



**HAL**  
open science

# Approximation -diffusion in stochastically forced kinetic equations

A Debussche, Julien Vovelle

► **To cite this version:**

A Debussche, Julien Vovelle. Approximation -diffusion in stochastically forced kinetic equations. 2017.  
hal-01567138v1

**HAL Id: hal-01567138**

**<https://hal.science/hal-01567138v1>**

Preprint submitted on 21 Jul 2017 (v1), last revised 17 Mar 2020 (v6)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Approximation - diffusion in stochastically forced kinetic equations

A. Debussche\* and J. Vovelle†

July 21, 2017

## Abstract

We derive the hydrodynamic limit of a kinetic equation where the interactions in velocity are modelled by a linear operator (Fokker-Planck or Linear Boltzmann) and the force in the Vlasov term is a stochastic process with high amplitude and short-range correlation. In the scales and the regime we consider, the hydrodynamic equation is a scalar second-order stochastic partial differential equation. Compared to the deterministic case, we also observe a phenomenon of enhanced diffusion.

**Keywords:** approximation-diffusion, kinetic equation, hydrodynamic limit

**MSC Number:** 35Q20 (35R60 60H15 35B40)

## Contents

|          |  |          |
|----------|--|----------|
| <b>1</b> | <b>Introduction</b>                                  | <b>2</b> |
| 1.1      | Kinetic equations . . . . .                          | 2        |
| 1.2      | Trajectories . . . . .                               | 3        |
| 1.3      | Main results . . . . .                               | 5        |
| <b>2</b> | <b>Mixing force field</b>                            | <b>7</b> |
| 2.1      | Mixing hypothesis . . . . .                          | 9        |
| 2.2      | Some consequences of the mixing hypothesis . . . . . | 11       |
| 2.3      | Covariance . . . . .                                 | 12       |
| 2.4      | Some simple examples . . . . .                       | 13       |
| 2.5      | Mixing force process . . . . .                       | 14       |

---

\*IRMAR, ENS Rennes, CNRS, UEB. av Robert Schuman, F-35170 Bruz, France. Email: arnaud.debussche@ens-rennes.fr, partially supported by the French government thanks to the the “Investissements d’Avenir” program ANR-11-LABX-0020-01

†Université de Lyon ; CNRS ; Université Lyon 1, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France. Email: vovelle@math.univ-lyon1.fr, partially supported by the ANR project STAB and the “Investissements d’Avenir” program LABEX MILYON ANR-10-LABX-0070

|          |   |           |
|----------|---|-----------|
| <b>3</b> | <b>Unperturbed equation: ergodic properties</b>                             | <b>14</b> |
| <b>4</b> | <b>Resolution of the kinetic equation</b>                                   | <b>20</b> |
| 4.1      | Cauchy Problem in the LB case . . . . .                                     | 20        |
| 4.2      | Cauchy Problem in the FP case . . . . .                                     | 22        |
| 4.3      | Markov property . . . . .   | 23        |
| <b>5</b> | <b>Deterministic convergence</b>  | <b>27</b> |
| 5.1      | Classical diffusion limit . . . . .   | 29        |
| 5.2      | Diffusion limit for the stochastically forced equation . . . . .            | 30        |
| 5.3      | The auxiliary test-function . . . . .                                       | 31        |
| 5.4      | Diffusion matrix . . . . .  | 32        |
| <b>6</b> | <b>Diffusion-approximation</b>  | <b>34</b> |
| 6.1      | Perturbed test-function . . . . .   | 35        |
| 6.1.1    | First corrector . . . . .   | 35        |
| 6.1.2    | Second corrector and limit generator . . . . .                              | 36        |
| 6.1.3    | First and second correctors . . . . .                                       | 37        |
| 6.2      | Bounds on the moments . . . . .   | 42        |
| 6.3      | Tightness . . . . .   | 43        |
| 6.4      | Convergence to the solution of a Martingale problem . . . . .               | 46        |
| 6.5      | Limit SPDE . . . . .  | 47        |
| 6.5.1    | Covariance . . . . .  | 47        |
| 6.5.2    | Limit equation . . . . .  | 49        |
| 6.5.3    | Resolution of the limit equation . . . . .                                  | 49        |
| 6.6      | Uniqueness for the limit martingale problem . . . . .                       | 51        |
| <b>A</b> | <b>Resolution of the unperturbed equation</b>                               | <b>54</b> |
| <b>B</b> | <b>Martingale characterization of Markov processes: quadratic variation</b> | <b>54</b> |

# 1 Introduction

## 1.1 Kinetic equations

Let  $N \in \mathbb{N}^*$ . We denote by  $\mathbb{T}^N$  the  $N$ -dimensional torus. Let  $\varepsilon > 0$ . We consider the following kinetic equation

$$\partial_t f + \varepsilon v \cdot \nabla_x f + \bar{E}(t, x) \cdot \nabla_v f = Qf, \quad t > 0, x \in \mathbb{T}^N, v \in \mathbb{R}^N, \quad (1.1)$$

which is a perturbation of the equation

$$\partial_t f + \bar{E}(t, x) \cdot \nabla_v f = Qf \quad t > 0, x \in \mathbb{T}^N, v \in \mathbb{R}^N. \quad (1.2)$$

The operator  $Q$  is either the linear Boltzmann operator

$$Q_{\text{LB}}f = \rho(f)M - f, \quad \rho(f) = \int_{\mathbb{R}^N} f(v)dv, \quad M(v) = \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{|v|^2}{2}\right), \quad (1.3)$$

or the Fokker-Planck operator

$$Q_{\text{FP}}f = \text{div}_v(\nabla_v f + vf). \quad (1.4)$$

The force field  $\bar{E}(t, x)$  in (1.2) is a Markov, stationary mixing process (see Section 2 for more details). We show in Section 3 that there is a unique, ergodic, invariant measure for (1.2) and that this invariant measure is the law of an invariant solution  $(x, v) \mapsto \rho(x)\bar{M}_t(x, v)$  parametrized by  $\rho(x)$ . See (3.6)-(3.7) for the definition of  $\bar{M}_t$ . Consider the solution  $f$  to (1.1) starting from a state

$$f_{\text{in}}(x, v) \approx \rho_{\text{in}}(x)\bar{M}_0(x, v). \quad (1.5)$$

Rescale over time intervals of order  $\varepsilon^{-2}$ :

$$f^\varepsilon(t, x, v) = f(\varepsilon^{-2}t, x, v). \quad (1.6)$$

Then  $f^\varepsilon$  is solution to the equation

$$\partial_t f^\varepsilon + \frac{v}{\varepsilon} \cdot \nabla_x f^\varepsilon + \frac{1}{\varepsilon^2} \bar{E}(\varepsilon^{-2}t, x) \cdot \nabla_v f^\varepsilon = \frac{1}{\varepsilon^2} Q f^\varepsilon, \quad t > 0, x \in \mathbb{T}^N, v \in \mathbb{R}^N. \quad (1.7)$$

On bounded time intervals  $[0, T]$ , we expect

$$f^\varepsilon(t, x, v) \approx \rho(x, t)\bar{M}_{\varepsilon^{-2}t}(x, v), \quad (1.8)$$

where  $\rho$  is solution to a given equation (the *hydrodynamic* equation) which we would like to identify. We do not prove (1.8), but find the limit equation satisfied by  $\rho = \lim_{\varepsilon \rightarrow 0} \rho^\varepsilon$ , where  $\rho^\varepsilon = \rho(f^\varepsilon)$ . We show in Theorem 1.2 that  $\rho$  satisfies a diffusion equation, where the drift term is a second order differential operator in divergence form with respect to the space-variable  $x$ . Showing that  $\rho^\varepsilon$  is close to  $\rho$  with  $\rho$  a diffusion (in infinite dimension) is therefore a result of diffusion-approximation (in infinite dimension). See Theorem 1.2 for the precise statement. Theorem 1.1 is concerned with the limit behaviour of the average  $\mathbb{E}\rho^\varepsilon$ , a deterministic issue. The proof of this result is related to characteristic equations associated to (1.1), which we discuss in the next section.

## 1.2 Trajectories

The phase space associated to (1.1) is  $\mathbb{T}^N \times \mathbb{R}^N$ . Consider the following systems of stochastic differential equations:

$$\begin{aligned} dX_t &= \varepsilon dV_t, \\ dV_t &= \bar{E}(t, X_t)dt + \text{jumps}, \end{aligned} \quad (1.9)$$

and

$$\begin{aligned} dX_t &= \varepsilon dV_t, \\ dV_t &= (\bar{E}(t, X_t) - V_t)dt + \sqrt{2}dB_t. \end{aligned} \tag{1.10}$$

In (1.9) the second equation describes the following piecewise deterministic Markov process (PDMP). Consider the Poisson process associated to the times  $(T_n)$  and to the probability measure  $Mdv$ : the increments  $T_{n+1} - T_n$  are i.i.d. with exponential law of parameter 1. At each time  $t = T_n$ ,  $V_t$  is jumping to a new value  $V_{T_n+}$  chosen at random, according to the probability law  $Mdv$ . Between each jump,  $(V_t)$  is evolving by the differential equation

$$\frac{dV_t}{dt} = E(t, X_t), \quad T_n < t < T_{n+1}, \tag{1.11}$$

which is coupled with the first equation of (1.9). In (1.10),  $B_t$  is an  $N$ -dimensional Wiener process. In both the LB case and the FP case, the extra stochastic processes which we introduce are independent on  $(\bar{E}(t))$ . In this context, the equation (1.1) gives the evolution of the density, with respect to the Lebesgue measure on  $\mathbb{T}_x^N \times \mathbb{R}_v^N$ , of the conditional law of  $(X_t, V_t)$ : let  $\mathcal{F}_t^E = \sigma((\bar{E}_s)_{0 \leq s \leq t})$ . If the law of  $(X_0, V_0)$  has density  $f_0$  with respect to the Lebesgue measure on  $\mathbb{T}_x^N \times \mathbb{R}_v^N$ , then

$$\mathbb{E}[\varphi(X_t, V_t) | \mathcal{F}_t^E] = \iint_{\mathbb{T}^N \times \mathbb{R}^N} \varphi(x, v) f_t(x, v) dx dv, \tag{1.12}$$

for all  $\varphi \in C_b(\mathbb{T}^N \times \mathbb{R}^N)$ . From (1.12), it follows that

$$\mathbb{E}[\varphi(X_t)] = \int_{\mathbb{T}^N} \varphi(x) \mathbb{E}\rho_t(x) dx, \quad \rho_t = \rho(f_t), \tag{1.13}$$

for all  $\varphi \in C_b(\mathbb{T}^N)$ . We have two main results. The first one, Theorem 1.1, gives the limit behaviour of  $\mathbb{E}\rho_{\varepsilon^{-2}t}$ . The second one, Theorem 1.2, describes the limit behaviour of  $\rho_{\varepsilon^{-2}t}$ . The first result should follow from the second one. However, there are various reasons for giving two separate statements:

1. we obtain the limit behaviour of  $\mathbb{E}\rho_{\varepsilon^{-2}t}$ , by proving the convergence in law of  $X_{\varepsilon^{-2}t}$  (hence focusing on the left-hand side of (1.13)),
2. on the contrary, the limit behaviour of  $\rho_{\varepsilon^{-2}t}$  is obtained by working at the level of the PDE (1.7),
3. the proof of Theorem 1.1 uses the central limit theorem for martingales. This approach to the limiting behaviour of (1.9) or (1.10) is very classical in a certain mathematical community (see, *e.g.*, the second paragraph of the introduction to [15], and also Chapter 13 of the same reference), but is certainly not familiar to a large group of analysts, and we wanted to emphasize these probabilistic aspects here,

4. we make a smallness hypothesis on the forcing stochastic field  $(\bar{E}(t))$ , hypothesis (1.24), in Theorem 1.2, which is necessary to ensure that the limit stochastic PDE (1.25) is well-posed. It should not be relevant in the context of Theorem 1.1 since there is no noise there (it is averaged out), and indeed, our alternative proof to Theorem 1.1 does not require any smallness hypothesis on  $(\bar{E}(t))$ ,
5. in the proof of Theorem 1.1, we introduce some tools and some results that are used later on in the proof of Theorem 1.2; with this progression, the proof of Theorem 1.2, which is quite long, is more gradual.

Note, however, that we establish Theorem 1.1 in the restrictive case of a field  $\bar{E}_t$  independent on  $x$ .

### 1.3 Main results

**Notations.** The three first moments of a function  $f \in L^1(\mathbb{R}^N, |v|^2 dv)$  are

$$\rho(f) = \int_{\mathbb{R}^N} f(v) dv, \quad J(f) = \int_{\mathbb{R}^N} v f(v) dv, \quad K(f) = \int_{\mathbb{R}^N} v \otimes v f(v) dv, \quad (1.14)$$

where  $a \otimes b$  is the  $N \times N$  rank-one matrix built on  $a, b \in \mathbb{R}^N$  with  $ij$ -th elements  $a_i b_j$ . We denote by  $K$  the second moment of  $M$  (due to the particular fact that  $M$  is a Maxwellian, this is simply the identity matrix of size  $N \times N$ ):

$$K = K(M) = \int_{\mathbb{R}^N} v \otimes v M(v) dv = \text{Id}_N. \quad (1.15)$$

For  $m \in \mathbb{N}$ , we denote by  $\bar{J}_m(f)$  the total  $m$ -th moment of  $f$ :

$$\bar{J}_m(f) = \iint_{\mathbb{T}^N \times \mathbb{R}^N} |v|^m f(x, v) dx dv. \quad (1.16)$$

Let us also introduce the Banach space

$$G_m = \{f \in L^1(\mathbb{T}^N \times \mathbb{R}^N); \bar{J}_0(f) + \bar{J}_m(f) < +\infty\}, \quad (1.17)$$

with norm  $\|f\|_{G_m} = \bar{J}_0(f) + \bar{J}_m(f)$ .

**Deterministic convergence.** Our first result gives the convergence of the average  $\mathbb{E}\rho^\varepsilon$ .

**Theorem 1.1.** *Let  $f_{\text{in}} \in G_3$  be non-negative. Let  $(\bar{E}_t)$  be a mixing force process according to Definition 2.2. Let  $f^\varepsilon \in C([0, T]; L^1(\mathbb{T}^N \times \mathbb{R}^N))$  be the mild solution to (1.7) with initial condition  $f_{\text{in}}$ , in the sense of Definition 4.1 or 4.2, depending on the nature of the collision operator  $Q$ . Let  $r^\varepsilon = \mathbb{E}\rho(f^\varepsilon)$ . Assume that  $f_{\text{in}}$  has the following structure:*

$$f_{\text{in}}(x, v) = \rho_{\text{in}}(x)g(v), \quad (1.18)$$

where  $g \in L^1(\mathbb{R}^N)$ ,  $\rho(g) = 1$ . Then,  $r^\varepsilon \rightarrow r$  in  $C([0, T]; L^2(\mathbb{T}^N) - \text{weak})$ , where  $r$  is the solution to the diffusion equation

$$\partial_t r - \operatorname{div}_x (K_{\sharp} \nabla_x r + \Psi r) = 0, \quad (1.19)$$

with initial condition

$$r(0) = r_{\text{in}}. \quad (1.20)$$

The coefficients in (1.19) have the following expression:

$$K_{\sharp} = K + \mathbb{E} [\bar{E}(0) \otimes [R_0(\bar{E}(0)) + (b-1)R_1(\bar{E}(0))]], \quad (1.21)$$

and

$$\Psi = \mathbb{E} [b\bar{E}(0) \cdot \nabla_x R_1(\bar{E}(0)) + [R_0(\bar{E}(0)) + (b-1)R_1(\bar{E}(0))] \operatorname{div}_x(\bar{E}(0))], \quad (1.22)$$

where  $b^{\text{LB}} = 2$  in the case  $Q = Q_{\text{LB}}$  and  $b^{\text{FP}} = 1$  in the case  $Q = Q_{\text{FP}}$ , and where the resolvent  $R_\lambda$  is defined by (2.20).

We show in (5.31) that  $K_{\sharp} \geq K$ . It is a remarkable fact that the stochastic forcing term  $\bar{E}_t$  has an influence on the diffusion matrix at the limit, and that it increases the diffusion effects. Note that the influence of stochastic mixing forcing terms in kinetic equations has also been investigated in [18, 12]. The context and the results in these two papers are different from the present one however. Indeed,

1. the starting kinetic equations in [18, 12] are not collisional,
2. In [18, 12], in the scaling that is considered, a collisional kinetic equation is obtained at the limit. The collision operator (an operator acting on functions of the variable  $v$  thus) is a diffusion operator. At the level of trajectories, this operator appears due to the convergence of the velocity  $V_t$  of particles to a diffusion like equation (1.10) with  $E = 0$ .

**Diffusion-approximation** Our main result of diffusion-approximation for  $\rho^\varepsilon$  is the following one.

**Theorem 1.2.** *Let  $f_{\text{in}}^\varepsilon \in G_3$  be non-negative. Let  $\bar{\sigma} > 2 + \frac{3}{2}N$ . Let  $(\bar{E}_t)$  be a mixing force field on  $H^{\bar{\sigma}}(\mathbb{T}^N; \mathbb{R}^N)$  according to Definition 2.1. Let  $f^\varepsilon \in C([0, T]; L^1(\mathbb{T}^N \times \mathbb{R}^N))$  be the mild solution to (1.7) with initial condition  $f_{\text{in}}^\varepsilon$ , in the sense of Definition 4.1 or 4.2, depending on the nature of the collision operator  $Q$ . Let  $\rho^\varepsilon = \rho(f^\varepsilon)$ . Assume the convergence*

$$\rho(f_{\text{in}}^\varepsilon) \rightarrow \rho_{\text{in}} \text{ in } L^1(\mathbb{T}^N), \text{ with } \rho_{\text{in}} \in L^2(\mathbb{T}^N). \quad (1.23)$$

Let  $K_{\sharp}$  and  $\Psi$  be defined by (1.21) and (1.22) respectively. There exists a constant  $\alpha_0 > 0$  such that, if the measure  $\alpha$  for the size of  $(\bar{E}_t)$  in (2.6) satisfies

$$\alpha < \alpha_0, \quad (1.24)$$

then, for all  $\bar{\sigma} \geq \sigma > 2 + \frac{3}{2}N$ ,  $(\rho^\varepsilon)$  converges in law on  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$  to  $\rho$ , the weak solution in the sense of Definition 6.1 of the stochastic equation

$$d\rho = \operatorname{div}_x(K_{\sharp}\nabla_x\rho + \Psi\rho)dt + \sqrt{2}\operatorname{div}_x(\rho S^{1/2}dW(t)), \quad (1.25)$$

with initial condition

$$\rho(0) = \rho_{\text{in}}. \quad (1.26)$$

In (1.25),  $W(t)$  is a cylindrical Wiener process on  $[L^2(\mathbb{T}^N)]^N$ , and  $S^{1/2}$  is the Hilbert-Schmidt operator on  $[L^2(\mathbb{T}^N)]^N$  defined in Section 6.5.1.

Note the weak mode of convergence of  $\rho^\varepsilon$ . It is weak in the probabilistic sense (convergence in law). This is inherent to the limit theorems (like the Donsker theorem) which lay the bases of diffusion-approximation results. The convergence is weak with respect to the space-variable also. We intend to improve this point, and to consider non-linear equations in a similar regime, in a future work.

The plan of the paper is the following one. In Section 2 we describe the type of forcing field  $\bar{E}(t)$  which we consider. In Section 3, we prove some mixing properties and compute the invariant measures for the unperturbed equation (1.2). In Section 4, we solve the Cauchy Problem for the kinetic equation (1.1). In Section 5, we prove Theorem 1.1 (deterministic limit). In Section 6, we establish our main result of approximation-diffusion, Theorem 1.2.

Note that the present paper is quite long. There are various reasons for these, first the fact that the whole proof of Theorem 1.2 requires many step. However, the heart of our diffusion-approximation result is the computations done by the perturbed test-function method in Section 6.1. An other reason for the length of the paper is that we have taken the care to present all the details of some intuitive facts, like the statements of Theorem 4.3 or Theorem B.1 for example. Indeed, semi-groups, generator and Markov processes in infinite dimension require some circumspection. With that regard, we have used in particular the references [10] and [19].

## 2 Mixing force field

Let  $\bar{\sigma} > 2 + \frac{3}{2}N$  and let  $F = H^{\bar{\sigma}}(\mathbb{T}^N; \mathbb{R}^N)$ . This is the state space for the mixing force field  $\bar{E}$ . Let  $(\bar{E}_t)_{t \geq 0}$  be a stationary, homogeneous Markov process of generator  $A$  over  $F$  (the generator is defined according to the theory developed in Appendix B). Let  $P(t, \mathbf{e}, B)$  be a transition function for  $(\bar{E}_t)$  associated to the filtration generated by  $(\bar{E}_t)$  (see, e.g., [10, p. 156] for the definition), satisfying the Chapman-Kolmogorov relation

$$P(t+s, \mathbf{e}, B) = \int_F P(s, \mathbf{e}_1, B)dP(t, \mathbf{e}, d\mathbf{e}_1), \quad (2.1)$$

for all  $s, t \geq 0$ ,  $\mathbf{e} \in F$ ,  $B$  Borel subset of  $F$ . Let  $\mathcal{P}(F)$  be the set of Borel probability measures on  $F$ . By [10, p. 157], up to a modification of the probability space  $(\Omega, \mathcal{F})$ , say



into a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}})$ , there exists a collection  $\{\mathbb{P}_\mu; \mu \in \mathcal{P}(F)\}$  of probability measures and some Markov processes  $(E(t, s))_{t \geq s}$  with transition function  $P$  such that,  $\mathbb{P}_\mu(E(s, s) \in D_0) = \mu(D_0)$  for all Borel subset  $D_0$  of  $F$ . When  $\mu$  is the Dirac mass  $\mu = \delta_{\mathbf{e}}$ , we use the shorter notation  $\mathbb{P}_{\mathbf{e}}$  instead of  $\mathbb{P}_{\delta_{\mathbf{e}}}$ . By [10, p. 157] additionally, for all  $D \in \mathcal{F}$ ,  $\mathbf{e} \mapsto \mathbb{P}_{\mathbf{e}}(D)$  is Borel measurable. Let  $\mathbf{e}_0$  be a random variable on  $F$  of law  $\mu$ . We do a slight abuse of notation and denote by  $(E(t, s; \mathbf{e}_0), \mathbb{P})$  the couple  $(E(t, s), \mathbb{P}_\mu)$ . This means that the finite-dimensional distribution of both processes are the same, *i.e.*

$$\mathbb{P}(E(t_1, s; \mathbf{e}_0) \in D_1, \dots, E(t_n, s; \mathbf{e}_0) \in D_n) = \mathbb{P}_\mu(E(t_1, s) \in D_1, \dots, E(t_n, s) \in D_n), \quad (2.2)$$

for all  $s \leq t_1 \leq \dots \leq t_n$ , and  $D_1, \dots, D_n$  Borel subsets of  $F$ . For simplicity, we use the notation  $E(t; \mathbf{e})$ , or  $E_t(\mathbf{e})$ , instead of  $E(t, 0; \mathbf{e})$ . Note that, by iteration of (2.1), we have

$$\begin{aligned} & \mathbb{P}(\bar{E}(0) \in D_0, \bar{E}(t_1) \in D_1, \dots, \bar{E}(t_n) \in D_n) \\ &= \int_{D_0} \dots \int_{D_{n-1}} P(t_n - t_{n-1}, \mathbf{e}_{n-1}, D_n) P(t_{n-1} - t_{n-2}, \mathbf{e}_{n-2}, d\mathbf{e}_{n-1}) \dots P(t_1, \mathbf{e}_0, d\mathbf{e}_1) d\nu(\mathbf{e}_0) \\ &= \mathbb{P}_\nu(E(t_1, 0) \in D_1, \dots, E(t_n, 0) \in D_n), \quad (2.3) \end{aligned}$$

where  $\nu$  is the law of  $\bar{E}(0)$ . Therefore  $\bar{E}_t$  and  $E_t(\bar{E}_0)$  have the same finite-dimensional distributions:  $\bar{E}_t$  is a version  $E_t(\bar{E}_0)$ . The probability space  $\tilde{\Omega}$  used in [10, p. 157] to define the probability measures  $\mathbb{P}_{\mathbf{e}}$  is the path-space  $F^{[0, +\infty)}$  (the  $\sigma$ -algebra  $\tilde{\mathcal{F}}$  is the product  $\sigma$ -algebra). Assume in addition that  $(\bar{E}_t)$  is càdlàg. Then it is clear that we can take the Skorohod space  $D([0, +\infty); F)$  as a path space to define  $\mathbb{P}_{\mathbf{e}}$ . The  $\sigma$ -algebra  $\tilde{\mathcal{F}}$  is then the trace of the product  $\sigma$ -algebra, which coincide with the Borel  $\sigma$ -algebra when the Skorohod topology is considered on  $D([0, +\infty); F)$ . In this context, it holds true that  $\mathbf{e} \mapsto \mathbb{P}_{\mathbf{e}}(D)$  is Borel measurable for all  $D \in \tilde{\mathcal{F}}$  (see the proof of Proposition 1.2 p. 158 in [10]). To sum up (see [22, Section I-3]), if  $(\bar{E}_t)$  is càdlàg, we can assume that  $t \mapsto E(t, s; \mathbf{e})$  is càdlàg, for all  $s \in \mathbb{R}$  and  $\mathbf{e} \in F$ . As a last remark, note that it is always possible, using the Kolmogorov extension theorem, to build a càdlàg stationary process  $(\check{E}(t))_{t \in \mathbb{R}}$  indexed by  $t \in \mathbb{R}$  with the finite-dimensional distributions

$$\begin{aligned} & \mathbb{P}(\check{E}(s) \in D_0, \check{E}(s + t_1) \in D_1, \dots, \check{E}(s + t_n) \in D_n) \\ &= \mathbb{P}(\bar{E}(0) \in D_0, \bar{E}(t_1) \in D_1, \dots, \bar{E}(t_n) \in D_n), \quad (2.4) \end{aligned}$$

for all  $s \in \mathbb{R}$ ,  $0 \leq t_1, \dots, t_n$ . Instead of adding a new notation  $(\check{E}(t))_{t \in \mathbb{R}}$ , we simply denote this process by  $(\bar{E}(t))_{t \in \mathbb{R}}$ . We also denote by  $(\mathcal{G}_t^E)$  the usual augmentation (*cf.* [22, Definition (4.13), Section I-4]) of the canonical filtration  $(\mathcal{F}_t)$  on  $D([0, +\infty); F)$  with respect to the family  $(\mathbb{P}_{\mathbf{e}})_{\mathbf{e} \in F}$ . In successive order,  $(\mathcal{F}_t)$  is the filtration generated by the evaluation maps  $(\pi_t)$ ,  $\pi_t(\omega) = \omega(t)$ ;  $\mathcal{F}_t^*$  is the intersection over  $\mathbf{e} \in F$  of the  $\sigma$ -algebras  $\mathcal{F}_t^{\mathbb{P}_{\mathbf{e}}}$  obtained by completing  $\mathcal{F}_t$  with  $\mathbb{P}_{\mathbf{e}}$ -negligible sets; and  $\mathcal{G}_t$  is  $\mathcal{F}_t^*$ :

$$\mathcal{G}_t = \bigcap_{s > t} \mathcal{F}_s^*. \quad (2.5)$$

**Definition 2.1** (Mixing force field). Let  $(\bar{E}_t)_{t \geq 0}$  be a càdlàg, stationary, homogeneous Markov process of generator  $A$  over  $F$ . We say that  $(\bar{E}_t)_{t \geq 0}$  is a mixing force field if the conditions (2.6), (2.7), (2.12), (2.19), (2.22) below are satisfied.

