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# A Disturbing Combination of Geometrical and Modular Rotations in the World of Arithmetic

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**Abstract :** In previous papers [1] [2], we have examined two types of 'rotations' in associated Stirling numbers of first and second kind at any order  $r$ , respectively :

1. *Geometrical rotations*, which helped to compact all those Stirling numbers in an arithmetical triangle's structure.
2. *Modular rotations* which determined their arithmetical properties using angles depending on  $r$ .

Those different types of rotations could be used independently from each other : each paper was devoted to one of them. In this article, we are going to use both rotations in order to obtain new results directly applicable to associated Stirling numbers as we find them written in the scientific literature, i.e. in "elastic stair steps", not in arithmetical triangles.

**Keywords :** Number Theory, Associated Stirling Numbers, Modular Rotations, Arithmetical Triangle.

**AMS Classification :** 11B73, 05A18.

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# 1 Transfer of modular properties to higher orders

Here, we want to show that some modular properties of Stirling numbers  $s_1(n, k)$  can be generalized to associated Stirling numbers  $s_r(n, k)$  of both kind at any order  $r$  [3], and conversely from the latter to the first. For this, we need the following recurrence relation on  $r$  :

**Relation 1.**  $\forall n, k \in \mathbb{N}$ ,

$$\begin{aligned} s_r(n, k) &= s_{r+1}(n, k) + \binom{n}{r} s_{r+1}(n-r, k-1) + \binom{2r}{r, r} \binom{n}{2r} s_{r+1}(n-2r, k-2) + \\ &\quad \binom{3r}{r, r, r} \binom{n}{3r} s_{r+1}(n-3r, k-3) + \dots + \underbrace{\binom{kr}{r, r, \dots, r}}_k \binom{n}{kr} s_{r+1}(n-kr, 0) \\ &= s_{r+1}(n, k) + \sum_{i=1}^k \underbrace{\binom{ir}{r, \dots, r}}_i \binom{n}{ir} s_{r+1}(n-ir, k-i) \end{aligned}$$

*Proof.* We will reason by enumeration. Let's remember that  $s_r(n, k)$  is the number of partitions of a set of  $n$  elements in  $k$  subsets of size at least equal to  $r$ . We want now to enumerate  $s_r(n, k)$  from the reunion of complementary partitions of a set of  $n$  elements in  $k$  subsets, with successively 0 subset of  $r$  elements (partitions noted  $P_0$ ), only 1 subset of  $r$  elements ( $P_1$ ), only 2 subsets of  $r$  elements ( $P_2$ ), ..., only  $k$  subsets of  $r$  elements ( $P_k$ ).

- As for the set of partitions  $P_0$ , its cardinal is simply  $s_{r+1}(n, k)$  : indeed, it contains no partition of  $r$  elements, thus every subset has at least  $r+1$  elements.

- As for  $P_1$  : we must first choose one subset of  $r$  elements among  $n$ . There is  $\binom{n}{r}$  possibilities. The  $k-1$  other subsets, chosen from a reduced set of  $n-r$  elements, are necessarily each of size at least equal to  $r+1$ . Then,  $Card(P_1) = \binom{n}{r} s_{r+1}(n-r, k-1)$ .

- As for  $P_2$  : those partitions have exactly two subsets of  $r$  elements, so we first have to choose  $2r$  elements from  $n$  : the number of choices is  $\binom{n}{2r}$ . Then, each subset of  $2r$  elements should be partitioned into two sub-subsets of  $r$  elements each : there is  $\binom{2r}{r, r} = \binom{2r}{r}$  choices. For each of those couple of subsets of  $r$  elements, we must partition the other  $n-2r$  elements in  $k-2$  subsets of at least  $r+1$  elements, which makes  $s_{r+1}(n-2r, k-2)$  possibilities. Then,  $Card(P_2) = \binom{2r}{r, r} \binom{n}{2r} s_{r+1}(n-2r, k-2)$ .

- ...

