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# *Time-Stepping Approximation of Rigid-Body Dynamics with Perfect Unilateral Constraints.*

## *I: The Inelastic Impact Case*

L. PAOLI

We consider a discrete mechanical system with a non-trivial mass matrix, subjected to perfect unilateral constraints described by the geometrical inequalities  $f_\alpha(q) \geq 0$ ,  $\alpha \in \{1, \dots, \nu\}$  ( $\nu \geq 1$ ). We assume that the transmission of the velocities at impact is governed by Newton's Law with a coefficient of restitution  $e = 0$  (so that the impact is inelastic). We propose a time-discretization of the second order differential inclusion describing the dynamics, which generalizes the scheme proposed in Paoli (J Differ Equ 211:247–281, 2005) and, for any admissible data, we prove the convergence of approximate motions to a solution of the initial-value problem.

### **1. Introduction**

We consider a discrete mechanical system subjected to perfect unilateral constraints. More precisely, let us denote by  $u \in \mathbb{R}^d$  the generalized coordinates of a typical configuration of the system. We assume that the set  $K$  of admissible configurations is described by  $\nu \geq 1$  geometrical inequalities

$$f_\alpha(u) \geq 0, \quad \alpha \in \{1, \dots, \nu\}$$

where  $f_\alpha$  is a smooth function (at least  $C^1$ ) such that  $\nabla f_\alpha(u)$  does not vanish in a neighbourhood of  $\{u \in \mathbb{R}^d; f_\alpha(u) = 0\}$ .

At each  $u \in \mathbb{R}^d$  we define the set of active constraints  $J(u)$  by

$$J(u) = \{\alpha \in \{1, \dots, \nu\}; f_\alpha(u) \leq 0\}.$$

In order to avoid some geometrical inconsistencies we assume, moreover, that the active constraints along  $\partial K$  are linearly independent, that is,  $(\nabla f_\alpha(u))_{\alpha \in J(u)}$  are linearly independent for all  $u \in K$ .

Then the dynamics is described by the following measure differential inclusion (see [17] or [6] for instance)

$$M(u)\ddot{u} - g(t, u, \dot{u}) \in -N_K(u) \quad (1)$$

where  $M(u)$  is the mass matrix of the system and  $N_K(u)$  is the normal cone to  $K$  at  $u$  given by

$$N_K(u) = \begin{cases} \{0\} & \text{if } u \in \text{Int}(K), \\ \left\{ \sum_{\alpha \in J(u)} \lambda_\alpha \nabla f_\alpha(u), \lambda_\alpha \leq 0 \forall \alpha \in J(u) \right\} & \text{if } u \in \partial K, \\ \emptyset & \text{if } u \notin K. \end{cases}$$

We also define the tangent cone to  $K$  at  $u$

$$T_K(u) = \left\{ w \in \mathbb{R}^d; (\nabla f_\alpha(u), w) \geq 0 \quad \forall \alpha \in J(u) \right\}$$

where  $(v, w)$  denotes the Euclidean scalar product of vectors  $v$  and  $w$  in  $\mathbb{R}^d$ . Since  $u(s) \in K$  for all  $s$ , we infer that

$$\dot{u}(t+0) \in T_K(u(t)), \quad \dot{u}(t-0) \in -T_K(u(t)) \quad (t > 0)$$

whenever  $\dot{u}(t \pm 0)$  exists. It follows that the velocities are discontinuous at impacts if  $\dot{u}(t-0) \notin T_K(u(t))$  and (1) implies that

$$M(u(t))(\dot{u}(t+0) - \dot{u}(t-0)) \in -N_K(u(t)).$$

This relation does not uniquely determine  $\dot{u}(t+0)$ , so we should add an impact law. Following MOREAU ([6] and [7], see also [14] or [16]) we assume that

$$\dot{u}(t+0) = \text{Proj}_{M(u(t))}(T_K(u(t)), \dot{u}(t-0)) \quad (2)$$

where  $\text{Proj}_{M(u)}$  denotes the projection relative to the Riemannian metric defined by the inertia operator  $M(u)$ .

More precisely, for admissible initial data  $(u_0, v_0) \in K \times T_K(u_0)$ , we consider the following problem:

**Problem (P)** Find  $u : [0, \tau] \rightarrow \mathbb{R}^d$  ( $\tau > 0$ ) such that:

- (P1)  $u$  is an absolutely continuous function from  $[0, \tau]$  to  $K$  and  $\dot{u} \in BV(0, \tau; \mathbb{R}^d)$ ,
- (P2) the differential inclusion

$$M(u)\ddot{u} - g(t, u, \dot{u}) \in -N_K(u)$$

is satisfied in the following sense: there exists a (non-unique) non-negative measure  $\mu$  such that the Stieltjes measure  $d\dot{u} = \ddot{u}$  and the usual Lebesgue measure  $dt$  admit densities with respect to  $d\mu$ , that is, there exist two  $d\mu$ -integrable functions  $v'_\mu$  and  $t'_\mu$  such that  $\ddot{u} = d\dot{u} = v'_\mu d\mu$ ,  $dt = t'_\mu d\mu$ , and such that

$$M(u(t))v'_\mu(t) - g(t, u(t), \dot{u}(t))t'_\mu(t) \in -N_K(u(t))d\mu \text{ almost everywhere,} \quad (3)$$

(P3) for all  $t \in (0, \tau)$

$$\dot{u}(t+0) = \text{Proj}_{M(u(t))} (T_K(u(t)), \dot{u}(t-0)) \quad (4)$$

(P4)  $u(0) = u_0, \dot{u}(0+0) = v_0$ .

Observe that the right-hand side of (3) is a cone, so that the differential inclusion remains true for any non-negative measure  $\mu$  with respect to which  $d\dot{u}$  and  $dt$  admit densities (see [7]).

For this model of impact, a very complete theoretical study has been performed by BALLARD in [1]: using existence results for both ordinary differential equations and variational inequalities, he proved the existence and uniqueness of a maximal solution for the initial value problem when the data are analytical. Some counter-examples show that uniqueness may be lost for less regular data (see [7] or [1] for instance) but existence results have still been established in the single constraint case (that is,  $\nu = 1$ ): see [3,4] and [9,12] for a trivial mass matrix (that is,  $M(u) \equiv \text{Id}_{\mathbb{R}^d}$ ), and [18] and [13,15] for a non-trivial mass matrix. All these results rely on the study of a sequence of approximate solutions constructed either by a penalty method [18] or by a time-stepping scheme [3,4,9,12,13,15].

For the multi-constraint case, these techniques encounter a new difficulty: in general, the motion is not continuous with respect to the data. Nevertheless, some sufficient conditions ensuring continuity on data have been established in [1] and [10]. In this framework, the convergence of the time-stepping scheme proposed in [9] has been extended to the multi-constraint case with inelastic shocks when the mass matrix is trivial and the set  $K$  is convex [11]. The aim of this paper is to relax these restricting conditions for the mass matrix and the set  $K$ , and to prove an analogous convergence result in a more general setting.

More precisely we assume the same kind of regularity for the data as in [10], that is,

- (H1)  $g$  is a continuous function from  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$  ( $T > 0$ ) to  $\mathbb{R}^d$ ;
- (H2) for all  $\alpha \in \{1, \dots, \nu\}$ , the function  $f_\alpha$  belongs to  $C^1(\mathbb{R}^d)$ ,  $\nabla f_\alpha$  is locally Lipschitz continuous and does not vanish in a neighbourhood of  $\{u \in \mathbb{R}^d; f_\alpha(u) = 0\}$ ;
- (H3) the set  $K$  is defined by

$$K = \left\{ u \in \mathbb{R}^d; f_\alpha(u) \geq 0, \alpha \in \{1, \dots, \nu\} \right\}$$

and the active constraints along  $\partial K$  are functionally independent, that is, the vectors  $(\nabla f_\alpha(u))_{J(u)}$  are linearly independent for all  $u \in K$ ;

- (H4)  $M$  is a mapping of class  $C^1$  from  $\mathbb{R}^d$  to the set of symmetric positive definite  $d \times d$  matrices.

With this last assumption, we may define  $M^{-1}(u)$ ,  $M^{1/2}(u)$  and  $M^{-1/2}(u)$  for all  $u \in \mathbb{R}^d$ ; the corresponding mappings are of class  $C^1$  from  $\mathbb{R}^d$  to the set of symmetric positive definite  $d \times d$  matrices.

Let  $F$  be a function such that

- (H5)  $F$  is continuous from  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times [0, h^*]$  ( $h^* > 0$ ) to  $\mathbb{R}^d$  and is consistent with respect to  $g$ , that is,

$$F(t, u, v, 0) = M^{-1}(u)g(t, u, v) \quad \forall (t, u, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

For admissible initial data  $(u_0, v_0) \in K \times T_K(u_0)$ , we consider the initial-value problem (P) and we define a time-stepping scheme as follows:

- the initial positions  $U^0$  and  $U^1$  are given by

$$U^0 = u_0, \quad U^1 \in \operatorname{Argmin}_{Z \in K} \|u_0 + hv_0 + hz(h) - Z\|_{M(u_0)} \quad (5)$$

with  $\lim_{h \rightarrow 0} z(h) = 0$ ,

- for all  $n \in \{1, \dots, \lfloor \frac{T}{h} \rfloor\}$ , let

$$W^n = 2U^n - U^{n-1} + h^2 F^n, \quad F^n = F\left(nh, U^n, \frac{U^n - U^{n-1}}{h}, h\right) \quad (6)$$

and

$$U^{n+1} \in \operatorname{Argmin}_{Z \in K} \|W^n - Z\|_{M(U^n)} \quad (7)$$

where  $\|\cdot\|_{M(U)}$  is the norm associated to the kinetic metric at  $U$  defined by  $\|Z\|_{M(U)}^2 = (Z, Z)_{M(U)}$  with

$$(Z, Z')_{M(U)} = (Z, M(U)Z') = (M(U)Z, Z')$$

for all  $(U, Z, Z') \in (\mathbb{R}^d)^3$ .

In the initialization procedure given at formula (5), the mapping  $h \mapsto z(h)$  can be chosen in such a way that the unconstrained dynamics is approximated at order  $p$ , with  $p \geq 1$ , by  $\tilde{U}^1 = u_0 + hv_0 + hz(h)$  at  $t_1 = h$ . For instance, the simplest choice  $z(h) \equiv 0$  leads to  $p = 1$ , while  $z(h) = \frac{h}{2}M^{-1}(u_0)g(0, u_0, v_0)$  leads to  $p = 2$ . Moreover we can observe that we obtain  $U^{n+1} = W^n$  when  $W^n \in K$  and thus

$$\frac{U^{n+1} - 2U^n + U^{n-1}}{h^2} = F^n \quad \text{when } W^n \in K,$$

which is a centered time-discretization of the unconstrained dynamics. Furthermore, if  $M(u) \equiv \operatorname{Id}_{\mathbb{R}^d}$  for all  $u \in \mathbb{R}^d$  and  $K$  is convex, we recognize the scheme introduced in [9] for the first time and whose convergence has been established in [9] when  $\partial K$  is smooth, and in [11] in the general case.

We now define the approximate solutions  $u_h$  by

$$u_h(t) = U^n + (t - nh) \frac{U^{n+1} - U^n}{h} \quad \forall t \in [nh, (n+1)h] \cap [0, T]$$

for all  $n \in \{0, \dots, \lfloor T/h \rfloor\}$  and  $h \in (0, h^*]$ .

Since the impact law (2) leads to some discontinuity with respect to the data if the active constraints at impacts create an obtuse angle (see [10]), we cannot expect convergence of the approximate motions unless we add some assumptions on the geometry of active constraints along  $\partial K$ .

So, for all  $u \in K$  and  $\alpha \in J(u)$ , let us define

$$e_\alpha(u) = \frac{M^{-1/2}(u)\nabla f_\alpha(u)}{|M^{-1/2}(u)\nabla f_\alpha(u)|}$$

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{R}^d$ , and assume that the ‘‘angle condition’’ given in [10], which ensures continuity on data in the case of inelastic shocks, holds. That is,

(H6) for all  $u \in \partial K$ , for all  $(\alpha, \beta) \in J(u)^2$ , such that  $\alpha \neq \beta$ , we have

$$(e_\alpha(u), e_\beta(u)) \leq 0.$$

This inequality can be interpreted geometrically: in the local momentum metric, defined by the matrix  $M^{-1}(u)$ , the active constraints create right or acute angles.

Then, under assumptions (H1)–(H6) we prove the convergence of a subsequence of the approximate solutions  $(u_h)_{h^* \geq h > 0}$  to a solution of problem (P).

The paper is organized as follows. In the next sections we establish a priori estimates for the discrete velocities and accelerations on a non-trivial time interval  $[0, \tau]$ , with  $0 < \tau \leq T$ . Then we pass to the limit when  $h$  tends to zero on  $[0, \tau]$ : using Ascoli’s and Helly’s theorems we obtain the convergence of a subsequence of  $(u_h)_{h^* \geq h > 0}$  to a limit  $u$  which satisfies (P1) and (P2). Next, for any instant  $t$  such that  $u(t) \in \partial K$ , we perform a precise local study of the approximate motions and we prove that the limit  $u$  also satisfies (P3) and (P4). Finally, we conclude the proof with some energy estimates which allow us to obtain global results.

## 2. A priori estimates for the discrete velocities

Let us begin with a priori estimates for the discrete velocities. Let  $\mathcal{B}$  be a given convex compact subset of  $\mathbb{R}^d$  such that  $\mathcal{B} \cap K \neq \emptyset$ . Possibly decreasing  $h^*$ , we may assume without loss of generality that

$$|z(h)| \leq 1 \quad \forall h \in (0, h^*].$$

In this section we consider a more general scheme for which the initialization procedure involves an initial time  $t_{0h} \in [0, T)$  depending on  $h$ . This modification will allow us, in the last section of the paper, to extend the a priori estimates of the discrete velocities by considering as ‘‘new’’ initial data the already constructed approximate positions at some time steps  $t_{0h}$  and  $t_{0h} + h$ .

So, let  $h \in (0, h^*]$ ,  $t_{0h} \in [0, T)$ ,  $U^0$  and  $U^1$  be given in  $\mathcal{B} \cap K$  and  $K$  respectively, and for all  $n \in \{1, \dots, \lfloor \frac{T-t_{0h}}{h} \rfloor\}$

$$U^{n+1} \in \text{Argmin}_{Z \in K} \|W^n - Z\|_{M(U^n)}$$

with

$$W^n = 2U^n - U^{n-1} + h^2 F^n, \quad F^n = F\left(t_{0h} + nh, U^n, \frac{U^n - U^{n-1}}{h}, h\right).$$

For all  $h \in (0, h^*]$  and  $n \in \{0, \dots, \lfloor \frac{T-t_{0h}}{h} \rfloor\}$ , we define

$$V^n = \frac{U^{n+1} - U^n}{h}.$$

First let us observe that

**Lemma 1.** For all  $h \in (0, h^*]$  and  $n \in \{1, \dots, \lfloor (T - t_{0h})/h \rfloor\}$ , we have

$$M(U^n)(V^{n-1} - V^n + hF^n) \in N_K(U^{n+1}).$$

**Proof.** Let  $h \in (0, h^*]$  and  $n \in \{1, \dots, \lfloor (T - t_{0h})/h \rfloor\}$ . By definition of  $U^{n+1}$  we have

$$\begin{aligned} \|W^n - U^{n+1}\|_{M(U^n)}^2 &\leq \|W^n - Z\|_{M(U^n)}^2 \\ &\leq \|W^n - U^{n+1}\|_{M(U^n)}^2 + 2(W^n - U^{n+1}, U^{n+1} - Z)_{M(U^n)} \\ &\quad + \|U^{n+1} - Z\|_{M(U^n)}^2 \end{aligned}$$

for all  $Z \in K$ , which yields

$$(W^n - U^{n+1}, Z - U^{n+1})_{M(U^n)} \leq \frac{1}{2} \|U^{n+1} - Z\|_{M(U^n)}^2 \quad \forall Z \in K. \quad (8)$$

If  $U^{n+1} \in \text{Int}(K)$ , we deduce from (8) that

$$W^n - U^{n+1} = h(V^{n-1} - V^n + hF^n) = 0$$

and

$$M(U^n)(V^{n-1} - V^n + hF^n) \in N_K(U^{n+1}) = \{0\}.$$

Assume now that  $U^{n+1} \in \partial K$  and let

$$\tilde{T}_K(U^{n+1}) = \left\{ w \in \mathbb{R}^d; \left( \nabla f_\alpha(U^{n+1}), w \right) > 0 \forall \alpha \in J(U^{n+1}) \right\}.$$