Our first hypothesis is that there exists a stable ball: there exists  $\alpha \geq 0$  such that: almost-surely, for all  $\mathbf{e}$  with  $\|\mathbf{e}\|_F \leq \alpha$ , for all  $t \geq 0$ ,

$$\|E(t; \mathbf{e})\|_F \leq \alpha. \quad (2.6)$$

Our second hypothesis is about the law  $\nu$  of  $\bar{E}_t$ . We assume that it is supported in the ball  $\bar{B}_\alpha$  of  $F$  (therefore, it has moments of all orders) and that it is centred:

$$\int_F \mathbf{e} d\nu(\mathbf{e}) = \mathbb{E} [\bar{E}_t] = 0, \quad (2.7)$$

for all  $t \geq 0$ . Note that a consequence of this hypothesis is that: almost-surely, for all  $t \geq 0$ ,

$$\|\bar{E}_t\|_F \leq \alpha. \quad (2.8)$$

Our third hypothesis is a mixing hypothesis: we assume that there exists a continuous, non-increasing, positive and integrable function  $\gamma_{\text{mix}} \in L^1(\mathbb{R}_+)$  such that, for all probability measures  $\mu, \mu'$  on  $F$ , for all random variables  $\mathbf{e}_0, \mathbf{e}'_0$  on  $F$  of law  $\mu$  and  $\mu'$  respectively, there is a coupling  $((E_t^*(\mathbf{e}_0))_{t \geq 0}, (E_t^*(\mathbf{e}'_0))_{t \geq 0})$  of  $((E_t(\mathbf{e}_0))_{t \geq 0}, (E_t(\mathbf{e}'_0))_{t \geq 0})$  such that

$$\mathbb{E} \|E_t^*(\mathbf{e}_0) - E_t^*(\mathbf{e}'_0)\|_F \leq \alpha \gamma_{\text{mix}}(t), \quad (2.9)$$

for all  $t \geq 0$ . Typically, we expect  $\gamma_{\text{mix}}$  to be of the form  $\gamma_{\text{mix}}(t) = C_{\text{mix}} e^{-\beta_{\text{mix}} t}$ ,  $\beta_{\text{mix}} > 0$  (see the example treated in Section 2.4 for instance). Written as in (2.9), our statement is a little bit incorrect. Indeed,  $(E_t(\mathbf{e}_0), \mathbb{P})$  is just a convention of notation for  $(E_t, \mathbb{P}_\mu)$  and, here in (2.9), two measures  $\mathbb{P}_\mu$  and  $\mathbb{P}_{\mu'}$  are involved. We need to give a more formal form to (2.9) therefore. This is done in the next paragraph, using a Monge-Kantorovitch problem. The precise mixing hypothesis is (2.12). However, regarding the mixing properties of  $(E_t)$ , one may jump directly to the paragraph 2.2, where we derive the consequences of (2.12) that are of interest to us.

## 2.1 Mixing hypothesis

Let  $W$  be the functional defined by

$$W(\mathbb{P}^1, \mathbb{P}^2) = \inf_{\Pi} \iint_{D(\mathbb{R}_+; F) \times D(\mathbb{R}_+; F)} c(E, E') d\Pi(E, E'), \quad (2.10)$$

$$c(E, E') = \sup_{t \geq 0} \sup_{0 < h < 1} \frac{1}{h} \int_t^{t+h} [\gamma_{\text{mix}}^{-1}(s) \|E(s) - E'(s)\|_F] ds, \quad (2.11)$$

for all  $\mathbb{P}^1, \mathbb{P}^2$  probability measures on  $D(\mathbb{R}_+; F)$ . The inf in (2.10) is on all probability measures  $\Pi$  on the product  $D(\mathbb{R}_+; F) \times D(\mathbb{R}_+; F)$  with respective marginals  $\mathbb{P}^1$  and  $\mathbb{P}^2$ .

Our mixing hypothesis is the following one: we assume that, for all  $\mu, \mu'$  probability measures on  $F$ ,

$$W(\mathbb{P}_\mu, \mathbb{P}_{\mu'}) \leq \alpha. \quad (2.12)$$

The cost

$$c: D(\mathbb{R}_+; F) \times D(\mathbb{R}_+; F) \rightarrow \mathbb{R} \cup \{+\infty\}$$

in (2.11) is a regularization of the cost

$$\bar{c}(E, E') := \sup_{t \geq 0} [\gamma_{\text{mix}}^{-1}(t) \|E(t) - E'(t)\|_F].$$

We use this regularization because the pointwise evaluation  $(E, E') \mapsto (E(t), E'(t))$  is not continuous on  $D(\mathbb{R}_+; F) \times D(\mathbb{R}_+; F)$  (which is endowed with the product topology of the Skorohod topology on each factor). On the other hand, the map

$$(E, E') \mapsto \int_t^{t+h} [\gamma_{\text{mix}}^{-1}(s) \|E(s) - E'(s)\|_F] ds$$

is continuous on  $D(\mathbb{R}_+; F) \times D(\mathbb{R}_+; F)$ . Consequently, the cost  $c$  is lower semi-continuous. By Theorem 4.1 in [23], the inf in (2.10) is reached for an optimal transport plan  $\Pi_{\mu, \mu'}^*$ . Let us give the interpretation of this result. Intuitively,  $W(\mathbb{P}_\mu, \mathbb{P}_{\mu'})$  is related to the optimal coupling problem for

$$\mathbb{E}c((E_t(\mathbf{e}_0))_{t \geq 0}, (E_t(\mathbf{e}'_0))_{t \geq 0}), \quad (2.13)$$

where  $\mathbf{e}_0$  has law  $\mu$  and  $\mathbf{e}'_0$  has law  $\mu'$ . Here, we use the notation introduced in (2.2). However, as already explained, (2.13) has no real meaning since  $(E_t(\mathbf{e}_0), \mathbb{P})$  is just a convention of notation for  $(E_t, \mathbb{P}_\mu)$  and that here in (2.13), two measures  $\mathbb{P}_\mu$  and  $\mathbb{P}_{\mu'}$  are involved. Nevertheless, let us do once again the following abuse of notation: we use the notation  $((E_t^*(\mathbf{e}_0))_{t \geq 0}, (E_t^*(\mathbf{e}'_0))_{t \geq 0}, \mathbb{P})$  to denote  $(E, E', \Pi_{\mu, \mu'}^*)$ . This means that, for all  $\Phi: D(\mathbb{R}_+; F) \times D(\mathbb{R}_+; F) \rightarrow \mathbb{R}$  continuous and bounded, we have

$$\mathbb{E}\Phi((E_t^*(\mathbf{e}_0))_{t \geq 0}, (E_t^*(\mathbf{e}'_0))_{t \geq 0}) = \iint_{D(\mathbb{R}_+; F) \times D(\mathbb{R}_+; F)} \Phi(E, E') d\Pi_{\mu, \mu'}^* \quad (2.14)$$

Intuitively, we have obtained (replacing the cost  $c$  by the cost  $\bar{c}$ ),

$$\mathbb{E} \sup_{t \geq 0} [\gamma_{\text{mix}}^{-1}(t) \|E_t^*(\mathbf{e}_0) - E_t^*(\mathbf{e}'_0)\|_F] \leq \alpha, \quad (2.15)$$

which is stronger than (2.9) ( $\mathbb{E} \sup_t$  versus  $\sup_t \mathbb{E}$ ). This remains however reasonable (see the examples in Section 2.4). Note that  $\mathbb{P}_\mu$  and  $\mathbb{P}_{\mu'}$  are supported in the set  $\Sigma_\alpha$  of the càdlàg functions  $\mathbb{R}_+ \rightarrow F$  taking values in the ball  $\bar{B}_\alpha$ . Consequently,  $\Pi_{\mu, \mu'}^*$  is supported in  $\Sigma_\alpha \times \Sigma_\alpha$ . This means that the factors  $(E_t^*(\mathbf{e}_0))_{t \geq 0}, (E_t^*(\mathbf{e}'_0))_{t \geq 0}$  of the coupling also satisfy (2.6).

## 2.2 Some consequences of the mixing hypothesis

Let  $t \geq 0$  be fixed. Take  $\Phi$  in (2.14) of the form

$$\Phi(E, E') = \frac{1}{h} \int_t^{t+h} \|E(s) - E'(s)\|_F ds.$$

By (2.12) and (2.14), we have

$$\mathbb{E} \left[ \frac{1}{h} \int_t^{t+h} \|E_s^*(\mathbf{e}_0) - E_s^*(\mathbf{e}'_0)\|_F ds \right] \leq \alpha \gamma_{\text{mix}}(t+h).$$

Taking the limit  $[h \rightarrow 0]$ , we obtain (by dominated convergence)

$$\mathbb{E} [\|E_{t+}^*(\mathbf{e}_0) - E_{t+}^*(\mathbf{e}'_0)\|_F] \leq \alpha \gamma_{\text{mix}}(t).$$

This being true for every  $t \geq 0$ , we have, actually, (2.9). If  $\varphi$  is a Lipschitz continuous function on  $F$ , now, we have

$$\mathbb{E} \varphi(E_t^*(\mathbf{e}_0)) = \langle e^{tA} \varphi, \mu \rangle$$

(where  $e^{tA}$  denote the semi-group associated to  $A$ :  $\mathbb{E}_{\mathbf{e}} \varphi(E_t) = e^{tA} \varphi(\mathbf{e})$ ). From (2.9), it follows that

$$|\langle e^{tA} \varphi, \mu \rangle - \langle e^{tA} \varphi, \mu' \rangle| \leq \|\varphi\|_{\text{Lip}} \alpha \gamma_{\text{mix}}(t), \quad (2.16)$$

for all  $t \geq 0$ . Let  $\nu$  denote the law of  $(\bar{E}(t))$  and let  $\mathbf{e} \in \bar{B}_\alpha$ . We will use (2.16) in particular when  $\mathbf{e}_0 = \mathbf{e}$  a.s. and  $\mathbf{e}'_0$  has law  $\nu$ . Then (2.16) gives the following mixing estimate:

$$\|e^{tA} \varphi(\mathbf{e}) - \langle \varphi, \nu \rangle\|_F \leq \alpha \|\varphi\|_{\text{Lip}} \gamma_{\text{mix}}(t), \quad (2.17)$$

for all  $t \geq 0$ , for all  $\mathbf{e} \in \bar{B}_\alpha$ . The estimate (2.17) has an extension to quadratic functionals: for all linear and continuous  $\Lambda: F \rightarrow \mathbb{R}$ , for all bi-linear and continuous  $q: F \times F \rightarrow \mathbb{R}$ , we have, for all  $\mathbf{e} \in \bar{B}_\alpha$ ,

$$\|e^{tA} [\Lambda + q](\mathbf{e}) - \langle \Lambda + q, \nu \rangle\|_F \leq \alpha (\|\Lambda\|_{B(F)} + 2\alpha \|q\|_{B(F \times F)}) \gamma_{\text{mix}}(t), \quad (2.18)$$

where  $\|\Lambda\|_{B(F)}$  is norm of linear form of  $\Lambda$  and  $\|q\|_{B(F \times F)}$  is the norm of bi-linear form of  $q$ . Note that, actually,  $\langle \Lambda, \nu \rangle = 0$  by (2.7). The factor  $\alpha$  in front of  $\|q\|_{B(F \times F)}$  in (2.18) is due to the decomposition (recall that  $\mathbf{e}_0 = \mathbf{e}$  a.s. and  $\mathbf{e}'_0$  has law  $\nu$ )

$$\begin{aligned} e^{tA} q(\mathbf{e}) - \langle q, \nu \rangle &= \mathbb{E} [q(E_t^*(\mathbf{e}_0), E_t^*(\mathbf{e}_0)) - q(E_t^*(\mathbf{e}_0), \mathbb{E}_t^*(\mathbf{e}'_0))] \\ &\quad + \mathbb{E} [q(E_t^*(\mathbf{e}_0), E_t^*(\mathbf{e}'_0)) - q(E_t^*(\mathbf{e}'_0), E_t^*(\mathbf{e}'_0))]. \end{aligned}$$

We have indeed

$$\begin{aligned} |e^{tA} q(\mathbf{e}) - \langle q, \nu \rangle| &\leq \|q\|_{B(F \times F)} \mathbb{E} [(\|E_t^*(\mathbf{e}_0)\|_F + \|E_t^*(\mathbf{e}'_0)\|_F) \|E_t^*(\mathbf{e}_0) - E_t^*(\mathbf{e}'_0)\|_F] \\ &\leq 2\alpha \|q\|_{B(F \times F)} \mathbb{E} \|E_t^*(\mathbf{e}_0) - E_t^*(\mathbf{e}'_0)\|_F \quad \text{by (2.6),} \\ &\leq 2\alpha^2 \|q\|_{B(F \times F)} \gamma_{\text{mix}}(t) \quad \text{by (2.9).} \end{aligned}$$

Without loss of generality (as we can rescale  $\gamma_{\text{mix}}$  if we rescale  $\alpha$ ), we assume

$$\|\gamma_{\text{mix}}\|_{L^1(\mathbb{R}_+)} = 1. \quad (2.19)$$

Using (2.17), the resolvent

$$R_\lambda \varphi(\mathbf{e}) := \int_0^\infty e^{-\lambda t} (e^{tA} \varphi)(\mathbf{e}) dt, \quad (2.20)$$

is well defined for all  $\lambda \geq 0$ ,  $\mathbf{e} \in \bar{B}_\alpha$  and all  $\varphi: F \rightarrow \mathbb{R}$  which is Lipschitz continuous and satisfies the cancellation condition  $\langle \varphi, \nu \rangle = 0$ . Using (2.7), we can therefore define  $R_\lambda \varphi(\mathbf{e})$  for  $\lambda \geq 0$ , where  $\varphi$  is the identity of  $F$ . By a slight abuse of notation, we write  $R_\lambda(\mathbf{e})$  in that case. By (2.9) (with  $\mathbf{e}_0 = \mathbf{e}$  a.s. and  $\mathbf{e}'_0 \sim \nu$ ) and (2.19), we have

$$\|R_0(\mathbf{e})\|_F \leq \alpha, \quad (2.21)$$

for all  $\mathbf{e}$  with  $\|\mathbf{e}\|_F \leq \alpha$ . We will assume that, there exists a constant  $C_\alpha^0 \geq 0$  such that, for all linear functional  $\Lambda: F \rightarrow \mathbb{R}$ , we have

$$|A[\Lambda(R_0)^2](\mathbf{e})| \leq C_\alpha^0 \|\Lambda\|_{B(F)}^2, \quad |A[\Lambda(R_0)](\mathbf{e})| \leq C_\alpha^0 \|\Lambda\|_{B(F)}, \quad (2.22)$$

for all  $\mathbf{e}$  with  $\|\mathbf{e}\|_F \leq \alpha$ .

### 2.3 Covariance

Our mixing hypothesis has the following consequence on the covariances of  $(E_t)$  and  $(\bar{E}_t)$ : let

$$\Gamma_{\mathbf{e}}(s, t) = \mathbb{E}[E_s(\mathbf{e}) \otimes E_t(\mathbf{e})], \quad \bar{\Gamma}(t) = \mathbb{E}[\bar{E}(t) \otimes \bar{E}(0)]. \quad (2.23)$$

Let  $t \geq s \geq r \geq 0$ . Conditioning on  $\mathcal{G}_{t-s}^E$ , we have

$$\Gamma_{\mathbf{e}}(t-r, t-s) = e^{(t-s)A} (e^{(s-r)A} \theta \otimes \theta)(\mathbf{e}), \quad \theta(\mathbf{e}) = \mathbf{e}$$

It follows from (2.18) that, for all  $\mathbf{e}$  with  $\|\mathbf{e}\|_F \leq \alpha$ ,

$$\|\Gamma_{\mathbf{e}}(t-r, t-s) - \bar{\Gamma}(s-r)\|_F \leq 2\alpha^2 \gamma_{\text{mix}}(t-s). \quad (2.24)$$

Note also that, for all  $i, j \in \{1, \dots, N\}$ ,  $x, y \in \mathbb{T}^N$ , we have

$$\bar{\Gamma}_{ij}(t, x, y) = \mathbb{E} \left( [\bar{E}_t(x)]_i [\bar{E}_0(y)]_j \right) = \int_{\bar{B}_\alpha} e^{tA} \pi_{x,i}(\mathbf{e}) \pi_{y,j}(\mathbf{e}) d\nu(\mathbf{e}),$$

where  $\pi_{x,i}(\mathbf{e}) := \mathbf{e}_i(x)$ . Consequently, (2.18) implies that

$$\|\bar{\Gamma}_{ij}(t)\|_{H^{\bar{\sigma}}(\mathbb{T}_x^N \times \mathbb{T}_y^N)} \leq 2\alpha^2 \gamma_{\text{mix}}(t), \quad (2.25)$$

for all  $i, j \in \{1, \dots, N\}$ .

## 2.4 Some simple examples

Let  $(E_n(\mathbf{e}))_{n \geq 0}$  be a Markov chain on  $F$  with  $E_0(\mathbf{e}) = \mathbf{e}$ , and let  $(N_t)_{t \geq 0}$  be a Poisson process of rate 1 ( $N_0 = 0$ ). Set

$$E(t, s; \mathbf{e}_0) = E_{N_{t-s}}(\mathbf{e}_0). \quad (2.26)$$

We assume that the ball  $\bar{B}_\alpha$  of  $F$  is stable by  $(E_n)$ . We set  $\bar{E}_t = E(t, 0; \bar{\mathbf{e}}_0)$ , where  $\bar{\mathbf{e}}_0$  is a random variable of law  $\nu$  independent on  $(E_n(\mathbf{e}))_{n \geq 0}$  and  $(N_t)_{t \geq 0}$ . Then  $(\bar{E}_t)$  is a stationary process (it is a time-homogeneous Markov process and is initially at equilibrium). With the definition (2.26) adopted above, two processes  $(E_t(\mathbf{e}_0))$  and  $(E_t(\mathbf{e}'_0))$  are coupled (this is a synchronous coupling). Indeed,  $E_t(\mathbf{e}_0) = E_t(\mathbf{e}'_0)$  as soon as  $t \geq T_1$ , where  $T_1$  is the time of occurrence of the first jump of  $N_t$ . Let  $\delta \in (0, 1)$  and let  $\gamma_{\text{mix}}(t) = C_{\text{mix}}e^{-\delta t}$ , where  $C_{\text{mix}}$  is a constant. We have

$$e^{\delta t} \|E_t(\mathbf{e}_0) - E_t(\mathbf{e}'_0)\|_F \leq 2\alpha e^{\delta t} \mathbf{1}_{t \leq T_1} \leq 2\alpha e^{\delta T_1}.$$

Since  $\mathbb{E}e^{\delta T_1} < +\infty$ , this gives (2.12) with a suitable choice of  $C_{\text{mix}}$ . In addition, the semi-group, generator and resolvent  $R_0$  have the explicit forms

$$e^{tA}\varphi(e) = e^{-t}\varphi(\mathbf{e}) + (1 - e^{-t})\langle \varphi, \nu \rangle,$$

and

$$A\varphi(\mathbf{e}) = \langle \varphi, \nu \rangle - \varphi(\mathbf{e}), \quad R_0\varphi(\mathbf{e}) = \mathbf{e}.$$

From these formula, we deduce the second inequality in (2.22) with  $C_\alpha^0 \geq \alpha$ . The first inequality in (2.22) is obtained with any  $C_\alpha^0 \geq 2\alpha^2$ .

An other instance of mixing force field is a function  $\eta(X_t)$ ,  $\eta: \mathbb{R}^m \rightarrow F$ , of an Ornstein-Uhlenbeck process  $(X_t)$  on  $\mathbb{R}^m$ :

$$dX_t = -X_t dt + \sqrt{2}dB_t, \quad (2.27)$$

where  $(B_t)$  is a Wiener process on  $\mathbb{R}^m$ . We choose  $\eta$  Lipschitz and taking values in the ball  $\bar{B}_\alpha$  of  $F$ . We will not develop that example much, but simply check that the mixing condition (2.12) is also satisfied here. Since  $\eta$  is Lipschitz, it is sufficient to check it directly on  $(X_t)$ . We use once again a synchronous coupling: let

$$X^*(t; X_0) = e^{-t}X_0 + \sqrt{2} \int_0^t e^{-(t-s)}dB_s, \quad X^*(t; X'_0) = e^{-t}X'_0 + \sqrt{2} \int_0^t e^{-(t-s)}dB_s$$

and let  $\gamma_{\text{mix}}(t) = C_{\text{mix}}e^{-t}$ , where  $C_{\text{mix}}$  is a constant. Then

$$e^t |X^*(t; X_0) - X^*(t; X'_0)| \leq |X_0| + |X'_0|,$$

hence (2.12) is satisfied provided we limit ourselves to initial laws  $\mu$  and  $\mu'$  with first moment below a given threshold. This is not a limitation since the invariant measure associated to (2.27), which is Gaussian, has a finite first moment.

## 2.5 Mixing force process

In Theorem 1.1 and Section 5, we consider the case where  $\bar{E}_t$  is independent on  $x$ . This means that the state space is  $\mathbb{R}^N$ , and  $(\bar{E}_t)$  is simply a process on  $\mathbb{R}^N$ . In this simpler framework, the notion of mixing force field is reduced to the following notion of mixing force process.

**Definition 2.2** (Mixing force process). Let  $(\bar{E}_t)$  be a càdlàg, stationary, homogeneous Markov process of generator  $A$  over  $\mathbb{R}^N$ . We say that  $(\bar{E}_t)$  is a mixing force process if the conditions (2.28), (2.29), (2.30) below are satisfied.

Condition (2.28) is the condition of localization

$$|E(t; \mathbf{e})| \leq \alpha, \quad (2.28)$$

almost-surely, for all  $t \geq 0$ , for all  $\mathbf{e} \in \bar{B}_\alpha$ , where  $\bar{B}_\alpha$  is the closed ball of center 0 and radius  $\alpha$  in  $\mathbb{R}^N$ . We require then that the invariant measure  $\nu$  of  $(E_t)$  is supported in  $\bar{B}_\alpha$  and that

$$\int_{\mathbb{R}^N} \mathbf{e} d\nu(\mathbf{e}) = \mathbb{E}[\bar{E}_t] = 0. \quad (2.29)$$

The mixing hypothesis is

$$W(\mathbb{P}_\mu, \mathbb{P}_{\mu'}) \leq \alpha \quad (2.30)$$

like in (2.12), except that now, in  $W$ ,  $D(\mathbb{R}_+; F)$  has to be replaced by the state space  $D(\mathbb{R}_+; \mathbb{R}^N)$  (and  $\mu$  and  $\mu'$  are probability measures on  $\mathbb{R}^N$ ). As in (2.9), taking  $\mathbf{e}_0$  of law  $\mu$  and  $\mathbf{e}'_0$  of law  $\mu'$ , this implies the existence of a coupling  $((E_t^*(\mathbf{e}_0))_{t \geq 0}, (E_t^*(\mathbf{e}'_0))_{t \geq 0})$  of  $((E_t(\mathbf{e}_0))_{t \geq 0}, (E_t(\mathbf{e}'_0))_{t \geq 0})$  such that

$$\mathbb{E}|E_t^*(\mathbf{e}_0) - E_t^*(\mathbf{e}'_0)| \leq \alpha \gamma_{\text{mix}}(t), \quad (2.31)$$

for all  $t \geq 0$ .

## 3 Unperturbed equation: ergodic properties

We consider first the equation

$$\partial_t f_t + \bar{E}(t) \cdot \nabla_v f_t = Q f_t \quad t > 0, v \in \mathbb{R}^N, \quad (3.1)$$

where  $Q = Q_{\text{LB}}$  or  $Q = Q_{\text{FP}}$ . In (3.1),  $\bar{E}(t)$  should stand for  $\bar{E}(x, t)$ , where  $(\bar{E}(t))$  is a mixing force field, since (3.1) is the instance of Equation 1.1 obtained for  $\varepsilon = 0$ . However,  $x$  is just a parameter and we may as well consider that  $(\bar{E}(t))$  is a mixing force process. Thus, *in all this section*,  $(\bar{E}(t))$  is a mixing force process in the sense of Definition 2.2.

To find the invariant measure for (3.1), we solve the equation starting from a given time  $s \in \mathbb{R}$ , and then let  $s \rightarrow -\infty$ . More precisely, given  $\mathbf{e} \in \mathbb{R}^N$ , we consider the following evolution equation:

$$\partial_t f_t + E(t, s; \mathbf{e}) \cdot \nabla_v f_t = Q f_t \quad t > s, v \in \mathbb{R}^N. \quad (3.2)$$

Let  $f \in L^1(\mathbb{R}^N)$  and  $s \in \mathbb{R}$ . The solution to (3.2) with initial condition  $f_{t=s} = f$  is

$$f_{s,t}^{\text{LB}}(v) = e^{-(t-s)} f \left( v - \int_s^t E(r, s; \mathbf{e}) dr \right) + \rho(f) \int_s^t e^{-(t-\sigma)} \left[ M \left( v - \int_\sigma^t E(r, s; \mathbf{e}) dr \right) \right] d\sigma, \quad (3.3)$$

when  $Q = Q_{\text{LB}}$ , and

$$f_{s,t}^{\text{FP}}(v) = e^{N(t-s)} \int_{\mathbb{R}^N} f \left( e^{(t-s)} v - \int_s^t e^{-(s-\sigma)} E(\sigma, s, \mathbf{e}) d\sigma + \sqrt{e^{2(t-s)} - 1} w \right) M(w) dw, \quad (3.4)$$

when  $Q = Q_{\text{FP}}$ . A brief explanation to (3.3) and (3.4) is given in Appendix A. By the term ‘‘solution to (3.2)’’, we mean weak solutions, *i.e.* functions  $f \in C([s, +\infty); L^1(\mathbb{R}^N))$  satisfying the identity

$$\langle f_t, \varphi \rangle = \langle f, \varphi \rangle + \int_s^t \langle f_\sigma, E(\sigma, t; \mathbf{e}) \cdot \nabla_v \varphi \rangle + \langle f_\sigma, Q^* \varphi \rangle d\sigma,$$

almost-surely, for all  $\varphi \in C_c^\infty(\mathbb{R}^N)$ , for all  $t \geq s$ . We may also consider mild solutions (this is equivalent, actually), as we do in Section 4. We do not need to be very specific on that point here. All that matter to us is to understand the limit behaviour of  $f_{s,t}$  defined by (3.3)-(3.4) when  $s \rightarrow -\infty$ . This is the content of the following result, Theorem 3.1.

**Theorem 3.1** (Invariant solutions). *Let  $(\bar{E}(t))$  be a mixing force process in the sense of Definition 2.2. Let  $f_{s,t}^{\text{LB}}$  and  $f_{s,t}^{\text{FP}}$  be defined by (3.3) and (3.4) respectively, with  $\mathbf{e} \in \bar{B}_\alpha$ . Then*

$$(f_{s,t}^{\text{LB}}, E(t, s; \mathbf{e})) \rightarrow (\rho(f) \bar{M}_t^{\text{LB}}, \bar{E}_t) \text{ and } (f_{s,t}^{\text{FP}}, E(t, s; \mathbf{e})) \rightarrow (\rho(f) \bar{M}_t^{\text{FP}}, \bar{E}_t) \quad (3.5)$$

in law on  $L^1(\mathbb{R}^N) \times \mathbb{R}^N$  when  $s \rightarrow -\infty$ , where  $\bar{M}_t^{\text{LB}}$  and  $\bar{M}_t^{\text{FP}}$  are defined by

$$\bar{M}_t^{\text{LB}} = \int_{-\infty}^t e^{-(t-\sigma)} \left[ M \left( v - \int_\sigma^t \bar{E}(r) dr \right) \right] d\sigma, \quad (3.6)$$

and

$$\bar{M}_t^{\text{FP}} = M \left( v - \int_{-\infty}^t e^{-(t-r)} \bar{E}(r) dr \right), \quad (3.7)$$

respectively.

We denote by  $\mu_\rho$  the invariant measure (parametrized by  $\rho$ ) defined by

$$\langle \varphi, \mu_\rho \rangle = \mathbb{E} \varphi(\rho \bar{M}_t, \bar{E}_t), \quad (3.8)$$

for all continuous and bounded function  $\varphi$  on  $L^1(\mathbb{R}^N) \times \mathbb{R}^N$ .



