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# Combinatorics for general kinetically constrained spin models

Laure MARÉCHÉ\*

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ABSTRACT. Bootstrap percolation is a well-known class of monotone cellular automata, in which sites may be infected or not and, at any step, a site becomes infected if a certain constraint is satisfied. Bootstrap percolation has a non-monotone stochastic counterpart, kinetically constrained models (KCM), which were introduced to model the liquid/glass transition, a major open problem of condensed matter physics. In KCM, the state of each site is resampled (independently) at rate 1 if the constraint is satisfied. A key problem for KCM is to determine the divergence of timescales as  $p \rightarrow 0$ , where  $p$  is the equilibrium density of infected sites. In this article we establish a combinatorial result which in turn allows to prove a lower bound on timescales for KCM.

## 1 Introduction

The study of KCM is linked to the study of bootstrap percolation models. A bootstrap percolation model (see [6]) is determined by the choice of an *update family*  $\mathcal{U} = \{X_1, \dots, X_m\}$  where  $m \in \mathbb{N}^*$  and the  $X_i$ , called *update rules*, are finite nonempty subsets of  $\mathbb{Z}^d \setminus \{0\}$ . It is a discrete time deterministic process  $(A_t)_{t \in \mathbb{N}}$ , where  $A_t$  is a set of vertices of  $\mathbb{Z}^d$ , considered as the set of *infected vertices* at time  $t$ .  $(A_t)_{t \in \mathbb{N}}$  is defined by choosing  $A_0$  (the set of initially infected vertices), and setting recursively

$$A_{t+1} = A_t \cup \{s \in \mathbb{Z}^d : s + X_i \subset A_t \text{ for some } i \in \{1, \dots, m\}\}$$

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for all  $t \in \mathbb{N}$ . In other words, a site  $s$  becomes infected at time  $t + 1$  if the translate by  $s$  of an update rule is infected at time  $t$ , and infected vertices remain infected forever.

KCM are a non-monotone, stochastic counterpart of bootstrap percolation, in which infected vertices can heal. It is a continuous time Markov process on  $\{0, 1\}^{\mathbb{Z}^d}$ : at any time  $t \in \mathbb{R}^+$ , each site of  $\mathbb{Z}^d$  is in state 0 or 1. The sites at state zero are the equivalent of infected vertices. Each site  $s$ , independently from the other sites, waits for a mean one exponential time, then tries to update its state, which means to replace it with a random variable that is 0 with probability  $p$  and 1 with probability  $1 - p$  ( $p \in (0, 1)$  is fixed). However, the update occurs only if there is an update rule  $X$  such that all the sites in  $s + X$  are at 0 (in other words, the state of a site can change only if “enough sites in its neighborhood” are at state zero). Then the site waits for another exponential time, tries again to update itself, and so on.

KCM were introduced by physicists to describe the liquid/glass transition, whose understanding is an important open problem of condensed matter physics (see for example [15, 4]). Indeed, their behavior when  $p$  tends to zero matches well the experimental observations, in particular the anomalous divergence of timescales. This is why both physicists and mathematicians are interested in studying the timescales of KCM when  $p$  tends to zero.

Various specific update families have been studied. Among them, we recall the Fredrickson-Andersen  $j$ -facilitated models (FA $j$ f) in  $\mathbb{Z}^d$ , with  $j \in \{1, \dots, d\}$ , for which  $\mathcal{U}$  is the collection of subsets of  $\{e_1, \dots, e_d, -e_1, \dots, -e_d\}$  of cardinal  $j$ , where  $\{e_1, \dots, e_d\}$  is the canonical basis (see [7, 13]). In words, a site can be updated if it has at least  $j$  nearest neighbors at state 0. The bootstrap percolation model corresponding to this update family is the classical  $j$ -neighbor percolation model (see [3] and references therein). Another important model is the East model on  $\mathbb{Z}$ , in which  $\mathcal{U} = \{-1\}$  (see [11] for a review of the East model). There are also results on its generalization to  $\mathbb{Z}^d$ , the East-like model, where  $\mathcal{U} = \{-e_1, \dots, -e_d\}$  (see [9]).

All of these models are particular cases of update families. However, there recently was a breakthrough in the study of bootstrap percolation (see [6, 5, 2]): beautiful universality results were proved for totally general update families. The main results concern the probability that every site is eventually infected and the mean time of infection of the origin,  $T_{\mathcal{U}}$ .

These results pave the way for understanding the KCM with general update families. A set of conjectures for the two-dimensional case was formulated in [14]. These conjectures concern a very natural timescale of the KCM:  $E_{\mu}(\tau_0)$ , the mean of the first time at which the origin is zero when each site is initially at state 0 with probability  $p$  independently of the others.

A technique to establish upper bounds on  $\mathbb{E}_\mu(\tau_0)$  when  $p$  tends to zero was introduced in [13], and was generalized in [12] to deal with all possible update families, proving an upper bound matching the conjectures of [14]. However, a robust technique is still missing for the lower bound; the only general result was proven in [7] and states that  $\mathbb{E}_\mu(\tau_0) \geq T_{\mathcal{U}}^{1-o(1)}$  when  $p$  tends to zero. Intuitively, this bound comes from the fact that in the KCM, the zeroes can not spread faster than the infection in the bootstrap percolation process does, as a site can change its state to zero in the KCM only if it can be infected in the next step of the bootstrap percolation process. But there are cases for which this lower bound is not the right scaling: in the East model, it was proven in [1] that  $\mathbb{E}_\mu(\tau_0) = e^{\Theta(\ln(1/p)^2)}$ , while  $T_{\mathcal{U}} = \Theta(1/p)$ .

This comes from the fact that a typical configuration in the East model has a “box” full of ones of size roughly  $1/p$  centered at the origin, hence to reach the origin, the zeroes have to “cross the box”. Moreover, a combinatorial result proven by Chung, Diaconis and Graham in [10] shows that in order to do that, the dynamics has to go through a configuration with  $\log_2(1/p)$  zeroes in the box. Such a configuration has probability  $e^{-\Theta(\ln(1/p)^2)}$ , therefore one has to wait a time  $e^{\Theta(\ln(1/p)^2)}$  for the dynamics to go through one of these, which yields the scaling.

It was conjectured in [14] that a large class of two-dimensional general KCM, called supercritical rooted models, have the same behavior as the East model, hence also verify  $\mathbb{E}_\mu(\tau_0) = e^{\Theta(\ln(1/p)^2)}$ . In this article, we establish a combinatorial result valid in an even larger class of models, stating that indeed, if we start from a configuration which contains only ones in a certain region of linear size  $\Omega(n2^n)$  centered at the origin, necessarily we have to go through a configuration with at least  $n$  zeroes in this region before a zero can appear at the origin. Our result allows to derive that  $\mathbb{E}_\mu(\tau_0) \geq e^{\Theta(\ln(1/p)^2)}$  for these models, using the argument introduced in [8] for the East model. Therefore it proves the lower bound in the conjecture of [14], not only for two-dimensional supercritical rooted models, but also for their natural generalization in any dimension.

Our result generalizes the result of [10] to all general models that are not in the supercritical unrooted class, and to any dimension (the East model studied in [10] is defined only in dimension 1). As the proof of [10] relies heavily on the orientation of the East model and the general models have a complete lack of orientation, our proof is completely different from theirs.

We will begin this article by explaining the notations and stating the result, then we will detail the one-dimensional case, in which the proofs are simpler, and finally we will present the proof in general dimension.

## 2 Notations and result

Fix  $d \in \mathbb{N}^*$  and set  $\mathcal{U} = \{X_1, \dots, X_m\}$  an update family.

We are interested in the configurations that are attainable by the dynamics of the KCM with update family  $\mathcal{U}$  in a finite domain  $\Lambda \subset \mathbb{Z}^d$  when we start with only ones in  $\Lambda$  and at most a bounded number of zeroes are allowed at the same time in  $\Lambda$ . We want to prove that none of these configurations can have a zero at the origin if  $\Lambda$  is chosen appropriately.

In order to state our result, we need some definitions and notations. First of all, we need to clarify what the dynamics of a KCM in a finite domain  $\Lambda$  is. It will be a dynamics on configurations in  $\{0, 1\}^\Lambda$  following the rules described in the introduction. However, these rules specify that a site  $s$  can be updated if one of the  $s + X_i$  contains only zeroes, and the  $s + X_i$  may not be entirely contained in the domain  $\Lambda$ . Consequently, one has to fix the states of the sites outside  $\Lambda$ . In this article, we will consider that they are zeroes (empty boundary conditions).

Set  $\Lambda \subset \mathbb{Z}^d$ . We denote by  $0_\Lambda$  the configuration which contains only zeroes in  $\Lambda$ , and by  $1_\Lambda$  the configuration which contains only ones in  $\Lambda$ . Furthermore, for all  $\eta \in \{0, 1\}^\Lambda$ ,  $x \in \Lambda$ , we define the configuration  $\eta^x$  by  $(\eta^x)_y = \begin{cases} 1 - \eta_x & \text{if } y = x \\ \eta_y & \text{if } y \neq x. \end{cases}$  Moreover, if  $\Lambda' \subset \Lambda$  and  $\eta \in \{0, 1\}^\Lambda$ , we denote by  $\eta_{\Lambda'}$  the configuration of  $\{0, 1\}^{\Lambda'}$  such that  $(\eta_{\Lambda'})_x = \eta_x$  for any  $x \in \Lambda'$ . In addition, if  $\Lambda' \subset \mathbb{Z}^d$  is disjoint from  $\Lambda$ , for all  $\eta \in \{0, 1\}^\Lambda$ ,  $\eta' \in \{0, 1\}^{\Lambda'}$ , we denote by  $\eta_\Lambda \eta'_{\Lambda'}$  the configuration on  $\Lambda \cup \Lambda'$  defined by  $(\eta_\Lambda \eta'_{\Lambda'})_x = \eta_x$  if  $x \in \Lambda$  and  $(\eta_\Lambda \eta'_{\Lambda'})_x = \eta'_x$  if  $x \in \Lambda'$ .

We say that a move from  $\eta \in \{0, 1\}^\Lambda$  to  $\eta' \in \{0, 1\}^\Lambda$  is legal if  $\eta' = \eta$ , or if  $\eta' = \eta^x$  with  $x \in \Lambda$  and there exists an update rule  $X \in \mathcal{U}$  such that  $(\eta_\Lambda 0_{\Lambda^c})_{x+X} = 0_{x+X}$ . That is, a move is legal if it is a possible move for the KCM with update family  $\mathcal{U}$ , in the domain  $\Lambda$  with empty boundary condition.

