LMI-based $\mathcal{H}\_\infty$ nonlinear state observer design for anaerobic digestion model

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***Abstract*— This note deals with the design of an $\mathscr{H}_\infty$ nonlinear state observer for the anaerobic digestion model. Positively, the designed observer is an unified one that can be used for different class of systems, mainly linear systems, Linear Parameter Varying (LPV) systems with known and bounded parameters, and nonlinear Lipschitz systems. Applying the Lyapunov theory and the $\mathscr{H}_\infty$ criterion, Linear Matrix Inequality (LMI) condition is synthesised and solved to obtain the designed observer gains. The novelty of our work consists in the relaxation of the synthesized LMI condition through the inclusion of additional decision variables. This was possible due to the use of a suitable reformulation of the Young's inequality. Stability, effectiveness and potency of the theoretical results are confirmed by the simulation results.**

***Index Terms*— Anaerobic digestion, LMIs, Observer design, Nonlinear systems, $\mathscr{H}_\infty$ Criterion.**

I. INTRODUCTION AND PRELIMINARIES

A. Introduction

One class of renewable energy sources with a very high potential for being integrated in virtual power plants (VPPs) are Biogas Plants (BPs), where biogas is generated by anaerobic digestion of bio-degradable materials. Indeed, BPs have the advantage of a high flexibility of power generation since the produced biogas can either be converted to electrical and heat energies or stored. However, the integration of BPs in VPPs requires a continuous supervision and an advanced monitoring of the anaerobic digestion process, which surely require on-line and adequate measurement devices. Nevertheless, the existing monitoring apparatus for the key variables of anaerobic digestion process, such as the different bacteria concentrations are very costly and need a lot of maintenance. Therefore, an alternative solution to reduce the ownership costs is to estimate the missing variables using a suitable mathematical model and the available cost effective measurements, which means the design of software sensors (state observers). Indeed, this has been an active area over the last decades, we can find in the literature different observer schemes for the anaerobic digestion processes, among them we can cite the asymptotic observer [3] which is quite simple and does not require the knowledge of some specific nonlinear functions. However, such an observer is very sensitive to model uncertainties and its convergence rate depends on the operational conditions. Therefore, it has been extended to interval observers [5] which have the advantage of using reliable measurements, which are nonlinear functions of the state vector. The interval observers estimate the interval where the state is lying when the system has large uncertainties. However, generally the rate of convergence is partially tunable and it is not easy to exploit the intervals for control. In order to enhance the convergence rate of the observers the Kalman filter has been designed repeatedly in the literature [20], [9], [8], which shows suitable results in different chemical applications, but unfortunately the convergence of estimation errors to zero is not guaranteed. The high gain observer [10], [14] converges rapidely to the model state variables, however its synthesis is complex and it is very sensitive to noise [15]. The invariant observer [6], [13] also convergences fast to the real state variables with the only use of simple and desirable measurements, however its robustness to measurement disturbances and model uncertainties is still an open question. Hence, to remedy limitations of the mentioned works we propose, in the current note, an $\mathscr{H}_\infty$ nonlinear state observer able to cope with model uncertainties and measurement disturbances. To ensure the estimation error $\mathscr{H}_\infty$ asymptotic stability, we synthesised an LMI condition whose feasibility is relaxed by introducing additional decision variables. This was possible thanks to the use of a suitable reformulation of the Young's inequality as it will be shown in the sequel.

The rest of this note is organized as follows: In Section II, we discuss the used mathematical model. Then, in Section III we give the general structure of the designed $\mathscr{H}_\infty$ observer. Later, in Section IV we provide the synthesised LMI condition and then, we apply the designed observer to the anaerobic digestion model and simulate it in Section V. Finally, we conclude the paper in Section VI.

B. Notation and Preliminaries

1) **Notation::** The following notations will be used throughout this paper:

- $(\star)$ is used for the blocks induced by symmetry;
- A^T represents the transposed matrix of A;
- $\mathbb{I}_r$ represents the identity matrix of dimension r;
- for a square matrix S, $S > 0$ $(S < 0)$ means that this matrix is positive definite (negative definite);
- the set $Co(x,y) = \{\lambda x + (1-\lambda)y, 0 \leq \lambda \leq 1\}$ is the convex hull of $\{x,y\}$;

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- $e_s(i) = \big(\underbrace{0,...,0,\overbrace{1}^{i\text{ th}},0,...,0}_{s\text{ components}}\big)^T \in \mathbb{R}^s, s \geq 1$ is a vector of the canonical basis of $\mathbb{R}^s$.