*Remark 3.1.* We call  $\bar{M}_t^{\text{LB}}$  and  $\bar{M}_t^{\text{FP}}$  the invariant solutions, since their laws are the invariant measure for (3.1). Note that  $(\bar{E}(r))$  in (3.6) and (3.7) is defined for all  $r \in \mathbb{R}$  (see the discussion and convention of notations around (2.4)).

*Remark 3.2.* Let  $\varphi$  be a bounded continuous function on  $\mathbb{R}^N \times \mathbb{R}^N$ . Similarly to (1.12), we have, by conditioning on the natural filtration  $(\mathcal{F}_t^E)$  of  $(E_t)$ :

$$\mathbb{E} [\varphi(V_{s,t}, E(t, s; \mathbf{e}))] = \mathbb{E} \int_{\mathbb{R}^N} f_{s,t}(v) \varphi(v, E(t, s; \mathbf{e})) dv, \quad (3.9)$$

where  $V_{s,t}$  is the solution to (1.9) or (1.10) (with  $\bar{E}(t)$  instead of  $\bar{E}(t, X_t)$ ) starting from  $V_s$  at time  $t = s$ , where  $V_s$  follows the law of density  $f$  with respect to the Lebesgue measure on  $\mathbb{R}^N$ . Since

$$\Phi: (f, \mathbf{e}) \mapsto \int_{\mathbb{R}^N} f(v) \varphi(v, \mathbf{e}) dv$$

is continuous and bounded on  $L^1(\mathbb{R}^N) \times \mathbb{R}^N$ , we deduce from Theorem 3.1 that

$$\lim_{s \rightarrow -\infty} \mathbb{E} [\varphi(V_{s,t}, E(t, s; \mathbf{e}))] = \langle \lambda_\rho, \varphi \rangle := \rho \mathbb{E} \int_{\mathbb{R}^N} \bar{M}_t(v) \varphi(v, \bar{E}_t) dv, \quad (3.10)$$

where  $\rho = \rho(f)$ .

The proof of Theorem 3.1 uses the estimates (3.13) and (3.14) in the following lemma.

**Lemma 3.2.** *For  $w, z \in \mathbb{R}^N$ , we have the estimates and identities*

$$\|M(\cdot - w)\|_{L^2(M^{-1})}^2 = e^{|w|^2}, \quad (3.11)$$

$$\|M(\cdot - w) - M(\cdot - z)\|_{L^2(M^{-1})}^2 = e^{|w|^2} + e^{|z|^2} - 2e^{w \cdot z}, \quad (3.12)$$

in  $L^2(M^{-1})$ , and

$$\|M(\cdot - w)\|_{L^1(\mathbb{R}^N)} = 1, \quad (3.13)$$

$$\|M(\cdot - w) - M(\cdot - z)\|_{L^1(\mathbb{R}^N)} \leq 2 \wedge \left[ \frac{|w - z|}{(1 - |w - z|)^+} \right]^{1/2} \quad (3.14)$$

in  $L^1(\mathbb{R}^N)$ .

*Proof of Lemma 3.2.* Standard manipulations and identities for Gaussian densities give (3.11), (3.12) and (3.13) (one can also use (3.15) below to prove (3.11) and (3.12)). By (3.13) and the triangular inequality, we have the bound by 2 in (3.14). To obtain the second estimate, we use the identity

$$\|M(\cdot - w) - M(\cdot - z)\|_{L^1(\mathbb{R}^N)} = \|M(\cdot - w + z) - M\|_{L^1(\mathbb{R}^N)},$$

and the expansion

$$M(v - w) = \frac{1}{(2\pi)^{N/2}} e^{-\frac{|v-w|^2}{2}} = M(w) \sum_{n \in \mathbb{N}^N} H_n(v) w^n, \quad (3.15)$$

where  $H_n$  is the  $n$ -th Hermite polynomial (see [16, Section 1.1.1]). This yields the inequality

$$\|M(\cdot - w) - M\|_{L^1(\mathbb{R}^N)} \leq M(w) \sum_{n \in \mathbb{N}^N \setminus \{0\}} \|H_n\|_{L^1(\mathbb{R}^N)} |w|^n.$$

Since  $\|H_n\|_{L^1(\mathbb{R}^N)} \leq \|H_n\|_{L^2(M^{-1})} = \frac{1}{\sqrt{n!}}$  (cf. [16, Lemma 1.1.1]), the Cauchy-Schwarz inequality yields, for  $|w| < 1$ ,

$$\|M(\cdot - w) - M\|_{L^1(\mathbb{R}^N)} \leq M(w) \left[ \frac{e^{|w|}|w|}{1 - |w|} \right]^{1/2} \leq \left[ \frac{|w|}{1 - |w|} \right]^{1/2}.$$

Indeed, setting  $a = |w|$ , we have  $a \in [0, 1]$  and

$$M(w)e^{|w|/2} = \left[ \frac{1}{(2\pi)^N} e^{a-a^2} \right]^{1/2} \leq \left[ \frac{1}{(2\pi)^N} e^{1/4} \right]^{1/2} \leq 1$$

since  $e^{1/4} \leq 2\pi$ .  $\square$

*Proof of Theorem 3.1.* Let  $\mathbf{e} \in \bar{B}_\alpha$   $t \in \mathbb{R}$ , let  $\Phi: L^1(\mathbb{R}^N) \times F \rightarrow \mathbb{R}$  be a bounded and uniformly continuous function and let  $\varepsilon > 0$ . Our aim is to show that

$$\mathbb{E}\Phi(f_{s,t}(v), E(t, s; \mathbf{e})) - \mathbb{E}\Phi(\rho\bar{M}_t, \bar{E}_t) < K\varepsilon, \quad (3.16)$$

for  $s < \min(0, t)$ ,  $|s|$  large enough, where  $K$  is a finite constant (it will turn out that  $K = 5+$ , but this does not matter). Note that it is sufficient to consider uniformly continuous functions in (3.16), cf. Proposition I-2.4 in [13]. We denote by  $\eta$  a modulus of uniform continuity of  $\Phi$  associated to  $\varepsilon$ .

**Step 1. Reduction to the case  $f \in L^2(M^{-1})$ .** The maps  $f \mapsto f_{s,t}$ ,  $f \mapsto \rho(f)\bar{M}_t$  are continuous on  $L^1$ , uniformly in  $s \leq t$ :

$$\|f_{s,t}\|_{L^1(\mathbb{R}^N)}, \|\rho(f)\bar{M}_t^{\text{LB}}\|_{L^1(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)}.$$

Using the uniform continuity of  $\Phi$  on  $K$ , we have

$$\mathbb{E}\Phi(f_{s,t}(v), E(t, s; \mathbf{e})) - \mathbb{E}\Phi(\rho\bar{M}_t, \bar{E}_t) < 2\varepsilon + \mathbb{E}\Phi(\tilde{f}_{s,t}, E(t, s; \mathbf{e})) - \mathbb{E}\Phi(\rho(\tilde{f})\bar{M}_t, \bar{E}_t)$$

if  $\|f - \tilde{f}\|_{L^1(\mathbb{R}^N)} < \eta$ . Therefore, to prove (3.16), we turn to the case  $f \in L^2(M^{-1})$ .

**Step 2. Cut-off after time  $s$ .** For  $s \leq t$ , introduce

$$\bar{M}_{s,t}^{\text{LB}} = \int_s^t e^{-(t-\sigma)} \left[ M \left( v - \int_\sigma^t \bar{E}(r) dr \right) \right] d\sigma, \quad (3.17)$$

and

$$\bar{M}_{s,t}^{\text{FP}} = M \left( v - \int_s^t e^{-(t-r)} \bar{E}(r) dr \right). \quad (3.18)$$

We have  $\|\bar{M}_{s,t}^{\text{LB}} - \bar{M}_t^{\text{LB}}\|_{L^1(\mathbb{R}^N)} \leq e^{-(t-s)}$  by a direct computation and

$$\|\bar{M}_{s,t}^{\text{FP}} - \bar{M}_t^{\text{FP}}\|_{L^1(\mathbb{R}^N)} \leq b \left( \int_{-\infty}^s e^{-(t-r)} \bar{E}(r) dr \right)$$

where  $b(|w - z|)$  is the right-hand side of (3.14). We use the bound

$$b(r) \leq \frac{\sqrt{5}}{2} r^{1/2} \quad (3.19)$$

and (2.6) to obtain, almost-surely,  $\|\bar{M}_{s,t}^{\text{FP}} - \bar{M}_t^{\text{FP}}\|_{L^1(\mathbb{R}^N)} \leq \frac{\sqrt{5}}{2} \alpha^{1/2} e^{-\frac{1}{2}(t-s)}$ . To sum up, in both the LB and FP case, we have a bound almost-sure on  $\|\bar{M}_{s,t} - \bar{M}_t\|_{L^1(\mathbb{R}^N)}$  by a deterministic quantity which tends to 0 when  $t - s \rightarrow +\infty$ . It follows that, for  $t - s$  large enough,

$$|\mathbb{E}\Phi(\rho(f)\bar{M}_t) - \mathbb{E}\Phi(\rho(f)\bar{M}_{s,t}, \bar{E}_t)| < \varepsilon.$$

In the next step we prove that

$$\mathbb{E}\Phi(f_{s,t}, E(t, s; \mathbf{e})) - \mathbb{E}\Phi(\rho(f)\bar{M}_{s,t}) < 2\varepsilon, \quad (3.20)$$

for  $t - s$  large enough.

**Step 3. Convergence in law.** Let  $\mathbf{e} \in \bar{B}_\alpha$ . Let  $\mathbf{e}_0 = \mathbf{e}$  a.s. and  $\mathbf{e}'_0 = \bar{E}_s$ . Since  $E(s, t; \mathbf{e})$  has the same law as  $E_{t-s}(\mathbf{e}_0)$  and  $\bar{E}(t)$  has the same law as  $E_{t-s}(\mathbf{e}'_0)$ , (2.31) gives a coupling

$$(E(s, t; \mathbf{e}), \bar{E}(t))_{t \geq s} \rightarrow (E^*(s, t; \mathbf{e}), \bar{E}_t^*)_{t \geq s}$$

such that

$$\mathbb{E}\|E^*(t, s; \mathbf{e}) - \bar{E}_t^*\|_F \leq \alpha \gamma_{\text{mix}}(t - s), \quad (3.21)$$

for all  $t \geq s$ . We have

$$\mathbb{E}\Phi(f_{s,t}, E(t, s; \mathbf{e})) - \mathbb{E}\Phi(\rho(f)\bar{M}_{s,t}, \bar{E}_t) = \mathbb{E}\Phi(f_{s,t}^*, E^*(s, t; \mathbf{e})) - \mathbb{E}\Phi(\rho(f)\bar{M}_{s,t}^*, \bar{E}_t^*), \quad (3.22)$$

where the superscript star in  $f_{s,t}$  and  $\bar{M}_{s,t}$  indicates that  $E(s, t; \mathbf{e})$  has been replaced by  $E^*(s, t; \mathbf{e})$  and  $\bar{E}(t)$  by  $\bar{E}_t^*$ . Since

$$\begin{aligned} & \mathbb{E}\Phi(f_{s,t}^*, E^*(s, t; \mathbf{e})) - \mathbb{E}\Phi(\rho(f)\bar{M}_{s,t}^*, \bar{E}_t^*) \\ & \leq \varepsilon + \|\Phi\|_{BC} \left[ \mathbb{P}(\|f_{s,t}^* - \rho(f)\bar{M}_{s,t}^*\|_{L^1(\mathbb{R}^N)} > \eta) + \mathbb{P}(\|E^*(s, t; \mathbf{e}) - \bar{E}_t^*\|_F > \eta) \right], \end{aligned}$$

it is sufficient to prove that  $f_{s,t}^* - \rho(f)\bar{M}_{s,t}^* \rightarrow 0$  and  $E^*(s, t; \mathbf{e}) - \bar{E}_t^* \rightarrow 0$  in probability on  $L^1(\mathbb{R}^N)$  and  $F$  respectively. We show the strongest (strongest, as is proved classically by means of the Markov inequality) property

$$\lim_{s \rightarrow -\infty} \mathbb{E}\|f_{s,t}^* - \rho(f)\bar{M}_{s,t}^*\|_{L^1(\mathbb{R}^N)} = 0, \quad \lim_{s \rightarrow -\infty} \mathbb{E}\|E^*(s, t; \mathbf{e}) - \bar{E}_t^*\|_F = 0. \quad (3.23)$$

The second limit in (3.23) is a consequence of (3.21). Let us prove the first limit. Consider first the LB case. Using (3.13) and the estimate  $|\rho(f)| \leq \|f\|_{L^1(\mathbb{R}^N)}$ , we have

$$\begin{aligned} \mathbb{E}\|f_{s,t}^{\text{LB},*} - \rho(f)\bar{M}_{s,t}^{\text{LB},*}\|_{L^1(\mathbb{R}^N)} &\leq \|f\|_{L^1(\mathbb{R}^N)}e^{-(t-s)} \\ &\quad + \|f\|_{L^1(\mathbb{R}^N)}\mathbb{E}\int_s^t e^{-(t-\sigma)}b\left(\int_\sigma^t |E^*(r,s,\mathbf{e}) - \bar{E}^*(r)|dr\right)d\sigma, \end{aligned}$$

where, as in (3.19), we denote by  $b(|w-z|)$  the right-hand side of (3.14). From (3.19) follows

$$2b(r) \leq \varepsilon + \frac{5}{4\varepsilon}r.$$

We deduce the estimate

$$\begin{aligned} \mathbb{E}\|f_{s,t}^{\text{LB},*} - \rho(f)\bar{M}_{s,t}^{\text{LB},*}\|_{L^1(\mathbb{R}^N)} &\leq \|f\|_{L^1(\mathbb{R}^N)}(e^{-(t-s)} + \varepsilon) \\ &\quad + \frac{5}{4\varepsilon}\|f\|_{L^1(\mathbb{R}^N)}\int_s^t e^{-(t-r)}\mathbb{E}|E^*(r,s,\mathbf{e}) - \bar{E}^*(r)|dr. \end{aligned}$$

By (3.21), this yields the following estimate:

$$\begin{aligned} \mathbb{E}\|f_{s,t}^{\text{LB},*} - \rho(f)\bar{M}_{s,t}^{\text{LB},*}\|_{L^1(\mathbb{R}^N)} &\leq \|f\|_{L^1(\mathbb{R}^N)}\left(e^{-(t-s)} + \varepsilon + \frac{5\alpha}{4\varepsilon}\int_s^t e^{-(t-r)}\gamma_{\text{mix}}(t-r)dr\right) \\ &= \|f\|_{L^1(\mathbb{R}^N)}\left(e^{-(t-s)} + \varepsilon + \frac{5\alpha}{4\varepsilon}\int_0^{t-s} e^{r-(t-s)}\gamma_{\text{mix}}(r)dr\right). \end{aligned} \tag{3.24}$$

We fix  $r_1$  such that  $\frac{5}{4}\alpha\int_{r_1}^\infty \gamma_{\text{mix}}(r)dr < \varepsilon^2$ . Then

$$\frac{5}{4}\alpha\int_0^{t-s} e^{r-(t-s)}\gamma_{\text{mix}}(r)dr \leq \varepsilon^2 + \frac{5}{4}\alpha\int_0^{r_1} \gamma_{\text{mix}}(r)dr e^{r_1-(t-s)} < 2\varepsilon^2$$

for  $t-s$  large enough and (3.23) follows from (3.24). In the FP case, we start first from the exponential estimate

$$\|f_{s,t}^{\text{FP}}|_{E\equiv 0} - \rho(f)M\|_{L^2(M^{-1})} \leq e^{s-t}\|f\|_{L^2(M^{-1})}. \tag{3.25}$$

In (3.25),  $f_{s,t}^{\text{FP}}|_{E\equiv 0}$  denote the function (3.4) obtained when  $E \equiv 0$ . The estimate (3.25) is a consequence of the dual estimate in  $L^2(M)$  for functions  $h$  such that  $\langle h, M \rangle_{L^2(\mathbb{R}^N)} = 0$ , cf. [1, p. 179]. It implies

$$\|f_{s,t}^{\text{FP}}|_{E\equiv 0} - \rho(f)M\|_{L^1(\mathbb{R}^N)} \leq e^{s-t}\|f\|_{L^2(M^{-1})}. \tag{3.26}$$

The translations

$$v \mapsto v - \int_s^t e^{-(t-\sigma)}\tilde{E}(\sigma,s,\mathbf{e})d\sigma, \quad v \mapsto v - \int_s^t e^{-(t-\sigma)}\tilde{E}_s^*(\sigma)d\sigma,$$

leave invariant the  $L^1$ -norm. Therefore (3.26) yields

$$\begin{aligned} & \mathbb{E} \|f_{s,t}^{\text{FP},*} - \rho(f) \bar{M}_t^{\text{FP},*}\|_{L^1(\mathbb{R}^N)} \leq e^{s-t} \|f\|_{L^2(M^{-1})} \\ & + |\rho(f)| \mathbb{E} \left\| M \left( \cdot - \int_s^t e^{-(t-\sigma)} \bar{E}^*(\sigma) d\sigma \right) - M \left( \cdot - \int_s^t e^{-(t-\sigma)} E^*(\sigma, s, \mathbf{e}) d\sigma \right) \right\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

We conclude as in the case  $Q = Q_{\text{LB}}$  by means of (3.14).  $\square$

## 4 Resolution of the kinetic equation

We consider the resolution of the Cauchy problem of (1.1) or (1.7) at fixed  $\varepsilon > 0$ . We set  $\varepsilon = 1$  for simplicity. Then (1.1) and (1.7) are the same equation

$$\partial_t f + v \cdot \nabla_x f + \bar{E}(t, x) \cdot \nabla_v f = Qf. \quad (4.1)$$

More generally, what matters to us is the dynamics given by  $(f, \mathbf{e}) \mapsto (f_t, E_t(\mathbf{e}))$ , where  $f_t$  is the solution to the equation

$$\partial_t f + v \cdot \nabla_x f + E(t, x) \cdot \nabla_v f = Qf, \quad (4.2)$$

with  $E(t, x) = E_t(\mathbf{e}(x))$ . Therefore, this is (4.2) which we solve. We simply assume that  $t \mapsto E(t, \cdot)$  is a càdlàg function with values in  $F$  (see Section 2 for the definition of the state space  $F$ ). In the particular case  $E(t, x) = E_t(\mathbf{e}(x))$ , we define in this way pathwise solutions. We solve the Cauchy Problem for (4.2) in the LB-case and in the FP-case in Section 4.1 and Section 4.2 respectively. Then, in Section 4.3, we establish the Markov property of the process  $(f_t, E_t(\mathbf{e}))$ , where the first component  $f_t$  is the solution to (4.2) with the forcing  $E(t, x) = E_t(\mathbf{e}(x))$ .

### 4.1 Cauchy Problem in the LB case

Let  $t \mapsto E(t, \cdot)$  be a càdlàg function with values in  $F$ . Let  $\Phi_t(x, v) = (X_t(x, v), V_t(x, v))$  denote the flow associated to the field  $(v, E(t, x))$ :

$$\begin{aligned} \dot{X}_t &= V_t, & X_0 &= x, \\ \dot{V}_t &= E(t, X_t), & V_0 &= v. \end{aligned}$$

The partial map  $(x, v) \mapsto \Phi_t(x, v)$  is a  $C^1$ -diffeomorphism of  $\mathbb{T}^N \times \mathbb{R}^N$ . We denote by  $\Phi^t$  the inverse application:  $\Phi^t \circ \Phi_t = \text{Id}$ . Note that  $\Phi^t$  and  $\Phi_t$  preserve the Lebesgue measure on  $\mathbb{T}^N \times \mathbb{R}^N$ .

**Definition 4.1** (Mild solution, LB case). Let  $f_{\text{in}} \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$ . Assume  $Q = Q_{\text{LB}}$ . A continuous function from  $[0, T]$  to  $L^1(\mathbb{T}^N \times \mathbb{R}^N)$  is said to be a mild solution to (4.2) with initial datum  $f_{\text{in}}$  if

$$f(t) = e^{-t} f_{\text{in}} \circ \Phi^t + \int_0^t e^{-(t-s)} [\rho(f(s)) M] \circ \Phi^{t-s} ds, \quad (4.3)$$

for all  $t \in [0, T]$ .

**Proposition 4.1** (The Cauchy Problem, LB case). *Let  $f_{\text{in}} \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$ . Assume (2.6). There exists a unique mild solution to (4.2) in  $C([0, T]; L^1(\mathbb{T}^N \times \mathbb{R}^N))$  with initial datum  $f_{\text{in}}$ . It satisfies*

$$\|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} \leq \|f_{\text{in}}\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} \quad \text{for all } t \in [0, T]. \quad (4.4)$$

*If  $f_{\text{in}} \geq 0$ , then  $f(t) \geq 0$  for all  $t \in [0, T]$  and (4.4) is an identity. In addition, if  $f_{\text{in}} \in W^{k,1}(\mathbb{T}^N \times \mathbb{R}^N)$  with  $k \leq 2$ , then*

$$\|f\|_{L^\infty(0, T; W^{k,1}(\mathbb{T}^N \times \mathbb{R}^N))} \leq C(k, T, f_{\text{in}}), \quad (4.5)$$

*where the constant  $C(k, T, f_{\text{in}})$  depends on  $k, T, N$ , and on the norms*

$$\sup_{t \in [0, T]} \|E(t, \cdot)\|_F \quad \text{and} \quad \|f_{\text{in}}\|_{W^{k,1}(\mathbb{T}^N \times \mathbb{R}^N)}$$

*only. Eventually, if  $f_{\text{in}} \in G_m$ , then  $f(t) \in G_m$  for all  $t \in [0, T]$ .*

*Proof of Proposition 4.1.* Let  $E_T$  denote the space of continuous functions from  $[0, T]$  to  $L^1(\mathbb{T}^N \times \mathbb{R}^N)$ . We use the norm

$$\|f\|_{E_T} = \sup_{t \in [0, T]} \|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)}$$

on  $E_T$ . Note that

$$\|\rho(f)\|_{L^1(\mathbb{T}^N)} \leq \|f\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)}. \quad (4.6)$$

Let  $f \in E_T$ . Assume that (4.3) is satisfied. Then, by (4.6), we have

$$\|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} \leq e^{-t} \|f_{\text{in}}\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} + \int_0^t e^{-(t-s)} \|f(s)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} ds.$$

By Gronwall's Lemma applied to  $t \mapsto e^t \|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)}$ , we obtain (4.4) as an a priori estimate. Besides, the  $L^1$ -norm of the integral term in (4.3) can be estimated by  $(1 - e^{-T}) \|f\|_{E_T}$ . Therefore existence and uniqueness of a solution to (4.3) in  $L^1(\Omega; E_T)$  follow from the Banach fixed point Theorem. To obtain the additional regularity (4.5), we do the same kind of estimates on the system satisfied by the derivatives and incorporate these estimates in the fixed-point space. To conclude the proof, let us assume  $f_{\text{in}} \geq 0$ . Since  $s \mapsto s^-$  (negative part) is convex and satisfies  $(a+b)^- \leq a^- + b^-$ , we deduce from (4.3) and the Jensen inequality that

$$f^-(t) \leq \int_0^t e^{-(t-s)} [\rho(f(s))M]^- \circ \Phi^{t-s} ds.$$

Since  $M \geq 0$  and  $\rho(f)^- \leq \rho(f^-)$ , (4.6) yields the estimate

$$\|f^-(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} \leq \int_0^t e^{-(t-s)} \|f^-(s)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} ds.$$

We conclude to  $f^- = 0$  by the Gronwall Lemma. Eventually, that  $f_{\text{in}} \in G_m$  implies  $f(t) \in G_m$  for all  $t \in [0, T]$  (propagation of moments) is proved in Proposition 6.3.  $\square$

## 4.2 Cauchy Problem in the FP case

Let  $K_t(x, v; y, w)$  denote the kernel associated to the kinetic Fokker-Planck equation

$$\partial_t f = Q_{\text{FP}} f - v \cdot \nabla_x f. \quad (4.7)$$

Let us recall some elementary facts about  $K_t$  (see [4] for more results about the analytical properties of  $K_t$ , and [21] for the probabilistic interpretation of  $K_t$ ). The function  $K_t(\cdot; y, w)$  is the density with respect to the Lebesgue measure on  $\mathbb{T}^N \times \mathbb{R}^N$  of the law  $\mu_t^{(y, w)}$  of the solution  $(X_t, V_t)$  to the SDE

$$dX_t = V_t dt, \quad X_0 = y, \quad (4.8)$$

$$dV_t = -V_t dt + \sqrt{2} dB_t, \quad V_0 = w. \quad (4.9)$$

where  $B_t$  is a Wiener process over  $\mathbb{R}^N$ . Therefore

$$K_t f(x, v) := \iint_{\mathbb{T}^N \times \mathbb{R}^N} K_t(x, y; y, w) f(y, w) dy dw$$

satisfies the identity

$$\langle K_t f, \varphi \rangle = \iint_{\mathbb{T}^N \times \mathbb{R}^N} \mathbb{E} \varphi(X_t, V_t) f(y, w) dy dw, \quad (4.10)$$

for  $f \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$  and  $\varphi: \mathbb{T}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  continuous and bounded. The solution to (4.8)-(4.9) is given explicitly by

$$\begin{aligned} X_t &= y + (1 - e^{-t})w + \int_0^t (1 - e^{-(t-s)}) dB_s, \\ V_t &= e^{-t}w + \int_0^t e^{-(t-s)} dB_s. \end{aligned} \quad (4.11)$$

The process  $(X_t^0, V_t^0)$  given by (4.11) when  $y = 0, w = 0$  is a Gaussian process with covariance matrix

$$Q_t := \begin{pmatrix} \int_0^t |1 - e^{-s}|^2 ds & \int_0^t e^{-s}(1 - e^{-s}) ds \\ \int_0^t e^{-s}(1 - e^{-s}) ds & \int_0^t e^{-2s} ds \end{pmatrix} \otimes I_N. \quad (4.12)$$

Using (4.12) and (4.10)-(4.11), one can show that  $K_t: L^p(\mathbb{T}^N \times \mathbb{R}^N) \rightarrow L^p(\mathbb{T}^N \times \mathbb{R}^N)$  with norm bounded by  $e^{\frac{N}{p}t}$ . We have also the estimate

$$\iint_{\mathbb{T}^N \times \mathbb{R}^N} |\nabla_w K_t(x, v; y, w)| dx dv \leq Ct^{-1/2}, \quad (4.13)$$

for all  $(y, w) \in \mathbb{T}^N \times \mathbb{R}^N, t \in [0, T]$ , with a constant  $C$  independent on  $(y, w)$  and  $T$ . The estimate (4.13) also follows from the estimate between (26) and (27) that can be found in [4].

**Definition 4.2** (Mild solution, FP case). Let  $t \mapsto E(t, \cdot)$  be a càdlàg function with values in  $F$ . Let  $p \in [1, +\infty[$ . Let  $f_{\text{in}} \in L^p(\mathbb{T}^N \times \mathbb{R}^N)$ . Assume  $Q = Q_{\text{FP}}$ . A continuous function from  $[0, T]$  to  $L^p(\mathbb{T}^N \times \mathbb{R}^N)$  is said to be a mild solution to (4.2) in  $L^p$  with initial datum  $f_{\text{in}}$  if

$$f(t) = K_t f_{\text{in}} + \int_0^t \nabla_w K_{t-s} [E(s) f(s)] ds, \quad (4.14)$$

for all  $t \in [0, T]$ .