- As for the general case  $P_i$  : we choose  $i$  subsets of  $r$  elements each one from  $n$  elements, which amounts first to choosing  $ir$  elements among  $n$  : it makes  $\binom{n}{ir}$  possibilities. From those  $ir$  elements, we can choose  $\underbrace{\binom{ir}{r, r, \dots, r}}_i$  subsets of  $r$  elements each. The  $k-i$  other subsets, chosen

from a reduced set of  $n - ir$  elements, are each of size at least equal to  $r + 1$ , which is by definition  $s_{r+1}(n - ir, k - i)$ . Then,  $Card(P_i) = \binom{ir}{r, \dots, r} \binom{n}{ir} s_{r+1}(n - ir, k - i)$ .

- ...

- As for  $P_k$  : it consists in exactly  $k$  subsets of  $r$  elements each chosen from  $n$  elements. As  $s_r(n, k)$  enumerates the partitions of  $k$  subsets of at least  $r$  elements, then  $P_k$  is not zero only for  $n = kr$ .

It does not matter if some terms are null : our reasoning applies to all cases  $n, k \in \mathbb{N}$ . Then, the total number of partitions of a set of  $n$  elements in  $k$  subsets of size at least equal to  $r$ , is :

$$s_r(n, k) = \sum_{i=0}^k Card(P_i) = s_{r+1}(n, k) + \binom{n}{r} s_{r+1}(n - r, k - 1) + \binom{2r}{r, r} \binom{n}{2r} s_{r+1}(n - 2r, k - 2) + \binom{3r}{r, r, r} \binom{n}{3r} s_{r+1}(n - 3r, k - 3) + \dots + \underbrace{\binom{kr}{r, r, \dots, r}}_k \binom{n}{kr} s_{r+1}(n - kr, 0)$$

□

**Note :** relation 1 can also be written as follows for suitable  $n, k$  :

**Relation 2.**

$$s_r(n, k) = s_{r+1}(n, k) + \sum_{i=1}^k \prod_{j=0}^{i-1} \binom{n - rj}{r} s_{r+1}(n - ir, k - i)$$

Indeed, we can easily verify that :

$$\binom{ir}{\underbrace{r, \dots, r}_i} \binom{n}{ir} = \prod_{j=0}^{i-1} \binom{n - rj}{r}$$

This formula allows us to inverse a theorem we proved in [2]. Let's remind that  $\pi_r$  is the arithmetical triangle's structure of the associated Stirling numbers of second kind at any order  $r$  [1]. Our method involves using the recurrence relation 1 above to transfer modular properties from any order  $r$  to the well-known case  $r = 1$ .

**Theorem 1.** We set  $r \geq 1$ . Let the arithmetical triangle  $\pi_r$  be, with  $\rho_r^0(n)$  the line in  $\pi_r$  associated with a natural integer  $n$ ,  $\eta_r^0(n)$  the horizontal line of  $\pi_r$  sharing its non-zero first term with  $\rho_r^0(n)$  and  $\delta_r^0(n)$  the line which forms an angle of  $-\frac{\pi}{4}$  with  $\eta_r^0(n)$ . Let finally  $\mathcal{A}_r^0(n)$  be the whole of the elements of  $\pi_r$  ranging in the area between  $\delta_r^0(n)$  and  $\rho_r^0(n)$ , except  $\delta_r^0(n)$ .

There is then for all prime numbers  $n \geq 2$  and for all  $r \geq 1$  :

$$\mathcal{A}_r^0(n) \equiv 0 \quad [n]$$

Inversely, if an integer  $n$  is such that  $n \wedge (r-1)! = 1$  and :

$$\mathcal{A}_r^0(n) \equiv 0 \quad [n]$$

then  $n$  is a prime number.

**Note :** The condition  $n \wedge (r-1)! = 1$  in theorem 1 is equivalent to saying that  $n$  has no prime factor  $p \leq r-1$ .

*Proof.* - sense  $\Rightarrow$  : The proof is in [2] (section 3). Yet, thanks to relation 1 and 2, we can avoid using the formula :

$$s_r(n, k) = \frac{1}{k!} \sum_{\substack{\alpha_1 \geq r \dots \alpha_k \geq r \\ \alpha_1 + \alpha_2 + \dots + \alpha_k = n}} \frac{n!}{\alpha_1! \alpha_2! \dots \alpha_k!}$$

in order to prove the property (\*) :

$$\forall n \text{ prime number}, \forall k, \forall r, s_r(n, k) \equiv 0[n].$$

Indeed, let  $n$  be a prime number. We will reason by recurrence on  $r$ .