2) **Preliminaries::** The preliminaries provided herein are very useful in the design of the synthesis conditions to ensure the asymptotic convergence of the state observer that will be designed in the sequel.

Theorem 1 (Mean value theorem [21]): Let $\varphi : \mathbb{R}^n \to \mathbb{R}^q$. Let $x, y \in \mathbb{R}^n$. We assume that φ is differentiable on $Co(x,y)$. Then, there are constant vectors $z_1, ..., z_q \in Co(x,y), z_i \neq x, z_i \neq y$ for $i = 1, ..., q$ such that :

$$\varphi(x) - \varphi(y) = \left(\sum_{i,j=1}^{q,n} e_q(i) e_n^T(j) \frac{\partial \varphi_i}{\partial x_j}(z_i) \right) (x - y). \quad (1)$$

Lemma 1.1 (*a variant of Lipschitz reformulation):* Let $\varphi : \mathbb{R}^n \to \mathbb{R}^q$ a differentiable function on $\mathbb{R}^n$. Then, the following items are equivalent:

- φ is a globally γ_φ-Lipschitz function;
- there exist finite and positive scalar constants a_{ij}, b_{ij} so that for all $x, y \in \mathbb{R}^n$ there exist $z_i \in Co(x,y), z_i \neq x, z_i \neq y$ and functions ψ_{ij}: $\mathbb{R}^n \to \mathbb{R}$ satisfying the following:

$$\varphi(x) - \varphi(y) = \sum_{i,j=1}^{i,j=q,n} \psi_{ij}(z_i) \mathscr{H}_{ij} \Big(x - y \Big) \quad (2)$$

$$a_{ij} \leq \psi_{ij}\Big(z_i\Big) \leq b_{ij}, \quad (3)$$

where

$$\psi_{ij}(z_i) = \frac{\partial \varphi_i}{\partial x_j}(z_i), \ \mathscr{H}_{ij} = e_q(i) e_n^T(j).$$

Notice that this lemma is obvious from the mean value theorem, but it is important to introduce it at this stage, under this formulation, in the aim to simplify the presentation of the proposed observer design method. Indeed, for our technique, we will exploit (2)-(3) instead of a direct use of Lipschitz property.

Lemma 1.2 (*[22]):* Let X and Y be two given matrices of appropriate dimensions. Then, for any symmetric positive definite matrix S of appropriate dimension, the following inequality holds:

$$X^T Y + Y^T X \leq \frac{1}{2} \Big[X + SY \Big]^T S^{-1} \Big[X + SY \Big]. \quad (4)$$

This lemma will be very useful for the main contributions of this paper. It allows providing less restrictive LMI conditions compared to the classical LMI techniques for the considered class of systems.

II. Mathematical Modeling of Anaerobic Digestion Process

Countless models of the anaerobic digestion process already exist. From first order model [1] to much more complex and higher order ones [7], [12], [17]. Our attention was mainly focused on two step models for continuous stirred tank reactors, due to their widespread application and popularity. Thus, among the existing ones [16], [18], [4], ...[19], we have selected the model studied by [11] due to its validation with experimental results and the comparison of its behaviour with the benchmark model ADM1. Moreover, the selected model has many advantages when it comes to control the produced biogas quality. Besides, being interested by the inclusion of BPs in VPPs, we have slightly modified the selected model by adding more freedom degree in the control of both biogas quantity and quality as depicted in Figure (1). The subsequent model is given by the following equations:

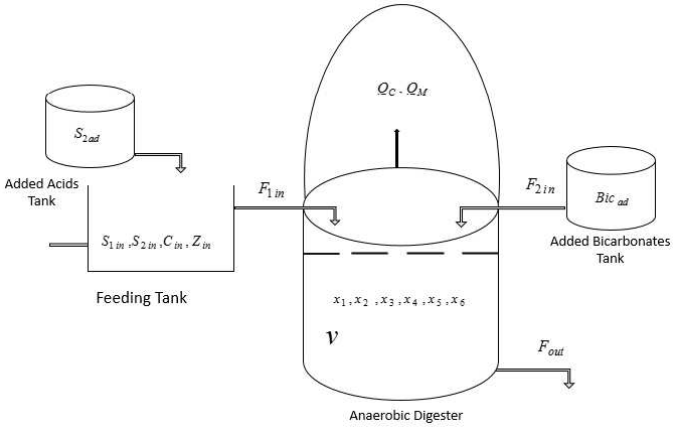

Fig. 1. Controlled anaerobic digestion process.

$$\begin{array}{rcl}
\dot{x}_1 & = & -k_1 \mu_1(x_1) x_2 + u_1 S_{1in} - u_{out} x_1 \\
\dot{x}_2 & = & (\mu_1(x_1) - \alpha) u_{out} x_2 \\
\dot{x}_3 & = & k_2 \mu_1(x_1) x_2 - k_3 \mu_2(x_3) x_4 + u_1 (S_{2in} + S_{2ad}) - \\
 & & u_{out} x_3 \\
\dot{x}_4 & = & (\mu_2(x_3) - \alpha) u_{out} x_4 \\
\dot{x}_5 & = & k_4 \mu_1(x_1) x_2 + k_5 \mu_2(x_3) x_4 + u_1 C_{in} + u_2 Bic_{ad} - \\
 & & u_{out} x_5 - Q_c(x) \\
\dot{x}_6 & = & u_1 Z_{in} + u_2 Bic_{ad} - u_{out} x_6
\end{array} \quad (5)$$

with

$$\begin{array}{l}
\mu_1(x_1) = \overline{\mu_1} \frac{x_1}{x_1 + k_{s_1}} \\
\mu_2(x_3) = \overline{\mu_2} \frac{x_3}{x_3 + k_{s3} + \frac{x_3^2}{k_{i3}}} \\
Q_m(x) = k_6 \mu_2(x_3) x_4 \\
co_2 = x_5 + x_3 - x_6 \\
Q_c(x) = \frac{RT \gamma CO_2}{P_T + RT\gamma(K_H P_T - co_2)} Q_m(x)
\end{array} \quad (6)$$

and

$$\begin{cases}
bic = x_6 - x_3 \\
co_2 = x_5 - bic \\
k_b = \frac{[H^+] bic}{co_2} \\
pH = -\log_{10}\left(k_b \frac{co_2}{bic}\right)
\end{cases} \quad (7)$$

where x_1 (g/l) is the organic substrate concentration to be consumed by the acidogenic bacteria x_2 (g/l) for growth and production of volatile fatty acids x_3 (mM) (which is supposed, in the model, to be pure acetate), x_4 (g/l) is the methanogenic bacteria concentration, x_5 (mM) represents the

inorganic carbon concentration and x_6 (mM) the alkalinity concentration. The control inputs are $u_1 = \frac{F_{1in}}{v}$ $(1/day)$ and $u_2 = \frac{F_{2in}}{v}$ $(1/day)$, where F_{1in} is the waste and the added stimulating acids (S_{2ad}) feeding rate and F_{2in} is the input flow rate of the added stimulating bicarbonates (Bic_{ad}) to the digester. Since the later volume (v) is constant the output flow rate $u_{out} = u_1 + u_2$. The gaseous outputs are $Q_m(x)$ and $Q_c(x)$ which represent the methane and carbon dioxide flow rates, respectively. All the rest of the used parameters in the model are defined in Table I.

III. Problem Statement

In order to make the results general and usable for other nonlinear models, we will present the methodology in a general way for a certain class of nonlinear models.

