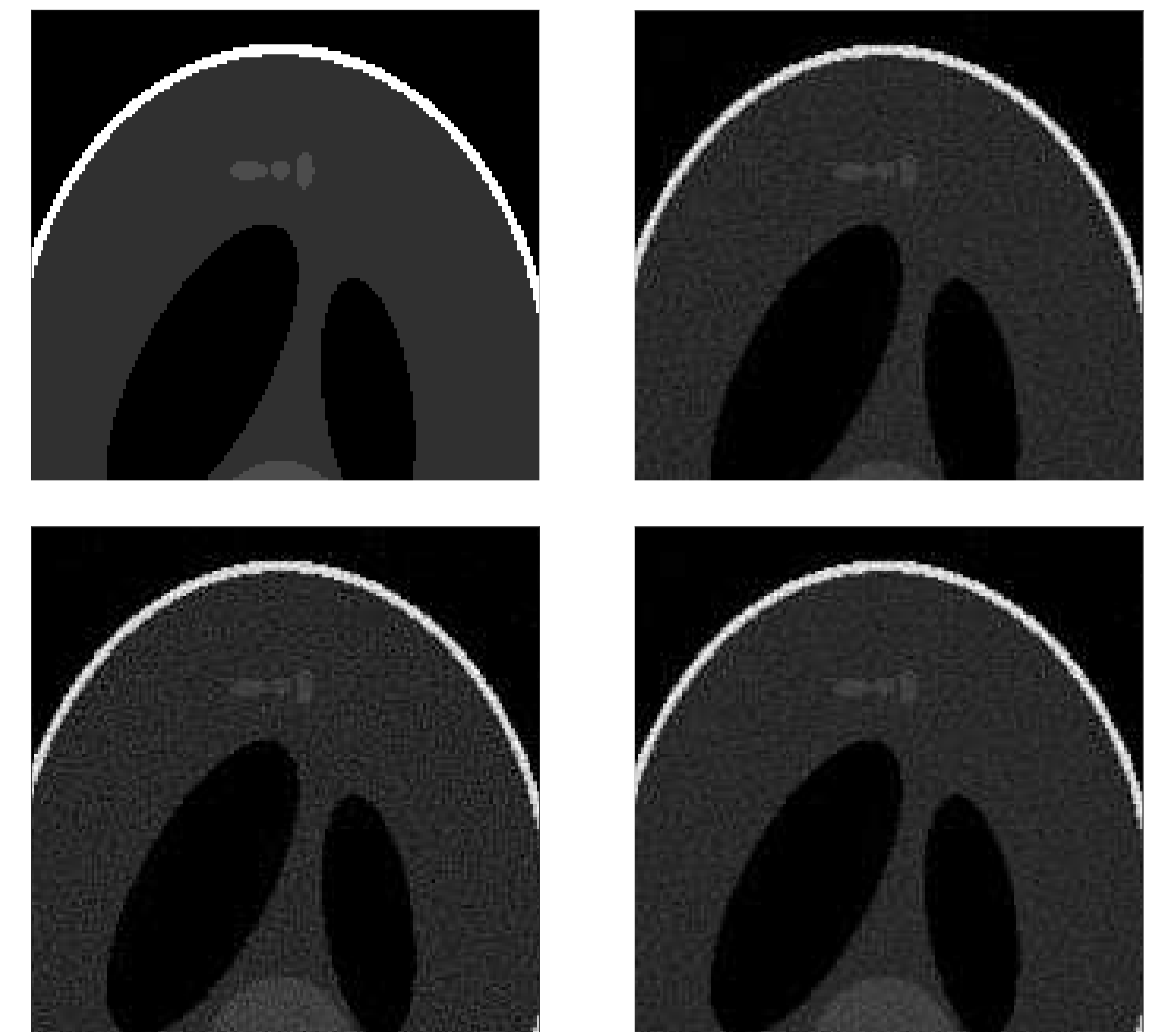


FIGURE 5 – The Shepp Logan phantom (64^3) is used as the original image. 128 projections are simulated uniformly between 0° and 180° . For the sparse representation of the image the multilevel Haar transform is used. Slice comparison between the original volume (top left) and JMAP reconstructed volumes St (top right), NIG (bottom left), \mathcal{VG} (bottom right). SNR=30dB.