1. For  $r = 1$  : (\*) is verified, since  $s_1(n, k)$  are the Stirling numbers of second kind [2].
2. We suppose (\*) for  $r \geq 1$ .
3. As  $n$  is prime, it divides  $\binom{n}{r}$ , and a fortiori  $\forall n, k$ , each term :

$$\prod_{j=0}^{i-1} \binom{n-rj}{r} s_{r+1}(n-ir, k-i)$$

in relation 2. So, by recurrence,  $n$  divides  $s_{r+1}(n, k)$ .

- sense  $\Leftarrow$  : Let's set  $r \geq 2$  and let  $n \geq 2$  be an integer such that  $n \wedge (r-1)!$  and  $\mathcal{A}_r^0(n) \equiv 0 \quad [n]$ . In particular,  $n$  divides  $\rho_r^0(n)$ , the line associated with  $n$  in  $\pi_r$ . By definition of  $\rho_r^0(n)$ ,  $n$  divides all the  $s_r(n, k)$  for  $k > 1$ . Moreover, as  $n$  is prime relatively to  $(r-1)!$ , it divides  $\binom{n}{r-1}$  and a fortiori all the terms

$$\prod_{j=0}^{i-1} \binom{n-(r-1)j}{r-1} s_r(n-i(r-1), k-i)$$

Now, as  $n$  divides all the right member of relation 2 for  $r-1$  :

$$s_{r-1}(n, k) = s_r(n, k) + \sum_{i=1}^k \prod_{j=0}^{i-1} \binom{n-(r-1)j}{r-1} s_r(n-i(r-1), k-i)$$

it divides all  $s_{r-1}(n, k)$  for  $k > 1$ . By recurrence on  $r$ ,  $n$  divides  $s_1(n, k)$ ,  $\forall k > 1$ .

Then, from the same properties used above of Stirling numbers of the second kind,  $n$  is a prime number.  $\square$

**Remark :** the same reasoning may be applied to the associated Stirling numbers of the first kind at any order. We just need to use in their case a similar recurrence relation as 1.

## 2 Extensions to other elastic structures

Now, let us generalize theorem 1 for all the structures  $\Pi_r^h$  as defined in [1] (definition 4) :

*For all positive integer  $r$  and all integer  $h$ , one will call respectively  $\Sigma_r^h$  and  $\Pi_r^h$  the arithmetical structures obtained after  $h$  iterations of  $R$  applied to the arithmetical triangles  $\sigma_r$  and  $\pi_r$ .*

We have :

**Theorem 2.** *One will preserve the notations of theorem 1 by adapting them to the  $\Pi_r^h$ , for any integer  $h$  and positive integer  $r$  fixed. Let  $\rho_r^h(n)$  be the half-line associated to an integer  $n$  in  $\Pi_r^h$  and  $\delta_r^h(n)$  the line starting from the origin of  $\rho_r^h(n)$  and forming an angle  $\theta$  with  $\rho_r^h(n)$  such as :*

$$\tan \theta = \frac{r}{1 + (h-1)(r+h-1)} \quad (1)$$

*We note down then  $\mathcal{A}_r^h(n)$  the set of the elements of  $\Pi_r^h$  included between  $\rho_r^h(n)$  and  $\delta_r^h(n)$  excluded (the latter thus plays the role of an 'asymptote' of the surface  $\mathcal{A}_r^h(n)$ ).*

*In this case, if  $n$  is a prime number, then :*

$$\mathcal{A}_r^h(n) \equiv 0 \quad [n]$$

*Inversely, if  $n$  is such that  $n \wedge (r-1)! = 1$  (or, in other words,  $n$  has no prime factor  $\leq r-1$ ) and, with the same notations :*

$$\mathcal{A}_r^h(n) \equiv 0 \quad [n]$$

*then  $n$  is a prime number.*

*Proof.* According to the results above, the application of the geometrical rotation  $R^h$  to the lines  $\rho_r^0(n)$  and  $\delta_r^0(n)$  adds a quantity  $h$  to their respective slopes  $r-1$  et  $-1$  and transforms them respectively into lines of angles  $\theta'$  et  $\theta''$  such as  $\tan \theta' = h-1+r$  et  $\tan \theta'' = h-1$ .