For all  $w \in \tilde{T}_K(U^{n+1})$  there exists a smooth curve  $t \mapsto \varphi(t)$  such that  $\varphi(0) = U^{n+1}$ ,  $\varphi'(0) = w$  and  $\varphi(t) \in K$  for all  $t$  in a right neighbourhood of 0. By choosing  $Z = \varphi(t)$  we infer that

$$(W^n - U^{n+1}, w)_{M(U^n)} \leq 0 \quad \forall w \in \tilde{T}_K(U^{n+1}).$$

Then the density of  $\tilde{T}_K(U^{n+1})$  in  $T_K(U^{n+1})$  leads to

$$(V^{n-1} - V^n + hF^n, w)_{M(U^n)} \leq 0 \quad \forall w \in T_K(U^{n+1}).$$

Finally, observing that  $T_K(u) = N_K(u)^\perp$  for all  $u \in K$ , we are able to conclude the proof.  $\square$

Let us introduce some notation. We define

$$\lambda_{\max}(u) = \|M(u)\|, \quad \lambda_{\min}(u) = \frac{1}{\|M^{-1}(u)\|} \quad \forall u \in \mathbb{R}^d.$$

Since  $u \mapsto M(u)$  is continuous with values in the set of symmetric positive definite matrices, the mappings  $u \mapsto \lambda_{\max}(u)$  and  $u \mapsto \lambda_{\min}(u)$  are well defined and continuous from  $\mathbb{R}^d$  to  $\mathbb{R}_+^*$ . Moreover

$$\lambda_{\min}(u)|w|^2 \leq \|w\|_{M(u)}^2 \leq \lambda_{\max}(u)|w|^2 \quad \forall w \in \mathbb{R}^d, \quad \forall u \in \mathbb{R}^d.$$

Since  $\mathcal{B}$  is compact, there exists  $\delta > 0$  such that, for all  $(q, q') \in \mathcal{B} \times \mathbb{R}^d$  such that  $|q - q'| \leq \delta$ , we have:

$$\begin{aligned} |\lambda_{\min}(q) - \lambda_{\min}(q')| &\leq \frac{1}{2} \inf_{u \in \mathcal{B}} \lambda_{\min}(u), \\ |\lambda_{\max}(q) - \lambda_{\max}(q')| &\leq \frac{1}{2} \sup_{u \in \mathcal{B}} \lambda_{\max}(u). \end{aligned}$$

We define

$$B_0 = \left\{ u \in \mathbb{R}^d; \text{dist}(u, \mathcal{B}) \leq \delta \right\}. \quad (9)$$

Then  $B_0$  is also a convex compact subset of  $\mathbb{R}^d$  and we have

$$\frac{1}{2} \inf_{u \in \mathcal{B}} \lambda_{\min}(u) \leq \inf_{u \in B_0} \lambda_{\min}(u), \quad \sup_{u \in B_0} \lambda_{\max}(u) \leq \frac{3}{2} \sup_{u \in \mathcal{B}} \lambda_{\max}(u).$$

We let

$$\lambda_{\min} = \frac{1}{2} \inf_{u \in \mathcal{B}} \lambda_{\min}(u), \quad \lambda_{\max} = \frac{3}{2} \sup_{u \in \mathcal{B}} \lambda_{\max}(u). \quad (10)$$

Of course we have

$$0 < \lambda_{\min}|w|^2 \leq \|w\|_{M(u)}^2 \leq \lambda_{\max}|w|^2 \quad \forall w \in \mathbb{R}^d \setminus \{0\}, \quad \forall u \in B_0.$$

Let  $C_0 > 0$  and  $C_F$  be given by

$$C_F = \sup \left\{ |F(t, u, v, h)|; t \in [0, T], u \in B_0 \cup B_1, |v| \leq C_0, h \in [0, h^*] \right\}, \quad (11)$$

where  $B_1 = \overline{B}(u_0, C_0 T + 1)$ . Since the mappings  $M, M^{-1}, M^{1/2}$  and  $M^{-1/2}$  are of class  $C^1$  on  $\mathbb{R}^d$ , they are Lipschitz continuous on  $B_0 \cup B_1$  and we denote by  $L_M, L_{M^{-1}}, L_{M^{1/2}}$  and  $L_{M^{-1/2}}$  the corresponding Lipschitz constants. Moreover, the functions  $\nabla f_\alpha$ ,  $1 \leq \alpha \leq \nu$ , are locally Lipschitzian and there exists also a positive real number  $L_f$  such that

$$|\nabla f_\alpha(Z) - \nabla f_\alpha(Z')| \leq L_f |Z - Z'| \quad \forall (Z, Z') \in (B_0 \cup B_1)^2, \quad \forall \alpha \in \{1, \dots, \nu\}.$$

Next, we obtain some rough estimates on the discrete velocities. More precisely, let us assume that

$$|V^l| \leq C_0 \quad \forall l \in \{0, \dots, n-1\}$$

for some  $n \geq 1$ . We obtain the following estimate on  $V^n$ :

**Proposition 1.** Let  $C_0 > 0$  and  $h_0^* \in (0, h^*]$  such that

$$h_0^* \leq \min\left(\frac{C_0}{C_F}, \frac{\delta}{8C_0} \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}}\right)$$

where  $C_F$  is defined by (11), and  $\lambda_{\min}, \lambda_{\max}$  are defined by (10). Let  $h \in (0, h_0^*]$ ,  $\tau_h = \min(\delta/(2C_0), T - t_{0h})$  and assume that there exists  $n \in \{1, \dots, \lfloor \tau_h/h \rfloor\}$  such that

$$|V^l| \leq C_0 \quad \forall l \in \{0, \dots, n-1\}.$$

Then

$$|V^n| \leq 4 \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0.$$

Moreover, for all  $l \in \{0, \dots, n\}$  such that  $J(U^{l+1}) \neq \emptyset$ ,

$$\left(\nabla f_\alpha(U^{l+1}), V^l\right) \leq \frac{L_f h}{2} |V^l|^2 \quad \forall \alpha \in J(U^{l+1}).$$

**Proof.** For all  $l \in \{0, \dots, n\}$  we have  $U^l \in B_0$ , since

$$|U^l - U^0| \leq \sum_{k=0}^{l-1} h |V^k| \leq lh C_0 \leq \tau_h C_0 \leq \delta \quad \forall l \in \{0, \dots, n\}.$$

By definition of  $U^{n+1}$  we have

$$\|W^n - U^{n+1}\|_{M(U^n)} \leq \|W^n - U^n\|_{M(U^n)}$$

since  $U^n \in K$  and

$$\begin{aligned} W^n - U^{n+1} &= h(V^{n-1} - V^n + hF^n), \\ W^n - U^n &= h(V^{n-1} + hF^n). \end{aligned}$$

Hence

$$\|V^n\|_{M(U^n)} \leq 2\|V^{n-1}\|_{M(U^n)} + 2h\|F^n\|_{M(U^n)}$$

and

$$|V^n| \leq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} (2C_0 + 2hC_F) \leq 4 \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0.$$

Now, we infer that  $U^{n+1} \in B_0$ . Indeed

$$|U^{n+1} - U^0| \leq |U^n - U^0| + h|V^n| \leq C_0 \tau_h + 4h \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0 \leq C_0 \tau_h + \frac{\delta}{2} \leq \delta.$$

Let  $l \in \{0, \dots, n\}$  such that  $J(U^{l+1}) \neq \emptyset$ . For all  $\alpha \in J(U^{l+1})$ , we have

$$0 \leq f_\alpha(U^l) - f_\alpha(U^{l+1}) = \int_0^1 \left( \nabla f_\alpha \left( U^{l+1} + t(U^l - U^{l+1}) \right), U^l - U^{l+1} \right) dt$$

and thus

$$\begin{aligned} \left( \nabla f_\alpha(U^{l+1}), V^l \right) &\leq - \int_0^1 \left( \nabla f_\alpha \left( U^{l+1} + t(U^l - U^{l+1}) \right) - \nabla f_\alpha \left( U^{l+1} \right), V^l \right) dt \\ &\leq \int_0^1 \left| \nabla f_\alpha \left( U^{l+1} + t(U^l - U^{l+1}) \right) - \nabla f_\alpha \left( U^{l+1} \right) \right| |V^l| dt. \end{aligned}$$

It follows that

$$\left( \nabla f_\alpha(U^{l+1}), V^l \right) \leq \frac{L_f h}{2} |V^l|^2.$$

□

Now we prove a more precise estimate on the discrete velocities. We have the following result:

**Proposition 2.** *Let  $C_0 > 0$  and assume that there exist  $C_0^* > 0$  and  $h_0^* \in (0, h^*]$  such that*

$$\begin{aligned} h_0^* &\leq \min \left( \frac{C_0}{C_F}, \frac{\delta}{8C_0} \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}} \right), \\ |V^0| &= \left| \frac{U^1 - U^0}{h} \right| \leq C_0^* < \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}} C_0 \quad \forall h \in (0, h_0^*], \end{aligned}$$

where  $C_F$  is defined by (11) and  $\lambda_{\min}, \lambda_{\max}$  are defined by (10). Then, there exists  $\tau_0 > 0$ , depending only on  $\mathcal{B}, C_0, C_0^*$  and the data, such that

$$|V^n| = \left| \frac{U^{n+1} - U^n}{h} \right| \leq C_0 \quad \forall nh \in [0, \min(\tau_0, T - t_{0h})], \quad \forall h \in (0, h_0^*].$$

**Proof.** Let us assume that  $h \in (0, h_0^*]$  and  $n \in \{1, \dots, \lfloor \tau_h/h \rfloor\}$  such that

$$|V^l| \leq C_0 \quad \forall l \in \{0, \dots, n-1\}$$

with  $\tau_h = \min(\delta/(2C_0), T - t_{0h})$ . Then, with Proposition 1, we know that  $U^l \in B_0$  for all  $l \in \{0, \dots, n+1\}$  and

$$|V^n| \leq 4 \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0.$$

Moreover, from Lemma 1 we know that

$$M(U^n) \left( V^{n-1} - V^n + hF^n \right) \in N_K(U^{n+1}).$$

It follows that

$$V^n = V^{n-1} + hF^n \quad \text{if } J(U^{n+1}) = \emptyset.$$

Let us assume now that  $J(U^{n+1}) \neq \emptyset$ .

Using once again, Proposition 1, for all  $\alpha \in J(U^{n+1})$  we have also

$$\left( \nabla f_\alpha(U^{n+1}), V^n \right) \leq \frac{L_f h}{2} |V^n|^2$$

and thus, with Lemma 7 (see Appendix),

$$\left( e_\alpha(U^{n+1}), M^{1/2}(U^{n+1})V^n \right) \leq \frac{L_f h}{2} \frac{|V^n|^2}{|M^{-1/2}(U^{n+1})\nabla f_\alpha(U^{n+1})|} \leq \frac{L_f h}{2m_{B_0}} |V^n|^2.$$

For sake of simplicity, denote  $J_n = J(U^{n+1})$  and  $e_\alpha^n = e_\alpha(U^{n+1})$  for all  $\alpha \in J_n$ . From assumption (H3) we know that  $(\nabla f_\alpha(U^{n+1}))_{\alpha \in J_n}$  is linearly independent. Thus  $(e_\alpha^n)_{\alpha \in J_n}$  is also linearly independent and, using Lemma 7, we know that there exist two dual bases  $(v_j(U^{n+1}))_{1 \leq j \leq d}$  and  $(w_j(U^{n+1}))_{1 \leq j \leq d}$  such that

$$\left| v_j(U^{n+1}) \right| = 1 \quad \forall j \in \{1, \dots, d\}, \quad v_j(U^{n+1}) = e_j^n \quad \forall j \in J_n$$

and

$$\left| w_j(U^{n+1}) \right| \leq C_{*, B_0} \quad \forall j \in \{1, \dots, d\}$$

where  $C_{*, B_0}$  depends only on the compact set  $B_0$  and the mappings  $f_\alpha$ ,  $\alpha \in \{1, \dots, v\}$ , and  $M$ .

Now, we define

$$w^n = -V^n + \frac{L_f h}{2m_{B_0}} |V^n|^2 \sum_{\alpha \in J_n} M^{-1/2}(U^{n+1})w_\alpha(U^{n+1}).$$

For all  $\alpha \in J_n$  we have clearly

$$\begin{aligned} & \left( \nabla f_\alpha(U^{n+1}), -w^n \right) \\ &= \left| M^{-1/2}(U^{n+1})\nabla f_\alpha(U^{n+1}) \right| \left( \left( e_\alpha^n, M^{1/2}(U^{n+1})V^n \right) - \frac{L_f h}{2m_{B_0}} |V^n|^2 \right) \leq 0, \end{aligned}$$

that is,  $w^n \in T_K(U^{n+1})$ . With Lemma 1 we get

$$\left( V^{n-1} - V^n + hF^n, w^n \right)_{M(U^n)} \leq 0$$

which yields

$$\left( V^n - V^{n-1} - hF^n, V^n - \frac{L_f h}{2m_{B_0}} |V^n|^2 \sum_{\alpha \in J_n} M^{-1/2}(U^{n+1})w_\alpha(U^{n+1}) \right)_{M(U^n)} \leq 0. \quad (12)$$

It follows that

$$\begin{aligned} \|V^n\|_{M(U^n)}^2 &\leq \frac{L_f h}{2m_{B_0}} |V^n|^2 \sum_{\alpha \in J_n} (M^{-1/2}(U^{n+1})w_\alpha(U^{n+1}), V^{n-1} - V^n + hF^n)_{M(U^n)} \\ &\quad + (V^n, V^{n-1} + hF^n)_{M(U^n)} \leq \|V^n\|_{M(U^n)} \|V^{n-1} + hF^n\|_{M(U^n)} \\ &\quad + \frac{L_f h}{2m_{B_0}} \frac{\lambda_{\max}}{\lambda_{\min}} |V^n| \|V^n\|_{M(U^n)} \sum_{\alpha \in J_n} |w_\alpha(U^{n+1})| |V^{n-1} - V^n + hF^n|. \end{aligned}$$

Using Proposition 1 we get

$$\begin{aligned} &|V^n| \sum_{\alpha \in J_n} |w_\alpha(U^{n+1})| |V^{n-1} - V^n + hF^n| \\ &\leq 4 \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \nu C_0 C_{*,B_0} \left( C_0 \left( 1 + 4 \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \right) + h C_F \right) \end{aligned}$$

if  $J_n \neq \emptyset$ . Recalling that

$$V^n = V^{n-1} + hF^n \quad \text{if } J_n = \emptyset$$

we obtain finally that

$$\begin{aligned} \|V^n\|_{M(U^n)} &\leq \|V^{n-1}\|_{M(U^n)} + h \sqrt{\lambda_{\max}} C_F \\ &\quad + \frac{2L_f h}{m_{B_0}} \left( \frac{\lambda_{\max}}{\lambda_{\min}} \right)^{3/2} \nu C_{*,B_0} C_0^2 \left( 2 + 4 \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \right) \end{aligned}$$

whenever  $J_n = \emptyset$  or  $J_n \neq \emptyset$ .

By using the Lipschitz property of  $M^{1/2}$  on  $B_0$ , we also have

$$\begin{aligned} \|V^{n-1}\|_{M(U^n)} &\leq \|V^{n-1}\|_{M(U^{n-1})} + \|M^{1/2}(U^n) - M^{1/2}(U^{n-1})\| |V^{n-1}| \\ &\leq \|V^{n-1}\|_{M(U^{n-1})} + L_{M^{1/2}h} |V^{n-1}|^2. \end{aligned}$$

We infer that

$$\|V^n\|_{M(U^n)} \leq \|V^{n-1}\|_{M(U^{n-1})} + C_1 h \quad (13)$$

with

$$C_1 = \sqrt{\lambda_{\max}} C_F + L_{M^{1/2}} C_0^2 + \frac{2L_f}{m_{B_0}} \left( \frac{\lambda_{\max}}{\lambda_{\min}} \right)^{3/2} \nu C_{*,B_0} C_0^2 \left( 2 + 4 \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \right).$$

It follows that

$$|V^n| \leq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} |V^0| + \frac{n C_1 h}{\sqrt{\lambda_{\min}}}.$$

Then, choosing  $\tau_0 \in (0, \delta/(2C_0)]$  such that

$$\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0^* + \frac{\tau_0 C_1}{\sqrt{\lambda_{\min}}} \leq C_0$$

and observing that

$$|V^0| \leq C_0^* < \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}} C_0 \leq C_0 \quad \forall h \in (0, h_0^*]$$

we may conclude the proof by induction on  $n$ .  $\square$

Let us now consider the initialization procedure given by formula (5), that is, let  $t_{0h} = 0$  and  $U^0$  and  $U^1$  be given by

$$U^0 = u_0, \quad U^1 \in \text{Argmin}_{Z \in K} \|u_0 + hv_0 + hz(h) - Z\|_{M(u_0)}, \quad \lim_{h \rightarrow 0} z(h) = 0,$$

for all  $h \in (0, h^*]$ . We can choose  $\mathcal{B} = \bar{B}(u_0, C + 1)$  with  $C \geq 0$ , and the previous results lead to an uniform estimate of the discrete velocities on a non-trivial time interval. More precisely, we obtain

**Theorem 1.** *For all  $C_0^* \geq 2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}( |v_0| + 1)$  and for all  $C_0 > \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0^*$ , there exist  $h_0^* \in (0, h^*]$  and  $\tau_0 > 0$ , depending only on  $\mathcal{B}$ ,  $C_0$ ,  $C_0^*$  and the data, such that*

$$|V^n| = \left| \frac{U^{n+1} - U^n}{h} \right| \leq C_0 \quad \forall nh \in [0, \min(\tau_0, T)], \quad \forall h \in (0, h_0^*].$$

**Proof.** Let  $C_0^* \geq 2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}( |v_0| + 1)$ ,  $C_0 > \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0^*$  and  $h_0^* \in (0, h^*]$  such that

$$h_0^* \leq \min \left( \frac{C_0}{C_F}, \frac{\delta}{8C_0} \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}} \right)$$

where  $C_F$  is defined by (11) and  $\lambda_{\min}$ ,  $\lambda_{\max}$  are defined by (10).