**Proposition 4.2** (The Cauchy Problem, FP case). Let  $t \mapsto E(t, \cdot)$  be a càdlàg function with values in  $F$ . Let  $p \in [1, +\infty[$ . Let  $f_{\text{in}} \in L^p(\mathbb{T}^N \times \mathbb{R}^N)$ . Then (4.2) has a unique mild solution  $f$  in  $L^p$  with initial datum  $f_{\text{in}}$ . If  $f_{\text{in}} \geq 0$ , then  $f(t) \geq 0$ , for all  $t \in [0, T]$ . In addition, for every  $k \leq 2$ , the regularity  $W^{k,p}(\mathbb{T}^N \times \mathbb{R}^N)$  is propagated:

$$\sup_{t \in [0, T]} \|f(t)\|_{W^{k,p}(\mathbb{T}^N \times \mathbb{R}^N)} \leq C(k, T) \|f_{\text{in}}\|_{W^{k,p}(\mathbb{T}^N \times \mathbb{R}^N)}, \quad (4.15)$$

where the constant  $C(k, T)$  depends on  $k$ ,  $T$ ,  $N$  and  $\sup_{t \in [0, T]} \|E(t, \cdot)\|_F$ . If  $p = 1$  and  $f_{\text{in}} \geq 0$ , then  $\|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} = \|f_{\text{in}}\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)}$ . If, more generally, there is no sign condition on  $f_{\text{in}} \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$ , then (4.4) is satisfied. Eventually, if  $f_{\text{in}} \in G_m$ , then  $f(t) \in G_m$  for all  $t \in [0, T]$ .

*Proof of Proposition 4.2.* The existence-uniqueness follows from the Banach fixed point Theorem using (4.13), in a manner similar to the proof of Proposition 4.1. To obtain (4.15) for  $k = 1$ , we assume first that  $f(t)$  is in  $W^{k,p}(\mathbb{T}^N \times \mathbb{R}^N)$  for all  $t$  and we use the relations

$$\begin{aligned} \nabla_x K_t(x, v; y, w) &= -\nabla_y K_t(x, v; y, w), \\ \nabla_v K_t(x, v; y, w) &= -(1 - e^{-t}) \nabla_y K_t(x, v; y, w) - e^{-t} \nabla_w K_t(x, v; y, w), \end{aligned}$$

and Gronwall's Lemma, to obtain (4.15). We can drop the a priori requirement that  $f(t)$  is in  $W^{k,p}(\mathbb{T}^N \times \mathbb{R}^N)$  for all  $t$  either by incorporating this in the fixed-point space, or by working with differential quotients. The case  $k = 2$  is obtained similarly. To prove that  $f_{\text{in}} \geq 0$  implies  $f(t) \geq 0$ , we use a duality argument: it is sufficient to prove the propagation of the sign for  $L^\infty$  solutions to the dual equation

$$\varphi(T) = \psi, \quad (4.16)$$

$$\partial_t \varphi = -v \cdot \nabla_x \varphi - \bar{E}_t \cdot \nabla_v \varphi - Q_{\text{FP}}^* \varphi, \quad 0 < t < T. \quad (4.17)$$

This follows from the maximum principle, since  $Q_{\text{FP}}^* \varphi = \Delta_v \varphi - v \cdot \nabla_v \varphi$ . The maximum principle for the solutions to (4.16)-(4.17) also yields the  $L^1$ -estimate (4.4). The propagation of moments is proved in Proposition 6.3.  $\square$

### 4.3 Markov property

We prove the following result.



**Theorem 4.3** (Markov property). *Let  $(\bar{E}(t))$  be a mixing force field in the sense of Definition 2.1. We denote by  $A$  the generator of  $(E_t)$ . Let  $\mathcal{X}$  denote the state space*

$$\mathcal{X} = L^1(\mathbb{T}^N \times \mathbb{R}^N) \times F. \quad (4.18)$$

For  $(f, \mathbf{e}) \in \mathcal{X}$ , let  $f_t$  denote the mild solution to (4.2) with initial datum  $f$  and forcing  $E_t(\mathbf{e})$ . Then  $(f_t, E_t(\mathbf{e}))_{t \geq 0}$  is a time-homogeneous Markov process over  $\mathcal{X}$ .

*Proof of Theorem 4.3.* For a synthetic treatment of the proof, we use the following notations:

$$G = L^1(\mathbb{T}^N \times \mathbb{R}^N), \quad H = W^{2,1}(\mathbb{T}^N \times \mathbb{R}^N).$$

For  $\Phi$  bounded and continuous on  $\mathcal{X}$ ,  $\mathbf{e} \in F$ , define

$$P_t \Phi(f, \mathbf{e}) = \mathbb{E}_{(f, \mathbf{e})} \Phi(f_t, E_t(\mathbf{e})). \quad (4.19)$$

Let  $(\mathcal{G}_t)$  be the filtration defined in (2.5). The process  $(f_t, E_t(\mathbf{e}))$  is  $(\mathcal{G}_t)$ -adapted. To prove this assertion, note that  $f_t$  is obtained as the solution of a fixed-point equation (see the proof of Proposition 4.1 and Proposition 4.2 respectively). Consequently  $(f_t)$  is the limit of the sequence obtained by iterating the fixed-point map, starting from  $f$ . It is simple to check that each element in this sequence is  $(\mathcal{G}_t)$ -adapted. Since  $(\mathcal{G}_t)$  is complete, this implies that  $(f_t)$  also is  $(\mathcal{G}_t)$ -adapted. Our aim, first, is to prove the following identity: for  $0 \leq s, t$ ,

$$\mathbb{E}_{(f, \mathbf{e})}(\Phi(f_{t+s}, E_{t+s}(\mathbf{e})) | \mathcal{G}_s) = (P_t \Phi)(f_s, E_s(\mathbf{e})). \quad (4.20)$$

We can use the propagation of the  $W^{2,1}$ -regularity stated in Proposition 4.1 and Proposition 4.2 and an argument of density to reduce the proof of (4.20) to the case where  $f \in H$ . In that case,  $(f_t, E_t)$  is seen as a process with state space  $\mathcal{Y} = H \times F$ . We apply this reduction because, when  $f_t$  has the regularity  $W^{2,1}(\mathbb{T}^N \times \mathbb{R}^N)$ , it is simple to prove that

$$f_t = \Psi_{0,t}(f, (E(\sigma))_{0 \leq \sigma \leq t}), \quad (4.21)$$

where  $\Psi_{0,t}(f, \cdot)$  is a *continuous* map from  $L^1([0, t]; F)$  to  $L^1(\mathbb{T}^N \times \mathbb{R}^N)$ . Indeed, if  $f_t^i$ ,  $i \in \{1, 2\}$  are two solutions to (4.2) corresponding to two different forcing terms  $E^i(t, x)$ ,  $i \in \{1, 2\}$ , we just need to write

$$[\partial_t + E^1 \cdot \nabla_v - Q](f_t^1 - f_t^2) = (E^2 - E^1) \cdot \nabla_v f_t^2,$$

multiply the equation by  $\text{sgn}(f_1 - f_2)$  and integrate, to obtain

$$\|f_t^1 - f_t^2\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} \leq C \int_0^t \|E^2(s) - E^1(s)\|_F ds,$$

where the constant  $C$  depends on the  $L_t^\infty L_{x,v}^1$ -norm of  $\nabla_v f_t^2$ . Similarly, we may introduce the solution map  $\Psi_{s,t}$ , which associates to the path  $(E(\sigma))_{s \leq \sigma \leq t}$  and to the datum  $f_\sigma$  at  $\sigma = s$ , the solution to (4.2) at time  $t$ . We have then the semi-group property

$$\Psi_{0,t+s}(f, (E(\sigma))_{0 \leq \sigma \leq t+s}) = \Psi_{s,t+s}(\Psi_{0,s}(f, (E(\sigma))_{0 \leq \sigma \leq s}), (E(\sigma))_{s \leq \sigma \leq t+s}). \quad (4.22)$$

Using (4.22), the relation (4.20) is equivalent to

$$\mathbb{E}\left(\bar{\Phi}\left[\Psi_{s,t+s}(\xi, (E_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s}), E_{t+s}(\mathbf{e})\right] \middle| \mathcal{G}_s\right) = (P_t \bar{\Phi})(\xi, E_s(\mathbf{e})). \quad (4.23)$$

where  $\xi = f_s$  is  $\mathcal{G}_s$ -measurable. The property (4.23) is true for every random variable  $\xi$  which is  $\mathcal{G}_s$ -measurable. The reason for this is that  $\Psi_{s,t}(\xi, (E_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s})$  is a functional of  $(E_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s}$ . The proof of (4.23) is clear if we replace  $(E_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s}$  by the piecewise constant path  $(\tilde{E}_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s}$ , where  $\tilde{E}_\sigma(\mathbf{e}) = E_{s+t_i}(\mathbf{e})$  for all  $\sigma \in [s+t_i, s+t_{i+1})$ , where

$$s = s + t_0 < s + t_1 < \cdots < s + t_N = s + t$$

is a subdivision of the interval  $[s, s+t]$ . Indeed, we have then

$$\bar{\Phi}\left[\Psi_{s,t+s}(\xi, (\tilde{E}_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s}), E_{t+s}(\mathbf{e})\right] = \bar{\Phi}_N(\xi; E_{s+t_0}(\mathbf{e}), \dots, E_{s+t_N}(\mathbf{e})),$$

where  $\bar{\Phi}_N$  is continuous in its arguments. By considering the finite-dimensional distributions of  $(E_t)$ , we obtain

$$\mathbb{E}\left[\bar{\Phi}_N(\xi; E_{s+t_0}(\mathbf{e}), \dots, E_{s+t_N}(\mathbf{e})) \middle| \mathcal{G}_s\right] = P_{N,t} \bar{\Phi}(\xi, E_s(\mathbf{e})), \quad (4.24)$$

where

$$\begin{aligned} P_{N,t} \bar{\Phi}(f, \mathbf{e}) &= \mathbb{E}_{\mathbf{e}} \bar{\Phi}_N(f; E_{t_0}(\mathbf{e}), \dots, E_{t_N}(\mathbf{e})) \\ &= \mathbb{E}_{\mathbf{e}} \bar{\Phi}\left[\Psi_{0,t}(f, (\hat{E}_\sigma(\mathbf{e}))_{0 \leq \sigma \leq t}), E_t(\mathbf{e})\right], \end{aligned}$$

where  $(\hat{E}_\sigma(\mathbf{e}))_{0 \leq \sigma \leq t}$  is the piecewise-constant path defined by  $\hat{E}_\sigma(\mathbf{e}) = E_{t_i}(\mathbf{e})$  for all  $\sigma \in [t_i, t_{i+1})$ ,  $i = 0, \dots, N-1$ . Since  $(E_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s}$  is càdlàg, we can approximate  $(E_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s}$  by  $(\tilde{E}_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s}$ , due to the continuous dependence of  $\Psi_{s,t+s}$  on  $(E_\sigma(\mathbf{e}))_{s \leq \sigma \leq t+s}$  with respect to the  $L^1([s, t+s]; F)$ -norm. When the size  $\max(t_{i+1} - t_i)$  of the subdivision tends to 0, we have  $P_{N,t} \bar{\Phi}(f, \mathbf{e}) \rightarrow P_t \bar{\Phi}(f, \mathbf{e})$ . This yields (4.20). From (4.20) then, it follows that  $(f_t, E_t(\mathbf{e}))_{t \geq 0}$  is a time-homogeneous  $(\mathcal{G}_t)$ -Markov process.

To achieve the proof, we use the theory of semigroups developed by Priola in [19]. We refer the reader to Appendix B on that subject. Denote by  $\text{BM}(\mathcal{X})$  the set of bounded measurable functions on  $\mathcal{X}$  and by  $\text{BC}(\mathcal{X})$  the set of bounded continuous functions on  $\mathcal{X}$ . Then  $\text{BC}(\mathcal{X})$  is  $\pi$ -dense in  $\text{BM}(\mathcal{X})$  (consider the class  $\mathcal{A}$  of Borel sets  $B$  such that the characteristic function  $\mathbf{1}_B$  is  $\pi$ -limits of a sequence of functions in  $\text{BC}(\mathcal{X})$ :  $\mathcal{A}$  contains closed sets since  $x \mapsto d(x, F)$  is continuous if  $F$  is closed,  $d$  being the metric on  $\mathcal{X}$ , and it is simple to check that  $\mathcal{A}$  is stable by countable union and by the operation  $B \mapsto B^c$ ). If  $\Phi \in \text{BM}(\mathcal{X})$  is the  $\pi$ -limit of a sequence  $(\Phi_n)$  of  $\text{BC}(\mathcal{X})$ , then

$$\begin{aligned} P_t \Phi_n(f_s, E_s(\mathbf{e})) &\rightarrow P_t \Phi(f_s, E_s(\mathbf{e})), \\ \mathbb{E}_{(f,\mathbf{e})}(\Phi_n(f_{t+s}, E_{t+s}(\mathbf{e})) \middle| \mathcal{G}_s) &\rightarrow \mathbb{E}_{(f,\mathbf{e})}(\Phi(f_{t+s}, E_{t+s}(\mathbf{e})) \middle| \mathcal{G}_s), \end{aligned} \quad (4.25)$$

by dominated convergence. Consequently (4.20) holds true when  $\Phi \in \text{BM}(\mathcal{X})$ . Assume that  $\text{BM}(\mathcal{X})$  is stable by  $P_t$ . Then, taking the expectancy in (4.20), we obtain the semi-group property  $P_{t+s} = P_s P_t$ . To prove that  $\text{BM}(\mathcal{X})$  is stable by  $P_t$ , it is sufficient to show that  $P_t$  sends  $\text{BC}(\mathcal{X})$  in  $\text{BM}(\mathcal{X})$ , since the latter is  $\pi$ -dense in the former. Let  $\Phi \in \text{BC}(\mathcal{X})$ . We have

$$P_t \Phi(f, \mathbf{e}) = \mathbb{E}_{\mathbf{e}} \Phi[\Psi_{0,t}(f, (E_s)_{0 \leq s \leq t}), E_t]. \quad (4.26)$$

We have seen that, if  $f \in H$ , then  $P_t \Phi(f, \mathbf{e})$  is the limit of the right-hand side of (4.26) where the path  $(E_s)_{0 \leq s \leq t}$  is replaced by a piecewise constant path. In that last case,  $\mathbf{e} \mapsto P_t \Phi(f, \mathbf{e})$  is measurable since (recall Section 2) for all Borel set  $D \in \mathcal{F}$ ,  $\mathbf{e} \mapsto \mathbb{P}_{\mathbf{e}}(D)$  is Borel-measurable. By approaching the càdlàg path  $(E_s)_{0 \leq s \leq t}$  by a piecewise constant paths, we can conclude therefore that, for all  $f \in H$ ,  $\mathbf{e} \mapsto P_t \Phi(f, \mathbf{e})$  is measurable. On the other hand, for all  $\mathbf{e} \in F$ , the map  $f \mapsto P_t \Phi(f, \mathbf{e})$  is continuous from  $G$  to  $\mathbb{R}$ . This is a consequence of the  $L^1$ -estimate (4.4). By approaching a function  $f$  in  $G$  by a sequence of more regular functions in  $H$ , we deduce that  $\mathbf{e} \mapsto P_t \Phi(f, \mathbf{e})$  is measurable for all  $f \in G$ . Consequently,  $P_t \Phi$  is a Carathéodory function. It is a classical fact then, that Carathéodory functions are measurable, [6, Lemma 1.2.3].  $\square$

Let us introduce the operators

$$\mathcal{L}_{\sharp} \varphi(f, \mathbf{e}) = A\varphi(f, \mathbf{e}) + (Qf - \mathbf{e} \cdot \nabla_v f, D_f \varphi(f, \mathbf{e})), \quad (4.27)$$

$$\mathcal{L}_{\flat} \varphi(f, \mathbf{e}) = - (v \cdot \nabla_x f, D_f \varphi(f, \mathbf{e})), \quad (4.28)$$

and  $\mathcal{L} = \mathcal{L}_{\sharp} + \mathcal{L}_{\flat}$ . Formally,  $\mathcal{L}$  is the generator associated to the Markov process  $(f_t, E_t)$ . We do not need to be much specific on that point here. Indeed, what is relevant to apply the perturbed test-function method in Section 6 (see (6.2)) are sufficient conditions for a test function to be both in the domain of  $\mathcal{L}_{\sharp}$  and in the domain of  $\mathcal{L}_{\flat}$ . We prove the following result.

**Proposition 4.4.** *Let  $(\bar{E}(t))$  be a mixing force field in the sense of Definition 2.1. Let  $A$  be the generator of  $(E_t)$ , let  $\mathcal{X}$  be the state space defined by (4.18), and let  $\mathcal{L}_{\sharp}$  and  $\mathcal{L}_{\flat}$  be defined by (4.27)-(4.28). Let  $\psi: \mathbb{R}^m \times F \rightarrow \mathbb{R}$  be a continuous function which is bounded on bounded set of  $\mathbb{R}^m \times F$  and satisfies the following properties:*

1. *for all  $u \in \mathbb{R}^m$ ,  $\mathbf{e} \mapsto \psi(u, \mathbf{e})$  is in the domain of  $A$  and  $(u, \mathbf{e}) \mapsto A\psi(u; \mathbf{e})$  is bounded on bounded sets of  $\mathbb{R}^m \times F$ ,*
2. *for all  $\mathbf{e} \in F$ ,  $u \mapsto \psi(u; \mathbf{e})$  is differentiable,  $(u, \mathbf{e}) \mapsto \nabla_u \psi(u; \mathbf{e})$  is bounded on bounded sets of  $\mathbb{R}^m \times F$  and continuous with respect to  $\mathbf{e}$ .*

Let  $\xi_1, \dots, \xi_m \in C_c^\infty(\mathbb{T}^N \times \mathbb{R}^N)$ . Then the test-function

$$\varphi: (f, \mathbf{e}) \mapsto \psi(\langle f, \xi_1 \rangle, \dots, \langle f, \xi_m \rangle; \mathbf{e}) \quad (4.29)$$

satisfies  $\mathcal{L}_{\sharp} \varphi(f, \mathbf{e}), \mathcal{L}_{\flat} \varphi(f, \mathbf{e}) < +\infty$  for all  $(f, \mathbf{e}) \in \mathcal{X}$  and  $\varphi$  is in the domain of  $\mathcal{L}$  in the sense that

$$P_t \varphi(f, \mathbf{e}) = \varphi(f, \mathbf{e}) + t \mathcal{L} \varphi(f, \mathbf{e}) + o(t), \quad (4.30)$$

for all  $(f, \mathbf{e}) \in \mathcal{X}$ .

*Proof of Proposition 4.4.* Let  $\xi = (\xi_i)_{1,m}$ . We have

$$\begin{aligned}\mathcal{L}_\# \varphi(f, \mathbf{e}) &= \{A\psi(u; \mathbf{e}) + \langle f, Q^* \xi + \mathbf{e} \cdot \nabla_v \xi \rangle \nabla_u \psi(u; \mathbf{e})\} \Big|_{u=\langle f, \xi \rangle}, \\ \mathcal{L}_\flat \varphi(f, \mathbf{e}) &= \langle f, v \cdot \nabla_x \xi \rangle \nabla_u \psi(u; \mathbf{e}) \Big|_{u=\langle f, \xi \rangle},\end{aligned}$$

therefore  $(f, \mathbf{e}) \mapsto (\mathcal{L}_\# \varphi(f, \mathbf{e}), \mathcal{L}_\flat \varphi(f, \mathbf{e}))$  is bounded on bounded sets of  $\mathcal{X}$ . To obtain (4.30), we use the decomposition of  $P_t \varphi(f, \mathbf{e}) - \varphi(f, \mathbf{e})$  into the sum of the terms

$$\mathbb{E}_{(f, \mathbf{e})} \varphi(f, E_t) - \varphi(f, \mathbf{e}) \tag{4.31}$$

and

$$\mathbb{E}_{(f, \mathbf{e})} [\varphi(f_t, E_t) - \varphi(f, E_t)]. \tag{4.32}$$

By item 1, we have the asymptotic expansion (4.31) =  $tA\psi(u; \mathbf{e}) \Big|_{u=\langle f, \xi \rangle} + o(t)$ . In addition, by (4.2), we have

$$u_t = u + t(\langle f, Q^* \xi + \mathbf{e} \cdot \nabla_v \xi \rangle + \langle f, v \cdot \nabla_x \xi \rangle) + o(t),$$

where  $u_t = \langle f_t, \xi \rangle$ ,  $u = \langle f, \xi \rangle$ . By item 1, we obtain the asymptotic expansion

$$(4.32) = t(\langle f, Q^* \xi + \mathbf{e} \cdot \nabla_v \xi \rangle + \langle f, v \cdot \nabla_x \xi \rangle) \nabla_u \psi(u; \mathbf{e}) \Big|_{u=\langle f, \xi \rangle} + o(t).$$

This concludes the proof.  $\square$

*Remark 4.1.* The result of Proposition 4.4 holds true if we consider some functions  $\xi_i$  not as smooth and localised as  $C_c^\infty$  functions, provided there is a sufficient balance with the regularity and integrability properties of  $f$ . For example, we apply Proposition 4.4 in Section 6.1.3 with  $\xi_i(x, v) = \hat{\xi}_i(x) \zeta_i(v)$ , where  $\hat{\xi}_i$  is in some Sobolev space  $H^s(\mathbb{T}^N)$  and  $\zeta_i(v)$  is a polynomial in  $v$  of degree less than two. In that case, we view  $(f_t, E_t)$  as a Markov process on  $\mathcal{X}_3 := G_3 \times F$  and the conclusion of Proposition 4.4 is valid for  $f \in G_3$ .

*Remark 4.2.* Note that, in the context of Proposition 4.4, the function  $|\psi|^2$  has the same properties (item 1 and item 2) as  $\psi$ . Therefore  $|\varphi|^2$  is also in the domain of  $\mathcal{L}$ .

## 5 Deterministic convergence

In this section, we prove Theorem 1.1. We establish the convergence

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \left| \int_{\mathbb{T}^N} r_t^\varepsilon(x) \varphi(x) dx - \int_{\mathbb{T}^N} r_t(x) \varphi(x) dx \right| = 0 \tag{5.1}$$

for all  $\varphi \in C(\mathbb{T}^N)$ . We use the two following results.

**Theorem 5.1** (Martingale characterization of Markov processes). *Let  $(X_t)$  be a  $\mathbb{R}^N$ -valued time-homogeneous Markov process of generator  $L$ . Assume that  $(X_t)$  is càdlàg. Then, for all continuous bounded  $\varphi: \mathbb{R}^N \rightarrow \mathbb{R}^N$  in the domain of  $L$ ,*

$$Z(t) := \varphi(X_t) - \varphi(X_0) - \int_0^t L\varphi(X_s)ds \quad (5.2)$$

*is an  $\mathbb{R}^N$ -valued  $(\mathcal{F}_t^X)$ -martingale. If  $\varphi^2$  is in the domain of  $L$ , then the quadratic variation of  $(Z(t))$  is*

$$[Z_i, Z_j]_t = \int_0^t (L\varphi_i\varphi_j - \varphi_i L\varphi_j - \varphi_j L\varphi_i)(X_s)ds. \quad (5.3)$$

**Theorem 5.2** (CLT for martingales). *Let  $(Z_t)$  be a  $\mathbb{R}^N$ -valued càdlàg martingale satisfying*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T/\varepsilon^2} |Z_t - Z_{t-}| \right] = 0. \quad (5.4)$$

*Assume*

$$\frac{1}{t}[Z, Z]_t \rightarrow \sigma^* \sigma \text{ in probability,} \quad (5.5)$$

*where  $\sigma$  is an  $N \times N$  matrix. Then*

$$\varepsilon Z_{t/\varepsilon^2} \rightarrow \sigma W_t \quad (5.6)$$

*in law on  $C([0, T]; \mathbb{R}^N)$ , where  $(W_t)$  is an  $N$ -dimensional Wiener process.*

*Proof of Theorem 5.1.* We apply [10, Proposition 1.7, p. 162]. Since  $(X_t)$  is càdlàg, it is progressive. We obtain the fact that  $(Z(t))$  is an  $(\mathcal{F}_t^X)$ -martingale. The proof of (5.3) is a consequence of Theorem B.1 in the appendix B.  $\square$

*Proof of Theorem 5.2.* Let  $(\varepsilon_n) \downarrow 0$ . We apply [10, Theorem 1.4, p. 339] to

$$M_n(t) = \varepsilon_n Z_{t/\varepsilon_n^2}, \quad c_{ij}(t) = (\sigma^* \sigma)_{ij} t.$$

Indeed, Condition (1.14) in [10, p. 340] is fulfilled by (5.4).  $\square$

*Remark 5.1.* Assume, in the context of Theorem 5.1, that  $(X_t)$  is ergodic, with invariant measure  $\lambda$  and let  $Z$  be defined by (5.2). We have then

$$\begin{aligned} \frac{1}{t}[Z_i, Z_j]_t &\rightarrow \langle L\varphi_i\varphi_j - \varphi_i L\varphi_j - \varphi_j L\varphi_i, \lambda \rangle \quad \text{in probability} \\ &= - \langle \varphi_i L\varphi_j + \varphi_j L\varphi_i, \nu \rangle \quad \text{since } L^* \lambda = 0. \end{aligned}$$

Therefore, we obtain (5.6) with

$$(\sigma^* \sigma)_{ij} = - \langle \varphi_i L\varphi_j + \varphi_j L\varphi_i, \lambda \rangle. \quad (5.7)$$

## 5.1 Classical diffusion limit

Let us first illustrate the application of Theorem 5.1 and Theorem 5.2 in the case where  $\bar{E} \equiv 0$  in (1.7) and  $Q = Q_{\text{LB}}$ . The equation (1.7) is deterministic then. An argument using a PDE theory approach gives the convergence of  $r^\varepsilon = \rho^\varepsilon$  to  $r$  solution to (1.19) in  $L^2(0, T; L^2(\mathbb{T}^N))$ , see [9, Theorem 1.1] for example. Using a probabilistic approach, we consider the jump process  $(V_t)$  associated to (1.2) (this is a pure jump process since we assume  $\bar{E} \equiv 0$ ). We obtain the convergence of the process  $(X_t^\varepsilon)$  that is behind  $r^\varepsilon = \rho^\varepsilon$ , and a convergence to  $r$  in  $C([0, T]; L^2(\mathbb{T}^N) - \text{weak})$ . Indeed, let us set

$$X_t^\varepsilon = X_0 + \left[ \varepsilon \int_0^{t/\varepsilon^2} V_s ds \right], \quad (5.8)$$

where  $X_0$  follows a law with density  $\rho_{\text{in}}$  with respect to the Lebesgue measure on  $\mathbb{T}^N$ . In (5.8),  $[Y]$  denote the equivalence class in  $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$  of an element  $Y \in \mathbb{R}^N$ . By (1.18),  $X_0$  and  $V_0$  are independent,  $V_0$  having the law of density  $g$  with respect to the Lebesgue measure on  $\mathbb{R}^N$ . Then  $r^\varepsilon$  is the density, with respect to the Lebesgue measure on  $\mathbb{T}^N$ , of the law of  $X_t^\varepsilon$ . For all  $\psi: \mathbb{R}^N \rightarrow \mathbb{R}$  continuous and bounded, we have

$$\int_{\mathbb{T}^N} r^\varepsilon(x, t) \psi(x) dx = \mathbb{E} \psi(X_t^\varepsilon). \quad (5.9)$$

The generator  $L$  of  $(V_t)$  is  $L\varphi(v) = \langle \varphi, M \rangle - \varphi$ . We apply Theorem 5.1 to the Markov process  $(V_t)$  with  $\varphi$  such that

$$L\varphi = -\text{Id}. \quad (5.10)$$

The solution to (5.10) is  $\varphi = \text{Id}$ . We have then

$$X_t^\varepsilon = X_0 + \varepsilon Z_{t/\varepsilon^2} - \varepsilon(V_{t/\varepsilon^2} - V_0) = X_0 + \varepsilon Z_{t/\varepsilon^2} + \mathcal{O}(\varepsilon), \quad (5.11)$$

where the  $\mathcal{O}(\varepsilon)$  is in  $L^1(\Omega)$  (and thus, in probability) since  $V_t$  has either the distribution of  $V_0$  or, if a jump has already occurred, the density  $M$  with respect to the Lebesgue measure on  $\mathbb{R}^N$ . We obtain, using (5.7), the convergence (5.5) with  $\sigma^* \sigma = 2K$ , where  $K$  is given by (1.15). By Theorem 5.2 thus, we have

$$X_t^\varepsilon \rightarrow X_t := X_0 - \sqrt{2K} W_t$$

in law on  $C([0, T]; \mathbb{R}^N)$ . Using (5.9), we obtain

$$\int_{\mathbb{T}^N} r^\varepsilon(x, \cdot) \psi(x) dx \rightarrow \int_{\mathbb{T}^N} r(x, \cdot) \psi(x) dx \quad \text{in } C([0, T]), \quad (5.12)$$

for all  $\psi: \mathbb{R}^N \rightarrow \mathbb{R}$  continuous and bounded, where  $r$  satisfies (1.19). Since  $(r^\varepsilon)$  is bounded uniformly in  $\varepsilon$  in  $C([0, T]; L^2(\mathbb{T}^N))$  (see [9]), the convergence (5.12) is satisfied for all  $\psi \in L^2(\mathbb{T}^N)$ . We conclude to the convergence of  $(r^\varepsilon)$  to  $r$  in  $C([0, T]; L^2(\mathbb{R}^N) - \text{weak})$ .