The difference  $\theta$  between these angles is such as :

$$\tan \theta = \tan(\theta' - \theta'') = \frac{\tan \theta' - \tan \theta''}{1 + \tan \theta' \tan \theta''} = \frac{r}{1 + (h-1)(r+h-1)}$$

By definition, there is thus :

$$\begin{aligned} R^h[\rho_r^0(n)] &= \rho_r^h(n) \\ R^h[\delta_r^0(n)] &= \delta_r^h(n) \end{aligned}$$

Thus, thanks to the classical property of composition of angles in geometrical rotations, the application of  $R^h$  to the surface  $\mathcal{A}_r^0(n)$  delimited by the lines  $\rho_r^0(n)$  and  $\delta_r^0(n)$  returns us another surface delimited by the new lines  $\rho_r^h(n)$  and  $\delta_r^h(n)$  : by definition, this surface is not other than  $\mathcal{A}_r^h(n)$ . Then, Theorem 1 simply applies to the latter.  $\square$

In figure 2 below, one gives an idea of the set  $\mathcal{A}_r^h(n)$  for  $h = 1$  and  $r = 5$  : then,  $\theta = 5$ . Besides, figure 3 shows different positions of the diagonal side of an arithmetical triangle like  $\pi_r$  ( $h = 0$ ) after  $h$  iterations of rotation  $R$ .

We deduce from Theorem 2 the following result :

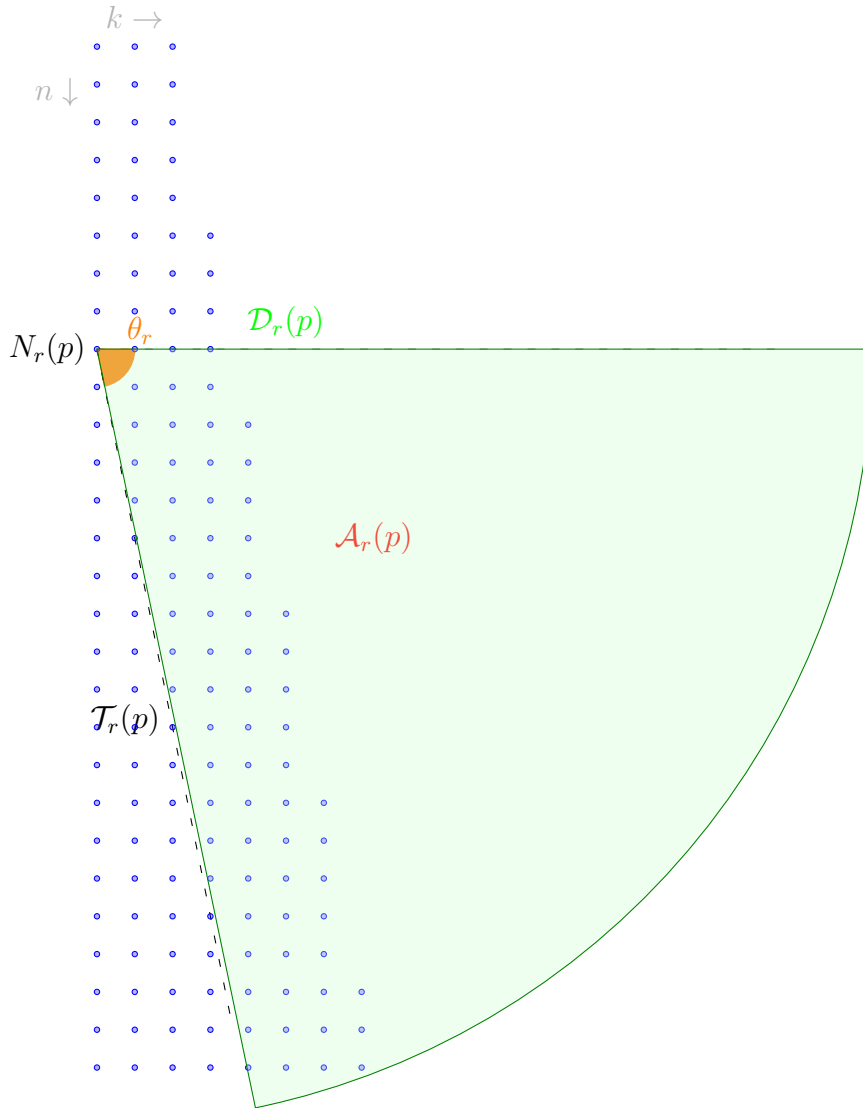
**Corollary 1.** *If  $p$  is prime, then it divides the set  $\mathcal{A}_r(p)$  of the terms of  $\mathcal{S}_r$  located between the  $p$ -nth horizontal non-zero line  $\mathcal{D}_r(p)$  and a half-line  $\mathcal{T}_r(p)$  which forms with it an angle of  $\theta_r = -\arctan(r)$ , except the terms located in the point  $N_r(p)$  and on  $\mathcal{T}_r(p)$  (see figure 1).*

