$\mathcal{H}\_\infty$ observer-based stabilization of switched discrete-time linear systems

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$\mathcal{H}_\infty$ Observer-Based Stabilization of Switched Discrete-Time Linear Systems

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***Abstract*— This paper deals with observer-based $\mathcal{H}_\infty$ controller design method via LMIs for a class of switched discrete-time linear systems with l_2-bounded disturbances. The main contribution of this note consists in a new and judicious use of the slack variables coming from Finsler's lemma. We clarify analytically how the proposed slack variables allows to eliminate some bilinear matrix coupling. The validity and effectiveness of the proposed design methodology are shown through a numerical example.**

***Index Terms*— Observer-based control; Linear matrix inequalities (LMIs); Switching Lyapunov Function (SLF); Finsler's lemma.**

I. Introduction

Many physical processes exhibit switched and hybrid behavior [1], [2], [3] and switching frequently occurs in many engineering applications such as motor engine control [4], networked control systems [5], etc. Stability of switching systems is widely investigated in the literature and becomes more and more a subject of constant evolution. An overview of some basic problems has been emphasized in [1]. Considerable and particular attention has been paid to the state estimation of linear switched systems [6], nonlinear switched systems [7] and Markov jump systems [8]. Theoretical explorations on stabilization and intelligent control for both switched linear systems and switched nonlinear systems have been addressed in the monograph [9].

On the other hand, most of the switched systems considered in the literature consist of linear subsystems or first-order nonlinear subsystems, and various types of complicated dynamics such as stochastic noises, unknown uncertainties are not taken into account. However, many industrial systems or physical systems cannot be described by simple switched system models, and thus the traditional control synthesis methods are not applicable for such systems. In this context, we aim to study a class of switching linear discrete-time systems affected by unknown disturbances. More precisely, we are interesting to $\mathcal{H}_\infty$ observer-based control design problem in the synchronous switching case, using LMI approach.

Control techniques by switching among different controllers have been applied extensively in recent years ([10], [11], [12], [2]). However, in this case, a fundamental prerequisite for the design of feedback control systems is full knowledge of the state that may be impossible or costly. This drawback is the main motivation to investigate the problem of estimating the state of switching systems by different observer structures [13], [11], [14], [15].

On the other hand, it is always required to design a control system which is not only stable, but also guarantees an adequate level of performance. This is way control systems design that can handle model uncertainties has been one of the most challenging problems, and has received considerable attention from control engineers and scientists [16], [17], [18]. Indeed, such a problem remains far from being solved especially when switched systems are concerned. Among the works dealing with the output feedback control for a class of switching discrete-time linear systems with parameters uncertainties, we can quote [19], [20], and [21], which constitute the main motivation of the proposed work.

The problem is first considered in [20], but without disturbances, using Finsler's lemma combined with switching Lyapunov function [10]. Unfortunately, an error has occurred when applying the Finsler lemma. A corrected version of the application is given in [21]. Our objective is to extend the study in [21], by taking into account the presence of disturbances in the state equations and in the output measurements, by introducing a more general structure of the slack variable coming from Finsler's lemma. The obtained result can be applied to robust observer-based $\mathcal{H}_\infty$ control design problem for polytopic uncertain linear time-varying systems. Indeed, asymptotic stability problem for switched linear systems with arbitrary switching is equivalent to the robust asymptotic stability problem for polytopic uncertain linear time-varying systems, for which several strong stability conditions are available in the literature [22]. In the goal to simplify the presentation of the new ideas in the paper and to focus on the new Finsler's inequality use, we will consider in this paper systems without parameter uncertainties in the presence of norm-bounded disturbances.

The rest of the paper is organized as follows. Section II is devoted to the problem statement. The main contribution is presented and proved in Section III. A numerical example is added in Section IV to demonstrate the validity and the effectiveness of the proposed methodology. Finally, we end the paper by a conclusion.

II. Formulation of the Problem

Let us consider the class of switching discrete-time linear systems described by:

$$x_{t+1} = A_{\sigma_t} x_t + B_{\sigma_t} u_t + E_{\sigma_t} w_t \tag{1a}$$

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$$y_t = C_{\sigma_t} x_t + S_{\sigma_t} w_t \tag{1b}$$

$$z_t = H_{\sigma_t} x_t + D_{\sigma_t} u_t + J_{\sigma_t} w_t \tag{1c}$$

where $t \in \mathbb{N}$, $x_t \in \mathbb{R}^n$ is the state vector, $y_t \in \mathbb{R}^p$ is the output measurement, and $u_t \in \mathbb{R}^m$ is the control signal, $w_t \in \mathbb{R}^v$ is an unknown exogenous disturbance, $z_t \in \mathbb{R}^q$ is the controlled output, and $\sigma : \mathbb{N} \to \Lambda = \{1, 2, \ldots, N\}$, $t \mapsto \sigma_t$, is a switching rule. If there is no ambiguity about σ_t and σ, we may juste write σ instead σ_t. A_σ, B_σ, E_σ, C_σ, S_σ, H_σ, D_σ, and J_σ, $\sigma \in \Lambda$, are $n\times n$, $n \times m$, $n \times v$, $p\times n$, $p \times v$, $q \times n$, $q \times m$ and $q \times v$ real matrices, respectively. The pairs (A_σ, B_σ) and (A_σ, C_σ) are assumed to be stabilizable and detectable, respectively. Throughout the paper, the coming assumptions are to build (see e.g. [20], [21]):

Assumption 1: The switching function, σ, is unknown a priori but its instantaneous values are available in real time.