By definition of  $U^1$ , we have

$$\left\| u_0 + hv_0 + hz(h) - U^1 \right\|_{M(u_0)} \leq \|u_0 + hv_0 + hz(h) - Z\|_{M(u_0)} \quad \forall Z \in K$$

and by choosing  $Z = u_0 = U^0$  we get

$$\|V^0\|_{M(U^0)} = \left\| \frac{U^1 - U^0}{h} \right\|_{M(u_0)} \leq 2 \|v_0 + z(h)\|_{M(u_0)}.$$

Thus,

$$|V^0| \leq 2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} |v_0 + z(h)| \leq 2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} (|v_0| + 1) \quad \forall h \in (0, h_0^*]$$

and

$$|V^0| \leq C_0^* < \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}} C_0 \quad \forall h \in (0, h_0^*].$$

It follows that we may apply Proposition 2, which yields the announced result.  $\square$

### 3. Convergence of the approximate solutions $(u_h)_{h^* \geq h > 0}$

Before passing to the limit as  $h$  tends to zero in the sequence  $(u_h)_{h^* \geq h > 0}$ , we prove an estimate for the discrete accelerations.

**Proposition 3.** *Let us now assume that there exist  $C_0 > 0$ ,  $\tau_0 > 0$ ,  $h_0^* \in (0, h^*]$  and a sequence  $(h_i)_{i \in \mathbb{N}}$ , decreasing to zero, such that*

$$|V^n| \leq C_0 \quad \forall nh_i \in [0, \min(\tau_0, T)], \quad \forall h_i \in (0, h_0^*]. \quad (14)$$

*Then there exist  $h_1^* \in (0, h_0^*]$  and  $C'_0 > 0$  such that, for all  $h_i \in (0, h_1^*]$*

$$\sum_{n=1}^N |V^n - V^{n-1}| \leq C'_0 \quad \text{with} \quad N = \left\lfloor \frac{\min(\tau_0, T)}{h_i} \right\rfloor.$$

**Proof.** The main ideas of the proof are the same as in proposition 2.4 in [11]. More precisely, let  $\mathcal{B} = \overline{B}(u_0, C + 1)$  with  $C \geq 0$ ,  $B_0$  be defined by (9) and  $C_F$  be defined by (11). Without loss of generality, possibly decreasing  $h_0^*$ , we assume that  $C_0 h_0^* \leq 1$  and  $C_F h_0^* \leq C_0$ . We denote  $K_1 = K \cap B_1 = K \cap \overline{B}(u_0, C_0 T + 1)$  and

$$\begin{aligned} \lambda_{\min, B_1} &= \inf_{u \in B_1} \lambda_{\min}(u) = \frac{1}{\sup_{u \in B_1} \|M^{-1}(u)\|}, \\ \lambda_{\max, B_1} &= \sup_{u \in B_1} \lambda_{\max}(u) = \sup_{u \in B_1} \|M(u)\|. \end{aligned} \quad (15)$$

Let  $h_i \in (0, h_0^*]$ . By definition of the scheme, we have  $U^n \in K$  for all  $n \in \{0, \dots, \lfloor T/h_i \rfloor + 1\}$ . Assumption (14) implies that

$$|U^n - U^0| \leq h_i \sum_{k=0}^{n-1} |V^k| \leq nh_i C_0 \leq C_0 T + 1 \quad \forall n \in \{0, \dots, N + 1\}$$

thus  $U^n \in K_1$  for all  $n \in \{0, \dots, N + 1\}$ .

By Lemma 8 (see Appendix), we infer that, for all  $q \in K_1$ , there exist  $a_q \in \mathbb{R}^d$  and two strictly positive numbers  $\delta_q$  and  $r_q$  such that, for all  $q' \in \overline{B}(q, 2\delta_q)$

$$\overline{B}(a_q, r_q) \subset T_K(q'). \quad (16)$$

It is obvious that  $K_1 \subset \bigcup_{q \in K_1} B(q, \delta_q)$ , and a compactness argument implies that there exists  $(q_j)_{1 \leq j \leq \ell}$  such that  $q_j \in K_1$  for all  $j \in \{1, \dots, \ell\}$  and

$$K_1 \subset \bigcup_{j=1}^{\ell} B(q_j, \delta_{q_j}).$$

In the remainder of the proof we will simply write  $\delta_j$ ,  $a_j$  and  $r_j$  instead of  $\delta_{q_j}$ ,  $a_{q_j}$  and  $r_{q_j}$ . We define

$$r = \min_{1 \leq j \leq \ell} r_j, \quad \delta' = \min_{1 \leq j \leq \ell} \delta_j, \quad \tau_1 = \frac{\delta'}{C_0}.$$

Let  $h_1^* \in (0, \min(h_0^*, \tau_1/2))$ ,  $h_i \in (0, h_1^*]$  and  $n \in \{0, \dots, N\}$ . Let  $j \in \{1, \dots, \ell\}$  be such that  $U^{n+1} \in B(q_j, \delta_j)$ . Then, for all  $m \in \{n+1, \dots, p\}$  with  $p = \min(N, n + \lfloor \tau_1/h_i \rfloor)$ , we have

$$\begin{aligned} |U^{m+1} - q_j| &\leq |U^{m+1} - U^{n+1}| + |U^{n+1} - q_j| \\ &\leq \left| \sum_{k=n+1}^m h_i V^k \right| + \delta_j \leq h_i C_0(m-n) + \delta_j \leq \delta' + \delta_j \leq 2\delta_j. \end{aligned}$$

By applying (16), we obtain that, for all  $m \in \{n+1, \dots, p\}$ , we have  $\overline{B}(a_j, r_j) \subset T_K(U^{m+1})$ . Thus

$$\overline{B}_{M(U^m)}(a_j, \sqrt{\lambda_{\min, B_1} r_j}) = \left\{ z \in \mathbb{R}^d; \|z - a_j\|_{M(U^m)} \leq \sqrt{\lambda_{\min, B_1} r_j} \right\} \subset \overline{B}(a_j, r_j).$$

Then, we use a classical result about contractions on Hilbert spaces due to MOREAU [5] and we infer that, for all  $z \in \mathbb{R}^d$

$$\begin{aligned} &\left\| z - \text{Proj}_{M(U^m)}(T_K(U^{m+1}), z) \right\|_{M(U^m)} \\ &\leq \frac{1}{2\sqrt{\lambda_{\min, B_1} r_j}} \left( \|z - a_j\|_{M(U^m)}^2 - \left\| \text{Proj}_{M(U^m)}(T_K(U^{m+1}), z) - a_j \right\|_{M(U^m)}^2 \right). \end{aligned}$$

With Lemma 1 we know that

$$M(U^m)(V^{m-1} - V^m + h_i F^m) \in N_K(U^{m+1}).$$

Since  $N_K(U^{m+1})$  and  $T_K(U^{m+1})$  are two closed convex polar cones, we get

$$\text{Proj}_{M(U^m)}(T_K(U^{m+1}), V^{m-1} - V^m + h_i F^m) = 0.$$

Hence

$$\begin{aligned} &\|V^{m-1} - V^m + h_i F^m\|_{M(U^m)} \\ &\leq \frac{1}{2\sqrt{\lambda_{\min, B_1} r_j}} \left( \|(V^{m-1} - V^m + h_i F^m) - a_j\|_{M(U^m)}^2 - \|a_j\|_{M(U^m)}^2 \right) \\ &\leq \frac{1}{2\sqrt{\lambda_{\min, B_1} r_j}} \left( \|V^{m-1} - V^m + h_i F^m\|_{M(U^m)}^2 - 2 \left( a_j, V^{m-1} - V^m + h_i F^m \right)_{M(U^m)} \right). \end{aligned}$$

It follows that

$$\begin{aligned} \|V^{m-1} - V^m\|_{M(U^m)} &\leq h_i \|F^m\|_{M(U^m)} + \frac{1}{2\sqrt{\lambda_{\min, B_1} r_j}} \left( \|V^{m-1} - V^m\|_{M(U^m)}^2 \right. \\ &\quad \left. + 2h_i \left( F^m, V^{m-1} - V^m \right)_{M(U^m)} + h_i^2 \|F^m\|_{M(U^m)}^2 \right. \\ &\quad \left. - 2 \left( a_j, V^{m-1} - V^m + h_i F^m \right)_{M(U^m)} \right). \end{aligned} \quad (17)$$

If  $J_m \neq \emptyset$ , we can reproduce the same computations as in Propositions 1 and 2 to obtain (see (12))

$$\left( V^m - V^{m-1} - h_i F^m, V^m - \frac{L_f h_i}{2m B_1} |V^m|^2 \sum_{\alpha \in J_m} M^{-1/2}(U^{m+1}) w_\alpha(U^{m+1}) \right)_{M(U^m)} \leq 0$$

which yields

$$\begin{aligned} - \left( V^{m-1}, V^m \right)_{M(U^m)} &\leq - \|V^m\|_{M(U^m)}^2 + h_i (F^m, V^m)_{M(U^m)} \\ &\quad + \frac{L_f h_i \lambda_{\max, B_1}}{2m B_1 \sqrt{\lambda_{\min, B_1}}} \nu C_{*, B_1} |V^m|^2 |V^{m-1} - V^m + h_i F^m| \end{aligned} \quad (18)$$

where  $C_{*, B_1}$  is the constant defined at Lemma 7.

Otherwise, if  $J_m = \emptyset$ , we have  $V^m = V^{m-1} + h_i F^m$ , hence

$$\begin{aligned} - \left( V^{m-1}, V^m \right)_{M(U^m)} &= (-V^m + h_i F^m, V^m)_{M(U^m)} \\ &= - \|V^m\|_{M(U^m)}^2 + h_i (F^m, V^m)_{M(U^m)} \end{aligned}$$

and (18) is still true.

Thus

$$\begin{aligned} \|V^{m-1} - V^m\|_{M(U^m)}^2 &\leq \|V^{m-1}\|_{M(U^m)}^2 - \|V^m\|_{M(U^m)}^2 + 2h_i (F^m, V^m)_{M(U^m)} \\ &\quad + \frac{L_f h_i \lambda_{\max, B_1}}{m B_1 \sqrt{\lambda_{\min, B_1}}} \nu C_{*, B_1} C_0^2 (2C_0 + h_i C_F). \end{aligned}$$

Going back to (17) and using the Lipschitz property of  $M$  on  $B_1$ , we obtain

$$\begin{aligned} \|V^{m-1} - V^m\|_{M(U^m)} &\leq h_i C'_1 + \frac{1}{2\sqrt{\lambda_{\min, B_1} r}} \left( \|V^{m-1}\|_{M(U^{m-1})}^2 - \|V^m\|_{M(U^m)}^2 \right. \\ &\quad \left. - 2 \left( a_j, V^{m-1} \right)_{M(U^{m-1})} + 2 \left( a_j, V^m \right)_{M(U^m)} \right) \end{aligned}$$

for all  $m \in \{n+1, \dots, p\}$ , where

$$\begin{aligned} C'_1 &= \sqrt{\lambda_{\max, B_1}} C_F \left( 1 + \frac{\sqrt{\lambda_{\max, B_1}} (C_0 + a)}{\sqrt{\lambda_{\min, B_1} r}} \right) + C_F^2 \frac{h_0^* \lambda_{\max, B_1}}{2\sqrt{\lambda_{\min, B_1} r}} \\ &\quad + \frac{C_0 + 2a}{2\sqrt{\lambda_{\min, B_1} r}} C_0^2 L_M + 3 \frac{L_f \lambda_{\max, B_1}}{2r m B_1 \lambda_{\min, B_1}} \nu C_{*, B_1} C_0^3 \end{aligned}$$

and  $a = \max_{1 \leq j \leq \ell} |a_j|$ . By summation we get

$$\begin{aligned} \sum_{m=n+1}^p \|V^{m-1} - V^m\|_{M(U^m)} &\leq (p-n) h_i C'_1 \\ &\quad + \frac{1}{2\sqrt{\lambda_{\min, B_1} r}} \left( \|V^n\|_{M(U^n)}^2 - \|V^p\|_{M(U^p)}^2 \right. \\ &\quad \left. + 4\lambda_{\max, B_1} C_0 a \right). \end{aligned}$$

Recalling that  $p = \min(N, n + \lfloor \tau_1/h_i \rfloor)$ , we infer that

$$\begin{aligned} \sum_{m=1}^N \|V^{m-1} - V^m\|_{M(U^m)} &\leq N h_i C'_1 + \frac{1}{2\sqrt{\lambda_{\min, B_1} r}} \left( \|V^0\|_{M(U^0)}^2 - \|V^N\|_{M(U^N)}^2 \right) \\ &\quad + (k_1 + 1) \frac{2\lambda_{\max, B_1} C_0 a}{\sqrt{\lambda_{\min, B_1} r}} \end{aligned}$$

where  $k_1 \in \mathbb{N}$  is such that

$$1 + k_1 \left\lfloor \frac{\tau_1}{h_i} \right\rfloor \leq N < (k_1 + 1) \left\lfloor \frac{\tau_1}{h_i} \right\rfloor.$$

Observing that  $k_1 \leq \min(\tau_0, T)/(\tau_1 - h_i)$  for all  $h_i \in (0, h_1^*]$ , and

$$|V^{m-1} - V^m| \leq \frac{1}{\sqrt{\lambda_{\min, B_1}}} \|V^{m-1} - V^m\|_{M(U^m)} \quad \forall m \in \{1, \dots, N\},$$

we can conclude the proof with

$$C'_0 = \frac{1}{\sqrt{\lambda_{\min, B_1}}} \left( T C'_1 + \frac{\lambda_{\max, B_1} C_0^2}{\sqrt{\lambda_{\min, B_1} r}} + \frac{2\lambda_{\max, B_1} C_0 a}{\sqrt{\lambda_{\min, B_1} r}} \left( \frac{T}{\tau_1 - h_1^*} + 1 \right) \right).$$

□

With these results we can now pass to the limit as  $h$  tends to zero. Let us recall the definition of the approximate solutions  $(u_h)_{h^* \geq h > 0}$ :

$$u_h(t) = U^n + (t - nh) \frac{U^{n+1} - U^n}{h} \quad \forall t \in [nh, (n+1)h] \cap [0, T] \quad (19)$$

and let us define

$$v_h(t) = V^n = \frac{U^{n+1} - U^n}{h} \quad \forall t \in [nh, (n+1)h] \cap [0, T] \quad (20)$$

for all  $n \in \{0, \dots, \lfloor T/h \rfloor\}$  and  $h \in (0, h^*]$ .