Motivated by the model of anaerobic digestion (5), we will investigate the general class of models described by the following equations:

$$\begin{cases} \dot{x} = A(\rho)x + B\gamma(x) + g(u,t) + Ew \\ y = Cx + Dw \end{cases} \tag{8}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^q$ is the input vector, $y \in \mathbb{R}^p$ is the linear output measurement, $w \in \mathbb{R}^z$ in the disturbance $\mathscr{L}_2$ bounded vector and $\rho \in \mathbb{R}^s$ is an $\mathscr{L}_\infty$ bounded and known parameter. The affine matrix $A(\rho)$ is expressed under the form

$$A(\rho) = A_0 + \sum_{j=1}^{s} \rho_j A_j$$

with

$$\rho_{j,\min} \leq \rho_j \leq \rho_{j,\max},$$

which means that the parameter ρ belongs to a bounded convex set for which the set of 2^s vertices can be defined by:

$$\mathbb{V}_\rho = \Big\{ \rho \in \mathbb{R}^s : \rho_j \in \{\rho_{j,\min}, \rho_{j,\max}\} \Big\}. \tag{9}$$

The matrices $A_i \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $E \in \mathbb{R}^{n\times z}$ and $D \in \mathbb{R}^{p\times z}$ are constant. The nonlinear function $\gamma : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is assumed to be globally Lipschitz and can always be written under the detailed form:

$$B\gamma(x) = \sum_{i=1}^{m} B_i \gamma_i(\overbrace{H_i x}^{\vartheta_i})$$

where $H_i \in \mathbb{R}^{n_i \times n}$ and B_i refers to the i^{th} column of the matrix B.

To estimate the unmeasurable sate variables of the model (8), we use the following observer scheme:

$$\dot{\hat{x}} = A(\rho)\hat{x} + \sum_{i=1}^{m} B_i \gamma_i(\hat{\vartheta}_i) + g(u,t) + L(\rho)\Big(y - C\hat{x}\Big) \tag{10}$$

with

$$\hat{\vartheta}_i = H_i \hat{x}, \quad L(\rho) = L_0 + \sum_{j=1}^{s} \rho_j L_j \tag{11}$$

where $\hat{x}$ is the estimate of x. The observer gains $L_i \in \mathbb{R}^{n\times p}$ will be calculated so that the estimation error $e = x - \hat{x}$, be $\mathscr{H}_\infty$ asymptotically stable.

Since $\gamma(.)$ is globally Lipschitz, then from Lemma 1.1 there exist $z_i \in Co(\vartheta_i, \hat{\vartheta}_i)$, functions

$$\phi_{ij} \ : \ \mathbb{R}^{n_i} \longrightarrow \ \mathbb{R}$$

and constants a_{ij}, b_{ij}, so that

$$\gamma(x) - \gamma(\hat{x}) = \sum_{i,j=1}^{m,n_i} \phi_{ij}(z_i)\mathscr{H}_{ij}\Big(\vartheta_i - \hat{\vartheta}_i\Big) \tag{12}$$

and

$$a_{ij} \leq \phi_{ij}\Big(z_i\Big) \leq b_{ij}, \tag{13}$$

where

$$\phi_{ij}(z_i) = \frac{\partial \gamma_i}{\partial \vartheta_i^j}(z_i), \ \mathscr{H}_{ij} = B_i e_{n_i}(j).$$

For shortness, we set $\phi_{ij} \triangleq \phi_{ij}\Big(z_i\Big)$. Without loss of generality, we assume that $a_{ij} = 0$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n_i$. For more details about this, we refer the reader to [2].

Since $\vartheta_i - \hat{\vartheta}_i = H_i e$, then we have

$$\gamma(x) - \gamma(\hat{x}) = \left(\sum_{i,j=1}^{m,n_i} \phi_{ij}\mathscr{H}_{ij}H_i \right) e \tag{14}$$

Therefore, the dynamic equation of the estimation error can be obtained as

$$\dot{e} = \left(\mathbb{A}_L(\rho) + \sum_{i,j=1}^{m,n_i} \Big[\phi_{ij}\mathscr{H}_{ij}H_i\Big] \right) e + \mathbb{E}_L(\rho)w \tag{15}$$

with

$$\mathbb{A}_L = A(\rho) - L(\rho)C, \ \mathbb{E}_L = E - L(\rho)D \tag{16}$$

Herein, the objective is to make the estimation error dynamics (15) $\mathscr{H}_\infty$ asymptotically stable and thus, satisfying the following $\mathscr{H}_\infty$ criterion:

$$\|e\|_{\mathscr{L}_2} \leq \sqrt{\mu\|w\|^2_{\mathscr{L}_2} + \nu\|e_0\|^2} \tag{17}$$

where $\sqrt{\mu}$ is the disturbance attenuation level and $\nu > 0$ a parameter to be determined.