Assumption 2: The switching of the observer for systems should coincide exactly with the switching of the system.

Assuming an arbitrary switching can be very useful in many practical applications such as the case when σ_t is computed via complex algorithms by a higher level supervisor or when it is generated by a human operator (for example the switch of gears in a car).

The observer-based controller we consider in this paper is under the form [23]:

$$\hat{x}_{t+1} = A_\sigma \hat{x}_t + B_\sigma u_t + L_\sigma \Big(y_t - C_\sigma \hat{x}_t \Big) \tag{2a}$$

$$u_t = K_\sigma \hat{x}_t \tag{2b}$$

where $\hat{x}_t \in \mathbb{R}^n$ is the estimate of x_t, and for each $\sigma \in \Lambda$, $L_\sigma \in \mathbb{R}^{n\times p}$ is the observer gain and $K_\sigma \in \mathbb{R}^{m\times n}$ is the control gain. Hence, we can write

$$\bar{x}_{t+1} = \underbrace{\begin{bmatrix} \Omega_{11}(\sigma) & \Omega_{12}(\sigma) \\ \Omega_{21}(\sigma) & \Omega_{22}(\sigma) \end{bmatrix}}_{\Omega_\sigma} \bar{x}_t + \underbrace{\begin{bmatrix} L_\sigma S_\sigma \\ L_\sigma S_\sigma - E_\sigma \end{bmatrix}}_{\Pi_\sigma} w_t \tag{3}$$

$$:= \Omega_\sigma \bar{x}_t + \Pi_\sigma w_t \tag{4}$$

where $\bar{x}_t = [\hat{x}_t^T \quad e_t^T]^T$, $e_t = \hat{x}_t - x_t$ represents the estimation error, and

$$\Omega_{11}(\sigma) = A_\sigma + B_\sigma K_\sigma \tag{5a}$$

$$\Omega_{12}(\sigma) = -L_\sigma C_\sigma \tag{5b}$$

$$\Omega_{21}(\sigma) = 0 \tag{5c}$$

$$\Omega_{22}(\sigma) = A_\sigma - L_\sigma C_\sigma. \tag{5d}$$

The aim is to design the gains K_σ and L_σ, $\sigma \in \Lambda$, such that the closed-loop system (3) is asymptotically stable, and meets performance requirement, under an arbitrary switching rule $\sigma \in \Lambda$. Our objective is to extend the study in [21], by taking into account the presence of disturbances in state equations, and by introducing a more general structure of the slack variable coming from Finsler's lemma. Let us denote $\xi(t) = [\xi_1(t), \xi_2(t), ..., \xi_N(t)]^T$, the indicator function which satisfies for each $i \in \Lambda$,

$$\xi_i(t) = \begin{cases} 1, & \sigma_t = i; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, system (3) with (1c) can be reformulated as:

$$\begin{bmatrix} \bar{x}_{t+1} \\ z_t \end{bmatrix} = \sum_{i=1}^{N} \xi_i(t) \begin{bmatrix} \Omega_i & \Pi_i \\ H_i + D_i K_i & -H_i \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ w_t \end{bmatrix}, \tag{6}$$

where the elements of Ω_i are defined by (5), when $\sigma = i$. For the closed-loop system (6), we consider the switching Lyapunov function defined as

$$V(\bar{x}_t, \xi(t)) = \bar{x}_t^T \hat{P}(\xi(t)) \bar{x}_t = \sum_{i=1}^{N} \xi_i(t) \bar{x}_t^T \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} \bar{x}_t. \tag{7}$$

Notice that the Lyapunov function (7) is well known in the literature, see e.g. [15], [24], [21], [23]. If we consider the switching Lyapunov function (7), we have, by assuming $\sigma_t = i$ and $\sigma_{t+1} = j$:

$$\begin{aligned} \Delta V_{ij}(t) :=& V(\overline{x}_{t+1}, \xi(t+1)) - V(\overline{x}_t, \xi(t)) \\ =& [\overline{x}_t^T \quad \overline{x}_{t+1}^T] \begin{bmatrix} -\hat{P}_i & 0 \\ 0 & \hat{P}_j \end{bmatrix} [\overline{x}_t^T \quad \overline{x}_{t+1}^T]^T \end{aligned} \tag{8}$$

for all $i, j \in \Lambda$, and hence the $\mathcal{H}_\infty$ performance criterion is achieved if the following requirement

$$W_{ij}(t) := \Delta V_{ij}(t) + z_t^T z_t - \mu^2 w_t^T w_t < 0, \tag{9}$$

holds for all $i, j \in \Lambda$ and $t \in \mathbb{N}$. Note that the criterion (9) can be deduced from [25] applied to switching systems case, see also [26]. Now, in order to linearize (9), we use Finsler's Lemma that we recall here for the sake of completness:

Lemma 1 (Finsler's Lemma): Let $x \in \mathbb{R}^n, P \in \mathbb{S}^{n\times n}$, and $H \in \mathbb{R}^{m\times n}$ such that rank $(H) = r < n$. The following statements are equivalent :

1) $x^T P x < 0$, $\forall Ux = 0, x \neq 0$,
2) $\exists X \in \mathbb{R}^{n\times m}$ such that $P + XU + U^T X^T < 0$.