Let us assume from now on that

(H7) there exist  $C_0 > 0$ ,  $\tau_0 > 0$ ,  $h_0^* \in (0, h^*]$  and a subsequence  $(h_i)_{i \in \mathbb{N}}$ , decreasing to zero, such that

$$|V^n| \leq C_0 \quad \forall nh_i \in [0, \min(\tau_0, T)] \quad \forall h_i \in (0, h_0^*].$$

We define  $\mathcal{B} = \overline{B}(u_0, C + 1)$  with  $C \geq 0$ . Let  $B_0$  and  $C_F$  be defined by (9) and (11), respectively. We assume (without loss of generality) that  $C_0 h_0^* \leq 1$  and  $C_F h_0^* \leq C_0$ . Let us denote  $\tau = \min(\tau_0, T)$ . From assumption (H7) and Proposition 3 we know that  $(u_{h_i})_{h_1^* \geq h_i > 0}$  is uniformly  $C_0$ -Lipschitz continuous on  $[0, \tau]$  and  $(v_{h_i})_{h_1^* \geq h_i > 0}$  is uniformly bounded in  $L^\infty(0, \tau; \mathbb{R}^d) \cap BV(0, \tau; \mathbb{R}^d)$ . It follows that  $(u_{h_i})_{h_1^* \geq h_i > 0}$  is equicontinuous and, using Ascoli's and Helly's theorems, we

infer that there exists a subsequence, still denoted  $(h_i)_{i \in \mathbb{N}}$ ,  $u \in C^0([0, \tau]; \mathbb{R}^d)$  and  $v \in BV(0, \tau; \mathbb{R}^d)$ , such that

$$u_{h_i} \rightarrow u \quad \text{strongly in } C^0([0, \tau]; \mathbb{R}^d), \quad (21)$$

and

$$v_{h_i} \rightarrow v \quad \text{pointwise in } [0, \tau]. \quad (22)$$

Moreover, we have

$$u_{h_i}(t) = u_0 + \int_0^t v_{h_i}(s) \, ds \quad \forall t \in [0, T], \quad \forall h_i \in (0, h^*].$$

Thus, with Lebesgue's theorem, we get

$$u(t) = \lim_{h_i \rightarrow 0} \left( u_0 + \int_0^t v_{h_i}(s) \, ds \right) = u_0 + \int_0^t v(s) \, ds \quad \forall t \in [0, \tau]. \quad (23)$$

We infer that  $u$  is  $C_0$ -Lipschitz continuous and

$$u_{h_i}(t), u(t) \in \overline{B}(u_0, C_0\tau) \subset B_1 = \overline{B}(u_0, C_0T + 1) \quad \forall t \in [0, \tau], \quad \forall h_i \in (0, h_1^*].$$

Moreover,  $u$  is absolutely continuous on  $[0, \tau]$ , thus  $u$  admits a derivative (in the classical sense) almost everywhere on  $[0, \tau]$  and  $\dot{u} \in L^1(0, \tau; \mathbb{R}^d)$ . From (23) we infer that  $\dot{u}(t) = v(t)$  for all  $t \in [0, \tau]$  such that  $v$  is continuous at  $t$ . Possibly modifying  $\dot{u}$  on a countable subset of  $[0, \tau]$ , we may assume without loss of generality that  $\dot{u} = v$ .

As usual, we adopt the convention

$$\begin{aligned} \dot{u}(0-0) &= v(0-0) = v(0) = \dot{u}(0), \\ \dot{u}(\tau+0) &= v(\tau+0) = v(\tau) = \dot{u}(\tau). \end{aligned} \quad (24)$$

Then we observe that

**Lemma 2.** For all  $t \in [0, \tau]$ ,  $u(t) \in K$ .

**Proof.** Let  $t \in [0, \tau]$ . For all  $h_i \in (0, h_1^*]$  there exists  $n \in \{0, \dots, \lfloor \frac{\tau}{h_i} \rfloor\}$  such that  $t \in [nh_i, (n+1)h_i)$ . Then, observing that  $U^n \in K$  we get

$$\begin{aligned} \text{dist}(u(t), K) &\leq |u(t) - U^n| \leq |u(t) - u_{h_i}(t)| + |u_{h_i}(t) - U^n| \\ &\leq |u(t) - u_{h_i}(t)| + (t - nh_i)|V^n| \leq \|u - u_{h_i}\|_{C^0([0, \tau]; \mathbb{R}^d)} + C_0 h_i. \end{aligned}$$

By passing to the limit as  $h_i$  tends to zero, we obtain  $\text{dist}(u(t), K) \leq 0$ , that is,  $u(t) \in K$ .  $\square$

### 3.1. Study of property (P2)

Now let us prove that  $u$  satisfies property (P2), that is, the differential inclusion (3). First, we observe that there exists at least one non-negative measure  $\mu$  such that the Stieltjes measure  $\ddot{u} = d\dot{u} = dv$  and the usual Lebesgue measure  $dt$  admit densities with respect to  $\mu$ . Indeed, let  $\mu$  be defined by  $d\mu = |d\dot{u}| + dt$ :  $\mu$  is non-negative and the measures  $\ddot{u} = d\dot{u}$  and  $dt$  are both absolutely continuous with respect to  $\mu$ .

Now, let  $\mu = |d\dot{u}| + dt$ . We denote by  $v'_\mu$  and  $t'_\mu$  the densities of  $d\dot{u} = dv$  and  $dt$  with respect to  $d\mu$ . We have to prove that

$$M(u(t))v'_\mu(t) - g(t, u(t), \dot{u}(t))t'_\mu(t) \in -N_K(u(t)) \quad d\mu \text{ almost everywhere.}$$

By Jeffery's theorem (see [2] or [4]) we know that there exists a  $d\mu$ -negligible set  $N \subset [0, \tau]$  such that, for all  $t \in [0, \tau] \setminus N$ :

$$v'_\mu(t) = \lim_{\varepsilon \rightarrow 0^+} \frac{d\dot{u}(I_\varepsilon)}{d\mu(I_\varepsilon)}, \quad t'_\mu(t) = \lim_{\varepsilon \rightarrow 0^+} \frac{dt(I_\varepsilon)}{d\mu(I_\varepsilon)}$$

with  $I_\varepsilon = [t, t + \varepsilon] \cap [0, \tau]$ .

We define

$$N' = \{t \in [0, \tau]; \dot{u}(t+0) = \dot{u}(t-0) \neq \dot{u}(t)\}$$

(we may observe that the convention (24) implies that  $0 \notin N'$  and  $\tau \notin N'$ ). Since  $\dot{u} = v$  belongs to  $BV(0, \tau; \mathbb{R}^d)$ ,  $N'$  is, at most, a countable subset of  $[0, \tau]$  and is negligible with respect to  $|d\dot{u}|$ .

Finally, let  $N_0 = \{t \in \{0\} \cup \{\tau\}; \dot{u} \text{ is continuous at } t\}$ . The set  $N_0$  is finite (it contains at most the two points  $t = 0$  and  $t = \tau$ ), so it is negligible with respect to  $|d\dot{u}|$ , and it follows that  $N \cup N' \cup N_0$  is also negligible with respect to  $d\mu$ . We have:

**Proposition 4.** *Let  $t \in [0, \tau] \setminus (N \cup N' \cup N_0)$  such that  $\dot{u}$  is continuous at  $t$ . Then*

$$M(u(t))v'_\mu(t) - g(t, u(t), \dot{u}(t))t'_\mu(t) \in -N_K(u(t)). \quad (25)$$

**Proof.** Let  $t \in [0, \tau] \setminus (N \cup N' \cup N_0)$  such that  $\dot{u}$  is continuous at  $t$ . Then  $t \in (0, \tau)$ ; for simplicity we will denote  $\bar{u} = u(t)$  in the remainder of the proof. By definition of  $N_K(\bar{u})$ , (25) is equivalent to

$$(g(t, \bar{u}, \dot{u}(t))t'_\mu(t) - M(\bar{u})v'_\mu(t), w) \leq 0$$

for all  $w \in T_K(\bar{u}) = \{w \in \mathbb{R}^d; (\nabla f_\alpha(\bar{u}), w) \geq 0, \forall \alpha \in J(\bar{u})\}$ .

First, let us observe that there exists  $r_{\bar{u}} > 0$  such that

$$J(q) \subset J(\bar{u}) \quad \forall q \in \overline{B}(\bar{u}, r_{\bar{u}}).$$

Indeed, for all  $\alpha \in \{1, \dots, v\} \setminus J(\bar{u})$  we have  $f_\alpha(\bar{u}) > 0$  and, by continuity of the mappings  $f_\alpha$  ( $1 \leq \alpha \leq v$ ), there exists  $r_{\bar{u}} > 0$  such that

$$f_\alpha(q) \geq \frac{f_\alpha(\bar{u})}{2} \quad \forall q \in \overline{B}(\bar{u}, r_{\bar{u}}), \quad \forall \alpha \in \{1, \dots, v\} \setminus J(\bar{u}).$$

Let us consider  $\tilde{T}_K(\bar{u})$  defined by

$$\tilde{T}_K(\bar{u}) = \begin{cases} \{w \in \mathbb{R}^d; (\nabla f_\alpha(\bar{u}), w) > 0 \forall \alpha \in J(\bar{u})\} & \text{if } J(\bar{u}) \neq \emptyset, \\ \mathbb{R}^d & \text{otherwise.} \end{cases}$$

Let  $w \in \tilde{T}_K(\bar{u})$ . If  $J(\bar{u}) \neq \emptyset$ , the continuity of the mappings  $\nabla f_\alpha$  ( $1 \leq \alpha \leq \nu$ ) implies that there exists  $r_w \in (0, r_{\bar{u}}]$  such that

$$(\nabla f_\alpha(q), w) \geq 0 \quad \forall \alpha \in J(\bar{u}), \quad \forall q \in \bar{B}(\bar{u}, r_w),$$

and thus  $w \in T_K(q)$  for all  $q \in \bar{B}(\bar{u}, r_w)$ . If  $J(\bar{u}) = \emptyset$ , we still have  $w \in T_K(q)$  for all  $q \in \bar{B}(\bar{u}, r_w)$  if we choose  $r_w = r_{\bar{u}}$ .

Using the continuity of  $u$  and the uniform convergence of  $(u_{h_i})_{i \in \mathbb{N}}$  to  $u$  on  $[0, \tau]$ , there exists  $\tilde{\varepsilon}_w \in (0, \min(t, \frac{\tau - t}{2}))$  such that, for all  $\varepsilon \in (0, \tilde{\varepsilon}_w]$ , there exists  $h_\varepsilon \in (0, h_1^*]$  such that

$$\begin{aligned} u(s) &\in \bar{B}\left(\bar{u}, \frac{r_w}{3}\right) \quad \forall s \in [t, t + \varepsilon], \\ h_\varepsilon &\leq \min\left(\frac{r_w}{3C_0}, \frac{\varepsilon}{3}\right), \quad \|u - u_{h_i}\|_{C^0([0, \tau]; \mathbb{R}^d)} \leq \frac{r_w}{3} \quad \forall h_i \in (0, h_\varepsilon]. \end{aligned}$$

It follows that for all  $\varepsilon \in (0, \tilde{\varepsilon}_w]$  and for all  $h_i \in (0, h_\varepsilon]$

$$u_{h_i}(s) \in \bar{B}\left(\bar{u}, \frac{2r_w}{3}\right) \quad \forall s \in [t, t + \varepsilon],$$

and

$$U^{n+1} \in \bar{B}(\bar{u}, r_w) \quad \forall nh_i \in [t, t + \varepsilon].$$

Now let  $\varepsilon \in (0, \tilde{\varepsilon}_w]$  and  $h_i \in (0, h_\varepsilon]$ . We define  $j$  and  $k$  by

$$j = \left\lfloor \frac{t}{h_i} \right\rfloor, \quad k = \left\lfloor \frac{t + \varepsilon}{h_i} \right\rfloor.$$

We have

$$0 < t_j = jh_i \leq t < t_{j+1} < \dots < t_k = kh_i \leq t + \varepsilon < t_{k+1} < \tau.$$

From Lemma 1 we know that, for all  $n \in \{j + 1, \dots, k\}$ , we have

$$\left( V^{n-1} - V^n + h_i F^n, w \right)_{M(U^n)} \leq 0$$

since  $w \in T_K(U^{n+1})$ , and by summation

$$\begin{aligned} &\sum_{n=j+1}^k h_i (M(U^n)F^n, w) + \sum_{n=j+1}^k \left( M(U^{n-1})V^{n-1} - M(U^n)V^n, w \right) \\ &+ \sum_{n=j+1}^k \left( M(U^n)V^{n-1} - M(U^{n-1})V^{n-1}, w \right) \leq 0. \end{aligned} \tag{26}$$

The last term can be easily estimated as  $\mathcal{O}(\varepsilon + h_i)$ . Indeed, the Lipschitz property of the mapping  $M$  on  $B_1$  implies that

$$\left\| M(U^n) - M(U^{n-1}) \right\| \leq L_M h_i |V^{n-1}| \leq h_i L_M C_0.$$

It follows that

$$\begin{aligned} & \left| \sum_{n=j+1}^k \left( M(U^n) V^{n-1} - M(U^{n-1}) V^{n-1}, w \right) \right| \\ & \leq \sum_{n=j+1}^k \left\| M(U^n) - M(U^{n-1}) \right\| C_0 |w| \\ & \leq (k-j) L_M h_i C_0^2 |w| \leq (\varepsilon + h_i) L_M C_0^2 |w|. \end{aligned}$$

The second term of the left-hand side of (26) is a telescopic sum which can be rewritten as

$$\begin{aligned} \left( M(U^j) V^j - M(U^k) V^k, w \right) &= \left( M(u_{h_i}(t)) (v_{h_i}(t) - v_{h_i}(t + \varepsilon)), w \right) \\ & \quad + \left( \left( M(U^j) - M(u_{h_i}(t)) \right) V^j, w \right) \\ & \quad + \left( \left( M(u_{h_i}(t)) - M(U^k) \right) V^k, w \right). \end{aligned}$$

Once again, the last two terms can be estimated by using the Lipschitz property of  $M$  on  $B_1$ :

$$\begin{aligned} & \left| \left( \left( M(U^j) - M(u_{h_i}(t)) \right) V^j + \left( M(u_{h_i}(t)) - M(U^k) \right) V^k, w \right) \right| \\ & \leq L_M C_0 |w| (|u_{h_i}(jh_i) - u_{h_i}(t)| + |u_{h_i}(t) - u_{h_i}(kh_i)|) \leq L_M C_0^2 |w| (h_i + \varepsilon). \end{aligned}$$

Moreover, with (21) and (22), we have

$$\begin{aligned} \lim_{h_i \rightarrow 0} M(u_{h_i}(t)) (v_{h_i}(t) - v_{h_i}(t + \varepsilon)) &= M(u(t)) (v(t) - v(t + \varepsilon)) \\ &= M(u(t)) (\dot{u}(t) - \dot{u}(t + \varepsilon)). \end{aligned}$$

Let us prove now that

$$\lim_{h_i \rightarrow 0} \sum_{n=j+1}^k h_i (M(U^n) F^n, w) = \int_t^{t+\varepsilon} (g(s, u(s), \dot{u}(s)), w) ds$$

for all  $\varepsilon \in (0, \tilde{\varepsilon}_w]$ .

Indeed, let  $\varepsilon \in (0, \tilde{\varepsilon}_w]$ . For all  $h_i \in (0, h_\varepsilon]$  and  $n \in \{j+1, \dots, k\}$ , we have  $F^n = F(nh_i, U^n, V^{n-1}, h_i)$  and with (H5)

$$F(nh_i, U^n, V^{n-1}, 0) = M^{-1}(U^n) g(nh_i, u_{h_i}(nh_i), v_{h_i}(s)) \quad \forall s \in [(n-1)h_i, nh_i).$$

It follows that

$$\begin{aligned} & \sum_{n=j+1}^k h_i (M(U^n)F^n, w) - \sum_{n=j+1}^k \int_{(n-1)h_i}^{nh_i} (g(nh_i, u_{h_i}(nh_i), v_{h_i}(s)), w) \, ds \\ &= \sum_{n=j+1}^k h_i \left( F(nh_i, U^n, V^{n-1}, h_i) - F(nh_i, U^n, V^{n-1}, 0), w \right)_{M(U^n)} \end{aligned} \quad (27)$$

In order to estimate the right-hand side of (27), we denote by  $\omega_F$  the modulus of continuity of  $F$  on the compact set  $[0, T] \times B_1 \times \overline{B}(0, C_0) \times [0, h^*]$  and we get

$$\begin{aligned} & \left| \sum_{n=j+1}^k h_i \left( F(nh_i, U^n, V^{n-1}, h_i) - F(nh_i, U^n, V^{n-1}, 0), w \right)_{M(U^n)} \right| \\ & \leq \sum_{n=j+1}^k h_i \|M(U^n)\| \omega_F(h_i) |w| \leq (\varepsilon + h_i) \lambda_{\max, B_1} \omega_F(h_i) |w|. \end{aligned} \quad (28)$$

Furthermore

$$\begin{aligned} & \sum_{n=j+1}^k \int_{(n-1)h_i}^{nh_i} (g(nh_i, u_{h_i}(nh_i), v_{h_i}(s)), w) \, ds \\ & - \int_t^{t+\varepsilon} (g(s, u_{h_i}(s), v_{h_i}(s)), w) \, ds \\ &= \sum_{n=j+1}^k \int_{(n-1)h_i}^{nh_i} (g(nh_i, u_{h_i}(nh_i), v_{h_i}(s)) - g(s, u_{h_i}(s), v_{h_i}(s)), w) \, ds \\ & - \int_{kh_i}^{t+\varepsilon} (g(s, u_{h_i}(s), v_{h_i}(s)), w) \, ds + \int_{jh_i}^t (g(s, u_{h_i}(s), v_{h_i}(s)), w) \, ds. \end{aligned} \quad (29)$$

Recalling that  $(u_{h_i}(s), v_{h_i}(s)) \in B_1 \times \overline{B}(0, C_0)$  for all  $s \in [0, \tau]$  and for all  $h_i \in (0, h_1^*]$ , we obtain the following estimates for the second and third terms of the right-hand side of (29):

$$\begin{aligned} & \left| \int_{kh_i}^{t+\varepsilon} (g(s, u_{h_i}(s), v_{h_i}(s)), w) \, ds \right| \leq |t + \varepsilon - kh_i| C_g |w| \leq h_i C_g |w|, \\ & \left| \int_{jh_i}^t (g(s, u_{h_i}(s), v_{h_i}(s)), w) \, ds \right| \leq |t - jh_i| C_g |w| \leq h_i C_g |w|, \end{aligned}$$

with  $C_g = \sup\{|g(s, q, v)|; (s, q, v) \in [0, T] \times B_1 \times \overline{B}(0, C_0)\}$ .