The $\mathscr{H}_\infty$ criterion (17) is satisfied if the following holds

$$\mathscr{W} \triangleq \dot{V}(e) + \|e\|^2 - \mu\|w\|^2 \leq 0. \tag{18}$$

where $\dot{V}(e)$ is the time derivative of the classical quadratic Lyapunov function $V(e) = e^T\mathbb{P}e, \quad \mathbb{P} = \mathbb{P}^T > 0$, which is commonly used to analyse the $\mathscr{H}_\infty$ stability of the estimation error. Thus, by calculating $\mathscr{W}$ along the trajectories of (15), we obtain:

$$\begin{aligned} \mathscr{W} = e^T \Bigg[& \mathbb{I}_n + \left(\mathbb{A}_L(\rho) + \sum_{i,j=1}^{m,n_i} \Big[\phi_{ij}\mathscr{H}_{ij}H_i\Big] \right)^T \mathbb{P} \\ & + \mathbb{P}\left(\mathbb{A}_L(\rho) + \sum_{i,j=1}^{m,n_i} \Big[\phi_{ij}\mathscr{H}_{ij}H_i\Big] \right) \Bigg] e \\ & + w^T\mathbb{E}_L^T(\rho)\mathbb{P}e + e^T\mathbb{P}\mathbb{E}_L(\rho)w - \mu w^T w. \end{aligned} \tag{19}$$

TABLE I
MODEL PARAMETERS

Acronyms	Definition	Units	Value
α	Proportion of dilution rate for bacteria	$mmol/l$	0.5
k_1	Yield for substrate (x_1) degradation	$g/(g$ of $x_2)$	$42,1$
k_2	Yield for VFA (x_3) production	$mmol/(g$ of $x_2)$	$116,5$
k_3	Yield for VFA consumption	$mmol/(g$ of $x_4)$	268
k_4	Yield for co_2 production	$mmol/g$	100
k_5	Yield for co_2 production	$mmol/g$	300
k_6	Yield for ch_4 production	$mmol/g$	302
$\overline{\mu}_1$	Maximum acidogenic bacteria (x_2) growth rate	$1/day$	$1,25$
$\overline{\mu}_2$	Maximum methanogenic bacteria (x_4) growth rate	$1/day$	$0,74$
k_{s_1}	Half saturation constant associated with x_1	g/l	$0,41$
k_{s3}	Half saturation constant associated with x_3	$mmol/l$	$8,42$
k_{i3}	Inhibition constant associated with x_3	$mmol/l$	247
k_b	Acidity constant of bicarbonate	mol/l	$6,5.10^{-7}$
K_H	Henry's constant	$mmole/(l.atm)$	27
R	Gas constant	$L.atm/(K.mol)$	$82,1$
P_T	Total preasure	atm	$1,013$
T	Temperature	Kelvin	308
γ	Dimensionless parameter introduced by Hess [11]	–	$0,025$

Hence, $\mathscr{W} < 0$ if the following holds:

$$\overbrace{\begin{bmatrix} \mathbb{A}_L^T(\rho)\mathbb{P}+\mathbb{P}\mathbb{A}_L(\rho)+\mathbb{I}_n & \mathbb{P}\mathbb{E}_L(\rho) \\ \mathbb{E}_L^T(\rho)\mathbb{P} & -\mu\mathbb{I}_z \end{bmatrix}}^{\Psi} + \sum_{i,j=1}^{m,n_i} \phi_{ij} \left(\overbrace{\begin{bmatrix} \mathbb{P}\mathscr{H}_{ij} \\ 0 \end{bmatrix}}^{\mathbb{X}_{ij}^T} \overbrace{\begin{bmatrix} H_i & 0 \end{bmatrix}}^{\mathbb{Y}_i} + \mathbb{Y}_i^T \mathbb{X}_{ij} \right) \leq 0. \tag{20}$$

Many methods can be applied to solve the LMI (20), however the resulted conditions may be conservative. Thus, we provide in the next section a suitable and enhanced LMI condition to cope with the conservatism issue.

IV. LMI SYNTHESIS CONDITION

The main contribution of this note may be summarised in the following theorem, which provides the new and enhanced LMI condition.