*Inversely, if  $n$  is an integer such that  $n \wedge (r - 1)! = 1$  (equivalently,  $n$  has no prime factor  $\leq r - 1$ ) and :*

$$\mathcal{A}_r(n) \equiv 0 \quad [n]$$

*then  $n$  is a prime number.*

figure 1



*Proof.* One obtains this corollary by setting  $h = 1 - r$  in Theorem 2 : the latter applies then to  $\Pi_r^{1-r}$  which is not other than the structure of the  $S_r(n, k)_{n, k \in \mathbb{N}}$  which is so much widespread in the literature. The line  $\rho_r^{1-r}(p)$  becomes indeed the  $p$ -nth line of  $\Pi_r^{1-r}$  of angle :

$$\arctan(1 - r + r - 1) = 0$$

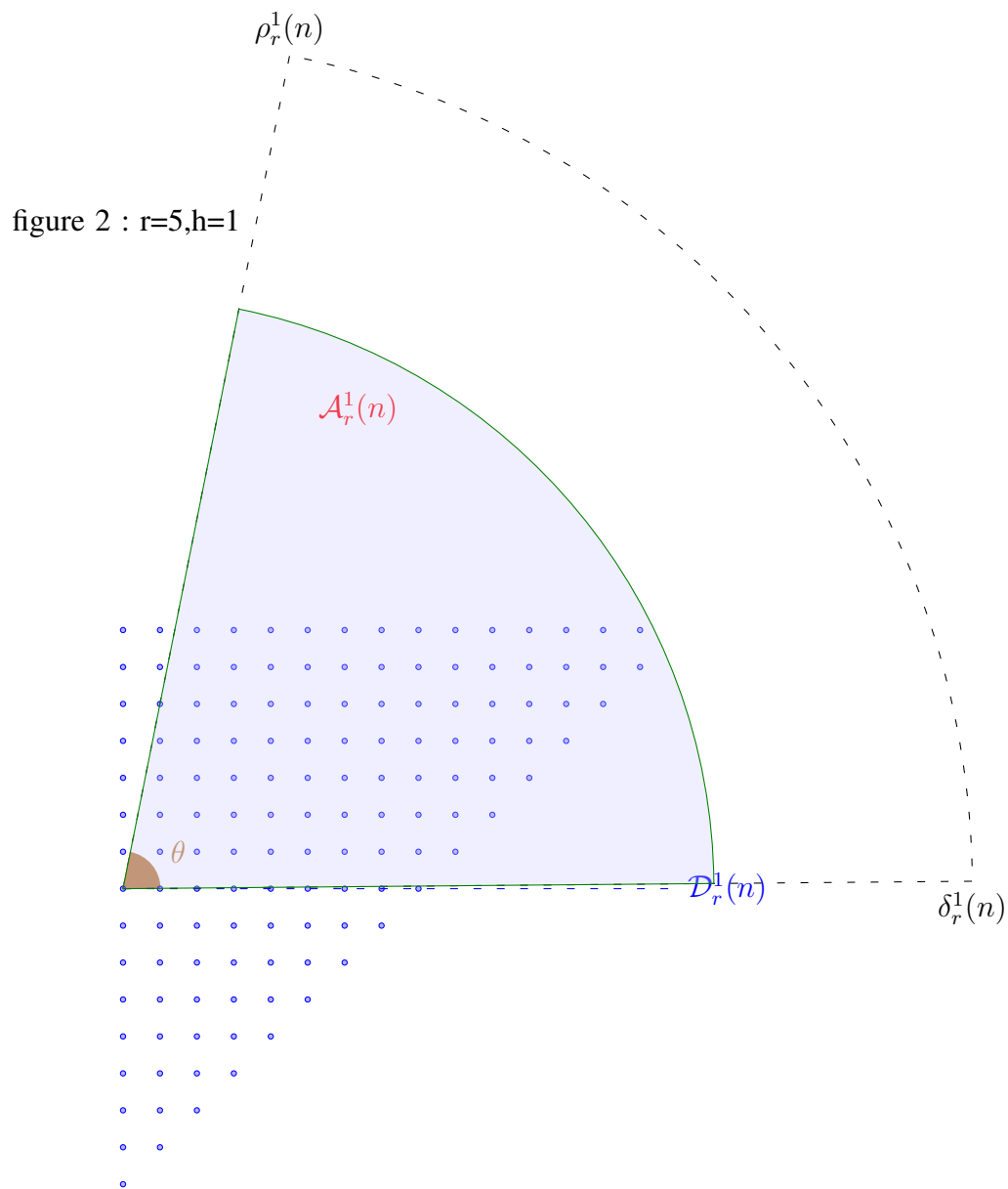
that is to say  $\mathcal{D}_r(p)$ , and  $\delta_r^{1-r}(p)$  becomes the asymptote of angle  $\arctan(1 - r - 1) = \arctan(r)$ .  $\square$