*Remark 5.2.* In the FP case, the proof is direct and does not require Theorem 5.1 and Theorem 5.2. Indeed, we have (5.8) where  $V_t$  satisfies (4.9). We replace  $V_s$  in the integral in (5.8) by  $-dV_s + \sqrt{2}dB_s$  to obtain

$$X_t^\varepsilon = X_0 + \varepsilon B_{t/\varepsilon^2} - \varepsilon(V_{t/\varepsilon^2} - V_0)$$

Since (4.11) gives a bound (in  $L^2(\Omega)$  for example) on  $(V_t)$ , we have (5.11) with  $Z_t = B_t$  and we conclude using the scale invariance of the Wiener process.

## 5.2 Diffusion limit for the stochastically forced equation

We consider now the general case of Equation (1.7) with a non-trivial mixing force process  $(\bar{E}_t)$  (cf. Definition 2.2). Recall that  $\mathcal{F}_t^E$  is the  $\sigma$ -algebra generated by  $(E_s)_{0 \leq s \leq t}$ . For  $\varphi: \mathbb{R}^N \rightarrow \mathbb{R}$  continuous and bounded, we have

$$\int_{\mathbb{T}^N} \rho^\varepsilon(x, t) \varphi(x) dx = \mathbb{E} \left[ \varphi(X_t^\varepsilon) | \mathcal{F}_{t/\varepsilon^2}^E \right],$$

and (5.9), where  $X_t^\varepsilon$  is like in (5.8) and  $(V_t)$  is either the stochastic process described in (1.9) in the case  $Q = Q_{\text{LB}}$  (given  $E$ , this is a PDMP), or the Ornstein-Uhlenbeck process (1.10) in the case  $Q = Q_{\text{FP}}$ . The process  $(V_t, E_t)$  is Markov and has the generator  $L$  given by

$$L\varphi(v, \mathbf{e}) = Q^* \varphi(v, \mathbf{e}) - \mathbf{e} \cdot \nabla_v \varphi(v, \mathbf{e}) + A\varphi(v, \mathbf{e}). \quad (5.13)$$

The proof of this result is analogous to (actually, simpler than, since the state space has finite dimension) the proof of Theorem 4.3. By Theorem 3.1 (see Remark 3.2),  $(V_t, E_t)$  is ergodic and has the invariant measure  $\lambda$  defined by

$$\langle \lambda, \varphi \rangle = \mathbb{E} \int_{\mathbb{R}^N} \bar{M}_t(v) \varphi(v, \bar{E}_t) dv.$$

This is a consequence of (3.10) and of the fact that  $(V_t, E_t)$  is time homogeneous. More precisely, denoting by  $(e^{tL})$  the semi-group generated by  $(V_t, E_t)$ , (3.10) gives, for all  $f \in L^1(\mathbb{R}^N)$ , and for all  $\varphi$  bounded and continuous on  $\mathbb{R}^N \times \mathbb{R}^N$ ,

$$\lim_{t \rightarrow +\infty} \int_{\mathbb{R}^N} f(v) e^{tL} \varphi(v, \mathbf{e}) dv = \langle \lambda, \varphi \rangle \int_{\mathbb{R}^N} f(v) dv. \quad (5.14)$$

To apply Theorem 5.1 and Theorem 5.2 like in Section 5.1, we need to solve the Poisson equation

$$L\varphi(v, \mathbf{e}) = -v. \quad (5.15)$$

Using (5.7) then, we need to compute

$$(\sigma^* \sigma)_{ij} = \langle \varphi_i v_j + \varphi_j v_i, \lambda \rangle. \quad (5.16)$$

### 5.3 The auxiliary test-function

Our aim is to find the solution  $\varphi$  to the Poisson equation  $L\varphi = \psi$  (with  $\psi(v) = -v$  in our case). An argument of perturbation (of the case  $E \equiv 0$ ) suggests the solution

$$\varphi(v, \mathbf{e}) = v + R_0(\mathbf{e}), \quad (5.17)$$

where the resolvent  $R_0(\mathbf{e})$  is defined in (2.20) and (2.21). The test-function (5.17) indeed satisfies (5.15) since  $Q^*v = -v$  and  $AR_0(\mathbf{e}) = -\mathbf{e}$ . However, there is a more systematic way to obtain (5.17) than just guessing, and we explain it below, since it involves some computations that are necessary later in Section 6. To solve  $L\varphi = \psi$  (with  $\psi(v) = -v$  in our case, thus), we use the resolvent formula

$$\varphi = - \int_0^\infty e^{tL} \psi dt. \quad (5.18)$$

This may work at least if  $\langle \psi, \lambda \rangle$ , which is equal to  $\lim_{t \rightarrow +\infty} e^{tL} \psi$  by ergodicity, vanishes. In our case  $\psi(v) = v$ , this condition is satisfied as we will see. Note first that, for  $f \in L^1(\mathbb{R}^N)$  and  $f_{0,t}$  is given either by (3.3) or (3.4) with  $s = 0$ , we have, by (3.9),

$$\int_{\mathbb{R}^N} f(v) e^{tL} \psi(v, \mathbf{e}) dv = \mathbb{E}_{\mathbf{e}} \int_{\mathbb{R}^N} f_{0,t}(v) \psi(v, \mathbf{e}) dv. \quad (5.19)$$

With  $\psi(v, \mathbf{e}) = -v$ , we obtain

$$\int_{\mathbb{R}^N} f(v) e^{tL} \psi(v, \mathbf{e}) dv = -\mathbb{E}_{\mathbf{e}} J(f_{0,t}). \quad (5.20)$$

**Lemma 5.3.** *Let  $f_{s,t}$  be equal either to (3.3), or to (3.4). The two first moments of  $f_{s,t}$  (see (1.14) for the definition of the moments) are, respectively,  $\rho(f_{s,t}) = \rho(f)$ , and*

$$J(f_{s,t}) = e^{-(t-s)} J(f) + \rho(f) \int_s^t e^{-(t-\sigma)} E(\sigma, s; \mathbf{e}) d\sigma. \quad (5.21)$$

*Proof of Lemma 5.3.* We use the formula

$$\int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ v^{\otimes 2} \end{pmatrix} M(v-w) dv = \begin{pmatrix} 1 \\ w \\ K + w^{\otimes 2} \end{pmatrix}, \quad (5.22)$$

where  $K$  is defined by (1.15). By (5.22) (and a change of variable in the FP-case), we obtain (5.21).  $\square$

**Corollary 5.4.** *Let  $f_{s,t}$  be equal either to (3.3), or to (3.4) and let  $\bar{M}_t^{\text{LB}}$  and  $\bar{M}_t^{\text{FP}}$  be defined by (3.6) and (3.7) respectively. We have then, for all  $\mathbf{e} \in \bar{B}_\alpha$ ,*

$$\int_0^\infty |\mathbb{E} J(f_{0,t})| dt < +\infty, \quad \int_0^\infty \mathbb{E} J(f_{0,t}) dt = J(f) + \rho(f) R_0(\mathbf{e}), \quad (5.23)$$

$$J(\bar{M}_0^{\text{LB}}) = J(\bar{M}_0^{\text{FP}}) = \int_{-\infty}^0 e^\sigma \bar{E}(\sigma) d\sigma. \quad (5.24)$$



*Proof of Corollary 5.4.* We take  $t = 0$ ,  $\rho = 1$  and the limit  $s \rightarrow -\infty$  in (5.21) to obtain (5.24). To prove (5.23), we use the following formula:

$$\int_0^\infty \int_0^t e^{-(t-r)} \mathbb{E} \varphi(E(0, r; \mathbf{e})) dr dt = \int_0^\infty \mathbb{E} \varphi(E(0, r; \mathbf{e})) dr.$$

□

Comparing (5.18)-(5.20) and (5.23), we obtain (5.17).

## 5.4 Diffusion matrix

Now that we have found  $\varphi$  solution to (5.15), let us come back to (5.16).

**Lemma 5.5.** *Let  $\bar{M}_t^{\text{LB}}$  and  $\bar{M}_t^{\text{FP}}$  be defined by (3.6) and (3.7) respectively. The expectation of the second moment of  $\bar{M}_0$  is*

$$\mathbb{E} [K(\bar{M}_0)] = K + b \mathbb{E} [\bar{E}(0) \otimes R_1(\bar{E}(0))], \quad (5.25)$$

where  $b^{\text{LB}} = 2$  and  $b^{\text{FP}} = 1$ .

*Proof of Lemma 5.5.* We compute, using (5.22),

$$K(\bar{M}_0^{\text{LB}}) = \int_{-\infty}^0 e^\sigma \left( K + \left[ \int_\sigma^0 \bar{E}(r) dr \right]^{\otimes 2} \right) d\sigma.$$

This gives

$$\mathbb{E} [K(\bar{M}_0^{\text{LB}})] = K + \int_{-\infty}^0 e^\sigma \int_\sigma^0 \int_\sigma^0 \bar{\Gamma}(r-s) dr ds d\sigma,$$

where  $\bar{\Gamma}(t)$  is the covariance of  $(\bar{E}(t))$  (see (2.23)). By symmetry, we have

$$\int_\sigma^0 \int_\sigma^0 \bar{\Gamma}(r-s) dr ds = 2 \int_\sigma^0 (r-\sigma) \bar{\Gamma}((r)) dr.$$

By two successive integration by parts, we have next

$$2 \int_{-\infty}^0 e^\sigma \int_\sigma^0 (r-\sigma) \bar{\Gamma}((r)) dr d\sigma = 2 \int_{-\infty}^0 e^\sigma \int_\sigma^0 \bar{\Gamma}((r)) dr d\sigma = 2 \int_{-\infty}^0 e^\sigma \bar{\Gamma}((\sigma)) d\sigma.$$

Coming back to the definition (2.23), we obtain

$$\begin{aligned} \int_{-\infty}^0 e^\sigma \bar{\Gamma}((\sigma)) d\sigma &= \int_{-\infty}^0 e^\sigma \mathbb{E} [\bar{E}(\sigma) \otimes \bar{E}(0)] d\sigma \\ &= \int_0^{+\infty} e^{-\sigma} \mathbb{E} [\bar{E}(-\sigma) \otimes \bar{E}(0)] d\sigma \\ &= \int_0^{+\infty} e^{-\sigma} \mathbb{E} [\bar{E}(0) \otimes \bar{E}(\sigma)] d\sigma = \mathbb{E} [\bar{E}(0) \otimes R_1(\bar{E}(0))], \end{aligned} \quad (5.26)$$

and we conclude to (5.25). Similarly, we have by (3.7) and (5.22),

$$K(\bar{M}_0^{\text{FP}}) = K + \left[ \int_{-\infty}^0 e^{\sigma} \bar{E}(\sigma) d\sigma \right]^{\otimes 2}.$$

To conclude to (5.25), we use the following Lemma 5.6. □

**Lemma 5.6** (Symmetry and positivity). *For  $\delta > 0$ , we have*

$$\mathbb{E} [R_{\delta}(\bar{E}(0)) \otimes \bar{E}(0)] = \delta \mathbb{E} \left[ \int_{-\infty}^0 e^{\delta\sigma} \bar{E}(\sigma) d\sigma \right]^{\otimes 2}. \quad (5.27)$$

*In particular, if  $\delta \geq 0$ , we have  $\mathbb{E} [R_{\delta}(\bar{E}(0)) \otimes \bar{E}(0)] = \mathbb{E} [\bar{E}(0) \otimes R_{\delta}(\bar{E}(0))]$  and this quantity is non-negative.*

*Proof of Lemma 5.6.* We compute

$$\begin{aligned} \mathbb{E} \left[ \int_{-\infty}^0 e^{\delta\sigma} \bar{E}(\sigma) d\sigma \right]^{\otimes 2} &= \int_{-\infty}^0 \int_{-\infty}^0 e^{\delta(\sigma+s)} \mathbb{E}[\bar{E}(s) \otimes \bar{E}(\sigma)] d\sigma ds \\ &= 2 \int_{-\infty}^0 \int_{\sigma=-\infty}^s e^{\delta(\sigma+s)} \mathbb{E}[\bar{E}(s) \otimes \bar{E}(\sigma)] d\sigma ds \\ &= 2 \int_{-\infty}^0 \int_{\sigma=-\infty}^s e^{\delta(\sigma+s)} \mathbb{E}[\bar{E}(0) \otimes \bar{E}(\sigma-s)] d\sigma ds \\ &= 2 \int_{-\infty}^0 \int_{\sigma=-\infty}^0 e^{\delta(\sigma+2s)} \mathbb{E}[\bar{E}(0) \otimes \bar{E}(\sigma)] d\sigma ds \\ &= \frac{1}{\delta} \int_{-\infty}^0 e^{\delta\sigma} \mathbb{E}[\bar{E}(0) \otimes \bar{E}(\sigma)] d\sigma \\ &= \frac{1}{\delta} \int_0^{\infty} e^{-\delta\sigma} \mathbb{E}[\bar{E}(\sigma) \otimes \bar{E}(0)] d\sigma = \frac{1}{\delta} \mathbb{E} [R_{\delta}(\bar{E}(0)) \otimes \bar{E}(0)], \end{aligned} \quad (5.28)$$

which gives the result. □

By (5.16) and (5.17), we have

$$\sigma^* \sigma = 2\mathbb{E} [K(\bar{M}_0)] + \mathbb{E} [R_0(\bar{E}(0)) \otimes J(\bar{M}_0) + J(\bar{M}_0) \otimes R_0(\bar{E}(0))].$$

We use (5.24) to derive the following expression

$$\mathbb{E} [R_0(\bar{E}_0) \otimes J(\bar{M}_0)] = \int_{-\infty}^0 e^{\sigma} \mathbb{E} [R_0(\bar{E}(0)) \otimes \bar{E}(\sigma)] d\sigma. \quad (5.29)$$

As in (5.26), this is  $\mathbb{E} [R_1 R_0(\bar{E}(0)) \otimes \bar{E}(0)]$  and thus

$$\sigma^* \sigma = 2\mathbb{E} [K(\bar{M}_0)] + \mathbb{E} [R_1 R_0(\bar{E}(0)) \otimes \bar{E}(0) + \bar{E}(0) \otimes R_1 R_0(\bar{E}(0))].$$

To obtain a more tractable expression of  $\sigma^* \sigma$ , we use the resolvent identity  $R_1 R_0 = R_0 - R_1$  which yields, by the symmetry property stated in Lemma 5.6,

$$\sigma^* \sigma = 2\mathbb{E} [K(\bar{M}_0)] + 2\mathbb{E} [\bar{E}(0) \otimes (R_0(\bar{E}(0)) - R_1(\bar{E}(0)))] .$$

Using (5.25), we obtain

$$\sigma^* \sigma = 2K + 2\mathbb{E} [\bar{E}(0) \otimes [R_0(\bar{E}(0)) + (b-1)R_1(\bar{E}(0))]] . \quad (5.30)$$

We have therefore (5.1) for all  $\varphi \in C(\mathbb{T}^N)$ , where  $r$  is the solution to (1.19) starting from  $r_{\text{in}}$ . Note that, as a consequence of Lemma 5.6, we have

$$K_{\sharp} \geq K, \quad (5.31)$$

in the sense of symmetric matrices.

## 6 Diffusion-approximation

In this section, we establish the limit behaviour of  $\rho^\varepsilon$  as stated in Theorem 1.2. We forget now the probabilistic origin of  $f^\varepsilon$ , the solution to (1.7). This probabilistic aspect has been used in the previous Section 5. Our main probabilistic object of study now is the process  $(f_t^\varepsilon)$  solution to the SPDE (1.7). More precisely, we consider the Markov process  $(f_t^\varepsilon, \bar{E}_t^\varepsilon)$  (see Theorem 4.3). The generator  $\mathcal{L}^\varepsilon$  of this process can be decomposed as

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L}_{\sharp} + \frac{1}{\varepsilon} \mathcal{L}_b,$$

where  $\mathcal{L}_{\sharp}$  and  $\mathcal{L}_b$  are defined by (4.27) and (4.28) respectively. For every  $\varphi$  in the domain of  $\mathcal{L}^\varepsilon$ , the process

$$M_\varphi^\varepsilon(t) := \varphi(f_t^\varepsilon, \bar{E}_t^\varepsilon) - \varphi(f_{\text{in}}, \bar{E}_0) - \int_0^t \mathcal{L}^\varepsilon \varphi(f_s^\varepsilon, \bar{E}_s^\varepsilon) ds \quad (6.1)$$

is a  $(\mathcal{G}_{t/\varepsilon^2})$ -martingale (this is a consequence of Theorem 4.3 and Theorem B.1 in Appendix B). The equation associated to the principal generator  $\mathcal{L}_{\sharp}$  is (1.2). It has been analysed in Section 3. Our approach to the proof of the convergence of  $(\rho^\varepsilon)$  uses the perturbed test-function method introduced by Papanicolaou, Stroock, Varadhan in [17] and adapted in the setting of hydrodynamic limits in [8]. Let us explain the main steps of the proof.

1. **Limit generator.** To find the limit generator  $\mathcal{L}$  associated to the equation satisfied by the limit  $\rho$  of  $(\rho^\varepsilon)$ , which acts on test functions  $\varphi(\rho)$ , we seek two correctors  $\varphi_1$  and  $\varphi_2$  such that, for the perturbed test function

$$\varphi^\varepsilon(f, \mathbf{e}) = \varphi(\rho) + \varepsilon \varphi_1(f, \mathbf{e}) + \varepsilon^2 \varphi_2(f, \mathbf{e}), \quad (6.2)$$

we may have  $\mathcal{L}^\varepsilon \varphi^\varepsilon = \mathcal{L} \varphi + o(1)$ . See Section 6.1.

2. **Tightness.** We prove the tightness of the sequence  $(\rho^\varepsilon)$  in an adequate space. First, we obtain some bounds uniform with respect to  $\varepsilon$  by perturbation of the functional which we try to estimate. See Section 6.2. Then we establish some uniform estimates on the time increments of  $(\rho^\varepsilon)$ . See Section 6.3.
3. **Convergence.** We use the characterization of (1.25)-(1.26) as a martingale problem to take the limit of the processes  $(\rho_\varepsilon)$ . This is a very classical approach to the convergence of stochastic processes, see the introduction to [14, Chapter III]. The class  $\Theta$  of test-functions  $\varphi(\rho)$  of the form

$$\varphi(\rho) = \psi \left( \langle \rho, \xi \rangle_{L^2(\mathbb{T}^N)} \right), \quad (6.3)$$

for  $\xi$  in a dense subset of  $L^2(\mathbb{T}^N)$  and  $\psi$  a Lipschitz function on  $\mathbb{R}$  such that  $\psi' \in C_b^\infty(\mathbb{R})$ , is a separating class in  $L^2(\mathbb{T}^N)$ : if two random variables  $\rho_1$  and  $\rho_2$  satisfy  $\mathbb{E}\varphi(\rho_1) = \mathbb{E}\varphi(\rho_2)$  for all  $\varphi$  as in (6.3), then  $\rho_1$  and  $\rho_2$  have the same laws (this follows from the fact that  $\Theta$  separates points and from Theorem 4.5 p. 113 in [10]). This is why we put a special emphasis in Section 6.1.3 on the application of the perturbed test-function method to test-functions as in (6.3).

## 6.1 Perturbed test-function

Let  $\varphi: L^2(\mathbb{T}^N) \rightarrow \mathbb{R}$  be a given test-function in the variable  $\rho$ . We specify its regularity later. Consider the perturbation (6.2). To obtain the approximation  $\mathcal{L}^\varepsilon \varphi^\varepsilon = \mathcal{L}\varphi + o(1)$ , we identify the powers in  $\varepsilon$  in each side of this equality. This gives, for the scale  $\varepsilon^{-2}$ , the first equation  $\mathcal{L}_\# \varphi = 0$ . This equation is satisfied since  $\varphi$  is independent on  $\mathbf{e}$ , hence  $A\varphi = 0$ , and

$$(Qf - \mathbf{e} \cdot \nabla_v f, D_f \varphi(\rho)) = (\rho(Qf - \mathbf{e} \cdot \nabla_v f), D_\rho \varphi(\rho)) = 0$$

since  $\rho(Qf) = 0$  and  $\rho(\mathbf{e} \cdot \nabla_v f) = 0$  separately. At the scale  $\varepsilon^{-1}$  and  $\varepsilon^0$ , we obtain the equation for the first corrector

$$\mathcal{L}_\# \varphi_1 + \mathcal{L}_b \varphi = 0 \quad (6.4)$$

and the equation for the second corrector

$$\mathcal{L}_\# \varphi_2 + \mathcal{L}_b \varphi_1 = \mathcal{L}\varphi, \quad (6.5)$$

respectively. If (6.4) and (6.5) are satisfied, then  $\mathcal{L}^\varepsilon \varphi^\varepsilon = \mathcal{L}\varphi + \varepsilon \mathcal{L}_b \varphi_2$ . We solve (6.4) and (6.5) by formal computations first, see Section 6.1.1 and Section 6.1.2. In Section 6.1.3 then, we discuss more rigorously the resolution of (6.4) and (6.5).

### 6.1.1 First corrector

We seek a solution to (6.4) by means of the resolvent formula

$$\varphi_1(f, \mathbf{e}) = \int_0^\infty \mathbb{E}_{(f, \mathbf{e})} \psi(f_t, E_t) dt, \quad \psi = \mathcal{L}_b \varphi,$$

where  $f_t$  is obtained either by (3.3) or (3.4) with  $s = 0$ . The right-hand side  $\psi$  is

$$\psi(f, \mathbf{e}) = \mathcal{L}_b \varphi(f, \mathbf{e}) = -(\operatorname{div}_x(vf), D_f \varphi(\rho)) = -(\operatorname{div}_x(J(f)), D_\rho \varphi(\rho)),$$

since  $\rho(vf) = J(f)$ . By (5.23), we obtain the candidate

$$\varphi_1(f, \mathbf{e}) = -(\operatorname{div}_x(H(f)), D_\rho \varphi(\rho)), \quad H(f) := J(f) + \rho(f)R_0(\mathbf{e}). \quad (6.6)$$

### 6.1.2 Second corrector and limit generator

Let  $\mu_\rho$  be the invariant measure parametrized by  $\rho$  associated to  $\mathcal{L}_\sharp$ , defined by (3.8). Since  $\mathcal{L}_\sharp^* \mu_\rho = 0$  and  $\langle \mathcal{L} \varphi, \mu_\rho \rangle = \mathcal{L} \varphi(\rho)$ , a necessary condition to (6.5) is that

$$\mathcal{L} \varphi(\rho) = \langle \mathcal{L}_b \varphi_1, \mu_\rho \rangle. \quad (6.7)$$

If (6.7) is satisfied, then we set

$$\varphi_2(f, \mathbf{e}) = \int_0^\infty (\mathbb{E}_{(f, \mathbf{e})} \mathcal{L}_b \varphi_1(f_t, E_t) - \langle \mathcal{L}_b \varphi_1, \mu_\rho \rangle) dt. \quad (6.8)$$

The equation (6.7) gives the limit generator  $\mathcal{L}$ . Since  $f \mapsto H(f)$  is linear, we have

$$\begin{aligned} \mathcal{L}_b \varphi_1(f, \mathbf{e}) &= -(\operatorname{div}_x(vf), D_f \varphi_1(f, \mathbf{e})) \\ &= (\operatorname{div}_x[H(\operatorname{div}_x(vf))], D_\rho \varphi(\rho)) + D_\rho^2 \varphi(\rho) \cdot (\operatorname{div}_x(H(f)), \operatorname{div}_x(J(f))), \end{aligned} \quad (6.9)$$

and thus

$$\mathcal{L} \varphi(\rho) = (\langle \psi, \mu_\rho \rangle, D_\rho \varphi(\rho)) + \int_{E \times F} D_\rho^2 \varphi(\rho) \cdot (\operatorname{div}_x(H(f)), \operatorname{div}_x(J(f))) d\mu_\rho(f, \mathbf{e}), \quad (6.10)$$

where  $\psi(f, \mathbf{e}) = \operatorname{div}_x(H(\operatorname{div}_x(vf)))$ . Let us compute the first term in the right-hand side of (6.10). Using (6.6), we have

$$\psi(f, \mathbf{e}) = D_x^2 : K(f) + \operatorname{div}_x [R_0(\mathbf{e}) \operatorname{div}_x(J(f))].$$

The part  $\langle D_x^2 : K(f), \mu_\rho \rangle = D_x^2 : [\rho \mathbb{E} K(\bar{M}_0)]$  is given by (5.25). To identify the contribution of the second part, we adapt (5.29) to obtain

$$\begin{aligned} \langle \operatorname{div}_x [R_0(\mathbf{e}) \operatorname{div}_x(J(f))], \mu_\rho \rangle &= \partial_{x_j} \int_{-\infty}^0 e^\sigma \mathbb{E} [R_0(E_j^*(0)) \partial_{x_i} (\rho E_i^*(\sigma))] \\ &= \partial_{x_j} \mathbb{E} [R_1 R_0(E_j^*(0)) \partial_{x_i} (\rho E_i^*(0))]. \end{aligned}$$

The first-order part in (6.10) is therefore  $(\langle \psi, \mu_\rho \rangle, D_\rho \varphi(\rho))$ , with

$$\langle \psi, \mu_\rho \rangle = D_x^2 : [\rho (K + \alpha \mathbb{E} [\bar{E}(0) \otimes R_1(\bar{E}(0))])] + \operatorname{div}_x \mathbb{E} [R_1 R_0(\bar{E}(0)) \operatorname{div}_x (\rho \bar{E}(0))].$$

This can be rewritten as

$$\langle \psi, \mu_\rho \rangle = \operatorname{div}_x (K_\sharp \nabla_x \rho + \Psi \rho), \quad (6.11)$$

where  $K_{\sharp}$  and  $\Psi$  are given in (1.21) and (1.22) respectively. Note that this is consistent with the result (5.30) obtained for a  $\bar{E}$  independent on  $x$  (indeed, the drift coefficient  $\Psi$  vanishes if  $\bar{E}$  is independent on  $x$ ). To compute the second-order part in (6.10), we have two terms to consider:  $\langle J(f) \otimes J(f), \mu_{\rho} \rangle$  and  $\langle R_0(\mathbf{e}) \otimes J(f), \mu_{\rho} \rangle$ . We have already established

$$\langle R_0(\mathbf{e}) \otimes J(f), \mu_{\rho} \rangle = \mathbb{E} [R_1 R_0(\bar{E}(0)) \otimes (\rho \bar{E}(0))].$$

By (5.24) and (5.27), we have also

$$\langle J(f) \otimes J(f), \mu_{\rho} \rangle = \mathbb{E} [(\rho R_1(\bar{E}(0))) \otimes (\rho \bar{E}(0))].$$

It follows by the resolvent identity  $R_1 R_0 = R_0 - R_1$  that

$$\begin{aligned} \int_{E \times F} D_{\rho}^2 \varphi(\rho) \cdot (\operatorname{div}_x(H(f)), \operatorname{div}_x(J(f))) d\mu_{\rho}(f, \mathbf{e}) \\ = \mathbb{E} D_{\rho}^2 \varphi(\rho) \cdot (\operatorname{div}_x(\rho R_0(\bar{E}(0))), \operatorname{div}_x(\rho \bar{E}(0))). \end{aligned} \quad (6.12)$$

### 6.1.3 First and second correctors

Recall (see (1.16), (1.17)) that

$$\bar{J}_m(f) = \iint_{\mathbb{T}^N \times \mathbb{R}^N} |v|^m f(x, v) dx dv, \quad G_m = \{f \in L^1(\mathbb{T}^N \times \mathbb{R}^N); \bar{J}_m(f) < +\infty\}.$$

Recall also that  $F = H^{\bar{\sigma}}(\mathbb{T}^N)$ . Let us introduce the following notations. We write  $a \lesssim b$  with the meaning that  $a \leq Cb$ , where the constant  $C$  may depend on  $\alpha$  (cf. (2.6)), on  $C_{\alpha}^0$  (cf. (2.22)), on various irrelevant constants, and on the dimension  $N$ .