In order to estimate the first term of the right-hand side of (29), we introduce  $\omega_g$  the modulus of continuity of  $g$  on  $[0, T] \times B_1 \times \overline{B}(0, C_0)$ . Observing that

$$|u_{h_i}(nh_i) - u_{h_i}(s)| \leq C_0 |nh_i - s| \leq C_0 h_i$$

for all  $s \in [(n-1)h_i, nh_i]$  and for all  $n \in \{j+1, \dots, k\}$ , we get

$$|g(nh_i, u_{h_i}(nh_i), v_{h_i}(s)) - g(s, u_{h_i}(s), v_{h_i}(s))| \leq \omega_g(C_0 h_i) + \omega_g(h_i)$$

for all  $s \in [(n-1)h_i, nh_i]$  and for all  $n \in \{j+1, \dots, k\}$ .

Hence

$$\begin{aligned} & \left| \sum_{n=j+1}^k \int_{(n-1)h_i}^{nh_i} (g(nh_i, u_{h_i}(nh_i), v_{h_i}(s)), w) ds \right. \\ & \quad \left. - \int_t^{t+\varepsilon} (g(s, u_{h_i}(s), v_{h_i}(s)), w) ds \right| \\ & \leq 2h_i C_g |w| + (\omega_g(C_0 h_i) + \omega_g(h_i)) (k-j)h_i |w| \\ & \leq 2h_i C_g |w| + (\omega_g(C_0 h_i) + \omega_g(h_i)) (\varepsilon + h_i) |w|. \end{aligned} \quad (30)$$

Then recalling that

$$u_{h_i}(s) \rightarrow_{h_i \rightarrow 0} u(s) \quad \text{for all } s \in [0, \tau],$$

and

$$v_{h_i}(s) \rightarrow_{h_i \rightarrow 0} v(s) = \dot{u}(s) \quad \text{for a.a. } s \in [0, \tau],$$

we infer from Lebesgue's theorem that

$$\lim_{h_i \rightarrow 0} \int_t^{t+\varepsilon} (g(s, u_{h_i}(s), v_{h_i}(s)), w) ds = \int_t^{t+\varepsilon} (g(s, u(s), \dot{u}(s)), w) ds. \quad (31)$$

Finally, combining (31), (30) and (29), we obtain

$$\begin{aligned} & \lim_{h_i \rightarrow 0} \sum_{n=j+1}^k \int_{(n-1)h_i}^{nh_i} (g(nh_i, u_{h_i}(nh_i), v_{h_i}(s)), w) ds \\ & = \int_t^{t+\varepsilon} (g(s, u(s), \dot{u}(s)), w) ds \end{aligned}$$

and with (27) and (28) we may conclude that

$$\lim_{h_i \rightarrow 0} \sum_{n=j+1}^k h_i (M(U^n)F^n, w) = \int_t^{t+\varepsilon} (g(s, u(s), \dot{u}(s)), w) ds.$$

Then, passing to the limit as  $h_i$  tends to zero in (26), we get

$$\begin{aligned} & \int_t^{t+\varepsilon} (g(s, u(s), \dot{u}(s)), w) ds + (M(u(t))(\dot{u}(t) - \dot{u}(t+\varepsilon)), w) \\ & \leq 2L_M \varepsilon C_0^2 |w| \end{aligned} \quad (32)$$

for all  $\varepsilon \in (0, \tilde{\varepsilon}_w]$ . Since  $\dot{u}$  is continuous at  $t$ , we have  $\dot{u}(t) = \dot{u}(t-0) = \dot{u}(t+0)$ . Moreover, since  $\dot{u} = v$  and  $v$  is continuous, except perhaps on a countable subset of  $[0, \tau]$ , we may choose a sequence  $(\varepsilon_i)_{i \in \mathbb{N}}$  decreasing to zero such that

$$\varepsilon_i \in (0, \tilde{\varepsilon}_w], \quad v(t + \varepsilon_i) = \dot{u}(t + \varepsilon_i) = \dot{u}(t + \varepsilon_i + 0) \quad \forall i \in \mathbb{N}.$$

It follows that

$$\dot{u}(t) - \dot{u}(t + \varepsilon_i) = \dot{u}(t-0) - \dot{u}(t + \varepsilon_i + 0) = -d\dot{u}([t, t + \varepsilon_i]) \quad \forall i \in \mathbb{N}.$$

Multiplying (32) by  $\frac{1}{d\mu([t, t + \varepsilon_i])}$  and passing to the limit as  $\varepsilon_i$  tends to zero, we obtain

$$(g(t, u(t), \dot{u}(t)), w) t'_\mu(t) - (M(u(t)) v'_\mu(t), w) \leq 0 \quad \forall w \in \tilde{T}_K(\bar{u}).$$

Finally, observing that  $\tilde{T}_K(\bar{u})$  is dense in  $T_K(\bar{u})$  we may conclude.  $\square$

Let us now consider  $t \in [0, \tau] \setminus (N \cup N' \cup N_0)$  such that  $\dot{u}$  is discontinuous at  $t$ . Then  $\dot{u}(t-0) \neq \dot{u}(t+0)$  and  $d\dot{u}$  possesses a Dirac mass at  $t$ . Thus  $\{t\}$  is not negligible anymore with respect to  $d\mu$  and (3) is equivalent to

$$M(u(t)) (\dot{u}(t+0) - \dot{u}(t-0)) \in -N_K(u(t)).$$

This property is a direct consequence of the following proposition:

**Proposition 5.** *For all  $t \in [0, \tau]$  we have*

$$M(u(t)) (\dot{u}(t+0) - \dot{u}(t-0)) \in -N_K(u(t)).$$

**Proof.** Let  $t \in [0, \tau]$  and denote, for simplicity,  $\bar{u} = u(t)$ . Thanks to the density of  $\tilde{T}_K(\bar{u})$  in  $T_K(\bar{u})$ , we only need to prove that

$$(M(\bar{u}) (\dot{u}(t-0) - \dot{u}(t+0)), w) \leq 0 \quad \forall w \in \tilde{T}_K(\bar{u}).$$

Let  $w \in \tilde{T}_K(\bar{u})$ . As in the proof of the previous proposition, we define  $r_w > 0$  such that

$$J(q) \subset J(\bar{u}) \quad \text{and} \quad w \in T_K(q) \quad \text{for all } q \in \overline{B}(\bar{u}, r_w).$$

We also define  $\tilde{\varepsilon}_w \in (0, \tau/2)$  such that for all  $\varepsilon \in (0, \tilde{\varepsilon}_w]$  we have

$$u(s) \in \overline{B}\left(\bar{u}, \frac{r_w}{3}\right) \quad \forall s \in [t - \varepsilon, t + \varepsilon] \cap [0, \tau],$$

and there exists  $h_\varepsilon \in (0, \min(h_1^*, r_w/(3C_0), \varepsilon/3)]$  such that

$$u_{h_i}(s) \in \overline{B}\left(\bar{u}, \frac{2r_w}{3}\right) \quad \forall s \in [t - \varepsilon, t + \varepsilon] \cap [0, \tau], \quad \forall h_i \in (0, h_\varepsilon],$$

and

$$U^{n+1} \in \overline{B}(\bar{u}, r_w) \quad \forall nh_i \in [t - \varepsilon, t + \varepsilon] \cap [0, \tau], \quad \forall h_i \in (0, h_\varepsilon].$$

Let  $\varepsilon \in (0, \tilde{\varepsilon}_w]$  and  $h_i \in (0, h_\varepsilon]$ . We define  $t_\varepsilon^- = \max(t - \varepsilon, 0)$ ,  $t_\varepsilon^+ = \min(t + \varepsilon, \tau)$  and

$$j = \left\lfloor \frac{t_\varepsilon^-}{h_i} \right\rfloor, \quad k = \left\lfloor \frac{t_\varepsilon^+}{h_i} \right\rfloor$$

that is, we have

$$0 \leq t_j = jh_i \leq t_\varepsilon^- < t_{j+1} < \cdots < t_k = kh_i \leq t_\varepsilon^+ \leq \tau.$$

It follows that

$$v_{h_i}(t_\varepsilon^-) = V^j, \quad v_{h_i}(t_\varepsilon^+) = V^k.$$

We have

$$\begin{aligned} & \left( M(u_{h_i}(t_\varepsilon^-)) v_{h_i}(t_\varepsilon^-) - M(u_{h_i}(t_\varepsilon^+)) v_{h_i}(t_\varepsilon^+), w \right) \\ &= \left( M(U^j)V^j - M(U^k)V^k, w \right) \\ &+ \left( \left( M(u_{h_i}(t_\varepsilon^-)) - M(U^j) \right) V^j, w \right) + \left( \left( M(U^k) - M(u_{h_i}(t_\varepsilon^+)) \right) V^k, w \right). \end{aligned} \quad (33)$$

Following the same ideas as in the previous proof, we use the Lipschitz continuity of  $M$  on  $B_1$  to estimate the last two terms of (33). More precisely,

$$\left| \left( \left( M(u_{h_i}(t_\varepsilon^-)) - M(U^j) \right) V^j, w \right) \right| \leq |w| |V^j|_{L_M} \left| u_{h_i}(t_\varepsilon^-) - U^j \right| \leq L_M |w| C_0^2 h_i,$$

and with similar computations

$$\left| \left( \left( M(U^k) - M(u_{h_i}(t_\varepsilon^+)) \right) V^k, w \right) \right| \leq L_M |w| C_0^2 h_i.$$

We rewrite the first term of (33) as

$$\begin{aligned} \left( M(U^j)V^j - M(U^k)V^k, w \right) &= \sum_{n=j+1}^k M(U^n)(V^{n-1} - V^n, w) \\ &+ \sum_{n=j+1}^k \left( \left( M(U^{n-1}) - M(U^n) \right) V^{n-1}, w \right), \end{aligned}$$

and, observing that  $w \in T_K(U^{n+1})$  for all  $n \in \{j+1, \dots, k\}$ , we infer from Lemma 1 that

$$\begin{aligned} \sum_{n=j+1}^k M(U^n)(V^{n-1} - V^n, w) &\leq \sum_{n=j+1}^k -h_i (M(U^n)F^n, w) \\ &\leq (k-j)h_i \lambda_{\max, B_1} C_F |w| \\ &\leq (2\varepsilon + h_i) \lambda_{\max, B_1} C_F |w|. \end{aligned}$$

It follows that

$$\begin{aligned} \left( M(U^j)V^j - M(U^k)V^k, w \right) &\leq \sum_{n=j+1}^k \left| \left( (M(U^{n-1}) - M(U^n)) V^{n-1}, w \right) \right| \\ &\quad + (2\varepsilon + h_i)\lambda_{\max, B_1} C_F |w| \end{aligned}$$

and, using once again the Lipschitz property of  $M$ , we get

$$\left( M(U^j)V^j - M(U^k)V^k, w \right) \leq (2\varepsilon + h_i)|w| \left( \lambda_{\max, B_1} C_F + L_M C_0^2 \right).$$

Finally, we obtain

$$\begin{aligned} & \left( M(u_{h_i}(t_\varepsilon^-)) v_{h_i}(t_\varepsilon^-) - M(u_{h_i}(t_\varepsilon^+)) v_{h_i}(t_\varepsilon^+), w \right) \\ & \leq 2L_M |w| C_0^2 h_i + (2\varepsilon + h_i)|w| \left( \lambda_{\max, B_1} C_F + L_M C_0^2 \right), \end{aligned}$$

for all  $h_i \in (0, h_\varepsilon]$  and for all  $\varepsilon \in (0, \tilde{\varepsilon}_w]$ .

Now, passing to the limit as  $h_i$  tends to zero, then as  $\varepsilon$  tends to zero, we may conclude.  $\square$

### 3.2. Transmission of the velocity at impacts

With the previous proposition, we observe that  $\dot{u}(t+0) = \dot{u}(t-0)$  for all  $t \in [0, \tau]$  such that  $J(u(t)) = \emptyset$ . That is,  $\dot{u}$  is continuous at  $t$  if  $u(t) \in \text{Int}(K)$  and, in this case, the impact law (4) is satisfied. Thus it remains only to prove that

$$\dot{u}(\bar{t}+0) = \text{Proj}_{M(u(\bar{t}))} (T_K(u(\bar{t})), \dot{u}(\bar{t}-0)) \quad (34)$$

for all  $\bar{t} \in (0, \tau)$  such that  $J(u(\bar{t})) \neq \emptyset$ .

In order to also obtain some information on  $\dot{u}(0+0)$ , we now consider  $\bar{t} \in [0, \tau)$  such that  $J(u(\bar{t})) \neq \emptyset$ . For simplicity, we denote  $\bar{u} = u(\bar{t})$  and  $\dot{u}^+ = \dot{u}(\bar{t}+0)$ ,  $\dot{u}^- = \dot{u}(\bar{t}-0)$ . With Proposition 5 we already know that  $M(\bar{u})(\dot{u}^- - \dot{u}^+) \in N_K(\bar{u})$ , that is, there exist non-positive real numbers  $(\mu_\alpha)_{\alpha \in J(\bar{u})}$  such that

$$M^{1/2}(\bar{u})(\dot{u}^- - \dot{u}^+) = \sum_{\alpha \in J(\bar{u})} \mu_\alpha e_\alpha(\bar{u}),$$

where we recall that

$$e_\alpha(\bar{u}) = \frac{M^{-1/2}(\bar{u})\nabla f_\alpha(\bar{u})}{|M^{-1/2}(\bar{u})\nabla f_\alpha(\bar{u})|} \quad \forall \alpha \in J(\bar{u}).$$

Moreover, since  $u(t) \in K$  for all  $t \in [0, \tau]$ , we have  $\dot{u}^+ \in T_K(\bar{u})$  and (34) reduces to

$$(\dot{u}^- - \dot{u}^+, \dot{u}^+)_{M(\bar{u})} = 0,$$

that is,

$$\left( e_\alpha(\bar{u}), M^{1/2}(\bar{u})\dot{u}^+ \right) = 0 \quad \text{for all } \alpha \in J(\bar{u}) \text{ such that } \mu_\alpha \neq 0.$$

From assumption (H3) we know that  $(\nabla f_\alpha(\bar{u}))_{\alpha \in J(\bar{u})}$  is linearly independent. It follows that  $(e_\alpha(\bar{u}))_{\alpha \in J(\bar{u})}$  is also linearly independent and there exist  $(e_\beta)_{\beta \in \{1, \dots, d\} \setminus J(\bar{u})}$  such that  $\{e_\alpha(\bar{u}); \alpha \in J(\bar{u})\} \cup \{e_\beta; \beta \in \{1, \dots, d\} \setminus J(\bar{u})\}$  is a basis of  $\mathbb{R}^d$  and  $|e_\beta| = 1$  for all  $\beta \in \{1, \dots, d\} \setminus J(\bar{u})$ .