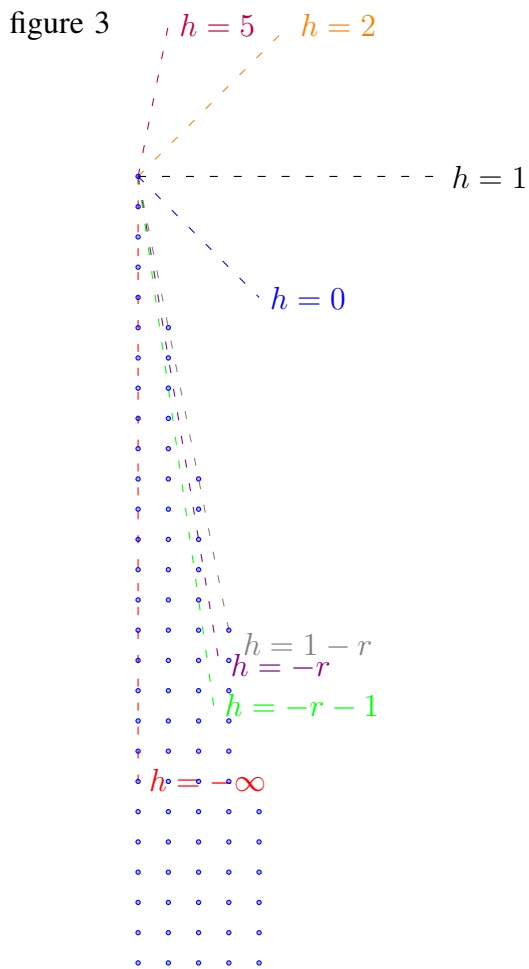
As for us, Corollary 1 is amazing in that it results from the composition of two "rotations" of a priori very different nature, namely a geometrical rotation and a modular one.

Moreover, the terms bound by a relation of recurrence in  $\Pi_r^{1-r}$  not being adjacent, it would have been very difficult to show directly Corollary 1 starting from this latter structure. One can indeed show that  $\Pi_r^{1-r} = S_r(n, k)_{n, k \in \mathbb{N}}$  has a contour in "stairs", such as one "descends" a walk



all the  $r$  lines (cf. figure 1). Then, the "edge effects" would have been terrible to manage, especially the asymptotic terms. Yet, the interesting point is that **the proof of Corollary 1 does not depend at all on the kind of final structure  $\Pi_r^{1-r}$  obtained after  $r - 1$  iterations of  $-R$ .**





### 3 Conclusion

Our future work will be devoted to interesting results and simulations, using Python code, of modular rotations in associated Stirling numbers. New confusing arithmetical phenomena will be explored and conjectures proposed.

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