**Definition 1.** If  $\eta, \eta' \in \{0, 1\}^\Lambda$ , a *legal path* from  $\eta$  to  $\eta'$  is a sequence of configurations  $(\eta^j)_{0 \leq j \leq m}$  such that  $m \in \mathbb{N}^*$ ,  $\eta^0 = \eta$ ,  $\eta^m = \eta'$ , and for all  $j \in \{0, \dots, m-1\}$ , the move from  $\eta^j$  to  $\eta^{j+1}$  is legal.

For any  $n \in \mathbb{N}^*$ , we say that  $(\eta^j)_{0 \leq j \leq m}$  is a *n-legal path* if for all  $j \in \{0, \dots, m\}$ ,  $\eta^j$  does not contain more than  $n$  zeroes in  $\Lambda$ .

A legal path is a possible path for the KCM dynamics; a *n-legal path* is a possible path such that at most  $n$  zeroes are allowed at the same time in  $\Lambda$ . In order to have lighter notations, we use the same notation  $\eta^j$  for the  $j$ -th step of a path and for the configuration that is equal to  $\eta$  everywhere except

at site  $j$ . In order to avoid confusion,  $\eta^0, \eta^j, \eta^{j+1}$  and  $\eta^m$  will always denote a step of a path, and no other index will be used to describe a step of a path.

For all  $n \in \mathbb{N}^*$ , we define  $V(n, \Lambda) = \{\eta \in \{0, 1\}^\Lambda \mid \text{there exists a } n\text{-legal path from } 1_\Lambda \text{ to } \eta\}$ .  $V(n, \Lambda)$  is the set of configurations of  $\{0, 1\}^\Lambda$  that are attainable from the configuration containing only ones using at most  $n$  zeroes.

$V(n, \Lambda)$  will be very different depending on the properties of  $\mathcal{U}$ . In this article, we will distinguish between two classes of update families. To define them, we recall the concept of stable directions introduced in [6]:

**Definition 2.** For any  $u \in S^{d-1}$ , let  $\mathbb{H}_u = \{x \in \mathbb{R}^d \mid \langle x, u \rangle < 0\}$  the half-space with boundary orthogonal to  $u$ . We say that  $u$  is a stable direction for the update family  $\mathcal{U}$  when there does not exist  $X \in \mathcal{U}$  such that  $X \subset \mathbb{H}_u$ .

This implies in particular that if we consider the KCM with update family  $\mathcal{U}$  in  $\mathbb{Z}^d$ , and if there are only ones in  $(\mathbb{H}_u)^c$ , then no zero can appear in  $(\mathbb{H}_u)^c$ . Indeed, a state change in a site  $s \in (\mathbb{H}_u)^c$  would require an update rule  $X$  such that the configuration is zero in  $s + X$ . Hence  $s + X$  would have to be in  $\mathbb{H}_u$ , which would imply  $\langle s + x, u \rangle < 0$  for all  $x \in X$ . But  $s \in (\mathbb{H}_u)^c$ , thus  $\langle s, u \rangle \geq 0$ , and we would deduce  $\langle x, u \rangle < 0$  for all  $x \in X$ , i.e.  $X \subset \mathbb{H}_u$ , which is impossible. Intuitively, it means that the zeroes of the configuration can not move towards direction  $u$ .

The following definition is an extension to the dimension  $d$  of the definition proposed in [14]:

**Definition 3.** We say that  $\mathcal{U}$  is supercritical unrooted if there exists a hyperplane of  $\mathbb{R}^d$  that contains all stable directions of  $\mathcal{U}$ .

Our combinatorial result (theorem 1) is valid in all models that *are not* supercritical unrooted, which actually covers many different behaviors. In particular, in two dimensions, according to the classification in [6] they include: supercritical models which have two non opposite stable directions, critical and subcritical models. Supercritical models which have two non opposite stable directions are those that are called supercritical rooted models in [14] and for which  $\mathbb{E}_\mu(\tau_0)$  is conjectured to scale as  $e^{\Theta(\ln(1/p)^2)}$ . Among the models mentioned in the introduction (FA $j$ f for  $j \in \{1, \dots, d\}$ , East and East-like), only FA1f is supercritical unrooted.

We now suppose that  $\mathcal{U}$  is not supercritical unrooted.

We are going to introduce notations that are necessary to state the result.

Firstly, there exists a basis  $\{v_1, \dots, v_d\}$  of  $\mathbb{R}^d$  and  $u_1, \dots, u_d \in S^{d-1}$  stable directions for  $\mathcal{U}$  such that  $\|v_i\|_2 = 1$  for any  $i \in \{1, \dots, d\}$ , such that in this

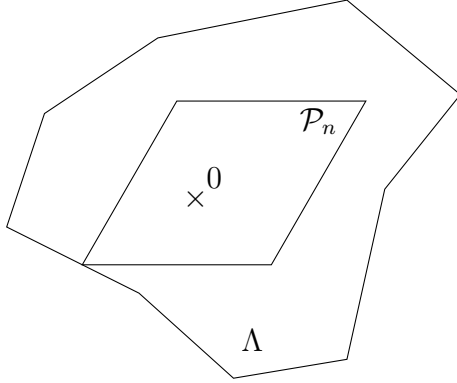


Figure 1: The setting of the theorem.

basis, for any  $i \in \{1, \dots, d\}$ ,  $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i = 0\}$  is orthogonal to  $u_i$ , and such that the half-space  $\mathbb{H}_{u_i}$  is  $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i > 0\}$ . The proof of this result may be found in the appendix.

From now on, we will use the coordinates of the basis  $\{v_1, \dots, v_d\}$ , but when we say a site is in  $\mathbb{Z}^d$ , we will mean that its coordinates in the canonical basis are integers.

For all  $n \in \mathbb{N}^*$ , we set  $a_n = r(2^n - 1)$  and  $b_n = rn2^{n-1}$ . We are going to consider a sequence  $(\mathcal{P}_n)_{n \in \mathbb{N}^*}$  of discrete boxes defined by

$$\mathcal{P}_n = \{x \in \mathbb{Z}^d \mid x = (x_1, \dots, x_d), \forall i \in \{1, \dots, d\}, -a_n \leq x_i \leq b_n\}.$$

Finally, we denote by  $r$  the range of the interactions:  $r = \max\{\|x - y\|_\infty \mid x, y \in X \cup \{0\}, X \in \mathcal{U}\}$  (beware: the range is usually defined in the canonical basis, while we define it in our new basis).

We can now state the

**Theorem 1.** *Let  $\mathcal{U}$  be any update family that is not supercritical unrooted. Fix  $n \in \mathbb{N}^*$  and  $\Lambda \subset \mathbb{Z}^d$  such that  $\mathcal{P}_n \subset \Lambda$ . Then for any  $\eta \in V(n, \Lambda)$ , it holds  $\eta_0 = 1$ .*

This means that if we consider the KCM with update family  $\mathcal{U}$  in a domain  $\Lambda$  large enough to contain  $\mathcal{P}_n$  (see figure 1), then more than  $n$  zeroes are necessary to bring a zero to the origin when we start with the configuration containing only ones.

*Remark.* • As  $\mathbb{Z}^d$  is translation invariant, the theorem implies that for any  $n \in \mathbb{N}^*$ , any domain  $\Lambda \subset \mathbb{Z}^d$ , any  $s \in \Lambda$  such that  $s + \mathcal{P}_n \subset \Lambda$ , for any  $\eta \in V(n, \Lambda)$ ,  $\eta_s = 1$ : for any site far enough from the boundary of the domain, one needs more than  $n$  zeroes to bring a zero at this site.

- Our theorem is stated for paths that are  $n$ -legal when all sites outside of  $\Lambda$  are zeroes (empty boundary conditions); it actually remains valid if we consider the  $n$ -legal paths for any boundary conditions. Indeed, if we consider that the sites of  $\Lambda^c$  are not all zeroes, the possible moves are more restricted, hence when one starts with the configuration containing only ones in the domain, it is harder to bring zeroes far from the boundary. Consequently, the theorem is also valid with other boundary conditions.
- Theorem 1 allows to prove that  $\mathbb{E}_\mu(\tau_0) \geq e^{\Theta(\ln(1/p)^2)}$  in all models that are not supercritical unrooted with the argument used in [8] for the East model. Indeed, in a typical configuration, there is a domain of diameter roughly  $(1/p)^{1/d}$  centered at the origin that is full of ones, hence theorem 1 implies that if a zero appears at the origin, the dynamics went through a configuration with at least  $\log_2((1/p)^{1/d}) = \frac{1}{d} \log_2(1/p)$  zeroes in the domain. Such a configuration has probability  $e^{-\Theta(\ln(1/p)^2)}$ , thus one has to wait a time  $e^{\Theta(\ln(1/p)^2)}$  for the dynamics to go through one of these, which yields the scaling. We stress that the theorem hence the lower bound for  $\mathbb{E}_\mu(\tau_0)$  are valid for all the models that are not supercritical unrooted, but that the scaling can be optimal only for a subclass of models. For example, in two dimensions, it is optimal for supercritical rooted models (see [12] for the matching upper bound) but it is not optimal for critical and subcritical models, where bootstrap percolation results ([6, 2]) imply that it is certainly not possible to infect the origin in a time  $e^{O(\ln(1/p)^2)}$ .
- If  $\mathcal{U}$  is supercritical unrooted instead, the behavior of the KCM is different: in particular in dimensions 1 and 2 one can prove (see sections 3.2 and 4.2) that there exists a finite  $k \in \mathbb{N}^*$  such that when we consider the KCM in any finite domain  $\Lambda$ , a zero can be brought to any  $s \in \Lambda$  using only  $k$  zeroes. We expect this result to hold in any dimension.

### 3 Warm-up: the one-dimensional case

We begin by examining how the definitions of the classes of models translate in this simpler setting. The following is an easy consequence of definition 3:

**Proposition 2.** *An update family  $\mathcal{U}$  is supercritical unrooted if and only if it includes an update rule contained in  $\mathbb{N}^*$  and an update rule contained in  $-\mathbb{N}^*$ .*



*Proof.* By definition 3, a supercritical unrooted model in  $\mathbb{Z}^d$  is a model whose stable directions are contained in an hyperplane of  $\mathbb{Z}^d$ . Here  $d = 1$ , and the only hyperplane of  $\mathbb{Z}$  is  $\{0\}$ . Therefore, a supercritical unrooted model is a model that has no stable direction.

Furthermore, in  $\mathbb{R}$  there are only two possible stable directions: 1 and  $-1$ . In addition,  $u$  is a stable direction when there is no update rule in  $\{x \in \mathbb{R} \mid \langle x, u \rangle < 0\}$ , hence  $\pm 1$  is a stable direction when there is no update rule in  $\mp \mathbb{N}^*$ .