Thus with the following parametrers,

$$\zeta_t = \begin{bmatrix} \overline{x}_t \\ \overline{x}_{t+1} \\ w_t \end{bmatrix}, \; P_{ij} = \begin{bmatrix} -\hat{P}_i & 0 & 0 & \Upsilon_i \\ (\star) & \hat{P}_j & 0 & 0 \\ (\star) & (\star) & -\mu^2 I & J_i^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix},$$

$$U_i = \begin{bmatrix} \Omega_i & -I & \Pi_i \end{bmatrix}, \; X_{i,j} = \begin{bmatrix} F_{i,j} \\ G_{i,j} \\ T_{i,j} \end{bmatrix},$$

where $\Upsilon_i := \begin{bmatrix} K_i^T D_i^T + H_i^T \\ -H_i^T \end{bmatrix}$, and $\hat{P}_i, \hat{P}_j \in \mathbb{R}^{2n\times 2n}$, $i, j \in \Lambda$, are symmetric positive definite matrices, it is then easy to find that $W_{ij}(t) < 0$ is equivalent to what we call Finsler's inequality:

$$P_{ij} + X_{i,j} U_i + U_i^T X_{i,j}^T < 0, \; \forall i, j \in \Lambda. \tag{10}$$

We replaced the choice of X_{ij}, U_i and P_{ij} in inequality (10), one obtains, after developing, the following detailed version of (10):

$$\begin{bmatrix} \Im_{ij} & -F_{ij} + \Omega_i^T G_{ij}^T & F_{ij}\Pi_i + \Omega_i^T T_{ij}^T & \Upsilon_i \\ (\star) & \hat{P}_j - \mathrm{He}(G_{ij}) & G_{ij}\Pi_i - T_{ij}^T & 0 \\ (\star) & (\star) & -\mu^2 I + \mathrm{He}(T_{ij}\Pi_i) & J_i^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} < 0, \tag{11}$$

for all $i, j \in \Lambda$, where $\Im_{ij} = \mathrm{He}(F_{ij}\Omega_i) - \hat{P}_i$, and $\mathrm{He}(Y) = Y + Y^T$, for any matrix Y.

III. Main contribution: New LMI design

Let us put

$$F_{ij} = \begin{bmatrix} F_{ij}^{11} & F_{ij}^{12} \\ F_{ij}^{21} & F_{ij}^{22} \end{bmatrix}, \; G_{ij} = \begin{bmatrix} G_{ij}^{11} & G_{ij}^{12} \\ G_{ij}^{21} & G_{ij}^{22} \end{bmatrix},$$
$$T_{ij} = \begin{bmatrix} T_{ij}^{1} & T_{ij}^{2} \end{bmatrix}, \; \hat{P}_i = \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix}. \tag{12}$$

Our problem consists in linearizing inequality (11) by choosing judiciously the matrices (12). By replacing (12) in (11), and after developing, we get the following detailed version:

$$\begin{bmatrix} \Psi_{ij} & \begin{bmatrix} \Upsilon_i \\ 0 \\ 0 \\ J_i^T \end{bmatrix} \\ (\star) & -I \end{bmatrix} < 0, \tag{13}$$

for all $i, j \in \Lambda$, where

$$\Psi_{ij} = \begin{bmatrix} \Omega_{11}^{ij} & \Omega_{12}^{ij} & \Omega_{13}^{ij} & \Omega_{14}^{ij} & \Omega_{15}^{ij} \\ (\star) & \Omega_{22}^{ij} & \Omega_{23}^{ij} & \Omega_{24}^{ij} & \Omega_{25}^{ij} \\ (\star) & (\star) & \Omega_{33}^{ij} & \hat{P}_j^{12} - G_{ij}^{12} - (G_{ij}^{21})^T & \Omega_{35}^{ij} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - \mathrm{He}\Big(G_{ij}^{22}\Big) & \Omega_{45}^{ij} \\ (\star) & (\star) & (\star) & (\star) & \Omega_{55}^{ij} \end{bmatrix},$$

$$\begin{aligned}
\Omega_{11}^{ij} =& -\hat{P}_i^{11} + \mathrm{He}\Big(F_{ij}^{11}A_i + F_{ij}^{11}B_iK_i\Big)\\
\Omega_{12}^{ij} =& -\hat{P}_i^{12} + F_{ij}^{12}A_i - (F_{ij}^{11} + F_{ij}^{12})L_iC_i\\
&+ K_i^TB_i^T(F_{ij}^{21})^T + A_i^T(F_{ij}^{21})^T\\
\Omega_{13}^{ij} =& -F_{ij}^{11} + A_i^T(G_{ij}^{11})^T + K_i^TB_i^T(G_{ij}^{11})^T,\\
\Omega_{14}^{ij} =& -F_{ij}^{12} + A_i^T(G_{ij}^{21})^T + K_i^TB_i^T(G_{ij}^{21})^T\\
\Omega_{15}^{ij} =& A_i^T(T_{ij}^1)^T + K_i^TB_i^T(T_{ij}^1)^T + (F_{ij}^{11} + F_{ij}^{12})L_iS_i\\
&- F_{ij}^{12}E_i,\\
\Omega_{22}^{ij} =& -\hat{P}_i^{22} + \mathrm{He}\Big(F_{ij}^{22}A_i - (F_{ij}^{22} + F_{ij}^{21})L_iC_i\Big),\\
\Omega_{23}^{ij} =& -F_{ij}^{21} - C_i^TL_i^T(G_{ij}^{11} + G_{ij}^{12})^T + A_i^T(G_{ij}^{12})^T\\
\Omega_{24}^{ij} =& -F_{ij}^{22} + A_i^T(G_{ij}^{22})^T - C_i^TL_i^T(G_{ij}^{22} + G_{ij}^{21})^T,\\
\Omega_{25}^{ij} =& -C_i^TL_i^T(T_{ij}^1 + T_{ij}^2)^T + A_i^T(T_{ij}^2)^T\\
&+ (F_{ij}^{21} + F_{ij}^{22})L_iS_i - F_{ij}^{22}E_i,\\
\Omega_{33}^{ij} =& \hat{P}_j^{11} - (G_{ij}^{11})^T - G_{ij}^{11},\\
\Omega_{35}^{ij} =& (G_{ij}^{11} + G_{ij}^{12})L_iS_i - G_{ij}^{12}E_i - (T_{ij}^1)^T,\\
\Omega_{45}^{ij} =& (G_{ij}^{21} + G_{ij}^{22})L_iS_i - G_{ij}^{22}E_i - (T_{ij}^2)^T,\\
\Omega_{55}^{ij} =& -\mu^2 I + \mathrm{He}\Big((T_{ij}^1 + T_{ij}^2)L_iS_i - T_{ij}^2E_i\Big).
\end{aligned}$$