**Proposition 6.1.** *Let  $\varphi$  be of the form (6.3), with  $\xi \in C^3(\mathbb{T}^N)$  and  $\psi$  a Lipschitz function on  $\mathbb{R}$  such that  $\psi' \in C_b^{\infty}(\mathbb{R})$ . Let  $\varphi_1, \varphi_2$  be the correctors defined by (6.4), (6.8) respectively. Then  $\varphi_1, \varphi_2$  satisfy  $\mathcal{L}_{\sharp} \varphi_i(f, \mathbf{e}) < +\infty$ ,  $\mathcal{L}_b \varphi_i(f, \mathbf{e}) < +\infty$  for all  $f \in G_3$ ,  $\mathbf{e} \in F$  and are in the domain of  $\mathcal{L}^{\varepsilon}$ . We have the estimates*

$$|\varphi_1(f, \mathbf{e})| \lesssim \|\psi\|_{W^{1,\infty}(\mathbb{R})} \|\xi\|_{C^1(\mathbb{T}^N)} (\bar{J}_0(f) + \bar{J}_1(f)), \quad (6.13)$$

and

$$|\mathcal{L}_b \varphi_1(f, \mathbf{e})| \lesssim \|\psi\|_{W^{2,\infty}(\mathbb{R})} \|\xi\|_{C^2(\mathbb{T}^N)}^2 (|\bar{J}_0(f)|^2 + |\bar{J}_2(f)|^2), \quad (6.14)$$

on  $\varphi_1$  and the following estimates on  $\varphi_2$ :

$$|\varphi_2(f, \mathbf{e})| \lesssim \|\psi\|_{W^{2,\infty}(\mathbb{R})} \|\xi\|_{C^2(\mathbb{T}^N)}^2 (|\bar{J}_0(f)|^2 + |\bar{J}_2(f)|^2), \quad (6.15)$$

and

$$|\mathcal{L}_b \varphi_2(f, \mathbf{e})| \lesssim \|\psi\|_{W^{3,\infty}(\mathbb{R})} \|\xi\|_{C^3(\mathbb{T}^N)}^3 (|\bar{J}_0(f)|^3 + |\bar{J}_3(f)|^3), \quad (6.16)$$

for all  $f \in G_3$ , for all  $\mathbf{e} \in F$  with  $\|\mathbf{e}\|_F \leq \alpha$ .

*Proof of Proposition 6.1.* Let us focus on the estimate (6.15) on  $|\varphi_2(f, \mathbf{e})|$ . Since

$$(h, D_f \varphi(\rho)) = \psi' \left( \langle \rho, \xi \rangle_{L^2(\mathbb{T}^N)} \right) \langle \rho(h), \xi \rangle_{L^2(\mathbb{T}^N)},$$

the equation (6.6) gives a first corrector

$$\varphi_1(f, e) = \psi' \left( \langle \rho, \xi \rangle_{L^2(\mathbb{T}^N)} \right) \langle J(f) + \rho(f)R_0(\mathbf{e}), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)}. \quad (6.17)$$

For simplicity, let us denote by  $\psi', \psi'', \dots$  the derivatives of  $\psi$  evaluated at the point  $\langle \rho, \xi \rangle_{L^2(\mathbb{T}^N)}$ . By (6.9), we have

$$\begin{aligned} \mathcal{L}_b \varphi_1(f, \mathbf{e}) &= \psi' \int_{\mathbb{T}^N} K(f) : D_x^2 \xi + J(f) \cdot \nabla_x [R_0(\mathbf{e}) \cdot \nabla_x \xi] dx \\ &\quad + \psi'' \langle J(f), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)} \langle J(f) + \rho(f)R_0(\mathbf{e}), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)}, \end{aligned} \quad (6.18)$$

and

$$\varphi_2(f, \mathbf{e}) = \int_0^\infty \mathbb{E}_{(f, \mathbf{e})} [\mathcal{L}_b \varphi_1(f_t, E_t) - \langle \mathcal{L}_b \varphi_1, \mu_\rho \rangle] dt, \quad (6.19)$$

where  $f_t$  is obtained either by (3.3) or (3.4) with  $s = 0$ . Consider the LB-case. There are two terms in  $f_t$  and three terms in  $\mathcal{L}_b \varphi_1$ , which makes at least six terms to consider. We find out more than six terms actually, because of the translations in  $v$ . Consider the first term in (3.3). By (5.22), and for

$$w_t := \int_0^t E_s(\mathbf{e}) ds,$$

we have

$$\begin{aligned} K(f(\cdot - w_t)) &= K(f) + J(f) \otimes w_t + w_t \otimes J(f) + \rho(f)w_t^{\otimes 2}, \\ J(f(\cdot - w_t)) &= J(f) + \rho(f)w_t. \end{aligned}$$

In (6.18)-(6.19), and regarding the linear terms with factor  $\psi'$ , this gives the contributions

$$\Phi_{2,a} = \psi' \int_{\mathbb{T}^N} \int_0^\infty e^{-t} \mathbb{E} [K(f) + 2J(f) \otimes w_t + \rho(f)w_t^{\otimes 2}] : D_x^2 \xi dt dx,$$

and

$$\Phi_{2,b} = \psi' \int_{\mathbb{T}^N} \int_0^\infty e^{-t} \mathbb{E} [(J(f) + \rho(f)w_t) \cdot \nabla_x [R_0(E_t(\mathbf{e})) \cdot \nabla_x \xi]] dt dx.$$

Using the bound  $\|w_t\|_F \leq t \sup_{s \in [0, t]} \|E_s(\mathbf{e})\|_F$  and (2.6), (2.21), we have

$$|\Phi_{2,a}|, |\Phi_{2,b}| \lesssim \|\psi'\|_{L^\infty(\mathbb{R})} \|\xi\|_{C^2(\mathbb{T}^N)} (\bar{J}_0(f) + \bar{J}_1(f) + \bar{J}_2(f)).$$

Since  $\bar{J}_1(f) \leq \frac{1}{2}\bar{J}_0(f) + \frac{1}{2}\bar{J}_2(f)$ , this gives us a bound by  $\|\psi'\|_{L^\infty(\mathbb{R})}\|\xi\|_{C^2(\mathbb{T}^N)}(\bar{J}_0(f) + \bar{J}_2(f))$ . Using (5.22) again, and still regarding the linear terms with factor  $\psi'$  only, we see that the second term in the expansion (3.3) of  $f_t^{\text{LB}}$  has the contributions

$$\Phi_{2,c} = \psi' \int_{\mathbb{T}^N} \int_0^\infty (\theta_c(t) - \theta_c(+\infty)) dt dx, \quad \Phi_{2,d} = \psi' \int_{\mathbb{T}^N} \int_0^\infty (\theta_d(t) - \theta_d(+\infty)) dt dx,$$

where

$$\theta_c(t) = \rho(f) \int_0^t e^{-(t-\sigma)} \left[ K + \mathbb{E} \left( \int_\sigma^t E_s(\mathbf{e}) ds \right)^{\otimes 2} \right] : D_x^2 \xi d\sigma, \quad (6.20)$$

$$\theta_d(t) = \rho(f) \int_0^t e^{-(t-\sigma)} \mathbb{E} \left[ \int_\sigma^t E_s(\mathbf{e}) ds \cdot \nabla_x [R_0(E_t(\mathbf{e})) \cdot \nabla_x \xi] \right] d\sigma, \quad (6.21)$$

By standard manipulations on the integrals in (6.20), we have

$$\theta_c(t) = \rho(f)(1 - e^{-t})K : D_x^2 \xi + 2\rho(f) \int_0^t e^{-\sigma} \int_0^\sigma \int_r^\sigma \Gamma_{\mathbf{e}}(t-r, t-s) : D_x^2 \xi ds dr d\sigma,$$

where the covariance  $\Gamma_{\mathbf{e}}$  is defined by (2.23). The most delicate term to estimate in  $\Phi_{2,c}$  is

$$\Phi_{2,c}^* = 2 \int_{\mathbb{T}^N} \rho(f) \int_0^\infty \int_0^t e^{-\sigma} \int_0^\sigma \int_r^\sigma [\Gamma_{\mathbf{e}}(t-r, t-s) - \bar{\Gamma}(s-r)] : D_x^2 \xi ds dr d\sigma dt dx.$$

The other terms are bounded by  $\|\xi\|_{C^2(\mathbb{T}^N)}(\bar{J}_0(f) + \bar{J}_2(f))$  using (2.6). Using also (2.24), we have

$$\begin{aligned} |\Phi_{2,c}^*| &\lesssim 2\bar{J}_0(f)\|\xi\|_{C^2(\mathbb{T}^N)} \int_0^\infty \int_0^t e^{-\sigma} \int_0^\sigma \int_r^\sigma \gamma_{\text{mix}}(t-s) ds dr d\sigma dt \\ &\lesssim 2\bar{J}_0(f)\|\xi\|_{C^2(\mathbb{T}^N)} \int_0^\infty \int_0^t s(e^{-s} - e^{-t}) \gamma_{\text{mix}}(t-s) ds dt. \end{aligned}$$

Neglecting the term  $-e^{-t}$  and using (2.19) gives a bound  $|\Phi_{2,c}^*| \lesssim 2\bar{J}_0(f)\|\xi\|_{C^2(\mathbb{T}^N)}$ . We have also

$$\theta_d(t) = \rho(f) \int_0^t e^{-\sigma} \int_0^\sigma \mathbb{E} [E_{t-s}(\mathbf{e}) \cdot \nabla_x [R_0(E_t(\mathbf{e})) \cdot \nabla_x \xi]] ds d\sigma.$$

Conditioning on  $\mathcal{G}_{t-s}$ , we see that

$$\mathbb{E} [E_{t-s}(\mathbf{e}) \otimes R_0(E_t(\mathbf{e}))] = e^{(t-s)A} [\psi \otimes e^{sA} R_0 \psi] (\mathbf{e}), \quad \psi(\mathbf{e}) = \mathbf{e}.$$



By (2.18), (2.21), (2.19), we obtain

$$\begin{aligned}
& \left\| \int_0^\infty \int_0^t e^{-\sigma} \int_0^\sigma \left[ e^{(t-s)A} [\psi \cdot \nabla_x (e^{sA} R_0 \psi \cdot \nabla_x \xi)](\mathbf{e}) \right. \right. \\
& \quad \left. \left. - \langle \psi \cdot \nabla_x (P_s R_0 \psi \cdot \nabla_x \xi), \nu \rangle \right] ds d\sigma dt \right\|_{C(\mathbb{T}^N)} \\
& \leq \alpha^2 \int_0^\infty \int_0^t e^{-\sigma} \int_0^\sigma \gamma_{\text{mix}}(t-s) ds d\sigma dt \|\xi\|_{C^1(\mathbb{T}^N)} \\
& \leq \alpha^2 \|\xi\|_{C^1(\mathbb{T}^N)}.
\end{aligned}$$

With this estimate, it is easy to prove that  $|\Phi_{2,d}| \lesssim \|\xi\|_{C^2(\mathbb{T}^N)} \bar{J}_0(f)$ . Let us look at the quadratic terms with factor  $\psi''$  now. There are two terms in (3.3), so four terms  $\Phi_{2,e}, \dots, \Phi_{2,h}$  to consider here. The first term in (3.3) has a factor  $e^{-t}$ , like in  $\Phi_a, \Phi_b$ . There is no contribution from  $\langle \mathcal{L}_b \varphi_1, \mu_\rho \rangle$  in  $\Phi_{2,e}, \Phi_{2,f}, \Phi_{2,g}$  hence, and the convergence of the integral in (6.19) is clear. Therefore, using the same arguments as above, we obtain the estimates

$$|\Phi_{2,e}|, |\Phi_{2,f}|, |\Phi_{2,g}| \lesssim \|\psi''\|_{L^\infty(\mathbb{R})} \|\nabla_x \xi\|_{C^1(\mathbb{T}^N)}^2 (|\bar{J}_0(f)|^2 + |\bar{J}_1(f)|^2). \quad (6.22)$$

Let us illustrate this on the example of  $\Phi_{2,g}$ . We have

$$\begin{aligned}
\Phi_{2,g} = \psi'' \int_0^\infty e^{-t} \int_0^t e^{-(t-\sigma)} \mathbb{E} \left[ \int_\sigma^t \langle \rho(f) E_r(\mathbf{e}), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)} dr \right. \\
\left. \times \langle J(f) + \rho(f) \int_0^t E_s(\mathbf{e}) ds + \rho(f) R_0(E_t), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)} \right],
\end{aligned}$$

which gives (6.22). The last term  $\Phi_{2,h}$  is

$$\Phi_{2,h} = \psi'' \int_0^\infty (\theta_h(t) - \theta_h(+\infty)) dt,$$

where

$$\begin{aligned}
\theta_h(t) = \mathbb{E} \int_0^t \int_\sigma^t \int_0^\sigma \int_{\sigma'}^t e^{-(t-\sigma)} e^{-(t-\sigma')} \langle \rho(f) E_s(\mathbf{e}), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)} \\
\times \langle \rho(f) E_{s'}(\mathbf{e}) + c(t)^{-1} \rho(f) R_0(E_t(\mathbf{e})), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)} ds' ds d\sigma' d\sigma.
\end{aligned}$$

The coefficient  $c(t)$  is

$$c(t) = \int_0^t \int_{\sigma'}^t e^{-(t-\sigma')} ds' d\sigma' = \int_0^t \sigma e^{-\sigma} = 1 - (t+1)e^{-t}.$$

The technique used to estimate the terms  $\Phi_{2,c}$  and  $\Phi_{2,d}$  applies here to give

$$|\Phi_{2,h}| \lesssim \|\psi''\|_{L^\infty(\mathbb{R})} \|\nabla_x \xi\|_{C^1(\mathbb{T}^N)}^2 |\bar{J}_0(f)|^2.$$

This concludes the estimate on  $\varphi_2$  in the LB-case. The estimate on  $\varphi_2$  in the FP-case is obtained by the same arguments. This follows from the expressions for  $K(f_t)$ ,  $J(f_t)$ , which involve various terms, similar to those estimated in the LB-case. For example, a careful computation based on (3.4) and (5.22) gives

$$K(f_t^{\text{FP}}) = \rho(f) \left[ (1 - e^{-2t})K + \left( \int_0^t e^{-(t-\sigma)} E_\sigma(\mathbf{e}) d\sigma \right)^{\otimes 2} \right] + e^{-2t}K(f) \\ + e^{-t} \left[ \int_0^t e^{-(t-\sigma)} E_\sigma(\mathbf{e}) d\sigma \otimes J(f) + J(f) \otimes \int_0^t e^{-(t-\sigma)} E_\sigma(\mathbf{e}) d\sigma \right].$$

A comparable expansion for  $J(f_t^{\text{FP}})$  gives the result, like in the LB-case. Using (2.22), a careful study of the terms composing  $\varphi_2$  shows that  $\varphi_1$  and  $\varphi_2$  are of the form (4.29) with some  $\xi_i$  as in Remark 4.1. By Proposition 4.4, we deduce that  $\mathcal{L}_\# \varphi_i(f, \mathbf{e}) < +\infty$ ,  $\mathcal{L}_\flat \varphi_i(f, \mathbf{e}) < +\infty$  for all  $f \in G_3$ ,  $\mathbf{e} \in F$  and that  $\varphi_1$  and  $\varphi_2$  are in the domain of  $\mathcal{L}^\varepsilon$ . There remains to prove (6.16). Compared to the development of  $\varphi_2$ , when computing  $\mathcal{L}_\flat \varphi_2$ , still more terms appear, which combine the derivatives of  $\psi$  up to the order three. However, all the questions of convergence of the integrals with respect to  $t$  have been dealt with in the estimate of  $\varphi_2$ . Although lengthy, it is not problematic, to prove (6.16): we do not expound that part thus.  $\square$

*Remark 6.1* (Linear test function). In Section 6.3, we apply Proposition 6.1 to a linear test-function  $\varphi(\rho) = \langle \rho, \xi \rangle_{L^2(\mathbb{T}^N)}$ , which means  $\psi' = 1$ ,  $\psi'' = 0$ . In that case, the bounds on the first corrector is a little bit simpler: we have

$$|\varphi_1(f, \mathbf{e})| \lesssim \|\xi\|_{C^1(\mathbb{T}^N)} (\bar{J}_0(f) + \bar{J}_1(f)), \quad (6.23)$$

and

$$|\mathcal{L}_\flat \varphi_1(f, \mathbf{e})| \lesssim \|\xi\|_{C^2(\mathbb{T}^N)} (\bar{J}_0(f) + \bar{J}_2(f)), \quad (6.24)$$

for all  $f \in G$ , for all  $\mathbf{e} \in F$  with  $\|\mathbf{e}\|_F \leq \alpha$ .

By Theorem 4.3, Remark 4.2 and Theorem B.1, we obtain the following corollary to Proposition 6.1.

**Corollary 6.2.** *Let  $\varphi$  be of the form (6.3), with  $\xi \in C^3(\mathbb{T}^N)$  and  $\psi$  a Lipschitz function on  $\mathbb{R}$  such that  $\psi' \in C_b^\infty(\mathbb{R})$ . Let  $\varphi_1, \varphi_2$  be the correctors defined by (6.4), (6.8) respectively. Let  $\theta$  be the correction of  $\varphi$  at order 0, 1 or 2:*

$$\theta \in \{\varphi, \varphi + \varepsilon \varphi_1, \varphi + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2\}.$$

Then

$$M_\theta^\varepsilon(t) := \theta(f_t^\varepsilon, \bar{E}_t^\varepsilon) - \theta(f_{\text{in}}, \bar{E}_0) - \int_0^t \mathcal{L}^\varepsilon \theta(f_s^\varepsilon, \bar{E}_s^\varepsilon) ds \quad (6.25)$$

is a  $(\mathcal{G}_{t/\varepsilon^2})$ -martingale with quadratic variation given by

$$\langle M_\theta^\varepsilon, M_\theta^\varepsilon \rangle_t = \int_0^t [\mathcal{L}^\varepsilon |\theta|^2 - 2\theta \mathcal{L}^\varepsilon \theta](f^\varepsilon(s), \bar{E}^\varepsilon(s)) ds,$$

for all  $t \geq 0$ .

## 6.2 Bounds on the moments

Recall that  $\bar{J}_m(f)$  denotes the  $m$ -th moment of  $f$  (see (1.16)) and that  $G_m$  is the state space of functions  $f \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$  such that  $\bar{J}_m(f) < +\infty$ .

**Proposition 6.3.** *Let  $f_0^\varepsilon \in G_m$ . Let  $(f_t^\varepsilon)$  be the unique mild solution to (1.7) on  $[0, T]$  given by Proposition 4.1 or 4.2. Then, for all  $m \in \mathbb{N}$ , almost-surely, for all  $t \geq 0$ ,*

$$\bar{J}_m(f_t^\varepsilon) \leq C(\alpha, m, t) [\bar{J}_m(f_0^\varepsilon) + \bar{J}_0(f_0^\varepsilon)], \quad (6.26)$$

where  $C(\alpha, m, t)$  is a constant which is bounded for  $t$  in a bounded set.

*Proof of Proposition 6.3.* By density, we can assume that  $f_{\text{in}} \in W^{2,1}(\mathbb{T}^N \times \mathbb{R}^N)$ . We can also replace  $v \mapsto |v|^m$  by  $v \mapsto |v|^m \chi_\eta(v)$ , where  $\chi_\eta$  is a function with compact support which converges pointwise to 1 when  $\eta \rightarrow 0$ . By the results of propagation of regularity given in Proposition 4.1 and Proposition 4.2, the following computations are licit then. For simplicity, we take directly  $\chi \equiv 1$ . First, we have

$$\frac{d}{dt} \bar{J}_{2m}(f_t^\varepsilon) = \frac{1}{\varepsilon^2} \left[ \bar{J}_{2m}(Q f_t^\varepsilon) + 2m \iint_{\mathbb{T}^N \times \mathbb{R}^N} |v|^{2(m-1)} v \cdot \bar{E}_t^\varepsilon f_t^\varepsilon(x, v) dx dv \right]. \quad (6.27)$$

If  $m = 0$ , then, for all  $t \geq 0$ , almost-surely,  $\bar{J}_0(f_t^\varepsilon) = \bar{J}_0(f_0^\varepsilon)$  since the equation is conservative. If  $m > 0$ , then we use the following inequality (which is a consequence of Young's inequality)

$$2m|v|^{2m-1} \leq \frac{1}{2\alpha} |v|^{2m} + [2\alpha(2m-1)]^{2m-1},$$

to infer, by (6.27) and (2.8), that

$$\frac{d}{dt} \bar{J}_{2m}(f_t^\varepsilon) \leq \frac{1}{\varepsilon^2} \left[ \bar{J}_{2m}(Q f_t^\varepsilon) + \frac{1}{2} \bar{J}_{2m}(f_t^\varepsilon) + \alpha [2\alpha(2m-1)]^{2m-1} \bar{J}_0(f_t^\varepsilon) \right].$$

We have, in the case  $Q = Q_{\text{LB}}$ ,

$$\bar{J}_{2m}(Q_{\text{LB}} f) = \bar{J}_{2m}(M) \bar{J}_0(f) - \bar{J}_{2m}(f).$$

If  $Q = Q_{\text{FP}}$ , then

$$\begin{aligned} \bar{J}_{2m}(Q_{\text{FP}} f) &= -2m \iint_{\mathbb{T}^N \times \mathbb{R}^N} |v|^{2(m-1)} v \cdot (\nabla_v f(x, v) + v f(x, v)) dx dv \\ &= (N + 2(m-1)) \bar{J}_{2(m-1)}(f) - 2m \bar{J}_{2m}(f). \end{aligned}$$

In the first case  $Q = Q_{\text{LB}}$ , we obtain

$$\bar{J}_{2m}(f_t^\varepsilon) \leq e^{-\frac{t}{2\varepsilon^2}} \bar{J}_{2m}(f_0^\varepsilon) + 2(1 - e^{-\frac{t}{2\varepsilon^2}}) [\bar{J}_{2m}(M) + \alpha [2\alpha(2m-1)]^{2m-1}] \bar{J}_0(f_0^\varepsilon).$$

This gives (6.26). If  $Q = Q_{\text{FP}}$ , we conclude similarly by a recursive argument on  $m$ .  $\square$

### 6.3 Tightness

For  $\sigma > 0$ , we denote by  $H^{-\sigma}(\mathbb{T}^N)$  the dual space of  $H^\sigma(\mathbb{T}^N)$ . Let  $J_1^\sigma = (\text{Id} - \Delta_x)^{-\sigma}$ . In the standard Fourier basis  $(w_k)$  of  $L^2(\mathbb{T}^N)$ ,  $J_1$  is given by

$$J_1^\sigma w_k = (1 + \lambda_k)^{-\sigma} w_k, \quad \lambda_k = 4\pi^2 |k|^2, \quad w_k(x) = \exp(2\pi i k \cdot x).$$

As  $J_1^{\sigma/2}$  is an isometry  $L^2(\mathbb{T}^N) \rightarrow H^\sigma(\mathbb{T}^N)$ , the norm on  $H^{-\sigma}(\mathbb{T}^N)$  is

$$\|\Lambda\|_{H^{-\sigma}(\mathbb{T}^N)} = \left[ \sum_{k \in \mathbb{Z}^d} |\langle \Lambda, J_1^{\sigma/2} w_k \rangle_{L^2(\mathbb{T}^N)}|^2 \right]^{1/2}. \quad (6.28)$$

Recall also that, by the Sobolev embedding, for all  $\sigma > 2 + \frac{N}{2}$ , there is a constant  $C(\sigma) \geq 0$  such that

$$\|\xi\|_{C^2(\mathbb{T}^N)} \leq C(\sigma) \|\xi\|_{H^\sigma(\mathbb{T}^N)}. \quad (6.29)$$

**Proposition 6.4** (Tightness). *Let  $f_0^\varepsilon \in G_3$ . Let  $(f_t^\varepsilon)$  be the unique mild solution to (1.7) on  $[0, T]$  given by Proposition 4.1 or 4.2. Let  $\sigma > 2 + \frac{3}{2}N$ . Then  $(\rho_t^\varepsilon)_{t \in [0, T]}$  is tight in the space  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$ .*

*Proof of Proposition 6.4.* Let us fix  $\sigma > \sigma_2 > \sigma_1$  such that

$$\sigma_1 > 2 + \frac{N}{2}, \quad \sigma_2 > \sigma_1 + \frac{N}{2}, \quad \sigma > \sigma_2 + \frac{N}{2}. \quad (6.30)$$

Let us introduce also

$$\zeta^\varepsilon = \rho^\varepsilon - \varepsilon \text{div}_x (J(f^\varepsilon) + \rho(f^\varepsilon) R_0(\bar{E}_t^\varepsilon)).$$

We show first that  $\rho^\varepsilon$  is close to  $\zeta^\varepsilon$  in  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$  and then prove that  $(\zeta^\varepsilon)$  is tight in  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$ .

**Step 1.**  $\rho^\varepsilon$  is close to  $\zeta^\varepsilon$ . Let  $\xi_k = J_1^{\sigma_2/2} w_k$ . Let  $\varphi_k(\rho) = \langle \rho, \xi_k \rangle_{L^2(\mathbb{T}^N)}$  and let

$$\varphi_{k,1}(f, \mathbf{e}) = \langle J(f) + \rho(f) R_0(\mathbf{e}), \nabla_x \xi_k \rangle_{L^2(\mathbb{T}^N)}$$

be the first corrector associated to  $\varphi_k$ . By the estimate on the first corrector in Proposition 6.1 and the estimates on the moments of  $f^\varepsilon$  in Proposition 6.3, we have

$$|\langle \rho^\varepsilon(t) - \zeta^\varepsilon(t), \xi_k \rangle_{L^2(\mathbb{T}^N)}| \lesssim \varepsilon \|\xi_k\|_{C^1(\mathbb{T}^N)} \lesssim \varepsilon \|\xi_k\|_{C^2(\mathbb{T}^N)}. \quad (6.31)$$

In (6.31), we have extended the notation  $a \lesssim b$  to denote the inequality  $a \leq Cb$ , where  $C$  depends on  $\alpha$ , on  $C_\alpha^0$ , on  $N$  and also on the constant  $C(\sigma_1)$  in (6.29), on  $\sup_{0 < \varepsilon < 1} \bar{J}_m(f_0^\varepsilon)$  for  $m = 0, \dots, 3$  and on  $T$ . Note that  $C$  should not depend on  $\varepsilon$ , nor on  $\omega$ .