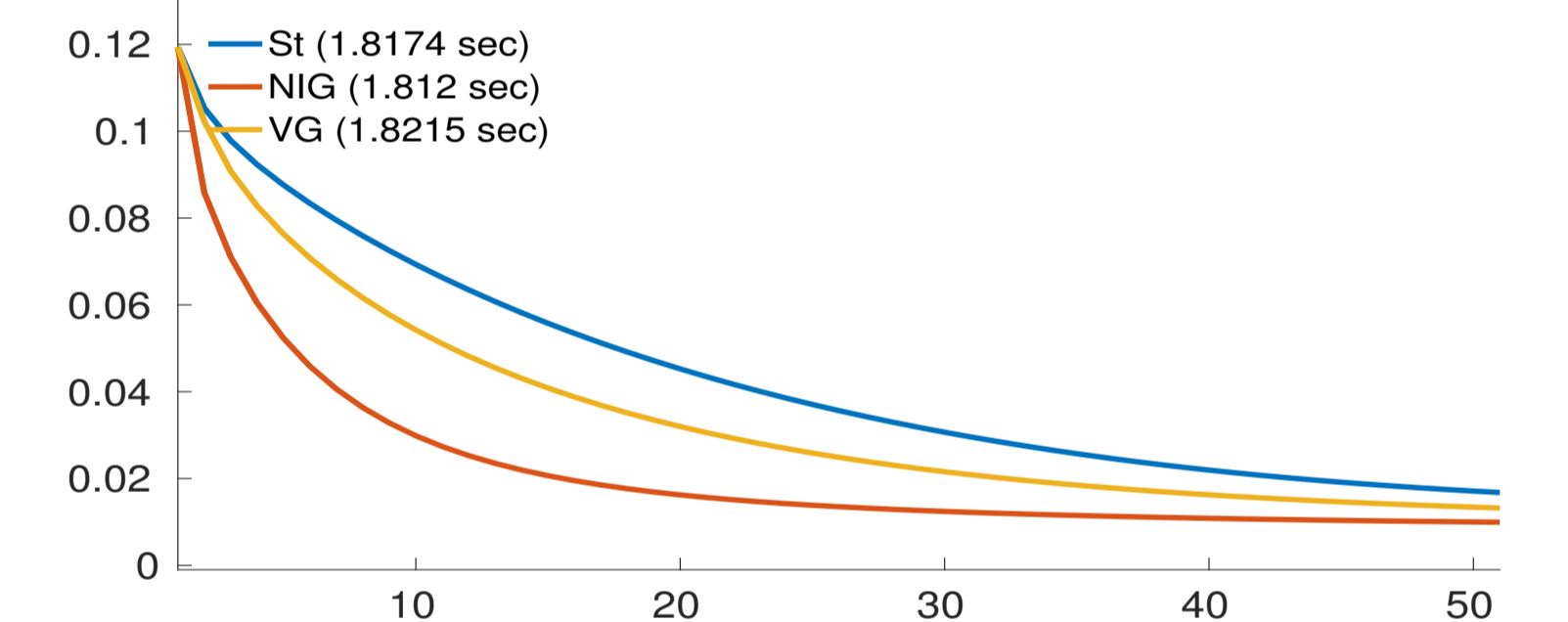


FIGURE 6 – Comparison of the normalized mean squared error (NMSE) during iterations for the three prior models : St , NIG and \mathcal{VG} .