Therefore, a supercritical unrooted model is a model with an update rule contained in  $\mathbb{N}^*$  and an update rule contained in  $-\mathbb{N}^*$ .  $\square$

Firstly, we will prove theorem 1 about non supercritical unrooted families, then we will show that the behavior of supercritical unrooted models is different.

### 3.1 Proof of theorem 1 for $d = 1$

Let  $\mathcal{U}$  be a non supercritical unrooted update family.

We begin by introducing some conventions that will lighten the notations. By definition of the basis  $\{v_1\}$ ,  $|v_1| = 1$ , hence we get  $v_1 = 1$  or  $v_1 = -1$ . Consequently,  $\mathbb{Z}$  is the same in the canonical basis and in  $\{v_1\}$ . Therefore, we may forget the canonical basis and use only  $\{v_1\}$ .

We notice that by definition of the basis  $\{v_1\}$ ,  $\{x \in \mathbb{R} \mid x > 0\} = \mathbb{H}_u$  with  $u$  a stable direction, thus there is no update rule contained in  $\{x \in \mathbb{R} \mid x > 0\}$ . We deduce that there is no update rule contained in  $\mathbb{N}^*$ .

We will prove the theorem by induction. For all  $n \in \mathbb{N}^*$ , we denote

$$\mathcal{H}_n = \text{“for any } \Lambda \subset \mathbb{Z} \text{ such that } \mathcal{P}_n = \{-a_n, \dots, b_n\} \subset \Lambda, \\ \text{for any } \eta \in V(n, \Lambda), \eta_0 = 1\text{”}.$$

Proving  $\mathcal{H}_n$  for all  $n \in \mathbb{N}^*$  will prove the theorem.

In order to do that, we will need the

**Lemma 1.** *Let  $n > 1$  and suppose  $\mathcal{H}_{n-1}$ . Then, for all  $\Lambda \subset \mathbb{Z}$  such that  $\mathcal{P}_n \subset \Lambda$ , for all  $\eta \in V(n, \Lambda) \setminus \{1_\Lambda\}$ ,  $\eta$  has at least one zero in  $\Lambda \setminus \mathcal{P}_{n-1}$ .*

This lemma means that if  $\mathcal{H}_{n-1}$  holds, in a large enough interval, any configuration attainable using no more than  $n$  zeroes must have one of its zeroes outside of  $\mathcal{P}_{n-1}$  (except the configuration containing only ones, that has no zero at all). This implies that there are at most  $n - 1$  zeroes in  $\mathcal{P}_{n-1}$ , which will allow us to use  $\mathcal{H}_{n-1}$  to prove that the origin can not be reached by zeroes.

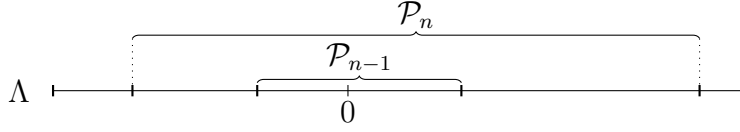


Figure 2: Proof of the theorem in the one-dimensional case: there must be a zero in  $\Lambda \setminus \mathcal{P}_{n-1}$ , hence there can be at most  $n - 1$  zeroes in  $\mathcal{P}_{n-1}$ . Thus  $\mathcal{H}_{n-1}$  implies that there is no zero at 0.

We first prove the theorem supposing lemma 1 holds; we will prove the lemma afterwards. As we just said, we will proceed by induction in order to prove that  $\mathcal{H}_n$  holds for any  $n \in \mathbb{N}^*$ .

**Case n=1** This is a simple case: there can be at most one zero in the configurations we consider, hence if they have a zero, this zero was created by the zeroes of the boundary conditions, thus it has to be near the boundary.

Let  $\Lambda \subset \mathbb{Z}$  such that  $\mathcal{P}_1 \subset \Lambda$ , and let  $\eta$  be in  $V(1, \Lambda)$ .

If  $\eta = 1_\Lambda$ , then  $\eta_0 = 1$  and there is nothing to prove. We now consider the case  $\eta \neq 1_\Lambda$ .

In this case,  $\eta$  has a single zero, at a site we denote by  $z$ , and there is a 1-legal path  $(\eta^j)_{0 \leq j \leq m}$  from  $\eta$  to  $1_\Lambda$ . Since  $\eta_z = 0$  and  $(1_\Lambda)_z = 1$ , there exists  $j \in \{0, \dots, m - 1\}$  such that  $\eta_z^j = 0$  and  $\eta_z^{j+1} = 1$ . Furthermore, there exists  $X \in \mathcal{U}$  such that  $(\eta_\Lambda^j 0_{\Lambda^c})_{z+X} = 0_{z+X}$ . Consequently, as  $\eta^j$  has no other zero than  $z$ ,  $z + X \subset \Lambda^c$ . We consider a site  $z' \in z + X$ ;  $z' \in \Lambda^c$  and  $|z' - z| \leq r$ . As  $z' \in \Lambda^c$ ,  $z' \notin \mathcal{P}_1$ , thus  $z' < -a_1$  or  $z' > b_1$ . Hence, if  $z' < -a_1$ , then  $z \leq z' + r < -a_1 + r = -r + r = 0$ , and if  $z' > b_1$ ,  $z \geq z' - r > b_1 - r = r - r = 0$ . In any case,  $z \neq 0$ , therefore  $\eta_0 = 1$ .

This proves  $\mathcal{H}_1$ .

**Induction** Let  $n > 1$ . We suppose  $\mathcal{H}_{n-1}$ . Let us show  $\mathcal{H}_n$ .

Let  $\Lambda \subset \mathbb{Z}$  such that  $\mathcal{P}_n \subset \Lambda$ . Let  $\eta$  be in  $V(n, \Lambda)$ .

Intuitively, lemma 1 implies that any configuration of  $V(n, \Lambda)$  but  $1_\Lambda$  has a zero in  $\Lambda \setminus \mathcal{P}_{n-1}$ , hence there can be at most  $n - 1$  zeroes in  $\mathcal{P}_{n-1}$ . Consequently, by  $\mathcal{H}_{n-1}$ ,  $\eta_0 = 1$  (see figure 2). We are going to write this argument rigorously.

If  $\eta = 1_\Lambda$ , then  $\eta_0 = 1$  and there is nothing to prove. Now, let us deal with the case  $\eta \neq 1_\Lambda$ .

As  $\eta \in V(n, \Lambda)$ , there exists a  $n$ -legal path  $(\eta^j)_{0 \leq j \leq m}$  from  $\eta$  to  $1_\Lambda$ . We will prove that  $(\eta_{\mathcal{P}_{n-1}}^j)_{0 \leq j \leq m}$  is a  $(n - 1)$ -legal path from  $\eta_{\mathcal{P}_{n-1}}^0 = \eta_{\mathcal{P}_{n-1}}$  to

$$\eta_{\mathcal{P}_{n-1}}^m = 1_{\mathcal{P}_{n-1}}.$$

Firstly, for all  $j \in \{0, \dots, m-1\}$ , the move from  $\eta_{\mathcal{P}_{n-1}}^j$  to  $\eta_{\mathcal{P}_{n-1}}^{j+1}$  is legal. Indeed:

- If  $\eta^{j+1} = \eta^j$  or if  $\eta^{j+1} = (\eta^j)^z$  with  $z \in \Lambda \setminus \mathcal{P}_{n-1}$ ,  $\eta_{\mathcal{P}_{n-1}}^{j+1} = \eta_{\mathcal{P}_{n-1}}^j$  and the move from  $\eta_{\mathcal{P}_{n-1}}^{j+1}$  to  $\eta_{\mathcal{P}_{n-1}}^j$  is legal.
- If  $\eta^{j+1} = (\eta^j)^z$  with  $z \in \mathcal{P}_{n-1}$ , then since the move from  $\eta^j$  to  $\eta^{j+1}$  is legal, there exists  $X \in \mathcal{U}$  such that  $(\eta_{\Lambda}^j 0_{\Lambda^c})_{z+X} = 0_{z+X}$ , and this implies that  $(\eta_{\mathcal{P}_{n-1}}^j 0_{(\mathcal{P}_{n-1})^c})_{z+X} = 0_{z+X}$ . Therefore, the move from  $\eta_{\mathcal{P}_{n-1}}^j$  to  $\eta_{\mathcal{P}_{n-1}}^{j+1}$  is legal.

Consequently, for all  $j \in \{0, \dots, m-1\}$ , the move from  $\eta_{\mathcal{P}_{n-1}}^j$  to  $\eta_{\mathcal{P}_{n-1}}^{j+1}$  is legal. Therefore  $(\eta_{\mathcal{P}_{n-1}}^j)_{0 \leq j \leq m}$  is a legal path.

Moreover, for all  $j \in \{0, \dots, m\}$ ,  $\eta_{\mathcal{P}_{n-1}}^j$  contains at most  $n-1$  zeroes. Indeed:

- If  $\eta^j = 1_{\Lambda}$ , then  $\eta_{\mathcal{P}_{n-1}}^j$  contains no zero at all.
- If  $\eta^j \neq 1_{\Lambda}$ , then since  $(\eta^\ell)_{j \leq \ell \leq m}$  is a  $n$ -legal path from  $\eta^j$  to  $1_{\Lambda}$ ,  $\eta^j \in V(n, \Lambda)$ , hence  $\eta^j \in V(n, \Lambda) \setminus \{1_{\Lambda}\}$ . Moreover, we suppose  $\mathcal{H}_{n-1}$ , thus we can apply lemma 1, which yields that  $\eta^j$  has at least one zero in  $\Lambda \setminus \mathcal{P}_{n-1}$ . Hence, as  $\eta^j$  contains at most  $n$  zeroes,  $\eta_{\mathcal{P}_{n-1}}^j$  contains at most  $n-1$  zeroes.

Therefore, for all  $j \in \{0, \dots, m\}$ ,  $\eta_{\mathcal{P}_{n-1}}^j$  contains at most  $n-1$  zeroes.

It follows that  $(\eta_{\mathcal{P}_{n-1}}^j)_{0 \leq j \leq m}$  is a  $(n-1)$ -legal path from  $\eta_{\mathcal{P}_{n-1}}$  to  $1_{\mathcal{P}_{n-1}}$ .

Thus  $\eta_{\mathcal{P}_{n-1}} \in V(n-1, \mathcal{P}_{n-1})$ .

Consequently, by  $\mathcal{H}_{n-1}$ ,  $\eta_0 = 1$ .

This proves  $\mathcal{H}_n$ .

Consequently,  $\mathcal{H}_{n-1}$  implies  $\mathcal{H}_n$  for all  $n > 1$ . Hence, as  $\mathcal{H}_1$  holds,  $\mathcal{H}_n$  holds for all  $n \in \mathbb{N}^*$ .