Theorem 2: If there exist symmetric positive definite matrices $\mathbb{P} \in \mathbb{R}^{n\times n}$, $\mathbb{S}_{ij} \in \mathbb{R}^{n_i\times n_i}$ for $j = 1,\dots\ n_i$, $i = 1,\dots\ m$, and matrices $\mathscr{R}_j \in \mathbb{R}^{p\times n}$, $j = 0,\dots\ s$ so that the following convex optimization problem is solvable:

$$\min(\mu) \quad \text{subject to (22)} \tag{21}$$

$$\begin{bmatrix} \begin{bmatrix} \mathbb{A}\Big(\mathbb{P},\mathscr{R}_j,\rho\Big) & \mathbb{E}\Big(\mathbb{P},\mathscr{R}_j,\rho\Big) \\ (\star) & -\mu I_z \end{bmatrix} & \overbrace{\begin{bmatrix} \Sigma_1 & \dots & \Sigma_m \end{bmatrix}}^{\Sigma} \\ (\star) & -\Lambda\mathbb{M} \end{bmatrix} \leq 0 \tag{22}$$

with

$$\mathbb{A}\Big(\mathbb{P},\mathscr{R}_j,\rho\Big) = A_0^T\mathbb{P}+\mathbb{P}A_0 - C^T\mathscr{R}_0 - \mathscr{R}_0^T C + \mathbb{I}_n + \sum_{j=1}^{s} \rho_j \Big(A_j^T\mathbb{P}+\mathbb{P}A_j - C^T\mathscr{R}_j - \mathscr{R}_j^T C \Big) \tag{23}$$

$$\mathbb{E}\Big(\mathbb{P},\mathscr{R}_j,\rho\Big) = \mathbb{P}E - \mathscr{R}_0^T D - \sum_{j=1}^{s} \rho_j \mathscr{R}_j^T D \tag{24}$$

and

$$\Sigma_i = \Big[\mathscr{N}_i^1\Big(\mathbb{P},\mathbb{S}_{i1}\Big) \dots\ \mathscr{N}_i^{n_i}\Big(\mathbb{P},\mathbb{S}_{in_i}\Big) \Big] \tag{25}$$

$$\mathscr{N}_i^j\Big(\mathbb{P},\mathbb{S}_{ij}\Big) = \begin{bmatrix} \mathbb{P}G\mathscr{H}_{ij} \\ 0 \end{bmatrix} + \begin{bmatrix} H_i^T \\ 0 \end{bmatrix} \mathbb{S}_{ij} \tag{26}$$

$$\Lambda = \text{block-diag}\Big(\Lambda_1,...,\Lambda_m\Big) \tag{27}$$

$$\Lambda_i = \text{block-diag}\left(\Lambda_i^1,\dots,\Lambda_i^{n_i}\right) \tag{28}$$

$$\Lambda_i^j = \frac{2}{b_{ij}}\mathbb{I}_{n_i} \tag{29}$$

$$\mathbb{M} = \text{block-diag}\Big(\mathbb{M}_1,\dots,\mathbb{M}_m\Big) \tag{30}$$

$$\mathbb{M}_i = \text{block-diag}\Big(\mathbb{S}_{i1},\dots,\mathbb{S}_{in_i}\Big) \tag{31}$$

then, the $\mathscr{H}_\infty$ criterion (17) is satisfied with $\nu = \lambda_{\max}(\mathbb{P})$. Hence, the observer gain L is computed as

$$L_j = \mathbb{P}^{-1}\mathscr{R}_j^T.$$

Proof: From Lemma 1.2, we deduce that for all symmetric positive definite matrices $\mathbb{S}_{ij}$ and scalars, we have

$$\mathbb{X}_{ij}^T\mathbb{Y}_i + \mathbb{Y}_i^T\mathbb{X}_{ij} \leq \frac{1}{2}\Big[\mathbb{X}_{ij}+\mathbb{S}_{ij}\mathbb{Y}_i\Big]^T\mathbb{S}_{ij}^{-1}\Big[\mathbb{X}_{ij}+\mathbb{S}_{ij}\mathbb{Y}_i\Big] \tag{32}$$