In what follows, we will discuss a manner of choosing the matrix F_{ij}, G_{ij}, T_{ij} and $\hat{P}_i$ in order to linearize Finsler's inequality (11) (or equivalently (13)). This problem is very complex, since the gain matrices L_i, $i \in \Lambda$, are attached to ten different matrices G_{ij}^{11}, G_{ij}^{12}, G_{ij}^{22}, G_{ij}^{21}, F_{ij}^{11}, F_{ij}^{12}, F_{ij}^{22}, F_{ij}^{21}, T_{ij}^1 and T_{ij}^2. We begin by dealing with the gain matrices K_i. In order to linearize the bilinear terms attached to K_i, we use the congruence principle. For this purpose, let us assume that F_{ij}^{11} is invertible, and independent of j, that is $F_{ij}^{11} = F_i^{11}$.

A. Linearization of (13) with respect to the gain K_i

In view of (13), the matrices G_{ij}^{11}, and G_{ij}^{22}, $i, j \in \Lambda$ are necessarily invertible. Applying the congruence principle to (13) with $\mathrm{diag}\Big((F_i^{11})^{-1}, I, (G_{ij}^{11})^{-1}, I\Big)$, and using the following changes of variables

$$(G_{ij}^{11})^{-1} = \tilde{G}_{ij}^{11}, \quad (F_i^{11})^{-1} = \tilde{F}_i^{11}, \; \tilde{K}_i = K_i(\tilde{F}_i^{11})^T$$

one obtains the following inequality:

$$\begin{bmatrix} \tilde{\Omega}_{11}^{ij} & \tilde{\Omega}_{12}^{ij} & \tilde{\Omega}_{13}^{ij} & \tilde{\Omega}_{14}^{ij} & \tilde{\Omega}_{15}^{ij} & \tilde{K}_i^TD_i^T + \tilde{F}_i^{11}H_i^T \\ (\star) & \Omega_{22}^{ij} & \tilde{\Omega}_{23}^{ij} & \Omega_{24}^{ij} & \Omega_{25}^{ij} & -H_i^T \\ (\star) & (\star) & \tilde{\Omega}_{33}^{ij} & \tilde{\Omega}_{34}^{ij} & \tilde{\Omega}_{35}^{ij} & 0 \\ (\star) & (\star) & (\star) & \Omega_{44}^{ij} & \Omega_{45}^{ij} & 0 \\ (\star) & (\star) & (\star) & (\star) & \Omega_{55}^{ij} & J_i^T \\ (\star) & (\star) & (\star) & (\star) & (\star) & -I \end{bmatrix} < 0, \tag{14}$$