This ends the proof of theorem 1, given lemma 1.

It remains to prove lemma 1.

*Proof of lemma 1.* Let  $\Lambda \subset \mathbb{Z}$  such that  $\mathcal{P}_n = \{-a_n, \dots, b_n\} \subset \Lambda$ .

We will consider a configuration  $\eta \in \{0, 1\}^{\Lambda}$ , different from  $1_{\Lambda}$ , containing at most  $n$  zeroes, such that all of its zeroes are in  $\mathcal{P}_{n-1}$ , and we will show that  $\eta \notin V(n, \Lambda)$ . This will imply that any element of  $V(n, \Lambda) \setminus \{1_{\Lambda}\}$  must contain at least a zero outside of  $\mathcal{P}_{n-1}$ , which is enough to prove the lemma.

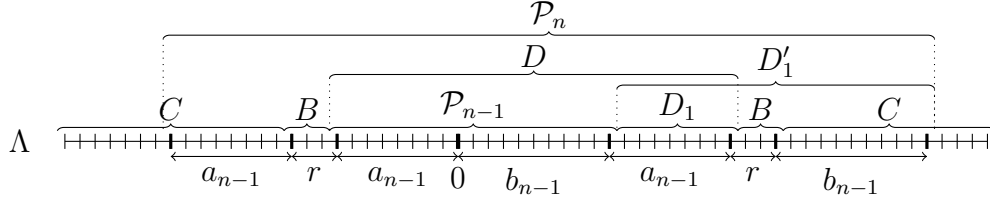


Figure 3: The setting of lemma 1.

Let  $(\eta^j)_{0 \leq j \leq m}$  be a  $n$ -legal path of configurations in  $\{0, 1\}^\Lambda$  with  $\eta^0 = \eta$ . We are going to show that  $\eta^m$  can not be  $1_\Lambda$ . This will imply that there does not exist a  $n$ -legal path from  $\eta$  to  $1_\Lambda$ , which means that  $\eta \notin V(n, \Lambda)$ .

To this end, we will denote (see figure 3):

$$B = \{-a_n + a_{n-1}, \dots, -a_n + a_{n-1} + r - 1\} \cup \{b_n - (b_{n-1} + r - 1), \dots, b_n - b_{n-1}\};$$

$$D = \{-a_n + a_{n-1} + r, \dots, b_n - (b_{n-1} + r)\}$$

$$D_1 = \{b_n - (b_{n-1} + a_{n-1} + r - 1), \dots, b_n - (b_{n-1} + r)\},$$

$$D'_1 = \{b_n - (b_{n-1} + a_{n-1} + r - 1), \dots, b_n\}$$

and  $C = \Lambda \setminus (B \cup D)$ .

We notice that

$$-a_n + a_{n-1} + r = -r(2^n - 1) + r(2^{n-1} - 1) + r = -r2^{n-1} + r = -a_{n-1}$$

and

$$\begin{aligned} b_n - (b_{n-1} + a_{n-1} + r) &= rn2^{n-1} - (r(n-1)2^{n-2} + r(2^{n-1} - 1) + r) \\ &= rn2^{n-2} - r2^{n-2} = r(n-1)2^{n-2} = b_{n-1} \end{aligned}$$

hence  $\mathcal{P}_{n-1} = \{-a_n + a_{n-1} + r, \dots, b_n - (b_{n-1} + a_{n-1} + r)\} = D \setminus D_1$ .

$B$  will be a “buffer zone”: we will prove that it remains full of ones and prevents the zeroes of  $C$  and  $D$  from interacting.

There will always be a zero in  $\mathcal{P}_{n-1}$ , because the leftmost zero  $z$  in  $\mathcal{P}_{n-1}$  would need an update rule full of zeroes to disappear. However, there is no zero in  $B$  and the thickness of  $B$  is larger than the range of the interactions, hence this update rule can not use zeroes in  $B$  or at the left of  $B$ . Thus it can use only zeroes in  $\mathcal{P}_{n-1}$  or at the right of  $\mathcal{P}_{n-1}$ , but  $z$  is the leftmost zero in  $\mathcal{P}_{n-1}$ . Therefore, the update rule would have to be completely contained in the right of  $z$ , which is impossible since the assumption made on the model implies that it has no update rule contained in  $\mathbb{N}^*$  (see the beginning of section 3).

Hence the leftmost zero in  $\mathcal{P}_{n-1}$  can not disappear. Consequently, there will always be a zero in  $\mathcal{P}_{n-1}$ , which implies  $\eta^m \neq 1_\Lambda$ .

More rigorously, we are going to prove by induction on  $j \in \{0, \dots, m\}$  that the property  $\mathcal{H}'_j$  holds, where  $\mathcal{H}'_j$  consists in:

( $P_1^j$ )  $\eta_{\mathcal{P}_{n-1}}^j$  contains a zero.

( $P_2^j$ )  $\eta_B^j = 1_B$ .

( $P_3^j$ )  $\eta_C^j 1_{\Lambda \setminus C} \in V(n-1, \Lambda)$

( $P_4^j$ )  $\eta_{D_1}^j 1_{D_1' \setminus D_1} \in V(n-1, D_1')$ .

The last two properties are necessary for the part of the argument that uses  $\mathcal{P}_{n-1}$  to show that the zeroes do not reach  $B$ .

If we can show  $\mathcal{H}'_j$  for all  $j \in \{0, \dots, m\}$ , in particular ( $P_1^m$ ) will imply that there is a zero in  $\eta_{\mathcal{P}_{n-1}}^m$ , thus  $\eta^m \neq 1_\Lambda$ , hence the lemma.

Let us prove  $\mathcal{H}'_j$  for all  $j \in \{0, \dots, m\}$ .

**Case  $j = 0$**

- ( $P_1^0$ ) is true, because  $\eta^0 = \eta \neq 1_\Lambda$ , thus  $\eta$  contains at least a zero, and by assumption all zeroes of  $\eta$  are in  $\mathcal{P}_{n-1}$ , hence  $\eta_{\mathcal{P}_{n-1}}$  contains at least one zero.
- ( $P_2^0$ ) is true, because  $\eta^0 = \eta$  has no zero in  $\Lambda \setminus \mathcal{P}_{n-1}$ , and  $B \subset \Lambda \setminus \mathcal{P}_{n-1}$ , hence  $\eta_B^0 = 1_B$ .
- ( $P_3^0$ ) is true, because  $C \subset \Lambda \setminus \mathcal{P}_{n-1}$ , thus  $\eta_C = 1_C$ , hence  $\eta_C^0 1_{\Lambda \setminus C} = 1_\Lambda \in V(n-1, \Lambda)$ .
- ( $P_4^0$ ) is true, because  $D_1 \subset \Lambda \setminus \mathcal{P}_{n-1}$ , thus  $\eta_{D_1} 1_{D_1' \setminus D_1} = 1_{D_1'}$ , which gives  $\eta_{D_1}^0 1_{D_1' \setminus D_1} \in V(n-1, D_1')$ .

Consequently,  $\mathcal{H}'_0$  holds.

**Induction** Let  $j$  be in  $\{0, \dots, m-1\}$ . We suppose that  $\mathcal{H}'_j$  holds. Let us show  $\mathcal{H}'_{j+1}$ .

As  $(\eta^j)_{0 \leq j \leq m}$  is a legal path, the move from  $\eta^j$  to  $\eta^{j+1}$  is legal, hence  $\eta^{j+1} = \eta^j$  or  $\eta^{j+1} = (\eta^j)^z$  where  $z \in \Lambda$  satisfies that there exists  $X \in \mathcal{U}$  with  $(\eta_\Lambda^j 0_{\Lambda^c})_{z+X} = 0_{z+X}$ .

If  $\eta^{j+1} = \eta^j$ ,  $\mathcal{H}'_{j+1}$  holds because  $\mathcal{H}'_j$  holds.

In the following, we deal with the case  $\eta^{j+1} = (\eta^j)^z$ . The arguments will depend on the position of  $z$ :

- Case  $z \in B$ .

We will show that  $z \in B$  is impossible: the buffer zone remains preserved at step  $j + 1$ .

If  $z$  was in  $B$ , we would have  $z + X \subset \Lambda$ . Furthermore, by  $(P_2^j)$   $\eta_B^j = 1_B$ , hence  $z + X \subset C \cup D$ . Moreover, if there existed  $x \in (z + X) \cap C$  and  $y \in (z + X) \cap D$ , then we would get  $|x - y| > r$ , which is impossible as  $r = \max\{|x - y| \mid x, y \in X \cup \{0\}, X \in \mathcal{U}\}$ . Therefore  $z + X \subset C$  or  $z + X \subset D$ . We are going to deal with the two cases separately:

- Case  $z + X \subset C$ . The move from  $\eta_C^j 1_{\Lambda \setminus C}$  to  $(\eta_C^j 1_{\Lambda \setminus C})^z$  would then be legal. In addition,  $\eta_{\mathcal{P}_{n-1}}^{j+1} = \eta_{\mathcal{P}_{n-1}}^j$  would contain at least a zero by  $(P_1^j)$ , and  $\eta^{j+1}$  contains at most  $n$  zeroes, hence  $(\eta_C^j 1_{\Lambda \setminus C})^z = \eta_{C \cup B}^{j+1} 1_{\Lambda \setminus (C \cup B)}$  would contain at most  $n - 1$  zeroes (by  $(P_2^j)$ ,  $\eta_B^j = 1_B$  hence  $\eta_B^{j+1} = (1_B)^z$ ). Finally, by  $(P_3^j)$ ,  $\eta_C^j 1_{\Lambda \setminus C} \in V(n - 1, \Lambda)$ . Since the move from  $\eta_C^j 1_{\Lambda \setminus C}$  to  $(\eta_C^j 1_{\Lambda \setminus C})^z$  would be legal,  $(\eta_C^j 1_{\Lambda \setminus C})^z$  would contain at most  $n - 1$  zeroes and  $\eta_C^j 1_{\Lambda \setminus C} \in V(n - 1, \Lambda)$ , we would deduce  $(\eta_C^j 1_{\Lambda \setminus C})^z \in V(n - 1, \Lambda)$ . However, this is impossible: since  $z \in B$  and  $\eta_B^j = 1_B$  by  $(P_2^j)$ ,  $(\eta_C^j 1_{\Lambda \setminus C})_z = 1$ , hence  $(\eta_C^j 1_{\Lambda \setminus C})^z$  would have a zero in  $z$ . But as  $z \in B$ ,  $z + \mathcal{P}_{n-1} \subset \Lambda$ , hence by  $\mathcal{H}_{n-1}$  and the invariance by translation of  $\mathbb{Z}$ , no configuration in  $V(n - 1, \Lambda)$  can have a zero in  $z$ , which yields a contradiction. Consequently,  $z + X \subset C$  is impossible.
- Case  $z + X \subset D$ . There will be different arguments depending on the position of  $z$ . Since  $z \in B$ ,  $z < -a_n + a_{n-1} + r$  or  $z > b_n - (b_{n-1} + r)$ .
  - \* If  $z < -a_n + a_{n-1} + r$ , we can use the assumption on the model. Indeed,  $z + X \subset D$  would imply that any site  $z' \in z + X \subset D$  would verify  $z' \geq -a_n + a_{n-1} + r$ . Consequently,  $z + X \subset z + \mathbb{N}^*$ , thus  $X \subset \mathbb{N}^*$ , which would contradict the assumption that the model is not supercritical unrooted.
  - \* If  $z > b_n - (b_{n-1} + r)$ , we will use  $\mathcal{H}_{n-1}$ . Indeed, it can be seen that  $z + X \subset D_1$ . Since  $z + X \subset D_1$ , the move from  $\eta_{D_1}^j 1_{D_1' \setminus D_1}$  to  $(\eta_{D_1}^j 1_{D_1' \setminus D_1})^z$  would be legal. Furthermore, by  $(P_1^j)$   $\eta_{\mathcal{P}_{n-1}}^{j+1} = \eta_{\mathcal{P}_{n-1}}^j$  would contain at least a zero, and  $\eta^{j+1}$  contains at most  $n$  zeroes, hence  $(\eta_{D_1}^j 1_{D_1' \setminus D_1})^z = \eta_{D_1 \cup (B \cap D_1')}^{j+1} 1_{D_1' \setminus (D_1 \cup B)}$  would contain at most  $n - 1$  zeroes. In addition, by  $(P_4^j)$   $\eta_{D_1}^j 1_{D_1' \setminus D_1} \in V(n - 1, D_1')$ . This would allow us to conclude that  $(\eta_{D_1}^j 1_{D_1' \setminus D_1})^z \in V(n - 1, D_1')$ . However, this is impossi-

ble:  $(\eta_{D_1}^j 1_{D_1 \setminus D_1})^z$  has a zero at  $z$ , and as  $z \in B$  one can check that  $z + \mathcal{P}_{n-1} \subset D_1'$ , which implies by  $\mathcal{H}_{n-1}$  that no configuration of  $V(n-1, D_1')$  can have a zero in  $z$ . Hence we get a contradiction.

We deduce a contradiction in both cases, therefore  $z + X \subset D$  is impossible.

Consequently,  $z \in B$  is impossible.

- Case  $z \in C$ .

If  $z \in C$ ,  $(P_1^{j+1})$  is true because  $\eta_{\mathcal{P}_{n-1}}^{j+1} = \eta_{\mathcal{P}_{n-1}}^j$ ,  $(P_2^{j+1})$  is true because  $\eta_B^{j+1} = \eta_B^j$ , and  $(P_4^{j+1})$  is true because  $\eta_{D_1}^{j+1} = \eta_{D_1}^j$ .

We are going to prove  $(P_3^{j+1})$ .

We observe that as  $z \in C$ ,  $z + X \subset \Lambda^c \cup C \cup B$ . Moreover,  $(P_2^j)$  implies  $\eta_B^j = 1_B$ , hence  $z + X \subset \Lambda^c \cup C$ . Therefore,  $(\eta_C^j 1_{\Lambda \setminus C} 0_{\Lambda^c})_{z+X} = 0_{z+X}$ , so the move from  $\eta_C^j 1_{\Lambda \setminus C}$  to  $(\eta_C^j 1_{\Lambda \setminus C})^z = \eta_C^{j+1} 1_{\Lambda \setminus C}$  is legal. Furthermore, by  $(P_1^{j+1})$   $\eta_{\mathcal{P}_{n-1}}^{j+1}$  contains a zero, and  $\eta^{j+1}$  contains at most  $n$  zeroes, hence  $\eta_C^{j+1} 1_{\Lambda \setminus C}$  contains at most  $n-1$  zeroes. In addition, by  $(P_3^j)$  we have  $\eta_C^j 1_{\Lambda \setminus C} \in V(n-1, \Lambda)$ . This allows to conclude that  $\eta_C^{j+1} 1_{\Lambda \setminus C} \in V(n-1, \Lambda)$ , which is  $(P_3^{j+1})$ . Therefore,  $(P_3^{j+1})$  is true.

Consequently,  $\mathcal{H}'_{j+1}$  holds.

- Case  $z \in D$ .

If  $z \in D$ ,  $(P_2^{j+1})$  is true because  $\eta_B^{j+1} = \eta_B^j$ , and  $(P_3^{j+1})$  is true because  $\eta_C^{j+1} = \eta_C^j$ .

Let us prove  $(P_1^{j+1})$ .

We want to prove that  $\eta_{\mathcal{P}_{n-1}}^{j+1}$  contains a zero. There will be two different cases:

- If  $z \in D_1$ , then  $\eta_{\mathcal{P}_{n-1}}^{j+1} = \eta_{\mathcal{P}_{n-1}}^j$ , hence by  $(P_1^j)$   $\eta_{\mathcal{P}_{n-1}}^{j+1}$  contains at least one zero.
- If  $z \in \mathcal{P}_{n-1}$ , then we notice that  $z + X \subset D \cup B$ . Moreover, by  $(P_2^j)$   $\eta_B^j = 1_B$ , thus  $z + X \subset D$ . Furthermore, by the assumption on the model,  $X$  can not be entirely contained in  $\mathbb{N}^*$ , and as it is an update rule it does not contains zero, hence it contains an element of  $-\mathbb{N}^*$ . Therefore, there exists  $z' \in z + X$  with  $z' < z$ . Since  $z' \in D$ ,  $z' \in \mathcal{P}_{n-1}$  and  $z' < z$ , we deduce  $z' \in \mathcal{P}_{n-1}$ . Then as  $\eta_{z+X}^j = 0_{z+X}$ ,  $\eta_{z'}^{j+1} = \eta_{z'}^j = 0$ , with  $z' \in \mathcal{P}_{n-1}$ . Consequently,  $\eta_{\mathcal{P}_{n-1}}^{j+1}$  contains a zero.

In both cases we conclude that  $\eta_{\mathcal{P}_{n-1}}^{j+1}$  contains a zero. Consequently,  $(P_1^{j+1})$  is true.

Now let us prove  $(P_4^{j+1})$ .

We are going to prove that  $\eta_{D_1}^{j+1} 1_{D_1' \setminus D_1} \in V(n-1, D_1')$ . As before, there will be two different cases:

- If  $z \in D_1$ , then  $z + X \subset D \cup B$ . Furthermore, by  $(P_2^j)$   $\eta_B^j = 1_B$ , hence we have  $z + X \subset D$ . Moreover, we observe that  $D_1' \setminus D_1$  is disjoint from  $D$ , thus we obtain  $(\eta_{D_1}^j 1_{D_1' \setminus D_1} 0_{(D_1')^c})_{z+X} = 0_{z+X}$ . This implies that the move from  $\eta_{D_1}^j 1_{D_1' \setminus D_1}$  to  $(\eta_{D_1}^j 1_{D_1' \setminus D_1})^z = \eta_{D_1}^{j+1} 1_{D_1' \setminus D_1}$  is legal. Furthermore,  $\eta_{\mathcal{P}_{n-1}}^{j+1}$  contains at least one zero by  $(P_1^{j+1})$ , and  $\eta^{j+1}$  contains at most  $n$  zeroes, hence  $\eta_{D_1}^{j+1} 1_{D_1' \setminus D_1}$  contains at most  $n-1$  zeroes. In addition,  $\eta_{D_1}^j 1_{D_1' \setminus D_1} \in V(n-1, D_1')$  by  $(P_4^j)$ . Consequently,  $\eta_{D_1}^{j+1} 1_{D_1' \setminus D_1} \in V(n-1, D_1')$ .
- If  $z \in \mathcal{P}_{n-1}$ , then  $\eta_{D_1}^{j+1} = \eta_{D_1}^j$ , and as  $\eta_{D_1}^j 1_{D_1' \setminus D_1} \in V(n-1, D_1')$  by  $(P_4^j)$ ,  $\eta_{D_1}^{j+1} 1_{D_1' \setminus D_1} \in V(n-1, D_1')$ .

In both cases we proved that  $\eta_{D_1}^{j+1} 1_{D_1' \setminus D_1} \in V(n-1, D_1')$ . Therefore  $(P_4^{j+1})$  is true.

This yields that  $\mathcal{H}'_{j+1}$  holds.

To conclude,  $\mathcal{H}'_{j+1}$  holds in all cases.

Consequently,  $\mathcal{H}'_j$  implies  $\mathcal{H}'_{j+1}$  for any  $j \in \{0, \dots, m-1\}$ , and  $\mathcal{H}'_0$  holds. This proves that  $\mathcal{H}'_j$  holds for any  $j \in \{0, \dots, m\}$ , which ends the proof of the lemma. □

### 3.2 Supercritical unrooted models

We are going to show that the hypothesis of theorem 1 is not restrictive: we set  $\mathcal{U}$  a supercritical unrooted update family and prove that theorem 1 is not true for  $\mathcal{U}$ . Indeed, a bounded number of zeroes is enough to bring a zero at any site of any finite domain; more precisely

**Proposition 3.** *There exists  $k \in \mathbb{N}^*$  such that for any finite domain  $\Lambda \subset \mathbb{Z}$  and any  $z \in \Lambda$ , there exists  $\eta \in V(k, \Lambda)$  such that  $\eta_z = 0$ .*

*Proof.* The idea of the proof is to build an interval of zeroes that can “move along  $\mathbb{Z}$ ”, and to transport it from the boundary conditions to  $z$ .



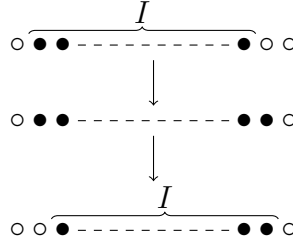


Figure 4: A move of  $I$  one step to the right. Zeroes are represented by  $\bullet$  and ones by  $\circ$ .

By assumption,  $\mathcal{U}$  is supercritical unrooted, hence by proposition 2 it includes an update rule  $X \subset \mathbb{N}^*$  and an update rule  $X' \subset -\mathbb{N}^*$ . We define  $\ell = \max\{x \in X\}$ ,  $\ell' = \min\{x \in X'\}$  and  $m = \max(\ell, -\ell')$ .