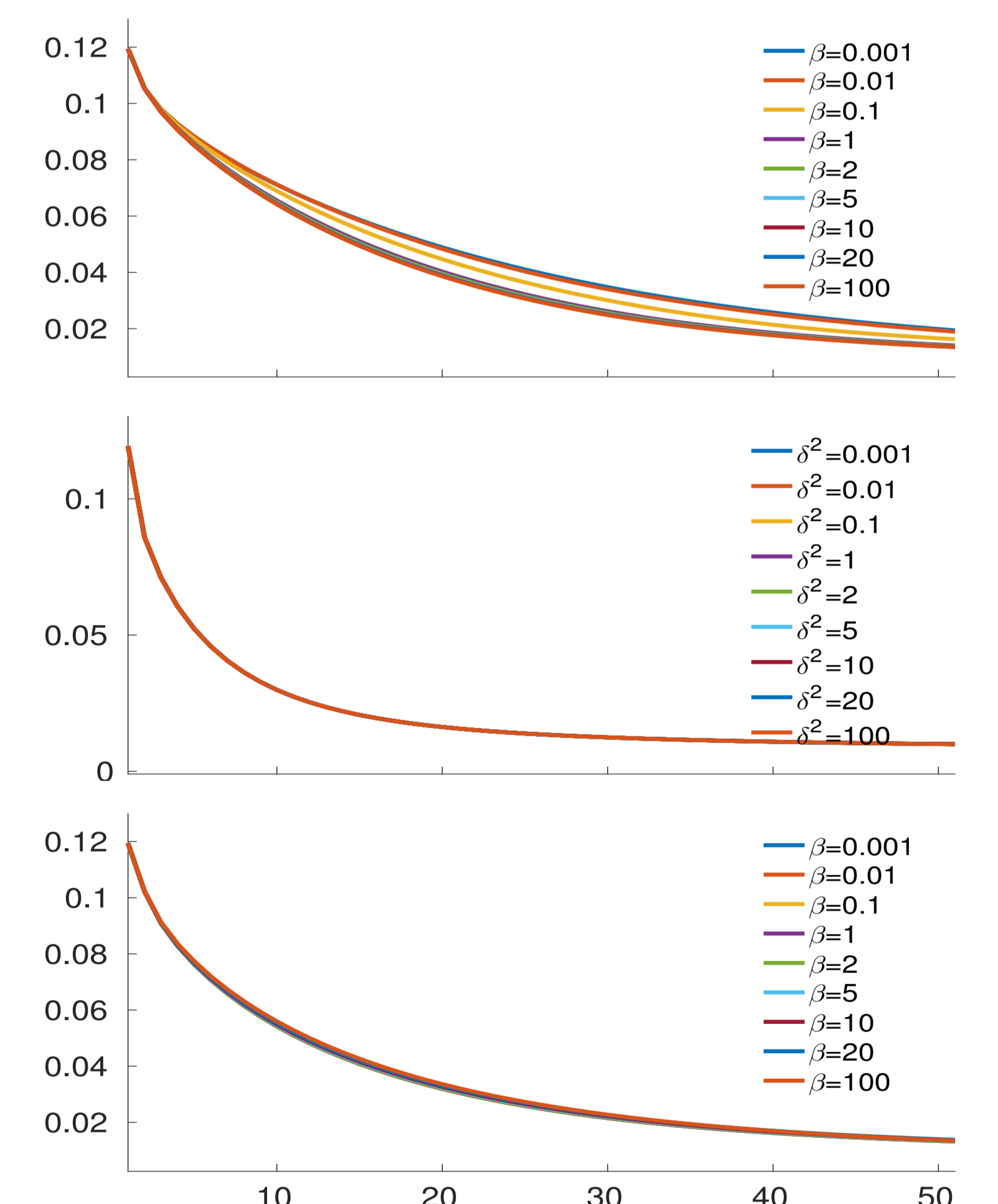


FIGURE 7 – Influence of the prior hyperparameters : NMSE vs. iterations for St (top) NIG (middle) \mathcal{VG} (bottom) priors

Conclusions Heavy tailed distributions expressed as Normal variance mixtures were considered in order to obtain analytical expressions for the unknowns of the model. The prior models and their corresponding JMAP iterative algorithms were developed and compared : the reconstruction accuracy is similar for the three prior models considered, but the rate of convergence is different and the sensibility to the prior hyperparameters is different. This results are encouraging. However we are now investigating methods to compute the Posterior mean via Variational Bayesian Approximation (VBA). The structure of the algorithms is the same but the computations need the diagonal elements of the covariance matrices which are too costly for 3D applications.