where

$$\begin{aligned}
\tilde{\Omega}_{11}^{ij} =& -\tilde{F}_i^{11}\hat{P}_i^{11}(\tilde{F}_i^{11})^T + \mathrm{He}\Big(A_i(\tilde{F}_i^{11})^T + B_i\tilde{K}_i\Big)\\
\tilde{\Omega}_{12}^{ij} =& \tilde{F}_i^{11}F_{ij}^{12}A_i - (I + \tilde{F}_i^{11}F_{ij}^{12})L_iC_i\\
&+ \tilde{K}_i^TB_i^T(F_{ij}^{21})^T + \tilde{F}_i^{11}A_i^T(F_{ij}^{21})^T - \tilde{F}_i^{11}\hat{P}_i^{12}\\
\tilde{\Omega}_{13}^{ij} =& -(\tilde{G}_{ij}^{11})^T + \tilde{F}_i^{11}A_i{}^T + \tilde{K}_i^TB_i^T\\
\tilde{\Omega}_{14}^{ij} =& -\tilde{F}_i^{11}F_{ij}^{12} + \tilde{F}_i^{11}A_i^T(G_{ij}^{21})^T + \tilde{K}_i^TB_i^T(G_{ij}^{21})^T\\
\tilde{\Omega}_{15}^{ij} =& \tilde{F}_i^{11}A_i^T(T_{ij}^1)^T + \tilde{K}_i^TB_i^T(T_{ij}^1)^T\\
&+ (I + \tilde{F}_i^{11}F_{ij}^{12})L_iS_i - \tilde{F}_i^{11}F_{ij}^{12}E_i,\\
\tilde{\Omega}_{23}^{ij} =& -F_{ij}^{21}(\tilde{G}_{ij}^{11})^T - C_i^TL_i^T(I + \tilde{G}_{ij}^{11}G_{ij}^{12})^T\\
&+ A_i^T(\tilde{G}_{ij}^{11}G_{ij}^{12})^T,\\
\tilde{\Omega}_{33}^{ij} =& \tilde{G}_{ij}^{11}\hat{P}_j^{11}(\tilde{G}_{ij}^{11})^T - \tilde{G}_{ij}^{11} - (\tilde{G}_{ij}^{11})^T,\\
\tilde{\Omega}_{34}^{ij} =& \tilde{G}_{ij}^{11}\hat{P}_j^{12} - \tilde{G}_{ij}^{11}G_{ij}^{12} - \tilde{G}_{ij}^{11}(G_{ij}^{21})^T,\\
\tilde{\Omega}_{35}^{ij} =& (I + \tilde{G}_{ij}^{11}G_{ij}^{12})L_iS_i - \tilde{G}_{ij}^{11}G_{ij}^{12}E_i - \tilde{G}_{ij}^{11}(T_{ij}^1)^T,\\
\Omega_{44}^{ij} =& \hat{P}_j^{22} - G_{ij}^{22} - (G_{ij}^{22})^T.
\end{aligned}$$

Inequality (14) is still a BMI with respect to $\tilde{K}_i$, since it's coupled with F_{ij}^{21}, G_{ij}^{21}, T_{ij}^1. So, we focus on the particular case where $G_{ij}^{21} = F_{ij}^{21} = 0$ and $T_{ij}^1 = T_{ij}^2 = 0$. Hence, applying a result of de Oliveira et al. [27] or [10], and using the notation $\tilde{P}_i^{11} = (\hat{P}_i^{11})^{-1}$, we obtain the equivalent form of (14):

$$\begin{bmatrix} \hat{\Omega}_{11}^{ij} & \hat{\Omega}_{12}^{ij} & \hat{\Omega}_{13}^{ij} & \hat{\Omega}_{14}^{ij} & \hat{\Omega}_{15}^{ij} & \tilde{K}_i^TD_i^T + \tilde{F}_i^{11}H_i^T \\ (\star) & \hat{\Omega}_{22}^{ij} & \hat{\Omega}_{23}^{ij} & \hat{\Omega}_{24}^{ij} & \hat{\Omega}_{25}^{ij} & -H_i^T \\ (\star) & (\star) & \tilde{\Omega}_{33}^{ij} & \hat{\Omega}_{34}^{ij} & \hat{\Omega}_{35}^{ij} & 0 \\ (\star) & (\star) & (\star) & \Omega_{44}^{ij} & \hat{\Omega}_{45}^{ij} & 0 \\ (\star) & (\star) & (\star) & (\star) & -\mu^2 I & J_i^T \\ (\star) & (\star) & (\star) & (\star) & (\star) & -I \end{bmatrix} < 0, \tag{15}$$

where

$$\begin{aligned}
\hat{\Omega}_{11}^{ij} =& \tilde{P}_i^{11} + \mathrm{He}\Big(-\tilde{F}_i^{11} + A_i(\tilde{F}_i^{11})^T + B_i\tilde{K}_i\Big),\\
\hat{\Omega}_{12}^{ij} =& \tilde{F}_i^{11}F_{ij}^{12}A_i - (I + \tilde{F}_i^{11}F_{ij}^{12})L_iC_i - \tilde{F}_i^{11}\hat{P}_i^{12},\\
\hat{\Omega}_{13}^{ij} =& -(\tilde{G}_{ij}^{11})^T + \tilde{F}_i^{11}A_i{}^T + \tilde{K}_i^TB_i^T\\
\hat{\Omega}_{14}^{ij} =& -\tilde{F}_i^{11}F_{ij}^{12}
\end{aligned}$$

$$
\begin{aligned}
\hat{\Omega}^{ij}_{15} &= (I + \tilde{F}^{11}_i F^{12}_{ij}) L_i S_i - \tilde{F}^{11}_i F^{12}_{ij} E_i,\\
\hat{\Omega}^{ij}_{22} &= -\hat{P}^{22}_i + \mathrm{He}\Big(F^{22}_{ij} A_i - F^{22}_{ij} L_i C_i \Big),\\
\hat{\Omega}^{ij}_{23} &= -C_i^T L_i^T (I + \tilde{G}^{11}_{ij} G^{12}_{ij})^T + A_i^T (\tilde{G}^{11}_{ij} G^{12}_{ij})^T,\\
\hat{\Omega}^{ij}_{24} &= -F^{22}_{ij} + A_i^T (G^{22}_{ij})^T - C_i^T L_i^T (G^{22}_{ij})^T,\\
\hat{\Omega}^{ij}_{25} &= F^{22}_{ij} L_i S_i - F^{22}_{ij} E_i,\\
\hat{\Omega}^{ij}_{33} &= \tilde{G}^{11}_{ij} \hat{P}^{11}_j (\tilde{G}^{11}_{ij})^T - \tilde{G}^{11}_{ij} - (\tilde{G}^{11}_{ij})^T,\\
\hat{\Omega}^{ij}_{34} &= \tilde{G}^{11}_{ij} \hat{P}^{12}_j - \tilde{G}^{11}_{ij} G^{12}_{ij},\\
\hat{\Omega}^{ij}_{35} &= (I + \tilde{G}^{11}_{ij} G^{12}_{ij}) L_i S_i - \tilde{G}^{11}_{ij} G^{12}_{ij} E_i,\\
\hat{\Omega}^{ij}_{45} &= G^{22}_{ij} L_i S_i - G^{22}_{ij} E_i.
\end{aligned}
$$