Let us prove that the sequence  $(\|\xi_k\|_{C^2(\mathbb{T}^N)})$  is square-summable. By (6.29), it is sufficient to consider the behaviour of  $(\|\xi_k\|_{H^{\sigma_1}(\mathbb{T}^N)})$ . This sequence is square summable by (6.30) since

$$\|\xi_k\|_{H^{\sigma_1}(\mathbb{T}^N)} = \|w_k\|_{H^{\sigma_1 - \sigma_2}} = \lambda_k^{\frac{\sigma_1 - \sigma_2}{2}},$$

with  $\lambda_k = 4\pi^2|k|^2$ . Summing the square of (6.31) over  $k$ , we obtain thus

$$\|\rho^\varepsilon - \zeta^\varepsilon\|_{C([0,T];H^{-\sigma}(\mathbb{T}^N))} \lesssim \varepsilon, \quad (6.32)$$

almost-surely.

**Step 2.**  $(\zeta^\varepsilon)$  is tight in  $C([0,T];H^{-\sigma}(\mathbb{T}^N))$ . For  $M > 0$ ,  $\delta \in (0, 3/4)$ , define the set

$$K_M = \left\{ \zeta \in C([0,T];H^{-\sigma}(\mathbb{T}^N)); \|\zeta\|_{C^\delta([0,T];H^{-\sigma}(\mathbb{T}^N))} + \|\zeta\|_{C([0,T];H^{-\sigma_1}(\mathbb{T}^N))} \leq M \right\}.$$

Since the injection  $H^{-\sigma_1}(\mathbb{T}^N) \hookrightarrow H^{-\sigma}(\mathbb{T}^N)$  is compact, and by the Ascoli theorem,  $K_M$  is compact in  $C([0,T];H^{-\sigma}(\mathbb{T}^N))$ . Our aim is to show that  $\mathbb{P}(\zeta^\varepsilon \notin K_M)$  is small for large  $M$ , uniformly with respect to  $\varepsilon$ . Using the Markov inequality, this follows from the bounds

$$\mathbb{E}\|\zeta^\varepsilon\|_{C^\delta([0,T];H^{-\sigma}(\mathbb{T}^N))}^4 \lesssim 1, \quad \mathbb{E}\|\zeta^\varepsilon\|_{C([0,T];H^{-\sigma_1}(\mathbb{T}^N))}^4 \lesssim 1. \quad (6.33)$$

The second bound in (6.33) is a direct consequence of the estimate on the moments of order zero and one of  $f^\varepsilon$ , cf. Proposition 6.3, and of the injection  $L^1(\mathbb{T}^N) \hookrightarrow H^{-\sigma_1}(\mathbb{T}^N)$  due to the condition (6.30) on  $\sigma_1$ . To obtain an estimate on the time increments of  $\zeta^\varepsilon$ , we introduce the perturbed test function  $\varphi_k^\varepsilon = \varphi_k + \varepsilon\varphi_{k,1}$  and the martingale (cf. (6.25))

$$M_k^\varepsilon(t) = \varphi_k^\varepsilon(f^\varepsilon(t), \bar{E}^\varepsilon(t)) - \varphi_k^\varepsilon(f^\varepsilon(0), \bar{E}^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon \varphi_k^\varepsilon(f^\varepsilon(s), \bar{E}^\varepsilon(s)) ds. \quad (6.34)$$

For  $0 \leq s \leq t \leq T$ , the time increment  $\langle \zeta^\varepsilon(t) - \zeta^\varepsilon(s), \xi_k \rangle_{L^2(\mathbb{T}^N)}$  reads

$$\langle \zeta^\varepsilon(t) - \zeta^\varepsilon(s), \xi_k \rangle_{L^2(\mathbb{T}^N)} = \int_s^t \mathcal{L}^\varepsilon \varphi_k^\varepsilon(f^\varepsilon(\sigma), \bar{E}^\varepsilon(\sigma)) d\sigma + M_k^\varepsilon(t) - M_k^\varepsilon(s).$$

This gives us two terms to estimate. Regarding the first term, we compute  $\mathcal{L}^\varepsilon \varphi_k^\varepsilon = \mathcal{L}_b \varphi_{k,1}$ . By Proposition 6.1 and (6.24), we obtain  $|\mathcal{L}^\varepsilon \varphi_k^\varepsilon(f, \mathbf{e})| \lesssim \|\xi_k\|_{C^2(\mathbb{T}^N)} (\bar{J}_0(f) + \bar{J}_2(f))$ . The estimates on the moments of  $f^\varepsilon$  yields the following bound:

$$\left| \int_s^t \mathcal{L}^\varepsilon \varphi_k^\varepsilon(f^\varepsilon(\sigma), \bar{E}^\varepsilon(\sigma)) d\sigma \right|^2 \lesssim |t-s|^2 \|\xi_k\|_{C^2(\mathbb{T}^N)}^2, \quad (6.35)$$

almost-surely. To estimate the time increments of the martingale  $M_k^\varepsilon(t)$ , we use the Burkholder, Davis, Gundy inequality, [5]:

$$\mathbb{E}|M_k^\varepsilon(t) - M_k^\varepsilon(s)|^4 \leq \mathbb{E} \sup_{s \leq r \leq t} |M_k^\varepsilon(r) - M_k^\varepsilon(s)|^4 \lesssim \mathbb{E} |\langle M_k^\varepsilon, M_k^\varepsilon \rangle_t - \langle M_k^\varepsilon, M_k^\varepsilon \rangle_s|^2.$$

By Corollary 6.2, the quadratic variation of  $M_k^\varepsilon$  is

$$\langle M_k^\varepsilon, M_k^\varepsilon \rangle_t = \int_0^t [\mathcal{L}^\varepsilon |\varphi_k^\varepsilon|^2 - 2\varphi_k^\varepsilon \mathcal{L}^\varepsilon \varphi_k^\varepsilon](f^\varepsilon(s), \bar{E}^\varepsilon(s)) ds.$$

Since  $D_f|\varphi_k^\varepsilon|^2 - 2\varphi_k^\varepsilon D_f\varphi_k^\varepsilon = 0$ , only the part  $\varepsilon^{-2}A$  of the generator  $\mathcal{L}^\varepsilon$  is contributing to the quadratic variation. Since, in addition,  $A|\varphi|^2 = 0$ ,  $A\varphi = 0$ , we obtain

$$\langle M_k^\varepsilon, M_k^\varepsilon \rangle_t = \int_0^t [A|\varphi_{k,1}|^2 - 2\varphi_{k,1}A\varphi_{k,1}] (f^\varepsilon(s), \bar{E}^\varepsilon(s)) ds. \quad (6.36)$$

As  $\varphi_{k,1}(f, \mathbf{e}) = c + \Lambda(R_0(\mathbf{e}))$ , where  $\Lambda(\mathbf{e}) = \langle \rho(f)\mathbf{e}, \nabla \xi_k \rangle_{L^2(\mathbb{T}^N)}$ , we have

$$[A|\varphi_{k,1}|^2 - 2\varphi_{k,1}A\varphi_{k,1}] (f, \mathbf{e}) = A|\Lambda(R_0(\mathbf{e}))|^2 - 2\Lambda(R_0(\mathbf{e}))A\Lambda(R_0(\mathbf{e})).$$

Since  $\|\Lambda\| \leq \bar{J}_0(f)\|\xi_k\|_{C^1(\mathbb{T}^N)}$ , the estimates (2.22) give, for  $\sigma \leq T$ ,

$$|[A|\varphi_{k,1}|^2 - 2\varphi_{k,1}A\varphi_{k,1}] (f^\varepsilon(\sigma), \bar{E}^\varepsilon(\sigma))| \lesssim \|\xi_k\|_{C^2(\mathbb{T}^N)}^2,$$

almost-surely, and we obtain

$$\mathbb{E}|M_k^\varepsilon(t) - M_k^\varepsilon(s)|^4 \lesssim |t - s|^2 \|\xi_k\|_{C^2(\mathbb{T}^N)}^4. \quad (6.37)$$

Gathering (6.35) and (6.37), we see that

$$\mathbb{E} \left| \langle \zeta^\varepsilon(t) - \zeta^\varepsilon(s), \xi_k \rangle_{L^2(\mathbb{T}^N)} \right|^4 \lesssim |t - s|^2 \|\xi_k\|_{C^2(\mathbb{T}^N)}^4,$$

for  $0 \leq s, t \leq T$ . By the Garsia - Rodemich - Rumsey inequality, [2, Theorem 7.34] with  $q = 4$ ,  $\alpha = \frac{1}{q} + \delta$ , we obtain

$$\mathbb{E} \left[ \sup_{s \neq t} \frac{\left| \langle \zeta^\varepsilon(t) - \zeta^\varepsilon(s), \xi_k \rangle_{L^2(\mathbb{T}^N)} \right|^4}{|t - s|^{4\delta}} \right] \lesssim C_\delta \|\xi_k\|_{C^2(\mathbb{T}^N)}^4. \quad (6.38)$$

We deduce that

$$\mathbb{E} \left[ \sup_{s \neq t} \frac{1}{|t - s|^{4\delta}} \sum_k \left| \langle \zeta^\varepsilon(t) - \zeta^\varepsilon(s), \xi_k \rangle_{L^2(\mathbb{T}^N)} \right|^4 \right] \lesssim C_\delta. \quad (6.39)$$

We have then

$$\begin{aligned} & \left| \sum_k \left| \langle \zeta^\varepsilon(t) - \zeta^\varepsilon(s), J_1^{\sigma/2} w_k \rangle_{L^2(\mathbb{T}^N)} \right|^2 \right|^2 \\ & \leq \sum_k \left| \langle \zeta^\varepsilon(t) - \zeta^\varepsilon(s), \xi_k \rangle_{L^2(\mathbb{T}^N)} \right|^4 \sum_k (1 + \lambda_k)^{\sigma_2 - \sigma}, \end{aligned} \quad (6.40)$$

since  $J_1^{\sigma/2} w_k = (1 + \lambda_k)^{\frac{\sigma_2 - \sigma}{2}} \xi_k$ . The second sum in (6.40) is finite by (6.30). It follows that  $\mathbb{E} \|\zeta^\varepsilon\|_{C^\delta([0, T]; H^{-\sigma}(\mathbb{T}^N))}^4 \lesssim 1$  and we conclude that  $(\rho^\varepsilon)$  is tight in  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$ .  $\square$

## 6.4 Convergence to the solution of a Martingale problem

Assume that the hypotheses of Proposition 6.4 are satisfied. Let  $\varepsilon_{\mathbb{N}} = \{\varepsilon_n; n \in \mathbb{N}\}$ , where  $(\varepsilon_n) \downarrow 0$ . By the Skorohod theorem [3, p. 70], there is a subset of  $\varepsilon_{\mathbb{N}}$ , which we still denote by  $\varepsilon_{\mathbb{N}}$ , a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ , some random variables  $\{\tilde{\rho}^\varepsilon; \varepsilon \in \varepsilon_{\mathbb{N}}\}$ ,  $\tilde{\rho}$  on  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$ , such that

1. for all  $\varepsilon \in \varepsilon_{\mathbb{N}}$ , the laws of  $\rho^\varepsilon$  and  $\tilde{\rho}^\varepsilon$  as  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$ -random variables coincide,
2.  $\tilde{\mathbb{P}}$ -a.s.,  $(\tilde{\rho}^\varepsilon)$  is converging to  $\tilde{\rho}$  in  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$  along  $\varepsilon_{\mathbb{N}}$ .

Let  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$  be the natural filtration of  $(\tilde{\rho}(t))_{t \in [0, T]}$ . Our aim is to show that the process  $(\tilde{\rho}(t))_{t \in [0, T]}$  is solution of the martingale problem associated to the limit generator  $\mathcal{L}$ .

**Proposition 6.5** (Martingale). *Let  $\sigma > 2 + \frac{3}{2}N$ ,  $\xi \in H^{\sigma+2}(\mathbb{T}^N)$ , and let  $\varphi$  be defined by  $\varphi(\rho) = \psi(\langle \rho, \xi \rangle_{H^{-\sigma}, H^\sigma})$ , where  $\psi$  is a Lipschitz function on  $\mathbb{R}$  such that  $\psi' \in C_b^\infty(\mathbb{R})$ . Then the process*

$$\tilde{M}_\varphi(t) := \varphi(\tilde{\rho}(t)) - \varphi(\tilde{\rho}(0)) - \int_0^t \mathcal{L}\varphi(\tilde{\rho}(s)) ds \quad (6.41)$$

is a continuous martingale with respect to  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ .

*Proof of Proposition 6.5.* Let  $0 \leq s \leq t \leq T$ . Let  $0 \leq t_1 < \dots < t_n \leq s$  and let  $\Theta$  be a continuous and bounded function on  $[H^{-\sigma}(\mathbb{T}^N)]^n$ . Note that  $\tilde{\mathcal{F}}_s$  is generated by the random variables  $\Theta(\tilde{\rho}(t_1), \dots, \tilde{\rho}(t_n))$ , for  $n \in \mathbb{N}^*$ ,  $(t_i)_{1, n}$  and  $\Theta$  as above. Our aim is therefore to prove that

$$\mathbb{E} \left[ (\tilde{M}_\varphi(t) - \tilde{M}_\varphi(s)) \Theta(\tilde{\rho}(t_1), \dots, \tilde{\rho}(t_n)) \right] = 0. \quad (6.42)$$

Let  $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$  be the second order correction of  $\varphi$ , with  $\varphi_1$  and  $\varphi_2$  given by Proposition 6.1. We start from the identity (see (6.25))

$$\mathbb{E} \left[ (M_\varphi^\varepsilon(t) - M_\varphi^\varepsilon(s)) \Theta(\rho^\varepsilon(t_1), \dots, \rho^\varepsilon(t_n)) \right] = 0, \quad (6.43)$$

where

$$M_\varphi^\varepsilon(t) := \varphi^\varepsilon(f^\varepsilon(t), \bar{E}_t^\varepsilon) - \varphi^\varepsilon(f_{\text{in}}, \bar{E}_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \varphi^\varepsilon(f^\varepsilon(s), \bar{E}_s^\varepsilon) ds, \quad (6.44)$$

Recall that  $\mathcal{L}^\varepsilon \varphi^\varepsilon = \mathcal{L}\varphi + \varepsilon\mathcal{L}_1\varphi_1 + \varepsilon^2\mathcal{L}_2\varphi_2$ . By (6.43), the estimates on the correctors (Proposition 6.1) and the uniform estimates on the moments of  $(f_t^\varepsilon)$  (Proposition 6.3), we have

$$\mathbb{E} \left[ (X_\varphi^\varepsilon(t) - X_\varphi^\varepsilon(s)) \Theta(\rho^\varepsilon(t_1), \dots, \rho^\varepsilon(t_n)) \right] = \mathcal{O}(\varepsilon),$$

where the process  $(X_\varphi^\varepsilon(t))$  is

$$X_\varphi^\varepsilon(t) = \varphi(\rho^\varepsilon(t)) - \varphi(\rho_{\text{in}}) - \int_0^t \mathcal{L}\varphi(\rho^\varepsilon(s)) ds.$$

By identities of the laws, it follows that

$$\tilde{\mathbb{E}} \left[ \left( \varphi(\tilde{\rho}^\varepsilon(t)) - \varphi(\tilde{\rho}^\varepsilon(s)) - \int_s^t \mathcal{L}\varphi(\tilde{\rho}^\varepsilon(s)) ds \right) \Theta(\tilde{\rho}^\varepsilon(t_1), \dots, \tilde{\rho}^\varepsilon(t_n)) \right] = \mathcal{O}(\varepsilon). \quad (6.45)$$

We must examine the convergence of each terms in (6.45). By a.s convergence of  $(\tilde{\rho}^\varepsilon)$  in  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$  along  $\varepsilon_{\mathbb{N}}$ , we have

$$\begin{aligned} \left[ \varphi(\tilde{\rho}^\varepsilon(t)) - \int_0^t \mathcal{L}\varphi(\tilde{\rho}^\varepsilon(s)) ds \right] \Theta(\tilde{\rho}^\varepsilon(t_1), \dots, \tilde{\rho}^\varepsilon(t_n)) \\ \rightarrow \left[ \varphi(\tilde{\rho}(t)) - \int_0^t \mathcal{L}\varphi(\tilde{\rho}(s)) ds \right] \Theta(\tilde{\rho}(t_1), \dots, \tilde{\rho}(t_n)) \end{aligned}$$

almost-surely when  $\varepsilon \rightarrow 0$  along  $\varepsilon_{\mathbb{N}}$ . Indeed,  $\mathcal{L}\varphi$  is continuous on  $H^{-\sigma}(\mathbb{T}^N)$ , in virtue of (6.10)-(6.11)-(6.12) and the fact that  $\xi \in H^{\sigma+2}(\mathbb{T}^N)$ . Since  $\Theta$  is bounded and  $\varphi(\tilde{\rho}^\varepsilon(t))$  and  $\mathcal{L}\varphi(\tilde{\rho}^\varepsilon(t))$  are a.s. bounded by a constant (a consequence of (6.26)), we can apply the dominated convergence theorem. This gives (6.42).  $\square$

## 6.5 Limit SPDE

### 6.5.1 Covariance

For  $i, j \in \{1, \dots, N\}$ ,  $x, y \in \mathbb{T}^N$ , we set

$$H(i, x, j, y) = \mathbb{E} \left( [R_0(\bar{E}_0(x))]_i [\bar{E}_0(y)]_j \right). \quad (6.46)$$

This defines a kernel on the product space  $[L^2(\mathbb{T}^N)]^N$ , and an associated operator  $S$ ,

$$S\rho_i(x) = \sum_{j=1}^N \int_{\mathbb{T}^N} H(i, x, j, y) \rho_j(y) dy. \quad (6.47)$$

**Proposition 6.6.** *The operator  $S$  is symmetric, non-negative and trace-class on the space  $[L^2(\mathbb{T}^N)]^N$ .*

*Proof of Proposition 6.6.* That  $S$  is non-negative means  $\langle S\rho, \rho \rangle \geq 0$ , where  $\langle \cdot, \cdot \rangle$  is the canonical scalar product on  $[L^2(\mathbb{T}^N)]^N$  given by

$$\langle \rho, \rho' \rangle = \sum_{i=1}^N \langle \rho_i, \rho'_i \rangle_{L^2(\mathbb{T}^N)}.$$

By Lemma 5.6 indeed, we have

$$\langle S\rho, \rho \rangle = \lim_{\delta \rightarrow 0} \delta \mathbb{E} \left| \int_0^\infty e^{-\delta t} \langle \rho, \bar{E}_t \rangle dt \right|^2 \geq 0.$$



Let  $(\zeta_k)$  be an orthonormal basis of  $[L^2(\mathbb{T}^N)]^N$ . Using the Bessel-Parseval identity, we have

$$\text{Trace}(S) = \sum_k \langle S\zeta_k, \zeta_k \rangle = \mathbb{E}\langle R_0(\bar{E}_0), \bar{E}_0 \rangle \lesssim 1,$$

therefore  $S$  is trace-class.  $\square$

**Proposition 6.7.** *The operator  $S$  admits a square-root  $S^{1/2}$  which is associated to a kernel  $H^{(1/2)}$ .*

*Proof of Proposition 6.7.* By the spectral theorem, there exists an orthonormal basis  $(\zeta_k)$  of  $[L^2(\mathbb{T}^N)]^N$  and some non-negative eigenvalues  $\lambda_k \geq 0$  such that

$$S = \sum_{k \in \mathbb{N}} \lambda_k \zeta_k \otimes \zeta_k, \text{ meaning that } S\rho = \sum_{k \in \mathbb{N}} \lambda_k \langle \rho, \zeta_k \rangle \zeta_k, \forall \rho.$$

It follows that the operator  $S^{1/2} := \sum_{k \in \mathbb{N}} \lambda_k^{1/2} \zeta_k \otimes \zeta_k$  is well-defined,  $S = [S^{1/2}]^2$ . In addition,  $H$  is given by

$$H(i, x, j, y) = \sum_{k \in \mathbb{N}} \lambda_k \zeta_k(i, x) \zeta_k(j, y)$$

and  $S^{1/2}$  is associated to the kernel  $H^{(1/2)}$ , with

$$H^{(1/2)}(i, x, j, y) = \sum_{k \in \mathbb{N}} \lambda_k^{1/2} \zeta_{k,i}(x) \zeta_{k,j}(y)$$

and

$$H(i, x, j, y) = \langle H^{(1/2)}(i, x, \cdot), H^{(1/2)}(j, y, \cdot) \rangle. \quad (6.48)$$

$\square$

**Proposition 6.8.** *Let  $\bar{\sigma} \geq \sigma \geq 0$ . The operator  $S^{1/2}$  can be extended as an operator  $[H^{-\sigma}(\mathbb{T}^N)]^N \rightarrow [L^2(\mathbb{T}^N)]^N$  which is bounded by  $\alpha$ :*

$$\left\| S^{1/2}(\rho) \right\|_{[L^2(\mathbb{T}^N)]^N} \leq \alpha \|\rho\|_{[H^{-\sigma}(\mathbb{T}^N)]^N}, \quad (6.49)$$

for all  $\rho \in [H^{-\sigma}(\mathbb{T}^N)]^N$ .

*Proof of Proposition 6.8.* The last line (5.28) in the proof of Lemma 5.6, gives us

$$\left\| S^{1/2}(\rho) \right\|_{[L^2(\mathbb{T}^N)]^N}^2 = \langle \rho, S(\rho) \rangle = \int_0^\infty \langle \bar{\Gamma}(t), \rho \otimes \rho \rangle dt,$$

where  $\bar{\Gamma}(t)$  is the covariance defined in Section 2.3. The result follows then from the estimate (2.25).  $\square$

### 6.5.2 Limit equation

Let  $(\beta_k(t))_{k \in \mathbb{N}}$  be some independent one-dimensional Wiener processes, let  $(\lambda_k, \zeta_k)$  denote the spectral elements of  $S$ , as in the Proof of Proposition 6.7 and let

$$W(t) = \sum_{k \in \mathbb{N}} \beta_k(t) \zeta_k \quad (6.50)$$

be a cylindrical Wiener process on  $[L^2(\mathbb{T}^N)]^N$ . Consider the equation (1.25) with the cylindrical Wiener process  $W(t)$  given by (6.50). In a first formal step, we test (1.25) against a test-function  $\xi$ . This gives

$$d\langle \rho_t, \xi \rangle_{H^{-\sigma}, H^\sigma} = b(\rho_t, \xi) dt + \sum_{k \in \mathbb{N}} \sigma_k(\rho_t, \xi) d\beta_k(t), \quad (6.51)$$

where

$$b(\rho, \xi) = \langle \rho, \operatorname{div}_x(K_\# \nabla_x \xi) - \Psi \nabla_x \xi \rangle_{H^{-\sigma}, H^\sigma}, \quad \sigma_k(\rho, \xi) = \sqrt{2} \lambda_k^{1/2} \langle \rho \nabla_x \xi, \zeta_k \rangle_{L^2(\mathbb{T}^N)}.$$

Since

$$\frac{1}{2} \sum_{k \in \mathbb{N}} |\sigma_k(\rho, \xi)|^2 = \left\| S^{1/2}(\rho \nabla_x \xi) \right\|_{[L^2(\mathbb{T}^N)]^N}^2, \quad (6.52)$$

the equation (1.25) is indeed the equation associated to the limit generator  $\mathcal{L}$ .

### 6.5.3 Resolution of the limit equation

**Definition 6.1.** Let  $\sigma$  satisfy  $\bar{\sigma} \geq \sigma \geq 0$ . Let  $m \in \mathbb{N}$  be greater than  $\sigma + 2$ . Let  $\rho_{\text{in}} \in H^{-\sigma}(\mathbb{T}^N)$ . Let  $W(t)$  be given by (6.50). An adapted process  $\rho \in C([0, T]; H^{-\sigma}(\mathbb{T}^N))$  is said to be a weak solution to (1.25)-(1.26) in  $H^{-\sigma}(\mathbb{T}^N)$  if

$$\langle \rho_t, \xi \rangle_{H^{-\sigma}, H^\sigma} = \langle \rho_{\text{in}}, \xi \rangle_{H^{-\sigma}, H^\sigma} + \int_0^t b(\rho_s, \xi) ds + \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\rho_s, \xi) d\beta_k(s), \quad (6.53)$$

for all  $\xi \in C^m(\mathbb{T}^N)$ , for all  $t \in [0, T]$ .

**Theorem 6.9.** Let  $\sigma$  satisfy  $\bar{\sigma} \geq \sigma \geq 0$ . Let  $S^{1/2}$  be the Hilbert-Schmidt operator defined in Section 6.5.1. Let  $W(t)$  be given by (6.50). There exists  $\alpha_0 > 0$  such that, under the smallness hypothesis (1.24), we have the following results.

1. Weak solutions to (1.25)-(1.26) in  $H^{-\sigma}(\mathbb{T}^N)$ ,  $\sigma \in \{0, \bar{\sigma}\}$  are unique.
2. There exists a solution to (1.25)-(1.26) in  $L^2(\mathbb{T}^N)$ .

We use the following corollary to items 1-2 in Theorem 6.9.

**Corollary 6.10.** Let  $\sigma$  satisfy  $\bar{\sigma} \geq \sigma \geq 0$ . Let  $S^{1/2}$  be the Hilbert-Schmidt operator defined in Section 6.5.1. Let  $\rho_{\text{in}} \in L^2(\mathbb{T}^N)$ . Let  $W(t)$  be given by (6.50). Assume (1.24). Then (1.25)-(1.26) admits a unique weak solution  $(\rho_t^*)$  in  $H^{-\sigma}(\mathbb{T}^N)$ , which coincides with the weak solution in  $L^2(\mathbb{T}^N)$ .