Using Lemma 7 (see Appendix) we know that there exists  $r_{\bar{u}} \in (0, r_{B_1}]$  such that

$$J(q) \subset J(\bar{u}) \quad \forall q \in \bar{B}(\bar{u}, r_{\bar{u}}) \quad (35)$$

and

$$\left| M^{-1/2}(q) \nabla f_\alpha(q) \right| \geq m_{B_1} > 0 \quad \forall q \in \bar{B}(\bar{u}, r_{\bar{u}}), \quad \forall \alpha \in J(\bar{u}).$$

Thus, for all  $\alpha \in \{1, \dots, d\}$  and for all  $q \in \bar{B}(\bar{u}, r_{\bar{u}})$  we define

$$v_\alpha(q) = \begin{cases} \frac{M^{-1/2}(q) \nabla f_\alpha(q)}{\left| M^{-1/2}(q) \nabla f_\alpha(q) \right|} & \text{if } \alpha \in J(\bar{u}), \\ e_\alpha & \text{if } \alpha \notin J(\bar{u}). \end{cases}$$

From (35) we infer that  $v_\alpha(q) = e_\alpha(q)$  for all  $\alpha \in J(q)$ , for all  $q \in B(\bar{u}, r_{\bar{u}}) \cap K$ . Moreover, the continuity of  $M^{-1/2}$  and  $\nabla f_\alpha$  ( $1 \leq \alpha \leq \nu$ ) implies that, possibly decreasing  $r_{\bar{u}}$ ,  $(v_\alpha(q))_{1 \leq \alpha \leq d}$  is a basis of  $\mathbb{R}^d$  for all  $q \in \bar{B}(\bar{u}, r_{\bar{u}})$ . We define the dual basis  $(w_\alpha(q))_{1 \leq \alpha \leq d}$  for all  $q \in \bar{B}(\bar{u}, r_{\bar{u}})$ . From Lemma 7, we know that the vectors  $(w_\alpha(q))_{1 \leq \alpha \leq d}$  are bounded independently of  $q$  by a constant  $C_{*, \bar{u}}$  and, since the mappings  $M^{-1/2}$  and  $\nabla f_\alpha$  ( $1 \leq \alpha \leq \nu$ ) are locally Lipschitz continuous, the mappings  $v_\alpha$  and  $w_\alpha$  ( $1 \leq \alpha \leq d$ ) are also Lipschitz continuous on  $\bar{B}(\bar{u}, r_{\bar{u}})$ ; we let  $L_{\bar{u}} \in \mathbb{R}_*^+$  be such that, for all  $\alpha \in \{1, \dots, d\}$  and for all  $(q, q') \in \bar{B}(\bar{u}, r_{\bar{u}})^2$

$$\left| v_\alpha(q) - v_\alpha(q') \right| \leq L_{\bar{u}} |q - q'|, \quad \left| w_\alpha(q) - w_\alpha(q') \right| \leq L_{\bar{u}} |q - q'|.$$

Finally, from the continuity of  $u$  and the uniform convergence of  $(u_{h_i})_{i \in \mathbb{N}}$  to  $u$  on  $[0, \tau]$ , we infer that there exist  $\bar{\varepsilon} \in (0, \frac{\tau - \bar{t}}{2}]$  and  $h_2^* \in (0, \min(h_1^*, \frac{\bar{\varepsilon}}{3}, \frac{r_{\bar{u}}}{3C_0})]$  such that

$$\begin{aligned} u(t) &\in \bar{B}\left(\bar{u}, \frac{r_{\bar{u}}}{3}\right) \quad \forall t \in [\bar{t} - \bar{\varepsilon}, \bar{t} + \bar{\varepsilon}] \cap [0, \tau], \\ \|u - u_{h_i}\|_{C^0([0, \tau]; \mathbb{R}^d)} &\leq \frac{r_{\bar{u}}}{3} \quad \forall h_i \in (0, h_2^*]. \end{aligned}$$

It follows that

$$U^{n+1}, U^n \in \bar{B}(\bar{u}, r_{\bar{u}}) \quad \forall nh_i \in [t - \bar{\varepsilon}, t + \bar{\varepsilon}] \cap [0, \tau], \quad \forall h_i \in (0, h_2^*]. \quad (36)$$

We begin with the following lemma.

**Lemma 3.** *Let  $\alpha \in J(\bar{u})$  such that  $\mu_\alpha \neq 0$ . Then, for all  $\varepsilon_1 \in (0, \bar{\varepsilon}]$  there exists  $h_{\varepsilon_1} \in (0, \min(h_2^*, \varepsilon_1/3)]$  such that for all  $h_i \in (0, h_{\varepsilon_1}]$ , there exists  $nh_i \in [\bar{t} - \varepsilon_1, \bar{t} + \varepsilon_1] \cap [0, \tau]$  such that  $f_\alpha(U^{n+1}) \leq 0$ .*

**Proof.** Let us assume that the announced result does not hold, that is, assume that there exists  $\varepsilon_1 \in (0, \bar{\varepsilon}]$  such that, for all  $h_{\varepsilon_1} \in (0, \min(h_2^*, \varepsilon_1/3)]$  there exists  $h_i \in (0, h_{\varepsilon_1}]$  such that  $f_\alpha(U^{n+1}) > 0$  for all  $nh_i \in [\bar{t} - \varepsilon_1, \bar{t} + \varepsilon_1] \cap [0, \tau]$ .

Hence, we can extract from  $(h_i)_{i \in \mathbb{N}}$  a subsequence denoted  $(h_{\varphi(i)})_{i \in \mathbb{N}}$  such that  $h_{\varphi(i)} \in (0, \min(h_2^*, \varepsilon_1/3)]$ ,  $(h_{\varphi(i)})_{i \in \mathbb{N}}$  decreases to zero and

$$f_\alpha(U^{n+1}) > 0 \quad \forall nh_{\varphi(i)} \in [\bar{t} - \varepsilon_1, \bar{t} + \varepsilon_1] \cap [0, \tau] \quad (37)$$

for all  $i \in \mathbb{N}$ .

For all  $\varepsilon \in (0, \varepsilon_1]$ , let us establish the following estimate:

$$\begin{aligned} & \left| \left( M^{1/2} (u_{h_{\varphi(i)}}(t_\varepsilon^-)) v_{h_{\varphi(i)}}(t_\varepsilon^-) - M^{1/2} (u_{h_{\varphi(i)}}(t_\varepsilon^+)) v_{h_{\varphi(i)}}(t_\varepsilon^+), w_\alpha(\bar{u}) \right) \right| \\ & \leq \mathcal{O} \left( \varepsilon + h_{\varphi(i)} + \|u - u_{h_{\varphi(i)}}\|_{C^0([0, \tau]; \mathbb{R}^d)} \right) \end{aligned}$$

where  $t_\varepsilon^- = \max(\bar{t} - \varepsilon, 0)$  and  $t_\varepsilon^+ = \min(\bar{t} + \varepsilon, \tau)$ . Then, by passing to the limit when  $i$  tends to  $+\infty$ , we will infer with (21) and (22) that

$$\left| \left( M^{1/2} (u(t_\varepsilon^-)) v(t_\varepsilon^-) - M^{1/2} (u(t_\varepsilon^+)) v(t_\varepsilon^+), w_\alpha(\bar{u}) \right) \right| \leq \mathcal{O}(\varepsilon)$$

and, when  $\varepsilon$  tends to zero, we will obtain

$$\left| \left( M^{1/2}(\bar{u}) (\dot{u}(\bar{t} - 0) - \dot{u}(\bar{t} + 0)), w_\alpha(\bar{u}) \right) \right| = |\mu_\alpha| \leq 0$$

which gives a contradiction.

Let  $\varepsilon \in (0, \varepsilon_1]$ . There exists  $i_\varepsilon \in \mathbb{N}$  such that, for all  $i \geq i_\varepsilon$  we have  $h_{\varphi(i)} \in (0, \varepsilon/2)$  and we define

$$n_i = \left\lfloor \frac{t_\varepsilon^-}{h_{\varphi(i)}} \right\rfloor, \quad p_i = \left\lfloor \frac{t_\varepsilon^+}{h_{\varphi(i)}} \right\rfloor.$$

Then,  $n_i + 1 < p_i$  and for all  $n \in \{n_i + 1, \dots, p_i\}$  we have  $nh_{\varphi(i)} \in [t_\varepsilon^-, t_\varepsilon^+]$ . We infer from Lemma 1 that, for all  $n \in \{n_i + 1, \dots, p_i\}$

$$M(U^n) \left( V^{n-1} - V^n + h_{\varphi(i)} F^n \right) \in N_K(U^{n+1}).$$

If  $J(U^{n+1}) \neq \emptyset$ , there exist non-positive real numbers  $(\mu_\beta^n)_{\beta \in J(U^{n+1})}$  such that

$$M(U^n) \left( V^{n-1} - V^n + h_{\varphi(i)} F^n \right) = \sum_{\beta \in J(U^{n+1})} \mu_\beta^n M^{1/2}(U^{n+1}) e_\beta(U^{n+1}).$$

From (36), we obtain  $e_\beta(U^{n+1}) = v_\beta(U^{n+1})$  for all  $\beta \in J(U^{n+1})$  and

$$\begin{aligned} |\mu_\beta^n| &= \left| \left( M(U^n) (V^{n-1} - V^n + h_{\varphi(i)} F^n), M^{-1/2}(U^{n+1}) w_\beta(U^{n+1}) \right) \right| \\ &\leq \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} (2C_0 + h_2^* C_F) C_{*, \bar{u}} \leq 3 \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} C_0 C_{*, \bar{u}} \quad \forall \beta \in J(U^{n+1}). \end{aligned}$$

From now on, let us denote

$$C'_2 = 3 \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} C_0 C_{*, \bar{u}}.$$

With (37) we know that  $\alpha \notin J(U^{n+1})$ , thus

$$\begin{aligned} & |(M^{1/2}(U^{n+1})(V^{n-1} - V^n + h_{\varphi(i)}F^n), w_\alpha(U^{n+1}))| \\ &= \left| \sum_{\beta \in J(U^{n+1})} \mu_\beta^n (M^{1/2}(U^{n+1})M^{-1}(U^n)M^{1/2}(U^{n+1})v_\beta(U^{n+1}), w_\alpha(U^{n+1})) \right| \\ &\leq \sum_{\beta \in J(U^{n+1})} \left| \mu_\beta^n \right| \|M^{1/2}(U^{n+1})\|^2 \|M^{-1}(U^{n+1}) - M^{-1}(U^n)\| |v_\beta(U^{n+1})| |w_\alpha(U^{n+1})| \\ &\leq \nu C'_2 C_{*, \bar{u}} \lambda_{\max, B_1} L_{M^{-1}} |U^{n+1} - U^n| \leq \nu C'_2 C_0 C_{*, \bar{u}} \lambda_{\max, B_1} L_{M^{-1}} h_{\varphi(i)} \end{aligned}$$

for all  $n \in \{n_i + 1, \dots, p_i\}$ , if  $J(U^{n+1}) \neq \emptyset$ .

If  $J(U^{n+1}) = \emptyset$ , this last inequality remains true since  $V^{n-1} - V^n + h_{\varphi(i)}F^n = 0$  if  $J(U^{n+1}) = \emptyset$ .

It follows that, for all  $n \in \{n_i + 1, \dots, p_i\}$

$$\begin{aligned} & \left( M^{1/2}(U^{n_i+1})V^{n_i} - M^{1/2}(U^{p_i+1})V^{p_i}, w_\alpha(\bar{u}) \right) \\ &= \sum_{n=n_i+1}^{p_i} \left( M^{1/2}(U^n)V^{n-1} - M^{1/2}(U^{n+1})V^n, w_\alpha(\bar{u}) \right) \\ &= \sum_{n=n_i+1}^{p_i} \left( M^{1/2}(U^{n+1})(V^{n-1} - V^n + h_{\varphi(i)}F^n), w_\alpha(U^{n+1}) \right) \\ &+ \sum_{n=n_i+1}^{p_i} \left( \left( M^{1/2}(U^n) - M^{1/2}(U^{n+1}) \right) V^{n-1}, w_\alpha(\bar{u}) \right) \\ &- \sum_{n=n_i+1}^{p_i} h_{\varphi(i)} \left( M^{1/2}(U^{n+1})F^n, w_\alpha(U^{n+1}) \right) \\ &+ \sum_{n=n_i+1}^{p_i} \left( M^{1/2}(U^{n+1})(V^{n-1} - V^n), w_\alpha(\bar{u}) - w_\alpha(U^{n+1}) \right) \end{aligned}$$

which yields

$$\begin{aligned} & \left| \left( M^{1/2}(U^{n_i+1})V^{n_i} - M^{1/2}(U^{p_i+1})V^{p_i}, w_\alpha(\bar{u}) \right) \right| \\ &\leq \sum_{n=n_i+1}^{p_i} \nu C'_2 C_0 C_{*, \bar{u}} \lambda_{\max, B_1} L_{M^{-1}} h_{\varphi(i)} \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=n_i+1}^{p_i} C_0 C_{*,\bar{u}} L_{M^{1/2}} h_{\varphi(i)} |V^n| + \sum_{n=n_i+1}^{p_i} h_{\varphi(i)} C_{*,\bar{u}} C_F \sqrt{\lambda_{\max, B_1}} \\
& + \sum_{n=n_i+1}^{p_i} \sqrt{\lambda_{\max, B_1}} |V^n - V^{n-1}| \left| w_\alpha(\bar{u}) - w_\alpha(U^{n+1}) \right|.
\end{aligned}$$

Then, observing that for all  $n \in \{n_i + 1, \dots, p_i\}$

$$\begin{aligned}
|\bar{u} - U^{n+1}| & \leq |u(\bar{t}) - u_{h_{\varphi(i)}}(\bar{t})| + |u_{h_{\varphi(i)}}(\bar{t}) - u_{h_{\varphi(i)}}(nh_{\varphi(i)})| + h_{\varphi(i)} |V^n| \\
& \leq \|u - u_{h_{\varphi(i)}}\|_{C^0([0, \tau]; \mathbb{R}^d)} + C_0(\varepsilon + h_{\varphi(i)}),
\end{aligned}$$

we can estimate  $|w_\alpha(\bar{u}) - w_\alpha(U^{n+1})|$  as

$$L_{\bar{u}} \left( \|u - u_{h_{\varphi(i)}}\|_{C^0([0, \tau]; \mathbb{R}^d)} + C_0(\varepsilon + h_{\varphi(i)}) \right).$$

Hence, with the estimate of the discrete accelerations obtained at Proposition 3, we get

$$\begin{aligned}
& \left| (M^{1/2}(U^{n_i+1})V^{n_i} - M^{1/2}(U^{p_i+1})V^{p_i}, w_\alpha(\bar{u})) \right| \\
& \leq (p_i - n_i) h_{\varphi(i)} C_{*,\bar{u}} \left( v C_2' C_0 \lambda_{\max, B_1} L_{M^{-1}} + C_0^2 L_{M^{1/2}} + C_F \sqrt{\lambda_{\max, B_1}} \right) \\
& \quad + \sqrt{\lambda_{\max, B_1}} L_{\bar{u}} \left( \|u - u_{h_{\varphi(i)}}\|_{C^0([0, \tau]; \mathbb{R}^d)} + C_0(\varepsilon + h_{\varphi(i)}) \right) \sum_{n=n_i+1}^{p_i} |V^n - V^{n-1}| \\
& = \mathcal{O} \left( \varepsilon + h_{\varphi(i)} + \|u - u_{h_{\varphi(i)}}\|_{C^0([0, \tau]; \mathbb{R}^d)} \right). \tag{38}
\end{aligned}$$

But  $V^{n_i} = v_{h_{\varphi(i)}}(t_\varepsilon^-)$ ,  $V^{p_i} = v_{h_{\varphi(i)}}(t_\varepsilon^+)$  and

$$\begin{aligned}
& \left| \left( M^{1/2} (u_{h_{\varphi(i)}}(t_\varepsilon^-)) v_{h_{\varphi(i)}}(t_\varepsilon^-) - M^{1/2} (u_{h_{\varphi(i)}}(t_\varepsilon^+)) v_{h_{\varphi(i)}}(t_\varepsilon^+), w_\alpha(\bar{u}) \right) \right. \\
& \quad \left. - \left( M^{1/2}(U^{n_i+1})V^{n_i} - M^{1/2}(U^{p_i+1})V^{p_i}, w_\alpha(\bar{u}) \right) \right| \\
& \leq \left( \left| M^{1/2} (u_{h_{\varphi(i)}}(t_\varepsilon^-)) - M^{1/2}(U^{n_i+1}) \right| \right. \\
& \quad \left. + \left| M^{1/2} (u_{h_{\varphi(i)}}(t_\varepsilon^+)) - M^{1/2}(U^{p_i+1}) \right| \right) C_0 C_{*,\bar{u}} \\
& \leq 2L_{M^{1/2}} C_0^2 C_{*,\bar{u}} h_{\varphi(i)}. \tag{39}
\end{aligned}$$

Finally, from (38), (39) we obtain

$$\begin{aligned}
& \left| \left( M^{1/2} (u_{h_{\varphi(i)}}(t_\varepsilon^-)) v_{h_{\varphi(i)}}(t_\varepsilon^-) - M^{1/2} (u_{h_{\varphi(i)}}(t_\varepsilon^+)) v_{h_{\varphi(i)}}(t_\varepsilon^+), w_\alpha(\bar{u}) \right) \right| \\
& = \mathcal{O} \left( \varepsilon + h_{\varphi(i)} + \|u - u_{h_{\varphi(i)}}\|_{C^0([0, \tau]; \mathbb{R}^d)} \right)
\end{aligned}$$

for all  $i \geq i_\varepsilon$  and for all  $\varepsilon \in (0, \varepsilon_1]$ , which enables us to conclude.  $\square$

Let us now prove that

$$\left( e_\alpha(\bar{u}), M^{1/2}(\bar{u})\dot{u}^+ \right) = 0 \quad \text{for all } \alpha \in J(\bar{u}) \text{ such that } \mu_\alpha \neq 0.$$

**Lemma 4.** *Let  $\alpha \in J(\bar{u})$  be such that  $\mu_\alpha \neq 0$ . Then*

$$\left( e_\alpha(\bar{u}), M^{1/2}(\bar{u})\dot{u}^+ \right) = 0.$$

**Proof.** Let  $\alpha \in J(\bar{u})$  such that  $\mu_\alpha \neq 0$ . Since  $\dot{u}^+ \in T_K(\bar{u})$  we have  $(M^{1/2}(\bar{u})\dot{u}^+, e_\alpha(\bar{u})) = (M^{1/2}(\bar{u})\dot{u}^+, v_\alpha(\bar{u})) \geq 0$  and it remains to prove that  $(M^{1/2}(\bar{u})\dot{u}^+, v_\alpha(\bar{u})) \leq 0$ . The main idea of the proof is to obtain an estimate of  $(M^{1/2}(u(\bar{t} + \varepsilon)), v_\alpha(u(\bar{t} + \varepsilon)))$  and to pass to the limit when  $\varepsilon$  tends to zero.