B. Linearization of (15) *with respect to the gain* L_i

Note that (15) is still a BMI because of presence of the coupling term $\tilde{G}^{11}_{ij} \hat{P}^{11}_j (\tilde{G}^{11}_{ij})^T$. On the other hand, the gain matrix L_i is attached to different variables, $(I + \tilde{F}^{11}_i F^{12}_{ij}) L_i C_i$, $(I + \tilde{G}^{11}_{ij} G^{12}_{ij}) L_i C_i$, $F^{22}_{ij} L_i C_i$ and $G^{22}_{ij} L_i C_i$, and since G^{22}_{ij} is invertible and L_i dependent of i, then we choose $G^{22}_{ij} = G^{22}_i$ independent of j. Instead of identified the two terms $(I + \tilde{F}^{11}_i F^{12}_i) = G^{22}_i$, $\ I + \tilde{G}^{11}_i G^{12}_i = G^{22}_i$, as already done in our paper [21], in this paper we identify the two terms $I + \tilde{F}^{11}_i F^{12}_{ij} = \ I + \tilde{G}^{11}_{ij} G^{12}_{ij} = F^{22}_{ij} = 0$, which amounts to put:

$$
F^{12}_{ij} = -F^{11}_i,\ G^{12}_{ij} = -G^{11}_{ij} \ \text{ and } \ F^{22}_i = 0,
$$

and due to the co-existence of $\tilde{F}^{11}_i \hat{P}^{12}_i$ and $\tilde{G}^{11}_{ij} \hat{P}^{12}_j$, we cannot use a change of variables. We can then assume that $\hat{P}^{12}_i = 0$. Finally, in view of the previous arguments, the structures of F_i and G_{ij} become

$$
F_i = \begin{bmatrix} F^{11}_i & -F^{11}_i \\ 0 & 0 \end{bmatrix},\ G_{ij} = \begin{bmatrix} G^{11}_{ij} & -G^{11}_{ij} \\ 0 & G^{22}_i \end{bmatrix}, \hat{P}^{12}_i = 0. \quad (16)
$$

The alternative choice (16) is more general than that in [21], since it involves both indices i and j. At this stage, we can introduce the change of variables $\hat{L}_i = G^{22}_i L_i$. Now, in order to deal with the bilinear term $\tilde{G}^{11}_{ij} \hat{P}^{11}_j (\tilde{G}^{11}_{ij})^T$, we use Schur's complement. Thus, (15) becomes

$$
\underbrace{\begin{bmatrix}
\Theta^i_{11} & \Theta^i_{12} & \Theta^{ij}_{13} & \Theta^i_{14} & E_i & \Theta^i_{16} & 0\\
(\star) & -\hat{P}^{22}_i & \Theta^i_{23} & \Theta^i_{24} & 0 & -H_i^T & 0\\
(\star) & (\star) & \Theta^{ij}_{33} & I & E_i & 0 & \tilde{G}^{11}_{ij}\\
(\star) & (\star) & (\star) & \Theta^{ij}_{44} & \Theta^i_{45} & 0 & 0\\
(\star) & (\star) & (\star) & (\star) & -\mu^2 I & J_i^T & 0\\
(\star) & (\star) & (\star) & (\star) & (\star) & -I & 0\\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\tilde{P}^{11}_j
\end{bmatrix}}_{\Xi_{ij}} < 0, \quad (17)
$$

$$
\begin{aligned}
\Theta^i_{11} &= \tilde{P}^{11}_i + \mathrm{He}\Big(-\tilde{F}^{11}_i + A_i (\tilde{F}^{11}_i)^T + B_i \tilde{K}_i \Big)\\
\Theta^i_{12} &= -A_i,\\
\Theta^{ij}_{13} &= -(\tilde{G}^{11}_{ij})^T + \tilde{F}^{11}_i A_i{}^T + \tilde{K}_i^T B_i^T +\\
\Theta^i_{14} &= I,\\
\Theta^i_{16} &= \tilde{K}_i^T D_i^T + \tilde{F}^{11}_i H_i^T,\\
\Theta^i_{23} &= -A_i^T\\
\Theta^i_{24} &= A_i^T (G^{22}_i)^T - C_i^T \hat{L}_i^T,\\
\Theta^{ij}_{33} &= -\tilde{G}^{11}_{ij} - (\tilde{G}^{11}_{ij})^T,\\
\Theta^{ij}_{44} &= \hat{P}^{22}_j - G^{22}_i - (G^{22}_i)^T,\\
\Theta^i_{45} &= \hat{L}_i S_i - G^{22}_i E_i.
\end{aligned}
$$

Hence, the following theorem is inferred.