*Proof of Theorem 6.9.* For simplicity, we consider only the case  $\Psi = 0$ . The adaptation to the case of a non-trivial  $\Psi$  is easy. The existence assertion in item 2 is proved in [7]. We can apply Theorem 6.24 p. 178 for example in the context of Example 6.23. To obtain item 1, we prove that

$$\sup_{t \in [0, T]} \mathbb{E} \left[ \|\rho_t\|_{H^{-\sigma}(\mathbb{T}^N)}^2 \right] \leq C \|\rho_{\text{in}}\|_{H^{-\sigma}(\mathbb{T}^N)}^2, \quad \sigma \in \{0, \bar{\sigma}\}. \quad (6.54)$$

Let us first establish the following estimate

$$\sum_{n \in \mathbb{Z}^N} \frac{|n|^2}{(1 + |m - n|^2)^{\bar{\sigma}} (1 + |n|^2)^{\sigma}} \leq C_1 \frac{1 + |m|^2}{(1 + |m|^2)^{\sigma}}, \quad \sigma \in \{0, \bar{\sigma}\}, \quad (6.55)$$

where  $C_1$  is a constant depending on  $\bar{\sigma}$  and  $N$ . If  $\sigma = 0$ , (6.55) follows from the fact that

$$|n|^2 \leq 2|m|^2 + 2|m - n|^2, \quad \sum_{n \in \mathbb{Z}^N} \frac{1 + |n|^2}{(1 + |n|^2)^{\bar{\sigma}}} < +\infty.$$

If  $\sigma = \bar{\sigma}$ , we make the distinction between the sum over  $|n| < \frac{1}{2}|m|$  and  $|n| \geq \frac{1}{2}|m|$ . In the first case, we have  $|m - n| \geq |n|$  and thus

$$\sum_{|n| < \frac{1}{2}|m|} \frac{|n|^2}{(1 + |m - n|^2)^{\bar{\sigma}} (1 + |n|^2)^{\bar{\sigma}}} \leq \frac{1}{4} \frac{|m|^2}{(1 + |m|^2)^{\bar{\sigma}}} \sum_{n \in \mathbb{Z}^N} \frac{1}{(1 + |n|^2)^{\bar{\sigma}}}.$$

In the second case, we use again the inequality  $|n|^2 \leq 2|m|^2 + 2|m - n|^2$  and the estimate

$$\sum_{|n| \geq \frac{1}{2}|m|} \frac{|m|^2 + |m - n|^2}{(1 + |m - n|^2)^{\bar{\sigma}} (1 + |n|^2)^{\bar{\sigma}}} \leq \frac{1 + |m|^2}{(1 + \frac{1}{4}|m|^2)^{\bar{\sigma}}} \sum_{n \in \mathbb{Z}^N} \frac{1 + |n|^2}{(1 + |n|^2)^{\bar{\sigma}}},$$

to conclude to (6.55). To prove (6.54), we use the Fourier basis  $\{w_n : x \mapsto \exp(2\pi i n \cdot x); n \in \mathbb{Z}^N\}$ . Taking  $\xi = w_n$  in (6.53), applying Itô's Formula, using the fact that  $K_{\sharp} \geq K = \text{Id}_N$  and using (6.52), we see that

$$\begin{aligned} & \frac{1}{2} \mathbb{E} |\langle \rho_t, w_n \rangle|^2 \\ & \leq \frac{1}{2} \mathbb{E} |\langle \rho_{\text{in}}, w_n \rangle|^2 - 4\pi^2 |n|^2 \int_0^t |\langle \rho_s, w_n \rangle|^2 ds + \int_0^t \left\| S^{1/2}(\rho_s \nabla_x w_n) \right\|_{[L^2(\mathbb{T}^N)]^N}^2 ds, \end{aligned}$$

where all the duality products  $\langle \cdot, \cdot \rangle$  are  $(H^{-\sigma}, H^{\sigma})$  duality products. By Proposition 6.8, we obtain

$$\begin{aligned} & \frac{1}{2} \mathbb{E} |\langle \rho_t, w_n \rangle|^2 \\ & \leq \frac{1}{2} \mathbb{E} |\langle \rho_{\text{in}}, w_n \rangle|^2 - 4\pi^2 |n|^2 \int_0^t |\langle \rho_s, w_n \rangle|^2 ds + \alpha^2 \int_0^t \|\rho_s \nabla_x w_n\|_{[H^{-\bar{\sigma}}(\mathbb{T}^N)]^N}^2 ds. \quad (6.56) \end{aligned}$$

The integrand in the last term in (6.56) is

$$\begin{aligned} \|\rho_s \nabla_x w_n\|_{[H^{-\bar{\sigma}}(\mathbb{T}^N)]^N}^2 &= \sum_{m \in \mathbb{Z}^N} (1 + |m|^2)^{-\bar{\sigma}} |\langle \rho_s, w_m \cdot \nabla_x w_n \rangle|^2 \\ &= 4\pi^2 \sum_{m \in \mathbb{Z}^N} (1 + |m|^2)^{-\bar{\sigma}} |n|^2 |\langle \rho_s, w_{m+n} \rangle|^2 \\ &= 4\pi^2 \sum_{m \in \mathbb{Z}^N} (1 + |m - n|^2)^{-\bar{\sigma}} |n|^2 |\langle \rho_s, w_m \rangle|^2. \end{aligned}$$

Here we use the remarkable identities  $\nabla w_n = 2\pi i n w_n$  and  $w_m w_n = w_{m+n}$ . If we multiply (6.56) by  $(1 + |n|^2)^{-\sigma}$  and sum the result over  $n \in \mathbb{Z}^N$ , (6.55) gives

$$\mathbb{E} \|\rho_t\|_{H^{-\sigma}(\mathbb{T}^N)}^2 \leq \mathbb{E} \|\rho_{\text{in}}\|_{H^{-\sigma}(\mathbb{T}^N)}^2 + C_1 \alpha^2 \int_0^t \mathbb{E} \|\rho_s\|_{H^{-\sigma}(\mathbb{T}^N)}^2 ds$$

if  $\alpha^2 C_1 < 1$ . Using the Gronwall Lemma, we conclude to (6.54).  $\square$

## 6.6 Uniqueness for the limit martingale problem

Denote by  $(\rho_t^*)$  the weak solution in  $L^2(\mathbb{T}^N)$  to (1.25)-(1.26) given by Corollary (6.10).

**Proposition 6.11** (Markov property). *Let  $(\mathcal{F}_t^W)$  be the filtration generated by the cylindrical Wiener process (6.50). For  $\varphi: L^2(\mathbb{T}^N) \rightarrow \mathbb{R}$  measurable and bounded, define*

$$P_t^* \varphi(\rho_{\text{in}}) = \mathbb{E} \varphi(\rho_t^*). \quad (6.57)$$

Then  $(P_t^*)_{t \geq 0}$  is a semi-group with generator  $\mathcal{L}$  and  $(\rho_t^*)$  is a Markov process with semi-group  $(P_t^*)_{t \geq 0}$ :

$$\mathbb{E} [\varphi(\rho_{t+s}^*) | \mathcal{F}_s^W] = (P_t^* \varphi)(\rho_s^*), \text{ a.s.}, \quad (6.58)$$

for all  $t, s \geq 0$ ,  $t + s \leq T$ .

*Proof of Proposition 6.11.* We only give the sketch of the proof. First, assume that we regularize the equation (1.25) into the following SPDE:

$$d\rho^\delta = \text{div}_x (K_\# \nabla_x \rho^\delta + \Psi \rho^\delta) dt + \sqrt{2} J_\delta \text{div}_x (\rho^\delta S^{1/2} dW(t)), \quad (6.59)$$

where  $J_\delta = (\text{Id} - \delta \Delta_x)^{-1/2}$ , and let us solve (6.59) starting from  $\rho_{\text{in}}$  (we apply Theorem 7.4 in [7]). We claim that the proof of Theorem 6.9 (case  $\sigma = 0$ ) can be adapted to prove, under the smallness hypothesis (1.24), that  $\rho^\delta \rightarrow \rho^*$  in  $C^\delta([0, T]; L^2(\mathbb{T}^N))$ . The process  $(\rho_t^\delta)_{t \in [0, T]}$  is Markov and satisfies the analog to (6.58) (see Theorem 9.8 in [7]). By taking the limit  $[\varepsilon \rightarrow 0]$ , we obtain (6.58) for  $\varphi$  continuous and bounded on  $L^2(\mathbb{T}^N)$ . The remaining arguments to conclude the proposition are then standard (see the proof of Theorem 4.3).  $\square$

We can now state the following uniqueness result. The proof is very similar to the proof of Theorem 4.1 p. 181-184 in [10], but is also different in many anecdotal aspects, so we expound on it.

**Proposition 6.12** (Uniqueness for the limit martingale problem). *Let  $(\rho_t^\sharp)$  be a process in  $C([0, T]; H^{-\sigma}(\mathbb{T}^N))$  on a probability space  $(\Omega^\sharp, \mathbb{P}^\sharp, \mathcal{F}^\sharp)$  with natural filtration  $(\mathcal{F}_t^\sharp)_{t \in [0, T]}$ . Let  $\sigma > 2 + \frac{3}{2}N$ ,  $\xi \in H^{\sigma+2}(\mathbb{T}^N)$ , and let  $\varphi$  be defined by*

$$\varphi(\rho) = \psi(\langle \rho, \xi \rangle_{H^{-\sigma}, H^\sigma}),$$

where  $\psi \in C_b^\infty(\mathbb{R})$ . Assume that the process  $(M_\varphi^\sharp(t))$  defined by

$$M_\varphi^\sharp(t) := \varphi(\rho^\sharp(t)) - \varphi(\rho^\sharp(0)) - \int_0^t \mathcal{L}\varphi(\rho^\sharp(s))ds \quad (6.60)$$

is a continuous martingale with respect to  $(\mathcal{F}_t^\sharp)_{t \in [0, T]}$ . Then

$$\mathbb{E}^\sharp \left[ \varphi(\rho_{t+s}^\sharp) | \mathcal{F}_t^\sharp \right] = P_s^* \varphi(\rho_t^\sharp), \quad (6.61)$$

for all  $s, t \geq 0$  with  $t + s \leq T$ .

*Proof of Proposition 6.12.* Without loss of generality, we may assume that  $T = +\infty$ . Let us first show that the martingale property for  $(M_\varphi^\sharp(t))$  implies the martingale property for

$$M_\psi^\sharp(t) := \psi(t, \rho^\sharp(t)) - \psi(0, \rho^\sharp(0)) - \int_0^t [(\partial_t + \mathcal{L})\psi](s, \rho^\sharp(s))ds, \quad (6.62)$$

where  $\psi(t, \rho) = \theta(t)\varphi(\rho)$  with  $\theta \in C^1(\mathbb{R}_+)$ . If  $(M(t))_{t \geq 0}$  is a continuous martingale, then

$$t \mapsto M(t)\theta(t) - \int_0^t M(\sigma)\theta'(\sigma)d\sigma$$

is a martingale. We apply this to  $(M_\varphi^\sharp(t))$  and use the Fubini theorem to obtain the fact that (6.62) is a continuous martingale. Taking now  $\theta(t) = e^{-\lambda t}$ ,  $\lambda > 0$  gives us

$$e^{-\lambda(t+r)} \mathbb{E}^\sharp[\varphi(\rho^\sharp(t+r)) | \mathcal{F}_t^\sharp] = e^{-\lambda t} \varphi(\rho^\sharp(t)) + \mathbb{E}^\sharp \left[ \int_t^{t+r} \lambda e^{-\lambda s} (\lambda^{-1} \mathcal{L} - \text{Id}) \varphi(\rho^\sharp(s)) ds \middle| \mathcal{F}_t^\sharp \right].$$

Doing the change of variable  $s = s' + t$  in the integral shows that

$$\varphi(\rho^\sharp(t)) = e^{-\lambda r} \mathbb{E}^\sharp[\varphi(\rho^\sharp(t+r)) | \mathcal{F}_t^\sharp] - \mathbb{E}^\sharp \left[ \int_0^r \lambda e^{-\lambda s} (\lambda^{-1} \mathcal{L} - \text{Id}) \varphi(\rho^\sharp(s+t)) ds \middle| \mathcal{F}_t^\sharp \right].$$

We let  $r \rightarrow +\infty$  to obtain

$$\varphi(\rho^\sharp(t)) = \mathbb{E}^\sharp \left[ \int_0^{+\infty} \lambda e^{-\lambda s} (\text{Id} - \lambda^{-1} \mathcal{L}) \varphi(\rho^\sharp(s+t)) ds \middle| \mathcal{F}_t^\sharp \right]. \quad (6.63)$$

The convergence is easy to justify since  $\varphi$  and  $\mathcal{L}\varphi$  are bounded. Compare (6.63) to the formula

$$R_\lambda^* = \int_0^\infty e^{-\lambda t} P_t^* dt \quad (6.64)$$

for the resolvent associated to  $(P_t^*)$ . Actually, both (6.63) and (6.64) can be written more concisely by introducing a probability space  $(\Omega^0, \mathbb{P}^0, \mathcal{F}^0)$  and a random variable  $\tau$  on  $\Omega^0$  with exponential distribution of parameter  $\lambda$ . We have then

$$J_\lambda^* \varphi := \lambda R_\lambda^* \varphi = \mathbb{E}^0 P_\tau^* \varphi, \quad \varphi(\rho^\sharp(t)) = \mathbb{E}^\sharp \left[ \mathbb{E}^0 [J_\lambda^*]^{-1} \varphi(\rho^\sharp(\tau + t)) \middle| \mathcal{F}_t^\sharp \right].$$

By iteration, we obtain, for  $k \geq 1$ ,

$$[J_\lambda^*]^k \varphi = \mathbb{E}^0 P_{\sigma_k}^* \varphi, \quad [J_\lambda^*]^k \varphi(\rho^\sharp(t)) = \mathbb{E} \left[ \mathbb{E}^0 \varphi(\rho^\sharp(\sigma_k + t)) \middle| \mathcal{F}_t^\sharp \right], \quad (6.65)$$

where  $\sigma_k = \tau_1 + \dots + \tau_k$  with  $\tau_1, \dots, \tau_k$  being some i.i.d. random variables of exponential distribution  $\mathcal{E}(\lambda)$ . We take  $\lambda = N$ , where  $N \rightarrow +\infty$  and  $k = [Ns]$  for a given  $s > 0$ . By the weak law of large numbers, we have

$$\sigma_k \rightarrow s$$

in probability. Taking the limit  $[k \rightarrow +\infty]$  in (6.65), we obtain

$$\lim_{N \rightarrow +\infty} [J_N^*]^{[Ns]} \varphi = P_s^* \varphi, \quad P_s^* \varphi(\rho^\sharp(t)) = \mathbb{E} \left[ \varphi(\rho^\sharp(s + t)) \middle| \mathcal{F}_t^\sharp \right].$$

This is the desired result.  $\square$

**Conclusion.** Using Proposition 6.5, we can apply Proposition 6.12 to  $\tilde{\rho}$ . Taking  $t = 0$  in (6.61), we obtain

$$\mathbb{E} \varphi(\tilde{\rho}_s) = P_s^*(\rho_{\text{in}}) = \mathbb{E} \varphi(\rho_s^*).$$

Since the class of test functions  $\varphi$  as in Proposition 6.12 is a separating class (*cf.* the discussion after (6.3)), we obtain the identity in law of  $(\tilde{\rho}_t)$  and  $(\rho_t^*)$ . By uniqueness of the limit, we also deduce that the whole sequence  $(\rho_t^\varepsilon)$  is converging in law. This concludes the proof of Theorem 1.2.

*Remark 6.2* (Identification of the limit by representation of martingale). There is an alternative approach to the identification of  $(\tilde{\rho}_t)$ . It consists in computing the quadratic variation of the martingale (6.41). Indeed, using Theorem B.1, we can show that

$$\langle \tilde{M}, \tilde{M} \rangle_t = 2 \int_0^t \left\| S^{1/2}(\tilde{\rho}_s \nabla_x \xi) \right\|_{[L^2(\mathbb{T}^N)]^N}^2 ds.$$

Then we use a theorem of representation for martingales (see, *e.g.*, Theorem 8.2 p. 220 in [7]) to introduce the cylindrical Wiener process  $W(t)$  and the equation 1.25. The interest of this method is that one only needs to show the limit martingale property of Proposition 6.5 for linear and quadratic test-functions. However, in the approach we have adopted here, we do not need any theorem of representation for martingales, and it may be considered more consistent that this alternative one, since we remain focused on the Martingale Problem.

This work was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program “Investissements d’Avenir” (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR) and by the LABEX Lebesgue Center of Mathematics, program ANR-11-LABX-0020-01.

## A Resolution of the unperturbed equation

Consider the LB case first. By integration with respect to  $v$  in the equation

$$\partial_t f_t + E(t, s; \mathbf{e}) \cdot \nabla_v f_t + f_t = \rho(f_t)M, \quad (\text{A.1})$$

one checks that  $\rho(f_t) = \rho(f)$  for all  $t \geq 0$ . Therefore, the formula (3.3) is simply the Duhamel formula associated to the PDE (A.1). In the FP case, instead of working on the PDE

$$\partial_t f_t + E(t, s; \mathbf{e}) \cdot \nabla_v f_t = Q_{\text{FP}} f_t, \quad (\text{A.2})$$

we work on the solution  $V_t$  to the equation

$$dV_t = (-V_t + E(t, s; \mathbf{e}))dt + \sqrt{2}dB_t, \quad t \geq s. \quad (\text{A.3})$$

If  $V_s$  has the law of density  $f$  with respect to the Lebesgue measure on  $\mathbb{R}^N$ , then by (1.12) (with no dependence on  $x$  here), we obtain, by explicit integration in (A.3),

$$\begin{aligned} & \int_{\mathbb{R}^N} \varphi(v) f_{s,t}^{\text{FP}}(v) dv \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi \left( e^{-(t-s)} v + \int_s^t e^{-(t-\sigma)} E(\sigma, s; \mathbf{e}) ds + \sqrt{1 - e^{-2(t-s)}} w \right) M(w) f(v) dw dv. \end{aligned}$$

A change of variable gives (3.4) then.

## B Martingale characterization of Markov processes: quadratic variation

This section is devoted to the proof of (B.5), for which we found no specific reference (a sketch of the proof is given in [11, Paragraph 6.9.1]). We recall the following notion from [19]. Let  $E$  be a Polish space. Let

$$\text{BC}(E) = \{ \varphi: E \rightarrow \mathbb{R}, \varphi \text{ continuous and bounded} \}, \quad \|\varphi\|_{\text{BC}(E)} = \sup_{x \in E} |\varphi(x)|. \quad (\text{B.1})$$

A family of operator  $P_t: \text{BC}(E) \rightarrow \text{BC}(E)$  indexed by  $\mathbb{R}_+$  is said to be a  $\pi$ -contraction semi-group if

1. the map  $t \ni \mathbb{R}_+ \mapsto P_t \varphi(x) \in E$  is continuous for every  $\varphi \in \text{BC}(E)$ ,  $x \in E$ ,
2. for any sequence  $(\varphi_n)$  of  $\text{BC}(E)$  which converges pointwise to  $\varphi \in \text{BC}(E)$  and satisfies  $\sup_n \|\varphi_n\|_{\text{BC}(E)} < +\infty$  (this mode of convergence is denoted  $\varphi_n \xrightarrow{\pi} \varphi$  in what follows<sup>1</sup>), for  $t \geq 0$ , we have  $P_t \varphi_n \xrightarrow{\pi} P_t \varphi$ ,
3.  $\|P_t \varphi\|_{\text{BC}(E)} \leq \|\varphi\|_{\text{BC}(E)}$  for all  $\varphi \in \text{BC}(E)$ .

---

<sup>1</sup>this is called the bp-convergence in [10], see [10, p. 111]

The generator  $\mathcal{L}$  associated to  $(P_t)$  is defined by

$$D(\mathcal{L}) = \left\{ \varphi \in \text{BC}(E); \exists \psi \in \text{BC}(E), \Delta_t \varphi \xrightarrow{\pi} \psi \text{ when } t \rightarrow 0+ \right\}, \quad (\text{B.2})$$

$$\mathcal{L}\varphi(x) = \lim_{t \rightarrow 0+} \Delta_t \varphi(x), \quad (\text{B.3})$$

where

$$\Delta_t = \frac{P_t - \text{Id}}{t}.$$

**Theorem B.1** (Martingale characterization of Markov processes). *Let  $E$  be a Polish space, let  $(\mathcal{G}_t)$  be a filtration. Let  $(X_t)$  be an  $E$ -valued time-homogeneous Markov process with respect to  $(\mathcal{G}_t)$ , with semi-group  $(P_t)$ . Assume that  $(P_t)$  is a  $\pi$ -contraction semi-group of generator  $\mathcal{L}$ . Assume that  $(X_t)$  is progressively measurable with respect to  $(\mathcal{G}_t)$ . Then, for all continuous bounded  $\varphi: \mathbb{E} \rightarrow \mathbb{R}$  in the domain of  $\mathcal{L}$ ,*

$$M_\varphi(t) := \varphi(X_t) - \varphi(X_0) - \int_0^t \mathcal{L}\varphi(X_s) ds \quad (\text{B.4})$$

is a  $(\mathcal{G}_t)$ -martingale. Furthermore, if  $(X_t)$  is càdlàg and stochastically continuous, and if  $\varphi^2$  is in the domain of  $\mathcal{L}$ , then the quadratic variation of  $(M_t)$  is

$$[M_\varphi, M_\varphi]_t = \int_0^t (\mathcal{L}|\varphi|^2 - 2\varphi\mathcal{L}\varphi)(X_s) ds. \quad (\text{B.5})$$

*Proof of Theorem B.1.* Let us prove first that  $(M_\varphi(t))$  is a martingale. Let  $0 \leq s \leq t$ . We have

$$\mathbb{E}[M_\varphi(t)|\mathcal{G}_s] - M_\varphi(s) = \mathbb{E}[M_\varphi(t) - M_\varphi(s)|\mathcal{G}_s] = P_{t-s}\varphi(X_s) - \varphi(X_s) - \int_s^t [P_{\sigma-s}\mathcal{L}\varphi](X_s) d\sigma.$$

We use the relation  $\frac{d}{dt}P_t\varphi(x) = P_t\mathcal{L}\varphi(x)$  (see [19, Proposition 3.2]) to conclude. Indeed, this gives

$$P_{t-s}\varphi - \varphi = \int_s^t P_{\sigma-s}\mathcal{L}\varphi d\sigma,$$

and thus  $\mathbb{E}[M_\varphi(t)|\mathcal{G}_s] - M_\varphi(s) = 0$ . To prove (B.5), we use the identity

$$\mathbb{E} \left| \int_0^t H_s dM_s \right|^2 = \mathbb{E} \int_0^t |H_s|^2 d[M, M]_s \quad (\text{B.6})$$

for the stochastic integral (see [20, 56-60] for the construction of the stochastic integral). In (B.6),  $(H_t)$  is an adapted bounded càdlàg process. Our aim, therefore, is to prove

$$\mathbb{E} \left| \int_0^t H_s dM_s \right|^2 = \mathbb{E} \int_0^t |H_s|^2 dA_s, \quad (\text{B.7})$$



where  $A_t$  is the right-hand side of (B.5). The process  $(A_t)$  has finite variation, therefore the right-hand side of (B.7) is a Stieltjes integral. It is sufficient for us to establish (B.7) for the trivial integrand  $H_t = 1$ . However, it is more natural to consider a general continuous, bounded, adapted process  $(H_t)$ . In that case (see [20, p. 41]), the Stieltjes integral in the right-hand side of (B.7) is the limit, when  $|\sigma| \rightarrow 0$ , of the increments

$$\mathbb{E} \sum_{i=0}^{n-1} |H_{t_i}|^2 (A_{t_{i+1}} - A_{t_i}), \quad (\text{B.8})$$

where  $\sigma = \{t_0 = 0, \dots, t_n = t\}$  is a subdivision of step  $|\sigma| = \max_{0 \leq i < n} (t_{i+1} - t_i)$ . The left-hand side of (B.7) is also the limit, when  $|\sigma| \rightarrow 0$  of

$$\mathbb{E} \left| \sum_{i=0}^{n-1} H_{t_i} (M_{t_{i+1}} - M_{t_i}) \right|^2. \quad (\text{B.9})$$

Therefore, it is sufficient to prove that (B.8) = (B.9) +  $o(1)$  when  $|\sigma| \rightarrow 0$ . To that purpose, we can develop the square in (B.9). By the martingale property, it is equal to

$$\mathbb{E} \sum_{i=0}^{n-1} |H_{t_i}|^2 |M_{t_{i+1}} - M_{t_i}|^2 = \mathbb{E} \sum_{i=0}^{n-1} |H_{t_i}|^2 \mathbb{E}[|M_{t_{i+1}} - M_{t_i}|^2 | \mathcal{G}_{t_i}].$$

We decompose

$$|M_{t_{i+1}} - M_{t_i}|^2 = |\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2 + R_{t_i, t_{i+1}},$$

where

$$R_{t_i, t_{i+1}} = \left| \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds \right|^2 - 2(\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})) \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds. \quad (\text{B.10})$$

We show later that  $R_{t_i, t_{i+1}}$  can be neglected. We have then

$$\begin{aligned} |\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2 &= M_{\varphi^2}(t_{i+1}) - M_{\varphi^2}(t_i) - 2\varphi(X_{t_i})(M_{\varphi}(t_{i+1}) - M_{\varphi}(t_i)) \\ &\quad + \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi^2(X_s) ds - 2\varphi(X_{t_i}) \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds. \end{aligned}$$

It follows that

$$\mathbb{E}[|\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2 | \mathcal{G}_{t_i}] = \int_{t_i}^{t_{i+1}} (\mathcal{L} \varphi^2(X_s) - 2\varphi(X_{t_i}) \mathcal{L} \varphi(X_s)) ds. \quad (\text{B.11})$$

Taking expectation in (B.11), we obtain first that

$$\mathbb{E}[|\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2] \leq C_{\varphi}(t_{i+1} - t_i), \quad (\text{B.12})$$

where  $C_\varphi = \|\mathcal{L}\varphi^2\|_{\text{BC}(E)} + 2\|\varphi\|_{\text{BC}(E)}\|\mathcal{L}\varphi\|_{\text{BC}(E)}$ . Consider now the cross-product term in the right-hand side of (B.10). Using Young's inequality with a parameter  $\eta > 0$ , we see that the term  $\mathbb{E}|R_{t_i, t_{i+1}}|$  can be bounded by

$$(1 + \eta^{-1})\mathbb{E}\left|\int_{t_i}^{t_{i+1}} \mathcal{L}\varphi(X_s)ds\right|^2 + \eta\mathbb{E}[|\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2],$$

and thus, taking  $\eta = (t_{i+1} - t_i)^{1/2}$ , bounded from above by  $C'_\varphi(t_{i+1} - t_i)^{3/2}$ . It follows that

$$\mathbb{E}\left|\sum_{i=0}^{n-1} R_{t_i, t_{i+1}}\right| = \mathcal{O}(|\sigma|^{1/2}).$$

Since  $(H_t)$  is bounded, we have, consequently, (B.8) = (B.9) +  $\mathbb{E}[r_\sigma] + \mathcal{O}(|\sigma|^{1/2})$  with

$$r_\sigma = 2\sum_{i=0}^{n-1} |H_i|^2 \int_{t_i}^{t_{i+1}} (\varphi(X_s) - \varphi(X_{t_i}))\mathcal{L}\varphi(X_s)ds.$$

There remains to prove that  $\mathbb{E}[r_\sigma] = o(1)$ . The random variable  $r_\sigma$  is bounded by a constant depending on  $H$  and  $\varphi$  (but not on  $\sigma$ ), therefore  $r_\sigma = o(1)$  in probability will imply the result. This latter estimate follows easily from the stochastic continuity of  $(X_t)$ .  $\square$

## References

- [1] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and Geometry of Markov Diffusion Operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.
- [2] Fabrice Baudoin. *Diffusion Processes and Stochastic Calculus*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2014.
- [3] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [4] F. Bouchut. Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions. *J. Funct. Anal.*, 111(1):239–258, 1993.
- [5] D. L. Burkholder, B. J. Davis, and R. F. Gundy. Integral inequalities for convex functions of operators on martingales. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pages 223–240. Univ. California Press, Berkeley, Calif., 1972.

- [6] C. Castaing, P. Raynaud de Fitte, and M. Valadier. *Young measures on topological spaces*, volume 571 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2004. With applications in control theory and probability theory.
- [7] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
- [8] A. Debussche and J. Vovelle. Diffusion limit for a stochastic kinetic problem. *Commun. Pure Appl. Anal*, 11(6):2305–2326, 2012.
- [9] P. Degond, T. Goudon, and F. Poupaud. Diffusion limit for nonhomogeneous and non-micro-reversible processes. *Indiana Univ. Math. J.*, 49(3):1175–1198, 2000.
- [10] S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.
- [11] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna. *Wave propagation and time reversal in randomly layered media*, volume 56 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2007.
- [12] T. Goudon and M. Rousset. Stochastic acceleration in an inhomogeneous time random force field. *Appl. Math. Res. Express. AMRX*, (1):1–46, 2009.
- [13] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic Differential Equations and Diffusion Processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, second edition, 1989.
- [14] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.
- [15] Tomasz Komorowski, Claudio Landim, and Stefano Olla. *Fluctuations in Markov Processes*, volume 345 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, 2012.
- [16] David Nualart. *The Malliavin Calculus and Related Topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.
- [17] G. C. Papanicolaou, D. Stroock, and S. R. S. Varadhan. Martingale approach to some limit theorems. In *Papers from the Duke Turbulence Conference (Duke Univ., Durham, N.C., 1976), Paper No. 6*, pages ii+120 pp. Duke Univ. Math. Ser., Vol. III. Duke Univ., Durham, N.C., 1977.
- [18] F. Poupaud and A. Vasseur. Classical and quantum transport in random media. *J. Math. Pures Appl. (9)*, 82(6):711–748, 2003.

- [19] E. Priola. On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions. *Studia Math.*, 136(3):271–295, 1999.
- [20] Philip E. Protter. *Stochastic Integration and Differential Equations*, volume 21 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2005.
- [21] M. Pulvirenti and C. Simeoni.  $L^\infty$ -estimates for the Vlasov-Poisson-Fokker-Planck equation. *Mathematical Methods in the Applied Sciences*, 23(10):923–935, 2000.
- [22] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [23] C. Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.