Now consider  $\Lambda \subset \mathbb{Z}$  a finite domain,  $I = \{a, \dots, a + m - 1\}$  with  $a \in \mathbb{Z}$ , and  $\eta \in \{0, 1\}^\Lambda$  such that  $\eta_I = 0_I$  (remember that the sites outside  $\Lambda$  are considered to be zeroes) and  $\eta_{\Lambda \setminus I} = 1_{\Lambda \setminus I}$ .

We will “move  $I$  one step to the right” with a  $(m + 1)$ -legal path by adding a zero at the right of  $I$  and removing the leftmost zero of  $I$  (see figure 4).

- if  $a + m \in \Lambda$ , the move from  $\eta$  to  $\eta^{a+m}$  is legal. Indeed, as  $X' \subset -\mathbb{N}^*$  and  $\ell' = \min\{x \in X'\}$ , we obtain  $X' \subset \{\ell', \dots, -1\} \subset \{-m, \dots, -1\}$ , hence  $a + m + X' \subset \{a, \dots, a + m - 1\} = I$ . Therefore  $\eta_{a+m+X'} = 0_{a+m+X'}$ , which implies that the move from  $\eta$  to  $\eta^{a+m}$  is legal. Thus we make the move from  $\eta$  to  $\eta^{a+m}$ , and we have  $(\eta^{a+m})_{a+m} = 0$ . If  $a + m \notin \Lambda$ , then we already have  $\eta_{a+m} = 0$  and we do not need to make the move. The configuration we obtain has zeroes in  $\{a, \dots, a + m\}$  and ones elsewhere in  $\Lambda$ .
- if  $a \in \Lambda$ , the move from  $\eta^{a+m}$  to  $(\eta^{a+m})^a$  is legal. Indeed, as  $X \subset \mathbb{N}^*$  and  $\ell = \max\{x \in X\}$ , we obtain  $X \subset \{1, \dots, \ell\} \subset \{1, \dots, m\}$ , hence  $a + X \subset \{a + 1, \dots, a + m\}$ . Therefore  $(\eta^{a+m})_{a+X} = 0_{a+X}$ , which implies that the move from  $(\eta^{a+m})$  to  $(\eta^{a+m})^a$  is legal. As it is legal, we make it. If  $a \notin \Lambda$ , we do not need to make this move as the zero in  $a$  does not count as a zero of the configuration. Moreover, since  $(\eta^{a+m})_a = \eta_a = 0$ ,  $((\eta^{a+m})^a)_a = 1$ , consequently the configuration we get has zeroes in  $\{a + 1, \dots, a + m\}$  and ones elsewhere in  $\Lambda$ .

The configuration  $(\eta^{a+m})^a$  has zeroes in  $\{a + 1, \dots, a + m\} = I + 1$  and ones elsewhere in  $\Lambda$ , thus we can consider that the interval of zeroes  $I$  moved one step to the right.

In addition, there are at most  $m$  zeroes in  $\Lambda$  for  $\eta$ ,  $m + 1$  zeroes in  $\Lambda$  for  $\eta^{a+m}$ , and  $m$  zeroes in  $\Lambda$  for  $(\eta^{a+m})^a$ , thus there are never more than  $m + 1$  zeroes in  $\Lambda$ . Consequently, one can move an interval  $I$  of zeroes of length  $m$  one step to the right using a  $(m + 1)$ -legal path.

Therefore, if  $\eta = 1_\Lambda$ , it is possible to start with  $I$  included in  $\Lambda^c$  (which contains an interval of length  $m$  since  $\Lambda$  is a finite domain); as we have empty boundary conditions,  $I$  is then an interval of zeroes. Then we move  $I$  to the right step by step using a  $(m + 1)$ -legal path, until  $I$  reaches  $z$ . This yields a  $(m + 1)$ -legal path from  $1_\Lambda$  to a configuration  $\eta'$  with  $\eta'_z = 0$ . Consequently, there exists a configuration  $\eta' \in V(m + 1, \Lambda)$  with  $\eta'_z = 0$ .

We conclude that the proposition is true with  $k = m + 1$ .  $\square$

## 4 The general case

### 4.1 Proof of theorem 1

We begin by making an observation about the basis  $\{v_1, \dots, v_d\}$  defined in section 2. It is constructed so that for any  $i \in \{1, \dots, d\}$ , there exists a stable direction  $u_i$  satisfying  $\mathbb{H}_{u_i} = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i > 0\}$ . By the definition of a stable direction, there is no update rule contained in  $\mathbb{H}_{u_i}$ . Therefore there is no update rule such that all its sites have a positive  $i$ -th coordinate.

The reasoning to prove theorem 1 in any dimension is the same as in dimension 1, with some subtleties due to a more complicated geometry.

We will again prove the theorem by induction: for all  $n \in \mathbb{N}^*$ , we denote

$$\mathcal{H}_n = \text{“for any } \Lambda \subset \mathbb{Z}^d \text{ such that } \mathcal{P}_n \subset \Lambda, \text{ for any } \eta \in V(n, \Lambda), \eta_0 = 1\text{”}.$$

Proving  $\mathcal{H}_n$  for all  $n \in \mathbb{N}^*$  will prove the theorem.

In order to do that, we again need the

**Lemma 2.** *Let  $n > 1$  and suppose  $\mathcal{H}_{n-1}$ . Then, for all  $\Lambda \subset \mathbb{Z}^d$  such that  $\mathcal{P}_n \subset \Lambda$ , for all  $\eta \in V(n, \Lambda) \setminus \{1_\Lambda\}$ ,  $\eta$  has at least one zero in  $\Lambda \setminus \mathcal{P}_{n-1}$ .*

We will prove the lemma 2 after the theorem.

**Case  $n=1$**  The case is completely similar to the dimension 1: if the configurations we consider have a zero, this zero was created by the zeroes of the boundary conditions, hence it has to be near the boundary. Therefore, we omit the proof.

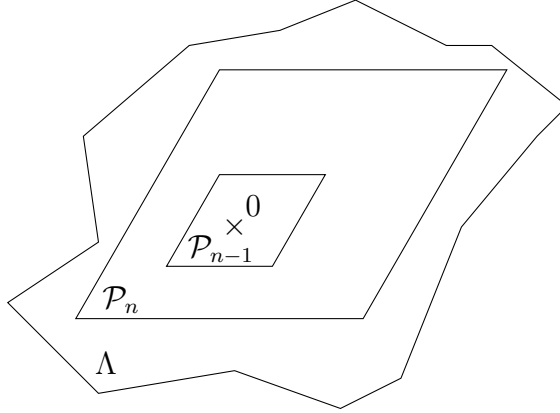


Figure 5: There must be a zero in  $\Lambda \setminus \mathcal{P}_{n-1}$ , hence there can be at most  $n - 1$  zeroes in  $\mathcal{P}_{n-1}$ . Thus  $\mathcal{H}_{n-1}$  implies that there is no zero at 0.

**Induction** Let  $n > 1$ . We suppose  $\mathcal{H}_{n-1}$ . Let us show  $\mathcal{H}_n$ .

Let  $\Lambda \subset \mathbb{Z}^d$  be such that  $\mathcal{P}_n \subset \Lambda$ , and let  $\eta$  be in  $V(n, \Lambda)$ .

If  $\eta = 1_\Lambda$ , then  $\eta_0 = 1$  and there is nothing to prove. Now, let us deal with the case  $\eta \neq 1_\Lambda$ .

The argument is the same as in dimension 1 (see figure 5). As  $\eta \in V(n, \Lambda)$ , there exists a  $n$ -legal path  $(\eta^j)_{0 \leq j \leq m}$  from  $\eta$  to  $1_\Lambda$ . The same reasoning as in the one-dimensional case yields that  $(\eta_{\mathcal{P}_{n-1}}^j)_{0 \leq j \leq m}$  is a  $(n - 1)$ -legal path from  $\eta_{\mathcal{P}_{n-1}}^0 = \eta_{\mathcal{P}_{n-1}}$  to  $\eta_{\mathcal{P}_{n-1}}^m = 1_{\mathcal{P}_{n-1}}$ . Consequently,  $\eta_{\mathcal{P}_{n-1}} \in V(n - 1, \mathcal{P}_{n-1})$ . By  $\mathcal{H}_{n-1}$ , this implies that  $\eta_0 = 1$ .

This proves  $\mathcal{H}_n$ .

Consequently,  $\mathcal{H}_{n-1}$  implies  $\mathcal{H}_n$  for all  $n > 1$ . Hence, as  $\mathcal{H}_1$  holds,  $\mathcal{H}_n$  holds for all  $n \in \mathbb{N}^*$ .

This ends the proof of theorem 1 given lemma 2.

We are now going to prove lemma 2.

*Proof of lemma 2.* Let  $\Lambda \subset \mathbb{Z}^d$  such that  $\mathcal{P}_n \subset \Lambda$ .

As in the one-dimensional case, we consider a configuration  $\eta \in \{0, 1\}^\Lambda$ , different from  $1_\Lambda$ , containing at most  $n$  zeroes, such that all of its zeroes are in  $\mathcal{P}_{n-1}$ , and we prove that  $\eta \notin V(n, \Lambda)$ . Let  $(\eta^j)_{0 \leq j \leq m}$  be a  $n$ -legal path with  $\eta^0 = \eta$ . We are going to show that  $\eta^m$  can not be  $1_\Lambda$ . This will imply that there can not be a  $n$ -legal path from  $\eta$  to  $1_\Lambda$ , which means that  $\eta \notin V(n, \Lambda)$ . This is enough to prove the lemma.