Theorem 1: For the closed-loop switched system (3), if there exist positive definite matrices $\tilde{P}^{11}_i, \hat{P}^{22}_i \in \mathbb{R}^{n\times n}$, invertible matrices G^{22}_i, $\tilde{G}^{11}_{ij}$, $\tilde{F}^{11}_i \in \mathbb{R}^{n\times n}$, matrices $\tilde{K}_i \in \mathbb{R}^{m\times n}$, $\hat{L}_i \in \mathbb{R}^{n\times p}$, for $i, j \in \Lambda$, so that the following convex optimization problem holds

$$
\begin{aligned}
&\min(\mu) \text{ subject to}\\
&\Xi_{ij} < 0, \text{ for all } i, j \in \Lambda,
\end{aligned} \quad (18)
$$

where Ξ_{ij} is given by (17), then the closed-loop switched system (3) is globally H_∞ asymptotically stable with a minimum attenuation level μ, under an arbitrary switching rule σ. The observer-based controller gains are given by

$$
K_i = \tilde{K}_i (\tilde{F}^{11}_i)^{-T}, \text{ and } \ L_i = (G^{22}_i)^{-1} \hat{L}_i,\ i \in \Lambda. \quad (19)
$$

Remark 1: In the previous analysis, we chose $\hat{P}_i$ as a diagonal matrix because $F^{11}_i \neq G^{11}_{ij}$. If one imposes the condition $F^{11}_i = G^{11}_i$, we can then choose $\hat{P}_i$ as a non-diagonal matrix (i.e. $\hat{P}^{12}_i \neq 0$).

IV. Illustrative Example

In this section, we present a numerical example to show the validity and effectiveness of the proposed design methodology. The example is described by the matrices in Table I below [28]:

i	A_i	B_i	Ci^T
1	$\begin{bmatrix} 0.7786 & 0.9908 & 0.1270 \\ 0.1616 & 0.8443 & 0.8144 \\ 0.9214 & 0.9747 & 0.7825 \end{bmatrix}$	$\begin{bmatrix} 0.2458 & 0.7409 \\ 0.2501 & 0.5257 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.3815 \\ 0.6916 \\ 0.7183 \end{bmatrix}$
2	$\begin{bmatrix} 0.3894 & 0.3263 & 0.7746 \\ 0.7806 & 0.9886 & 0.1297 \\ 0.8814 & 0.4718 & 0.3110 \end{bmatrix}$	$\begin{bmatrix} 0.2722 & 0.6055 \\ 0.1576 & 0.1580 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.0591 \\ 0.8258 \\ 0.4354 \end{bmatrix}$
3	$\begin{bmatrix} 0.3049 & 0.4247 & 0.8979 \\ 0.8448 & 0.2485 & 0.6921 \\ 0.7558 & 0.9160 & 0.3636 \end{bmatrix}$	$\begin{bmatrix} 0.4945 & 0.3020 \\ 0.9237 & 0.9118 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.5204 \\ 0.8010 \\ 0.9708 \end{bmatrix}$
4	$\begin{bmatrix} 0.1194 & 0.3964 & 0.2454 \\ 0.1034 & 0.2515 & 0.4983 \\ 0.6981 & 0.8655 & 0.2403 \end{bmatrix}$	$\begin{bmatrix} 0.9894 & 0.7205 \\ 0.1709 & 0.1519 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.6995 \\ 0.3081 \\ 0.8767 \end{bmatrix}$

TABLE I

System parameters

All the matrices A_1, A_2, A_3 and A_4 are clearly unstable. Assume that the system is disturbed by a noise $w_t = \chi(t)\omega_t$, where ω_t is an uniformly distributed random variable on the interval $[0\ 1]$, and $\chi(.)$ is defined by

$$
\chi(t) = \begin{cases} 2 & \text{if } t \in [0, 20[;\\ -2 & \text{if } t \in [40, 70[;\\ 0 & \text{elsewhere,} \end{cases}
$$

with adequate weighted matrices given in Table II:

Mode i	1	2	3	4
E_i	$\begin{bmatrix}0.1\\0.2\\0\end{bmatrix}$	$\begin{bmatrix}0.2\\0.1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0.1\\0.1\end{bmatrix}$	$\begin{bmatrix}0.1\\-0.2\\0.1\end{bmatrix}$
H_i^T	$\begin{bmatrix}0\\0.1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0.2\\0\end{bmatrix}$	$\begin{bmatrix}0.2\\0\\0.1\end{bmatrix}$	$\begin{bmatrix}0.1\\0.1\\0\end{bmatrix}$
J_i	0.2	0.2	0.3	0.5
S_i	0.1	0.5	0.2	0.6
D_i	0	0	0	0

TABLE II

THE MATRICES RELATED TO DISTURBANCES IN THE SYSTEM.

After solving the LMI (18), we get the optimal disturbance attenuation level $\mu_{\min} = 0.6144$, and the observer-based controller gains:

$$K_1 = \begin{bmatrix} 10.1862 & 0.7576 & -5.3030 \\ -4.9168 & -1.9387 & 1.1747 \end{bmatrix}, \; L_1 = \begin{bmatrix} 0.4639 \\ 0.8315 \\ 0.9666 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -5.5306 & -11.4018 & 1.6525 \\ 1.2740 & 4.3284 & -2.2230 \end{bmatrix}, \; L_2 = \begin{bmatrix} 0.6820 \\ 0.9787 \\ 0.6032 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -4.9708 & -4.9824 & -5.9156 \\ 4.3113 & 5.0517 & 5.3305 \end{bmatrix}, \; L_3 = \begin{bmatrix} 0.6070 \\ 0.6126 \\ 0.6508 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} -4.1119 & -4.3603 & 10.2364 \\ 4.8140 & 4.4365 & -14.5480 \end{bmatrix}, \; L_4 = \begin{bmatrix} 0.4675 \\ -0.1066 \\ 0.6191 \end{bmatrix}.$$

The simulation results corresponding to these observer-based controller gains are given in Figures 1-4. These simulations are performed for an horizon $T = 110$, with $x_0 = \begin{bmatrix}1 & 2 & 5\end{bmatrix}^T$, $\hat{x}_0 = \begin{bmatrix}-1 & 3 & 6\end{bmatrix}^T$.