More precisely, let  $\varepsilon \in (0, \bar{\varepsilon}]$ . Then

$$v(\bar{t} + \varepsilon) = \lim_{h_i \rightarrow 0} v_{h_i}(\bar{t} + \varepsilon) = \lim_{h_i \rightarrow 0} V^{p_i}$$

with  $p_i = \lfloor \frac{\bar{t} + \varepsilon}{h_i} \rfloor$  for all  $i \in \mathbb{N}$ . Observing that

$$\begin{aligned} \left| u(\bar{t} + \varepsilon) - U^{p_i+1} \right| &\leq \left| u(\bar{t} + \varepsilon) - u_{h_i}(\bar{t} + \varepsilon) \right| + \left| u_{h_i}(\bar{t} + \varepsilon) - u_{h_i}((p_i + 1)h_i) \right| \\ &\leq \|u - u_{h_i}\|_{C^0([0, \tau]; \mathbb{R}^d)} + C_0 h_i \end{aligned}$$

the continuity of  $v_\alpha$  and  $M^{1/2}$  on  $\bar{B}(\bar{u}, r_{\bar{u}})$  implies that

$$\begin{aligned} &\left( M^{1/2}(u(\bar{t} + \varepsilon)) v(\bar{t} + \varepsilon), v_\alpha(u(\bar{t} + \varepsilon)) \right) \\ &= \lim_{h_i \rightarrow 0} \left( M^{1/2}(U^{p_i+1}) v_{h_i}(\bar{t} + \varepsilon), v_\alpha(U^{p_i+1}) \right) \\ &= \lim_{h_i \rightarrow 0} \left( M^{1/2}(U^{p_i+1}) V^{p_i}, v_\alpha(U^{p_i+1}) \right), \end{aligned}$$

and we will prove that

$$\left( M^{1/2}(U^{p_i+1}) V^{p_i}, v_\alpha(U^{p_i+1}) \right) \leq \mathcal{O}\left(\varepsilon + h_i + \|u - u_{h_i}\|_{C^0([0, \tau]; \mathbb{R}^d)}\right). \quad (40)$$

Let us apply Lemma 3: for all  $i$  such that  $h_i \in (0, h_\varepsilon]$  we define  $N_i$  as the last time step in  $[\bar{t} - \varepsilon, \bar{t} + \varepsilon] \cap [0, \tau]$  where the constraint  $f_\alpha$  is active, that is,

$$N_i = \max \left\{ n \in \mathbb{N}; nh_i \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon] \cap [0, \tau] \text{ and } f_\alpha(U^{n+1}) \leq 0 \right\}.$$

Since  $\alpha \in J(U^{N_i+1})$ , we infer, as in Proposition 1, that

$$\begin{aligned} \left( v_\alpha(U^{N_i+1}), M^{1/2}(U^{N_i+1}) V^{N_i} \right) &= \left( e_\alpha(U^{N_i+1}), M^{1/2}(U^{N_i+1}) V^{N_i} \right) \\ &\leq \frac{L_f h_i}{2m_{B_1}} |V^{N_i}|^2 \leq \frac{L_f C_0^2}{2m_{B_1}} h_i. \end{aligned}$$

Moreover, with the same computations as in Lemma 3, for all  $nh_i \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon] \cap [0, \tau]$  such that  $n \geq 1$  we have

$$M(U^n)(V^{n-1} - V^n + h_i F^n) = \sum_{\beta \in J(U^{n+1})} \mu_\beta^n M^{1/2}(U^{n+1}) v_\beta(U^{n+1}) \quad (41)$$

with

$$-3 \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} C_0 C_{*, \bar{u}} = -C'_2 \leq \mu_\beta^n \leq 0 \quad \forall \beta \in J(U^{n+1}). \quad (42)$$

Thus, for all  $h_i \in (0, h_\varepsilon]$  we get

$$\begin{aligned} & \left( M^{1/2}(U^{p_i+1}) V^{p_i}, v_\alpha(U^{p_i+1}) \right) \\ &= \left( M^{1/2}(U^{N_i+1}) V^{N_i}, v_\alpha(U^{N_i+1}) \right) \\ &+ \left( M^{1/2}(U^{N_i+1}) V^{N_i}, v_\alpha(U^{p_i+1}) - v_\alpha(U^{N_i+1}) \right) \\ &+ \sum_{n=N_i+1}^{p_i} \left( M^{1/2}(U^{n+1}) V^n - M^{1/2}(U^n) V^{n-1}, v_\alpha(U^{p_i+1}) \right) \\ &\leq \frac{L_f C_0^2}{2m_{B_1}} h_i + \left( M^{1/2}(U^{N_i+1}) V^{N_i}, v_\alpha(U^{p_i+1}) - v_\alpha(U^{N_i+1}) \right) \\ &+ \sum_{n=N_i+1}^{p_i} \left( \left( M^{1/2}(U^{n+1}) - M^{1/2}(U^n) \right) V^{n-1}, v_\alpha(U^{p_i+1}) \right) \\ &+ \sum_{n=N_i+1}^{p_i} h_i \left( M^{1/2}(U^{n+1}) F^n, v_\alpha(U^{p_i+1}) \right) \\ &+ \sum_{n=N_i+1}^{p_i} \left( M^{1/2}(U^{n+1})(V^n - V^{n-1} - h_i F^n), v_\alpha(U^{p_i+1}) \right). \end{aligned}$$

Using the Lipschitz property of  $M^{1/2}$  on  $B_1$  and recalling that the mappings  $v_\alpha$  ( $\alpha \in \{1, \dots, \nu\}$ ) are  $L_{\bar{u}}$ -Lipschitz continuous on  $\bar{B}(\bar{u}, r_{\bar{u}})$ , we get

$$\begin{aligned} & \left( M^{1/2}(U^{p_i+1}) V^{p_i}, v_\alpha(U^{p_i+1}) \right) \\ &\leq \sum_{n=N_i+1}^{p_i} h_i (L_{M^{1/2}} C_0^2 + \sqrt{\lambda_{\max, B_1}} C_F) + \frac{L_f C_0^2}{2m_{B_1}} h_i \\ &+ 2\varepsilon \sqrt{\lambda_{\max, B_1}} C_0^2 L_{\bar{u}} + \sum_{n=N_i+1}^{p_i} \left( M^{1/2}(U^{n+1})(V^n - V^{n-1} - h_i F^n), v_\alpha(U^{p_i+1}) \right). \end{aligned} \quad (43)$$

There remains the task of estimating the last term. Using (41) and (42) we rewrite it as follows

$$\begin{aligned}
& \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} (-\mu_\beta^n) (M^{1/2}(U^{n+1})M^{-1}(U^n)M^{1/2}(U^{n+1})v_\beta(U^{n+1}), v_\alpha(U^{p_i+1})) \\
& \leq \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} C'_2 \|M^{1/2}(U^{n+1})\|^2 \|M^{-1}(U^{n+1}) - M^{-1}(U^n)\| \\
& \quad + \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} (-\mu_\beta^n) (v_\beta(U^{n+1}), v_\alpha(U^{p_i+1})). \tag{44}
\end{aligned}$$

By definition of  $N_i$  we have  $\alpha \notin J(U^{n+1})$  for all  $n \in \{N_i + 1, \dots, p_i\}$ . Moreover, from assumption (H6) we have

$$(v_\beta(\bar{u}), v_\alpha(\bar{u})) = (e_\beta(\bar{u}), e_\alpha(\bar{u})) \leq 0 \quad \forall \beta \in J(\bar{u}) \setminus \{\alpha\}$$

and (35) and (36) imply that  $J(U^{n+1}) \subset J(\bar{u})$  for all  $nh_i \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon] \cap [0, \tau]$ .

It follows that

$$\begin{aligned}
& \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} (-\mu_\beta^n) (v_\beta(U^{n+1}), v_\alpha(U^{p_i+1})) \\
& \leq \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} (-\mu_\beta^n) \left( (v_\beta(U^{n+1}), v_\alpha(U^{p_i+1})) - (v_\beta(\bar{u}), v_\alpha(\bar{u})) \right) \\
& \leq \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} |\mu_\beta^n| L_{\bar{u}} (|U^{n+1} - \bar{u}| + |U^{p_i+1} - \bar{u}|).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} (-\mu_\beta^n) (M^{1/2}(U^{n+1})M^{-1}(U^n)M^{1/2}(U^{n+1})v_\beta(U^{n+1}), v_\alpha(U^{p_i+1})) \\
& \leq \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} C'_2 \lambda_{\max, B_1} L_{M^{-1}} C_0 h_i \\
& \quad + \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} 2 |\mu_\beta^n| L_{\bar{u}} \left( \|u - u_{h_i}\|_{C^0([0, \tau]; \mathbb{R}^d)} + C_0(\varepsilon + h_i) \right).
\end{aligned}$$

But

$$\begin{aligned}
|\mu_\beta^n| &= \left| \left( M^{-1/2}(U^{n+1})M(U^n)(V^{n-1} - V^n + h_i F^n), w_\beta(U^{n+1}) \right) \right| \\
&\leq \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} \left( |V^{n-1} - V^n| + h_i C_F \right) C_{*, \bar{u}}
\end{aligned}$$

for all  $\beta \in J(U^{n+1})$ , for all  $n \in \{N_i + 1, \dots, p_i\}$ . Hence, with the estimate of the discrete accelerations obtained at Proposition 3

$$\begin{aligned}
& \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} (-\mu_\beta^n) (M^{1/2}(U^{n+1})M^{-1}(U^n)M^{1/2}(U^{n+1})v_\beta(U^{n+1}), v_\alpha(U^{p_i+1})) \\
& \leq 2\nu \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} C_{*, \bar{u}} (C'_0 + 2\varepsilon C_F) L_{\bar{u}} \left( \|u - u_{h_i}\|_{C^0([0, \tau]; \mathbb{R}^d)} + C_0(\varepsilon + h_i) \right) \\
& \quad + 2\varepsilon \nu C'_2 \lambda_{\max, B_1} L_{M^{-1}} C_0.
\end{aligned} \tag{45}$$

Finally, combining (43), (44) and (45), we obtain

$$\begin{aligned}
& \left( M^{1/2}(U^{p_i+1})V^{p_i}, v_\alpha(U^{p_i+1}) \right) \\
& \leq \frac{L_f C_0^2}{2m_{B_1}} h_i + 2\varepsilon C_0^2 \sqrt{\lambda_{\max, B_1}} L_{\bar{u}} \\
& \quad + 2\varepsilon (L_{M^{1/2}} C_0^2 + \sqrt{\lambda_{\max, B_1}} C_F + \nu C'_2 \lambda_{\max, B_1} L_{M^{-1}} C_0) \\
& \quad + 2\nu \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} C_{*, \bar{u}} (C'_0 + 2\varepsilon C_F) L_{\bar{u}} \left( \|u - u_{h_i}\|_{C^0([0, \tau]; \mathbb{R}^d)} + C_0(\varepsilon + h_i) \right)
\end{aligned}$$

for all  $h_i \in (0, h_\varepsilon]$ , for all  $\varepsilon \in (0, \bar{\varepsilon}]$ , which proves (40). Passing to the limit as  $h_i$  tends to zero, then when  $\varepsilon$  tends to zero, we may conclude the proof.  $\square$

### 3.3. Study of the initial conditions

We can now prove quite easily that property (P4) is satisfied.

**Lemma 5.** *The initial conditions  $(u_0, v_0)$  are satisfied in the following sense:*

$$u(0) = u_0, \quad \dot{u}(0+0) = v_0.$$

**Proof.** Since the sequence  $(u_{h_i})_{i \in \mathbb{N}}$  converges uniformly to  $u$  on  $[0, \tau]$ , we have

$$u(0) = \lim_{h_i \rightarrow 0} u_{h_i}(0).$$

But  $u_{h_i}(0) = U^0 = u_0$  for all  $h_i \in (0, h^*]$ , and thus  $u(0) = u_0$ . From the results of the previous subsection we already know that

$$\dot{u}(0+0) = \text{Proj}_{M(u(0))} (T_K(u(0)), \dot{u}(0-0))$$

where  $\dot{u}(0-0) = \dot{u}(0) = v(0)$  (see (24)). It follows that

$$\dot{u}(0+0) = \text{Proj}_{M(u_0)} (T_K(u_0), v(0)). \tag{46}$$

Since the sequence  $(v_{h_i})_{i \in \mathbb{N}}$  converges pointwise to  $v$  on  $[0, \tau]$  we have

$$v(0) = \lim_{h_i \rightarrow 0} v_{h_i}(0).$$

Let us prove now that  $\lim_{h_i \rightarrow 0} v_{h_i}(0) = v_0$ . For all  $h_i \in (0, h^*]$  we have

$$v_{h_i}(0) = V^0 = \frac{U^1 - U^0}{h_i}$$

and the definition of  $U^1$  implies that

$$h_i \|v_0 + z(h_i) - v_{h_i}(0)\|_{M(u_0)} \leq \|u_0 + h_i v_0 + h_i z(h_i) - Z\|_{M(u_0)} \quad \forall Z \in K,$$

which yields

$$\|v_0 - v_{h_i}(0)\|_{M(u_0)} \leq 2 \|z(h_i)\|_{M(u_0)} + \|v_0 - v\|_{M(u_0)}$$

for all  $v \in \mathbb{R}^d$  such that  $u_0 + h_i v \in K$ .

If  $u_0 \in \text{Int}(K)$ , we infer that there exists  $r_{u_0} > 0$  such that  $u_0 + h_i v_0 \in K$  for all  $h_i \in (0, \frac{r_{u_0}}{|v_0|+1}]$  and thus

$$\|v_0 - v_{h_i}(0)\|_{M(u_0)} \leq 2 \|z(h_i)\|_{M(u_0)} \quad \forall h_i \in \left(0, \min\left(\frac{r_{u_0}}{|v_0|+1}, h^*\right)\right].$$

Since  $\lim_{h \rightarrow 0} z(h) = 0$ , we get  $\lim_{h_i \rightarrow 0} v_{h_i}(0) = v(0) = v_0$ .