To this end, we denote for all  $i \in \{1, \dots, d\}$  (see figure 6):

$$D = \{(x_1, \dots, x_d) \in \mathbb{Z}^d \mid \forall j, -a_n + a_{n-1} + r \leq x_j \leq b_n - (b_{n-1} + r)\},$$

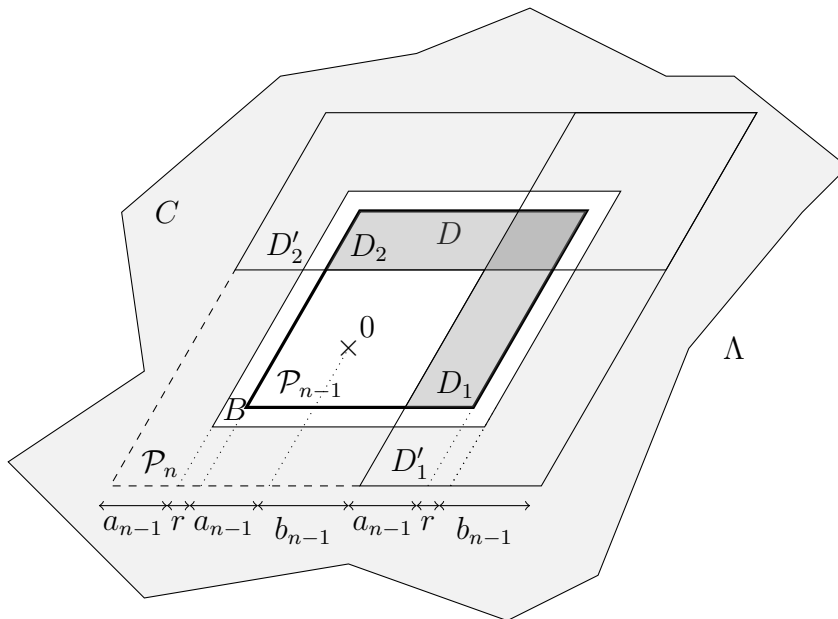


Figure 6: The setting of lemma 2.  $C$  is in light gray;  $D_1$  and  $D_2$  are in darker gray.  $D$  is the region with the thick outline.

$$B = \{(x_1, \dots, x_d) \in \mathbb{Z}^d \mid \forall j, -a_n + a_{n-1} \leq x_j \leq b_n - b_{n-1}\} \setminus D,$$

$$D_i = \{(x_1, \dots, x_d) \in D \mid x_i > b_n - (b_{n-1} + a_{n-1} + r)\},$$

$$D'_i = \{(x_1, \dots, x_d) \in \mathcal{P}_n \mid x_i > b_n - (b_{n-1} + a_{n-1} + r)\}$$

and  $C = \Lambda \setminus (B \cup D)$ . We also notice that the same calculations as in dimension 1 yield that

$$\mathcal{P}_{n-1} = \{(x_1, \dots, x_d) \in \mathbb{Z}^d \mid \forall j, -a_n + a_{n-1} + r \leq x_j \leq b_n - (b_{n-1} + a_{n-1} + r)\}$$

hence  $\mathcal{P}_{n-1} = D \setminus (\bigcup_{i=1}^d D_i)$ .

As in the one-dimensional case,  $B$  will be a buffer zone. In that case, the main reason for which no zero could appear in  $B$  was that a zero remained trapped in  $\mathcal{P}_{n-1}$ , hence there were at most  $n - 1$  zeroes elsewhere, and  $\mathcal{H}_{n-1}$  limited their possible positions.

Here we can not keep a zero in  $\mathcal{P}_{n-1}$ , but we can keep a zero in all the  $D \setminus D_i$ , because initially there is at least a zero in  $\mathcal{P}_{n-1} \subset D \setminus D_i$ , and at any time, a zero of  $D \setminus D_i$  with the lowest  $i$ -th coordinate among the zeroes of  $D \setminus D_i$  will need, in order to disappear, a zero with a  $i$ -th coordinate as low as its own because there is no update rule such that all of its zeroes have positive  $i$ -th coordinate (this is the reason for which we work in the basis  $\{v_1, \dots, v_d\}$ ).

This zero can not be in  $B$  since  $B$  remains full of ones, hence it is in  $D \setminus D_i$  and so remains in  $D \setminus D_i$  at the next step of the path.

This will have the same practical consequences as the zero trapped in  $\mathcal{P}_{n-1}$  had in the one-dimensional case: the presence of a zero in each of the  $D \setminus D_i$  prevents  $\eta^m$  to be  $1_\Lambda$ ; the  $n - 1$  zeroes that any of the  $D_i$ , or  $C$ , may contain will not escape the  $D_i$  or  $C$ . Moreover, for any  $i \in \{1, \dots, d\}$ , the argument that in dimension 1 prevented the zeroes of  $\mathcal{P}_{n-1}$  from escaping to the left part of  $B$  because there were no update rule contained in  $\mathbb{N}^*$  will here prevent zeroes to enter  $B$  via the face of  $D$  with the lowest  $i$ -th coordinate, since there is no update rule such that its zeroes all have positive  $i$ -th coordinates. Therefore the buffer zone  $B$  will be preserved.

More precisely, we are going to prove by induction on  $j \in \{0, \dots, m\}$  that the property  $\mathcal{H}'_j$  holds, where  $\mathcal{H}'_j$  consists in:

( $P_1^j$ ) For all  $i \in \{1, \dots, d\}$ ,  $\eta_{D \setminus D_i}^j$  contains a zero.

( $P_2^j$ )  $\eta_B^j = 1_B$ .

( $P_3^j$ )  $\eta_C^j 1_{\Lambda \setminus C} \in V(n - 1, \Lambda)$ .

( $P_4^j$ ) For all  $i \in \{1, \dots, d\}$ ,  $\eta_{D_i}^j 1_{D_i \setminus D_i} \in V(n - 1, D_i)$ .

If we can show  $\mathcal{H}'_j$  for all  $j \in \{0, \dots, m\}$ , in particular ( $P_1^m$ ) will imply that there is a zero in  $\eta_D^m$ , thus  $\eta^m \neq 1_\Lambda$ , hence the lemma.

Let us prove  $\mathcal{H}'_j$  for all  $j \in \{0, \dots, m\}$ .

**Case  $j = 0$**  It works exactly as in the one-dimensional case. Consequently,  $\mathcal{H}'_0$  holds.

**Induction** Let  $j$  be in  $\{0, \dots, m - 1\}$ . We suppose that  $\mathcal{H}'_j$  holds. Let us show  $\mathcal{H}'_{j+1}$ .

As  $(\eta^j)_{0 \leq j \leq m}$  is a legal path, the move from  $\eta^j$  to  $\eta^{j+1}$  is legal, hence  $\eta^{j+1} = \eta^j$  or  $\eta^{j+1} = (\eta^j)^z$  where  $z \in \Lambda$  and satisfies that there exists  $X \in \mathcal{U}$  with  $(\eta_\Lambda^j 0_{\Lambda^c})_{z+X} = 0_{z+X}$ .

If  $\eta^{j+1} = \eta^j$ ,  $\mathcal{H}'_{j+1}$  holds because  $\mathcal{H}'_j$  holds.

In the following, we deal with the case  $\eta^{j+1} = (\eta^j)^z$ . As in the one-dimensional case, the arguments will depend on the position of  $z$ :

- Case  $z \in B$ .

We will show that  $z \in B$  is impossible.

If  $z$  was in  $B$ , we would get  $z + X \subset \Lambda$ . Furthermore, by ( $P_2^j$ )  $\eta_B^j = 1_B$ , hence  $z + X \subset C \cup D$ . Moreover, if there existed  $x \in (z + X) \cap C$  and

$y \in (z + X) \cap D$ , then we would get  $\|x - y\|_\infty > r$ , which is impossible as  $r = \max\{\|x - y\|_\infty \mid x, y \in X \cup \{0\}, X \in \mathcal{U}\}$ . Therefore  $z + X \subset C$  or  $z + X \subset D$ . We are going to deal with the two cases separately:

- Case  $z + X \subset C$ . This case is proven impossible in the same way as in dimension 1: by showing that  $(\eta_C^j 1_{\Lambda \setminus C})^z$  would be a configuration of  $V(n - 1, \Lambda)$  that has a zero in  $z \in \Lambda$  with  $z + \mathcal{P}_{n-1} \subset \Lambda$ , which contradicts  $\mathcal{H}_{n-1}$ .
- Case  $z + X \subset D$ . Again, there will be different arguments depending on the position of  $z$ . Since  $z \in B$ , if we denote by  $(z_1, \dots, z_d)$  the coordinates of  $z$ , there would exist  $i \in \{1, \dots, d\}$  such that  $z_i < -a_n + a_{n-1} + r$  or  $z_i > b_n - (b_{n-1} + r)$ .

\* If  $z_i < -a_n + a_{n-1} + r$ , we can use the assumption on the model. Indeed,  $z + X \subset D$  would imply that the  $i$ -th coordinate of any site of  $z + X$  would be greater than or equal to  $-a_n + a_{n-1} + r$ . Consequently,  $z + X \subset z + \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i > 0\}$ , hence  $X \subset \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i > 0\}$ , which is impossible.

\* If  $z_i > b_n - (b_{n-1} + r)$ , we will use  $\mathcal{H}_{n-1}$ . Indeed,  $z \in D'_i$ . Moreover,  $D$  is disjoint from  $D'_i \setminus D_i$ , hence  $(\eta_{D'_i}^j 1_{D'_i \setminus D_i} 0_{(D'_i)^c})_{z+X} = 0_{z+X}$ , thus the move from  $\eta_{D'_i}^j 1_{D'_i \setminus D_i}$  to  $(\eta_{D'_i}^j 1_{D'_i \setminus D_i})^z$  would be legal. Using the same arguments as in the one-dimensional case, we would conclude that the configuration  $(\eta_{D'_i}^j 1_{D'_i \setminus D_i})^z$  would be in  $V(n - 1, D'_i)$ . However, since  $z \in B$  and  $z_i > b_n - (b_{n-1} + r)$ , it can be seen that  $z + \mathcal{P}_{n-1} \subset D'_i$ , and  $(\eta_{D'_i}^j 1_{D'_i \setminus D_i})^z$  has a zero in  $z$ , hence this would contradict  $\mathcal{H}_{n-1}$ .

We deduce a contradiction in both cases, therefore  $z + X \subset D$  is impossible.

Consequently,  $z \in B$  is impossible.

- Case  $z \in C$ .

As in the one-dimensional case, if  $z \in C$ ,  $(P_1^{j+1})$  is true because  $\eta_{D \setminus D_i}^{j+1} = \eta_{D \setminus D_i}^j$  for any  $i \in \{1, \dots, d\}$ ,  $(P_2^{j+1})$  is true because  $\eta_B^{j+1} = \eta_B^j$ , and  $(P_4^{j+1})$  is true because  $\eta_{D_i}^{j+1} = \eta_{D_i}^j$  for any  $i \in \{1, \dots, d\}$ . Furthermore, the proof of  $(P_3^{j+1})$  is the same as in dimension 1. Therefore,  $(P_3^{j+1})$  is true.

Consequently,  $\mathcal{H}'_{j+1}$  holds.

- Case  $z \in D'$ .

As in the one-dimensional case, if  $z \in D$ ,  $(P_2^{j+1})$  is true because  $\eta_B^{j+1} = \eta_B^j$ , and  $(P_3^{j+1})$  is true because  $\eta_C^{j+1} = \eta_C^j$ .

Let us prove  $(P_1^{j+1})$ .