V. Conclusion

In this paper, new LMI conditions have been developed for the problem of the stabilization of a class of switching discrete-time linear systems with l_2-bounded disturbances. We have shown that a judicious choice of slack variables coming from Finsler's lemma leads to less conservative LMIs. Analytical proofs have been provided to clarify how the proposed choice allows to eliminate some bilinear matrix coupling. The validity of the proposed design method is shown through a numerical example. In future work, we hope to extend our technique to switching systems with unknown switching modes and to linear parameter varying systems with inexact parameters.

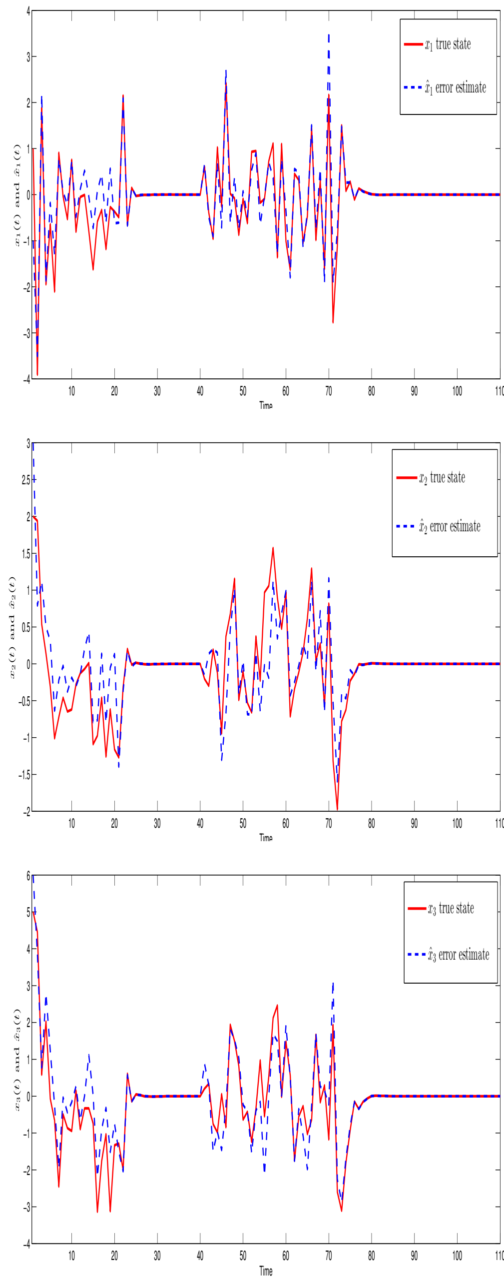

Fig. 1. Behavior of x and $\hat{x}$

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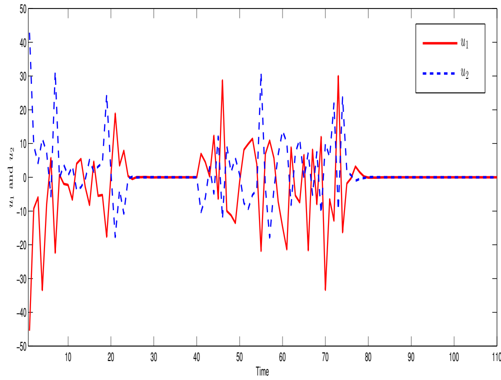

Fig. 2. Behavior of u_1 and u_2

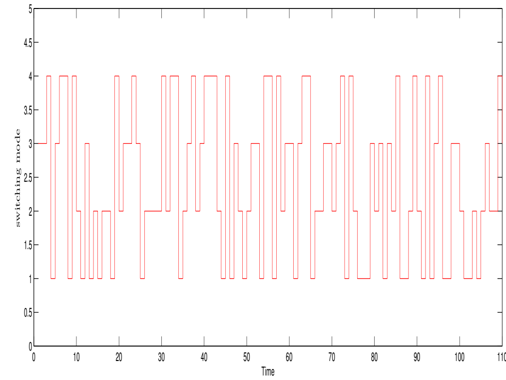

Fig. 4. Switching mode σ_t

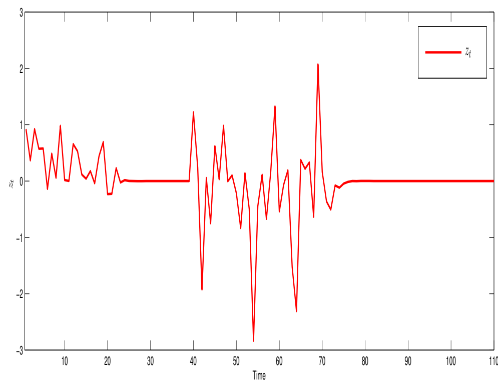

Fig. 3. Control output z_t

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