Consequently, from (13) and the fact that without loss of generality $a_{ij} = 0$, inequality (20) is satisfied if

$$\Psi + \sum_{i,j=1}^{m,n_i} \left(\mathscr{N}_i^j \left(\mathbb{P}, \mathbb{S}_{ij} \right) \left(\Lambda_i^j \mathbb{M}_i^j \right)^{-1} \times \mathscr{N}_i^j \left(\mathbb{P}, \mathbb{S}_{ij} \right)^T \right) \leq 0. \quad (33)$$

Therefore, from Schur lemma, inequality (33) is equivalent to

$$\begin{bmatrix} \Psi & \begin{bmatrix} \Sigma_1 & \dots & \Sigma_m \end{bmatrix} \\ (\star) & -\Lambda\mathbb{M} \end{bmatrix} \leq 0 \quad (34)$$

Finally, with the change of variable $\mathscr{R}_j = L_j^T \mathbb{P}$, the inequality (34) becomes identical to (22). Hence, the $\mathscr{H}_\infty$ criterion (17) is satisfied with the minimum μ obtained by (21). This ends the proof. ■

V. Simulation Results

It is obvious to verify the assumptions on the LPV parameter ρ and the global Lipschitz property of $\gamma(.)$ corresponding to (5). Hence, the anaerobic digestion model (5) is easily written under the form (8) with the following parameters:

$$\rho = u_{out}, \; A(\rho) = \rho \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; \gamma(x) = \begin{bmatrix} \mu_1(x_1)x_2 \\ \mu_2(x_3)x_4 \end{bmatrix}$$

$$g(u,t) = \begin{bmatrix} u_1 S_{1in} \\ 0 \\ u_1(S_{2in} + S_{2ad}) \\ 0 \\ u_1 C_{in} + u_2 Bic_{ad} - Q_c(x) \\ u_1 Z_{in} + u_2 Bic_{ad} \end{bmatrix}, \quad B = \begin{bmatrix} -k_1 & 0 \\ 1 & 0 \\ k_2 & -k_3 \\ 0 & 1 \\ k_4 & k_5 \\ 0 & 0 \end{bmatrix}$$

Let's assume that the the linear output $y = [x_1, x_3, x_6]^T$, and thus the matrix $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.
For simulation, we assume that the model is affected by a sinusoidal disturbance whose amplitude and frequency varies in a finite time interval as depicted in Figure 2. The matrices E and D corresponding to the model (8) are as the following:

$$E = \begin{bmatrix} 0.1 & 0.2 & 1 & 0.1 & 1 & 0.5 \end{bmatrix}^T, \quad D = \begin{bmatrix} 0.1 & 0.5 & 1 \end{bmatrix}^T.$$

Concerning the observer design we have, $m = 2$, $s = 1$, $n_i = 2$, $\gamma_1(x) = \mu_1(x_1)x_2$, $\gamma_2(x) = \mu_2(x_3)x_4$,

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -k_1 & 1 & k_2 & 0 & k_4 & 0 \end{bmatrix}^T, \; B_2 = \begin{bmatrix} 0 & 0 & -k_3 & 1 & k_5 & 0 \end{bmatrix}^T$$

Simulation have been run for 100 days, with a varying control ($\rho_{min} = 0$, $\rho_{max} = 0.62$), $Bic_{ad} = 20$, $S_{2ad} = 11$, $S_{1in} = 12.19$, $S_{2in} = 6.7$, $C_{in} = 58.08$, $Z_{in} = 31$, $u_2 = 0.2$ and the parameter values given in Table I. The system and the observer were initialized by $x(0) = [2, 0.15, 4, 0.6, 65, 62]^T$ and $\hat{x}(0) = [2, 0.2, 4, 1, 55, 62]^T$, respectively. Using the LMI toolbox of MATLAB to solve the problem (21), the observer parameters have been found to be

$$L_0 = \begin{bmatrix} 61.3104 & -54.7744 & 21.3289 \\ -1.6623 & 1.3492 & -0.3639 \\ -93.7694 & 435.7117 & -207.7567 \\ -0.5194 & -1.8127 & 1.0305 \\ -235.3312 & -185.3113 & 116.9083 \\ -0.1957 & 0.4014 & 0.1800 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 38.0125 & -33.9604 & 13.2229 \\ -1.0306 & 0.8365 & -0.2256 \\ -58.1368 & 270.1411 & -128.8089 \\ -0.3220 & -1.1239 & 0.6389 \\ -145.9061 & -114.8913 & 72.4886 \\ -0.1213 & 0.2489 & 0.1116 \end{bmatrix}$$