We set  $i \in \{1, \dots, d\}$ . Let us prove that  $\eta_{D \setminus D_i}^{j+1}$  contains a zero. There will be two different cases:

- If  $z \in D_i$ , then  $\eta_{D \setminus D_i}^{j+1} = \eta_{D \setminus D_i}^j$ , hence by  $(P_1^j)$   $\eta_{D \setminus D_i}^{j+1}$  contains at least one zero.
- If  $z \in D \setminus D_i$ , then we observe that  $z + X \subset D \cup B$ . Moreover, by  $(P_2^j)$   $\eta_B^j = 1_B$ , thus  $z + X \subset D$ . Furthermore, by assumption on the model  $X \not\subset \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i > 0\}$ , hence  $z + X \not\subset \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i > 0\}$ , which implies that there exists a site  $z' \in z + X$  such that the  $i$ -th coordinate of  $z'$  is lesser than or equal to the  $i$ -th coordinate of  $z$ , hence lesser than or equal to  $b_n - a_{n-1} - b_{n-1} - r$  since  $z \in D \setminus D_i$ . Therefore, as  $z' \in D$  and the  $i$ -th coordinate of  $z'$  is lesser than or equal to  $b_n - a_{n-1} - b_{n-1} - r$ ,  $z' \in D \setminus D_i$ . Furthermore, we have  $\eta_{z'}^{j+1} = \eta_{z'}^j = 0$ . Consequently,  $\eta_{D \setminus D_i}^{j+1}$  contains a zero.

In both cases we conclude that  $\eta_{D \setminus D_i}^{j+1}$  contains a zero. Consequently,  $\eta_{D \setminus D_i}^{j+1}$  contains a zero for all  $i \in \{1, \dots, d\}$ , hence  $(P_1^{j+1})$  is true.

Now let us prove  $(P_4^{j+1})$ .

We let  $i \in \{1, \dots, d\}$ . We are going to prove that  $\eta_{D_i}^{j+1} 1_{D_i' \setminus D_i} \in V(n-1, D_i')$ . As before, there will be two different cases:

- If  $z \in D_i$ , then we observe that  $z + X \subset D \cup B$ . Furthermore, by  $(P_2^j)$   $\eta_B^j = 1_B$ , hence we have  $z + X \subset D$ . Moreover,  $D_i' \setminus D_i$  is disjoint from  $D$ , thus we obtain  $(\eta_{D_i}^j 1_{D_i' \setminus D_i} 0_{(D_i')^c})_{z+X} = 0_{z+X}$ . This implies that the move from  $\eta_{D_i}^j 1_{D_i' \setminus D_i}$  to  $(\eta_{D_i}^j 1_{D_i' \setminus D_i})^z = \eta_{D_i}^{j+1} 1_{D_i' \setminus D_i}$  is legal. Furthermore,  $\eta_{D \setminus D_i}^{j+1}$  contains at least one zero by  $(P_1^{j+1})$ , and  $\eta^{j+1}$  contains at most  $n$  zeroes, hence  $\eta_{D_i}^{j+1} 1_{D_i' \setminus D_i}$  contains at most  $n-1$  zeroes. In addition,  $\eta_{D_i}^j 1_{D_i' \setminus D_i} \in V(n-1, D_i')$  by  $(P_4^j)$ . Consequently,  $\eta_{D_i}^{j+1} 1_{D_i' \setminus D_i} \in V(n-1, D_i')$ .
- If  $z \in D \setminus D_i$ , then  $\eta_{D_i}^{j+1} = \eta_{D_i}^j$ , and as  $\eta_{D_i}^j 1_{D_i' \setminus D_i} \in V(n-1, D_i')$  by  $(P_4^j)$ ,  $\eta_{D_i}^{j+1} 1_{D_i' \setminus D_i} \in V(n-1, D_i')$ .

In both cases we proved that  $\eta_{D_i}^{j+1} 1_{D_i' \setminus D_i} \in V(n-1, D_i')$ . This implies that  $\eta_{D_i}^{j+1} 1_{D_i' \setminus D_i} \in V(n-1, D_i')$  for any  $i \in \{1, \dots, d\}$ . Therefore  $(P_4^{j+1})$  is true.

This yields that  $\mathcal{H}'_{j+1}$  holds.

To conclude,  $\mathcal{H}'_{j+1}$  holds in all cases.

Consequently,  $\mathcal{H}'_j$  implies  $\mathcal{H}'_{j+1}$  for any  $j \in \{0, \dots, m-1\}$ , and  $\mathcal{H}'_0$  holds. This proves that  $\mathcal{H}'_j$  holds for any  $j \in \{0, \dots, m\}$ , which ends the proof of the lemma. □

## 4.2 Supercritical unrooted models

As in the one dimensional case, we wish to prove that the hypothesis of our theorem 1 is not restrictive, namely that the result is not valid in supercritical unrooted models. In the one-dimensional case, we proved that a bounded number of zeroes was enough to bring a zero anywhere in any finite domain by constructing an interval of zeroes that could move along  $\mathbb{Z}$ , and by making it move from the boundary conditions to the target site. In  $d = 2$  we can extend the construction using the bootstrap results of [6], and we expect the idea to extend to higher dimensions.

For  $d = 2$ , [6] explains how, if we have a semicircle containing no stable direction, we can build a “droplet” of zeroes that can grow in the direction given by the middle of the semicircle; as any change of site state can be reversed in a KCM, the droplet will also be able to shrink in that direction. If we have a supercritical unrooted model, its stable directions are contained in a hyperplane of  $\mathbb{R}^2$ , which means a straight line, hence there are at most two stable directions, and they must then be opposite. Therefore, there exists two opposite semicircles containing no stable direction, with middles  $u$  and  $-u$ . The construction of [6] can then be used to build a droplet that can grow and shrink in directions  $u$  and  $-u$ . When it grows in one direction, we have it shrink in the other, which allows to keep it below a bounded size and to keep a bounded number of zeroes. Furthermore, this “grow in one direction/shrink in the other” mechanism makes the droplet move along that direction. Hence we can move this droplet from the boundary conditions to the target site. This allows to bring a zero to the target site using only a bounded number of zeroes.

## Appendix: existence of the basis $\{v_1, \dots, v_d\}$

By assumption, the update family  $\mathcal{U}$  is not supercritical unrooted, hence its stable directions are not contained in any hyperplane of  $\mathbb{R}^d$ . Therefore, there exists stable directions  $u_1, \dots, u_d$  of  $\mathcal{U}$  that form a basis of  $\mathbb{R}^d$ .



For any  $u \in S^{d-1}$ , we denote  $\mathcal{H}_u$  the hyperplane orthogonal to  $u$ :  $\mathcal{H}_u = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = 0\}$ .

Then, for any  $i \in \{1, \dots, d\}$ ,  $\bigcap_{j \neq i} \mathcal{H}_{u_j}$  is a straight line. Indeed,  $\bigcap_{j \neq i} \mathcal{H}_{u_j}$  is the intersection of  $d-1$  hyperplanes in  $\mathbb{R}^d$ , hence it contains a straight line. Furthermore,  $\bigcap_{j \neq i} \mathcal{H}_{u_j}$  is orthogonal to the  $u_j$ ,  $j \neq i$ , and since  $\{u_1, \dots, u_d\}$  is a basis of  $\mathbb{R}^d$ ,  $\{u_j : j \neq i\}$  generate a vector space of dimension  $d-1$ . Therefore  $\bigcap_{j \neq i} \mathcal{H}_{u_j}$  is orthogonal to a vector space of dimension  $d-1$ . Consequently, it is at most a straight line. This yields that  $\bigcap_{j \neq i} \mathcal{H}_{u_j}$  is a straight line.

For any  $i \in \{1, \dots, d\}$ , we define  $v_i$  as a vector of  $\ell^2$ -norm 1 in  $\bigcap_{j \neq i} \mathcal{H}_{u_j}$ . We are going to show that  $\{v_1, \dots, v_d\}$  is a basis of  $\mathbb{R}^d$ . For any vector set  $\{w_1, \dots, w_m\} \subset \mathbb{R}^d$ , we denote by  $\text{Vect}\{w_1, \dots, w_m\}$  the vector space generated by  $\{w_1, \dots, w_m\}$ . It will be enough to prove that  $\text{Vect}\{v_1, \dots, v_d\}$  is  $\mathbb{R}^d$ . In order to do that, we take  $v \in \mathbb{R}^d$  a vector orthogonal to the space  $\text{Vect}\{v_1, \dots, v_d\}$ . We are going to show that  $v$  must be the null vector. For all  $i \in \{1, \dots, d\}$ ,  $v$  is orthogonal to  $v_i$ . Moreover, the vector space orthogonal to  $v_i$  has dimension  $d-1$ . Furthermore,  $v_i \in \bigcap_{j \neq i} \mathcal{H}_{u_j}$ , hence the  $u_j$ ,  $j \neq i$  are orthogonal to  $v_i$ . Hence, as the  $u_j$ ,  $j \neq i$  are  $d-1$  linearly independent vectors, the vector space orthogonal to  $v_i$  is  $\text{Vect}\{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_d\}$ . This implies that  $v$  belongs to  $\text{Vect}\{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_d\}$ , for any  $i \in \{1, \dots, d\}$ . As  $\{u_1, \dots, u_d\}$  is a basis of  $\mathbb{R}^d$ , this yields  $v = 0$ . Consequently, the vector space orthogonal to  $\text{Vect}\{v_1, \dots, v_d\}$  is reduced to  $\{0\}$ . We deduce  $\text{Vect}\{v_1, \dots, v_d\} = \mathbb{R}^d$ , thus  $\{v_1, \dots, v_d\}$  is a basis of  $\mathbb{R}^d$ .

Said basis verifies that for any  $i \in \{1, \dots, d\}$ ,  $\mathcal{H}_{u_i} = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i = 0\}$ . Indeed, the latter hyperplane is generated by the vectors  $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_d$ , which are  $d-1$  linearly independent vectors belonging to the hyperplane  $\mathcal{H}_{u_i}$  of  $\mathbb{R}^d$ , hence they generate  $\mathcal{H}_{u_i}$ .

Consequently, we obtain that for any  $i \in \{1, \dots, d\}$ ,  $\|v_i\|_2 = 1$  and in the basis  $\{v_1, \dots, v_d\}$ ,  $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i = 0\}$  is orthogonal to  $u_i$ .

We also want the half-space  $\mathbb{H}_{u_i}$  to be  $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i > 0\}$  for all  $i \in \{1, \dots, d\}$ . We just showed that  $\{x \in \mathbb{R}^d \mid \langle x, u_i \rangle = 0\} = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i = 0\}$ , therefore  $\mathbb{H}_{u_i}$  is either  $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i > 0\}$  or  $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i < 0\}$ . If  $\mathbb{H}_{u_i} = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i < 0\}$ , we replace  $v_i$  with  $-v_i$ . Thus we get  $\mathbb{H}_{u_i} = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i > 0\}$ .

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