Estimation of the missing state variables with the designed nonlinear observer has been compared with another estimation using the same structure of the observer but without including the $\mathscr{H}_\infty$ criterion (18), the results are depicted in Figures 3, 4 and 5. Looking at the later figures, we see clearly that the designed observer is robust against the injected disturbance and converges relatively fast to the real state variables despite the large initial estimation error. Moreover, we notice a blatant difference in the estimation robustness when including and excluding the H_∞ criterion. This can only confirm the effectiveness of proposed methodology.

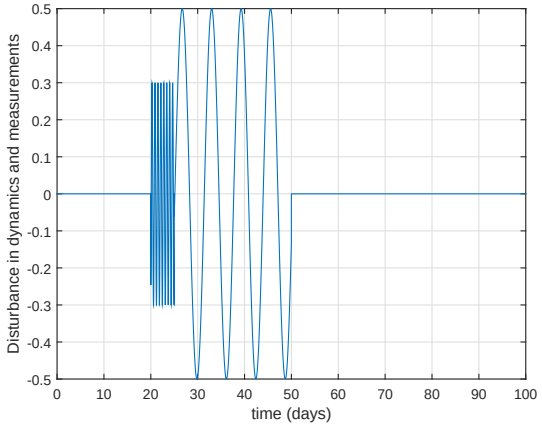

Fig. 2. Disturbance in dynamics and measurements (w).

VI. Conclusion

In this note, we have designed an $\mathscr{H}_\infty$ nonlinear state observer for the anaerobic digestion model. The $\mathscr{H}_\infty$ asymptotic stability of the estimation error was analysed using

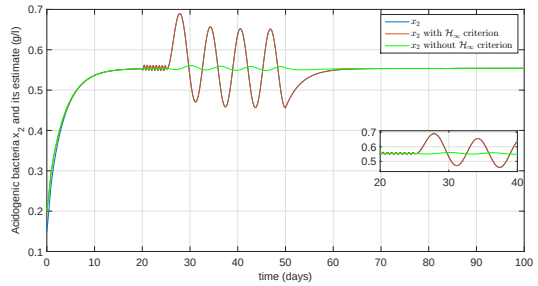

Fig. 3. Acidogenic bacteria x_2 and its estimate $\hat{x}_2(g/l)$.

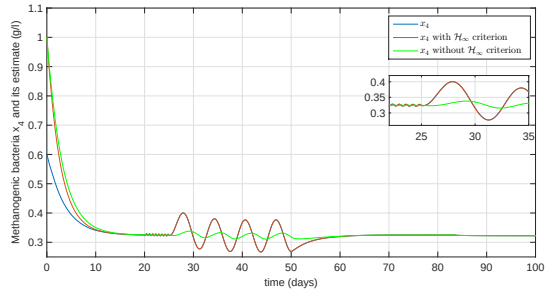

Fig. 4. Mathenogenic bacteria x_4 and its estimate $\hat{x}_4(g/l)$.

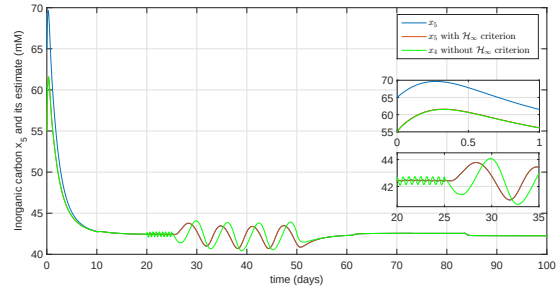

Fig. 5. Inorganic carbon x_5 and its estimate $\hat{x}_5(mM)$.

LMI techniques. We have used a suitable reformulation of the Young's inequality to enhance the feasibility of the synthesised LMI condition. Besides, the theoretical results have been provided in a general way in order to facilitate the use of the designed observer for other applications. For simulation, we have compared the proposed observer with another observer which has the same structure but which is not including the $\mathscr{H}_\infty$ criterion. The results being promising, we target in the near future to extend the methodology for nonlinear outputs, and later to synthesise an optimal control of the biogas quality and quantity based on the designed observer.

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