Let us assume now that  $u_0 \in \partial K$ . Since  $v_0 \in T_K(u_0)$  and  $\tilde{T}_K(u_0)$  is dense in  $T_K(u_0)$ , we may consider a sequence  $(v_p)_{p \in \mathbb{N}^*}$  converging to  $v_0$  and such that

$$v_p \in \tilde{T}_K(u_0) = \left\{ w \in \mathbb{R}^d; (\nabla f_\alpha(u_0), w) > 0 \quad \forall \alpha \in J(u_0) \right\} \quad \forall p \in \mathbb{N}^*.$$

It follows that  $(|v_p|)_{p \in \mathbb{N}^*}$  remains bounded, and we consider  $M \in \mathbb{R}_+^*$  such that  $M \geq |v_p|$  for all  $p \in \mathbb{N}^*$ . Using the continuity of the mappings  $f_\alpha$ ,  $\alpha \in \{1, \dots, v\}$ , there exists  $r_{u_0} > 0$  such that

$$f_\alpha(q) \geq \frac{f_\alpha(u_0)}{2} \quad \forall \alpha \notin J(u_0), \quad \forall q \in \overline{B}(u_0, r_{u_0}).$$

Let  $p \in \mathbb{N}^*$ . From the definition of  $r_{u_0}$ , we infer that

$$f_\alpha(u_0 + tv_p) \geq 0 \quad \forall \alpha \notin J(u_0), \quad \forall t \in \left(0, \frac{r_{u_0}}{M}\right).$$

Moreover, if  $\alpha \in J(u_0)$

$$\begin{aligned} & f_\alpha(u_0 + tv_p) \\ &= f_\alpha(u_0) + t (\nabla f_\alpha(u_0), v_p) + t \int_0^1 (\nabla f_\alpha(u_0 + stv_p) - \nabla f_\alpha(u_0), v_p) \, ds. \end{aligned}$$

If  $t \in (0, 1/M]$ , we get  $|tv_p| \leq 1$  and thus  $u_0 + stv_p \in B_1$  for all  $s \in [0, 1]$ , which yields

$$f_\alpha(u_0 + tv_p) \geq t (\nabla f_\alpha(u_0), v_p) - t^2 \frac{L_f |v_p|^2}{2} \quad \forall \alpha \in J(u_0).$$

It follows that there exists  $t_p \in (0, \min(1, r_{u_0})/M]$  such that  $f_\alpha(u_0 + tv_p) \geq 0$  for all  $\alpha \in \{1, \dots, \nu\}$  and for all  $t \in (0, t_p]$ . Thus

$$\begin{aligned} \|v_0 - v_{h_i}(0)\|_{M(u_0)} &\leq 2 \|z(h_i)\|_{M(u_0)} + \|v_0 - v_p\|_{M(u_0)} \\ \forall h_i &\in (0, \min(h^*, t_p)], \quad \forall p \in \mathbb{N}^*. \end{aligned}$$

Then, passing to the limit when  $h_i$  tends to zero, we get

$$\|v_0 - v(0)\|_{M(u_0)} \leq \|v_0 - v_p\|_{M(u_0)} \quad \forall p \in \mathbb{N}^*$$

and recalling that the sequence  $(v_p)_{p \in \mathbb{N}^*}$  converges to  $v_0$ , we obtain  $v_0 = v(0)$ . Finally, using (46) and recalling that  $v_0 \in T_K(u_0)$ , we get

$$\dot{u}(0+0) = \text{Proj}_{M(u_0)}(T_K(u_0), v_0) = v_0.$$

□

With the previous results, we can state the following theorem:

**Theorem 2.** *Let us assume that there exist  $C_0 > 0$ ,  $\tau_0 > 0$ ,  $h_0^* \in (0, h^*]$  and a subsequence of the approximate positions defined by (5)–(7) such that*

$$|V^n| = \left| \frac{U^{n+1} - U^n}{h_i} \right| \leq C_0 \quad \forall nh_i \in [0, \min(\tau_0, T)], \quad \forall h_i \in (0, h_0^*]$$

with  $(h_i)_{i \in \mathbb{N}}$  decreasing to zero. Let  $u_h$  and  $v_h$  be defined by (19) and (20). Then, there exist a subsequence still denoted  $(h_i)_{i \in \mathbb{N}}$  and  $(u, v) \in C^0([0, \min(\tau_0, T)]; \mathbb{R}^d) \times BV(0, \min(\tau_0, T); \mathbb{R}^d)$  such that

$$\begin{aligned} u_{h_i} &\rightarrow u \quad \text{strongly in } C^0([0, \min(\tau_0, T)]; \mathbb{R}^d), \\ v_{h_i} &\rightarrow v \quad \text{pointwise in } [0, \min(\tau_0, T)], \end{aligned}$$

with

$$u(t) = u_0 + \int_0^t v(s) \, ds \quad \text{for all } t \in [0, \min(\tau_0, T)],$$

and  $u$  is a solution of problem (P) on  $[0, \min(\tau_0, T)]$ .

By combining Theorem 2 with the a priori estimate of the discrete velocities obtained in Theorem 1, we immediately obtain a local convergence result for the numerical scheme, and thus a local existence result for problem (P).

#### 4. Energy estimates and global results

In order to establish global convergence results, we now state an energy estimate for the solutions of problem (P).

**Proposition 6.** *Let  $C > \|v_0\|_{M(u_0)}$ . Then there exists  $\tau(C) > 0$  such that, for any solution  $u$  of problem (P) defined on  $[0, \tau]$  (with  $\tau \in (0, T]$ ), we have*

$$\begin{aligned} |u(t) - u_0| &\leq C \quad \forall t \in [0, \min(\tau(C), \tau)], \\ \|\dot{u}(t)\|_{M(u(t))} &\leq C \quad dt \text{ almost everywhere on } [0, \min(\tau(C), \tau)]. \end{aligned}$$

**Proof.** Let us define the kinetic energy  $E$  by

$$E = \frac{1}{2} (\dot{u}, M(u)\dot{u}).$$

Since  $\dot{u} \in BV(0, \tau; \mathbb{R}^d)$  and  $u$  is absolutely continuous from  $[0, \tau]$  to  $\mathbb{R}^d$ ,  $E$  belongs to  $BV(0, \tau; \mathbb{R}^d)$ . Moreover (see [8])

$$dE = \left( d\dot{u}, M(u) \left( \frac{\dot{u}^+ + \dot{u}^-}{2} \right) \right) + \frac{1}{2} (\dot{u}, (dM(u)\dot{u}) \dot{u}) dt.$$

Let  $[t_1, t_2] \subset [0, \tau]$ . Then

$$\begin{aligned} E(t_2 + 0) - E(t_1 + 0) &= \int_{(t_1, t_2]} \left( d\dot{u}, M(u) \left( \frac{\dot{u}^+ + \dot{u}^-}{2} \right) \right) \\ &\quad + \frac{1}{2} \int_{t_1}^{t_2} (\dot{u}(t), (dM(u(t)) \dot{u}(t)) \dot{u}(t)) dt. \end{aligned}$$

Let us define  $D = \{t \in (t_1, t_2]; \dot{u}(t + 0) \neq \dot{u}(t - 0)\}$ . The set  $D$  is at most denumerable and

$$\begin{aligned} \int_{(t_1, t_2]} \left( d\dot{u}, M(u) \left( \frac{\dot{u}^+ + \dot{u}^-}{2} \right) \right) &= \int_{(t_1, t_2] \setminus D} (d\dot{u}, M(u)\dot{u}^+) \\ &\quad + \sum_{t \in D} \frac{1}{2} \left( \|\dot{u}(t + 0)\|_{M(u(t))}^2 - \|\dot{u}(t - 0)\|_{M(u(t))}^2 \right). \end{aligned}$$

But, with property (P2), we have also

$$\begin{aligned} M(u(t)) v'_\mu(t) - g(t, u(t), \dot{u}(t)) t'_\mu(t) &\in -N_K(u(t)) \\ d\mu &\text{ almost everywhere on } [0, \tau] \end{aligned}$$

where  $d\mu = |d\dot{u}| + dt$  and  $v'_\mu$  and  $t'_\mu$  are, respectively, the densities of  $d\dot{u}$  and  $dt$  with respect to  $\mu$ . Thus

$$\begin{aligned} \int_{(t_1, t_2] \setminus D} (d\dot{u}, M(u)\dot{u}^+) &= \int_{(t_1, t_2] \setminus D} (g(t, u, \dot{u}), \dot{u}^+) t'_\mu d\mu \\ &\quad + \int_{(t_1, t_2] \setminus D} (M(u)v'_\mu - g(t, u, \dot{u})t'_\mu, \dot{u}^+) d\mu. \end{aligned}$$

Since  $\dot{u}(t+0) = \dot{u}(t-0) \in (T_K(u(t))) \cap (-T_K(u(t)))$  for all  $t \in (t_1, t_2] \setminus D$ , the last term vanishes and we get

$$\begin{aligned} E(t_2+0) - E(t_1+0) &= \frac{1}{2} \int_{t_1}^{t_2} (\dot{u}(t), (dM(u(t)) \dot{u}(t)) \dot{u}(t)) dt \\ &\quad + \int_{(t_1, t_2] \setminus D} (g(t, u, \dot{u}), \dot{u}^+) t'_\mu d\mu \\ &\quad + \sum_{t \in D} \frac{1}{2} \left( \|\dot{u}(t+0)\|_{M(u(t))}^2 - \|\dot{u}(t-0)\|_{M(u(t))}^2 \right). \end{aligned}$$

But, with property (P3), we know that

$$2E(t+0) = \|\dot{u}(t+0)\|_{M(u(t))}^2 \leq \|\dot{u}(t-0)\|_{M(u(t))}^2 = 2E(t-0) \quad \forall t \in (0, \tau)$$

and finally

$$\begin{aligned} E(t_2+0) - E(t_1+0) &\leq \int_{t_1}^{t_2} (g(t, u(t), \dot{u}(t)), \dot{u}(t)) dt \\ &\quad + \frac{1}{2} \int_{t_1}^{t_2} (\dot{u}(t), (dM(u(t)) \dot{u}(t)) \dot{u}(t)) dt. \end{aligned}$$

In particular, for all  $t \in [0, \tau)$

$$\begin{aligned} E(t+0) &\leq E(0+0) + \int_0^t (g(s, u(s), \dot{u}(s)), \dot{u}(s)) ds \\ &\quad + \frac{1}{2} \int_0^t (\dot{u}(s), (dM(u(s)) \dot{u}(s)) \dot{u}(s)) ds. \end{aligned}$$

Observing that  $C^2 > 2E(0+0)$ , the continuity of  $u$  on  $[0, \tau]$  and the right continuity of  $E(\cdot+0)$  on  $[0, \tau)$  imply that there exists  $\bar{\tau} \in (0, \tau)$  such that

$$|u(t) - u(0)| \leq C, \quad E(t+0) \leq \frac{C^2}{2} \quad \forall t \in [0, \bar{\tau}]. \quad (47)$$

We define

$$\tau_{max} = \sup \{ \bar{\tau} \in (0, \tau) \text{ such that (47) holds} \}.$$

Since  $u$  is continuous on  $[0, \tau]$ , we have

$$|u(t) - u(0)| \leq C \quad \forall t \in [0, \tau_{max}], \quad (48)$$

and

$$E(t+0) = \frac{1}{2} \left| M^{1/2}(u(t)) \dot{u}(t+0) \right|^2 \leq \frac{C^2}{2} \quad \forall t \in [0, \tau_{max}]. \quad (49)$$

It follows that

$$\|\dot{u}(t)\|_{M(u(t))}^2 = \left| M^{1/2}(u(t)) \dot{u}(t) \right|^2 \leq C^2 dt \text{ almost everywhere on } [0, \tau_{max}].$$

If  $\tau_{\max} = \tau$  there is nothing to prove. Otherwise, we define

$$\alpha = \sup \left\{ \left| \left( g(t, q, M^{-1/2}(q)w), M^{-1/2}(q)w \right) \right| ; t \in [0, T], q \in \overline{B}(u_0, C), w \in \overline{B}(0, C) \right\},$$

$$\beta = \sup \left\{ \frac{1}{2} \left\| M^{-1/2}(q) \left( dM(q)M^{-1/2}(q)w \right) M^{-1/2}(q) \right\|, q \in \overline{B}(u_0, C), w \in \overline{B}(0, C) \right\}$$

$$\gamma = \sup \left\{ \left\| M^{-1/2}(q) \right\|, q \in \overline{B}(u_0, C) \right\}$$

With (48) and (49) we obtain that for all  $t \in [0, \tau_{\max}]$

$$E(t+0) - E(0+0) \leq \int_0^t \left( \alpha + \beta \left| M^{1/2}(u(s)) \dot{u}(s) \right|^2 \right) ds \leq (\alpha + \beta C^2) t$$

and

$$|u(t) - u_0| \leq \int_0^t |\dot{u}(s)| ds \leq \int_0^t \gamma \left| M^{1/2}(u(s)) \dot{u}(s) \right| ds \leq \gamma C t.$$

Then, the continuity of  $u$  on  $[0, \tau]$  and the right continuity of  $E(\cdot + 0)$  on  $[0, \tau]$  imply that  $\tau_{\max} \geq \min(\tau, \tau(C))$  where  $\tau(C)$  is defined by

$$\tau(C) = \begin{cases} \min \left( \frac{1}{\gamma}, \frac{C^2 - 2E(0+0)}{2(\alpha + \beta C^2)} \right) & \text{if } \alpha \neq 0 \text{ or } \beta \neq 0, \\ \frac{1}{\gamma} & \text{if } \alpha = 0 \text{ and } \beta = 0. \end{cases}$$

□

Now we can prove that

**Theorem 3.** *Let  $C > \|v_0\|_{M(u_0)}$  and  $\tau(C) > 0$  such that, for any solution  $u$  of problem (P) defined on  $[0, \tau]$  (with  $\tau \in (0, T)$ ), we have*

$$\begin{aligned} |u(t) - u_0| &\leq C \quad \forall t \in [0, \min(\tau(C), \tau)], \\ \|\dot{u}(t)\|_{M(u(t))} &\leq C \quad dt \text{ almost everywhere on } [0, \min(\tau(C), \tau)]. \end{aligned}$$

*Let  $u_h$  and  $v_h$  be the approximate positions and velocities defined by (19) and (20). Then, there exists a subsequence  $(u_{h_i}, v_{h_i})_{i \in \mathbb{N}}$ ,  $\tau \in [\min(\tau(C), T), T]$  and  $(u, v) \in C^0([0, \tau]; \mathbb{R}^d) \times BV(0, \tau; \mathbb{R}^d)$  such that*

$$\begin{aligned} u_{h_i} &\rightarrow u \quad \text{strongly in } C^0([0, \tau]; \mathbb{R}^d), \\ v_{h_i} &\rightarrow v \quad \text{pointwise in } [0, \tau], \end{aligned}$$

with

$$u(t) = u_0 + \int_0^t v(s) ds \quad \forall t \in [0, \tau]$$

and  $u$  is a solution of problem (P) on  $[0, \tau]$ .

**Proof.** Let  $C > \|v_0\|_{M(u_0)}$ . We define  $\mathcal{B} = \overline{\mathcal{B}}(u_0, C + 1)$  and  $B_0, \lambda_{\min}, \lambda_{\max}$  by (9) and (10), respectively. Let us choose  $C_0$  such that

$$C_0 > \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0^*$$

with

$$C_0^* = \max\left(2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}( |v_0| + 1), C' \right)$$

$$C' = (C + 1) \sup\left\{\|M^{-1/2}(q)\|; q \in \overline{\mathcal{B}}(u_0, C + 1)\right\}$$

and let  $C_F$  be defined by (11).

Then, from Theorem 1, we know that there exists  $h_0^* \in (0, h^*]$  and  $\tau_0 > 0$ , depending only on  $\mathcal{B}, C_0, C_0^*$  and the data, such that

$$|V^n| = \left| \frac{U^{n+1} - U^n}{h} \right| \leq C_0 \quad \forall nh \in [0, \min(\tau_0, T)], \quad \forall h \in (0, h_0^*]. \quad (50)$$

Moreover, from Proposition 2, we know also that, for all  $t_{0h} \in [0, T)$  and for all  $(\hat{U}^0, \hat{U}^1) \in (\mathcal{B} \cap K) \times K$  such that

$$|\hat{U}^1 - \hat{U}^0| \leq hC_0^* \quad \forall h \in (0, h_0^*]$$

the approximate positions defined by

$$\hat{U}^{n+1} \in \operatorname{Argmin}_{Z \in K} \|\hat{W}^n - Z\|_{M(\hat{U}^n)}$$

with

$$\hat{W}^n = 2\hat{U}^n - \hat{U}^{n-1} + h^2 \hat{F}^n, \quad \hat{F}^n = F\left(t_{0h} + nh, \hat{U}^n, \frac{\hat{U}^n - \hat{U}^{n-1}}{h}, h\right)$$

for all  $n \in \{1, \dots, \lfloor \frac{T-t_{0h}}{h} \rfloor\}$  and for all  $h \in (0, h_0^*]$ , satisfy

$$\left| \frac{\hat{U}^{n+1} - \hat{U}^n}{h} \right| \leq C_0 \quad \forall nh \in [0, \min(\tau_0, T - t_{0h})], \quad \forall h \in (0, h_0^*].$$

Let  $\tau(h) = m(h)h$  be the maximal discrete time step such that estimate (50) holds, that is, for all  $h \in (0, h_0^*]$

$$m(h) = \max\left\{n \in \{0, \dots, \lfloor T/h \rfloor\}; |V^k| \leq C_0 \quad \forall k \in \{0, \dots, n\}\right\}.$$

We define  $\tau_1 = \liminf_{h \rightarrow 0} \tau(h) = \liminf_{h \rightarrow 0} m(h)h$ . Theorem 1 implies that  $\tau_1 \geq \tau' = \min(\tau_0, T)$ . Let us now distinguish two subcases.

Case 1:  $\tau(C) < T$ .

Let us prove that  $\tau_1 > \tau(C) = \min(\tau(C), T)$ . Indeed, assume that  $\tau_1 \leq \tau(C)$  and let  $\varepsilon \in (0, \tau'/8)$ . Then, there exists a subsequence  $(h_i)_{i \in \mathbb{N}}$ , decreasing to zero, such that  $(\tau(h_i))_{i \in \mathbb{N}}$  converges to  $\tau_1$  and there exists  $h_\varepsilon^* \in (0, \min(h_0^*, \tau'/8)]$  such that  $m(h_i)h_i \geq \tau_1 - \varepsilon$  for all  $h_i \in (0, h_\varepsilon^*]$ . We may apply theorem 2 with  $h_0^* := h_\varepsilon^*$  and  $\tau_0 := \tau_1 - \varepsilon$ ; we infer that there exists a subsequence, still denoted  $(h_i)_{i \in \mathbb{N}}$ , such that  $(u_{h_i}, v_{h_i})_{i \in \mathbb{N}}$  converges to a solution of problem (P) on  $[0, \tau_1 - \varepsilon]$ . Thus, with Proposition 6 we get

$$\begin{aligned} |u(t) - u_0| &\leq C \quad \forall t \in [0, \tau_1 - \varepsilon], \\ \|\dot{u}(t)\|_{M(u(t))} &\leq C \quad dt \text{ almost everywhere on } [0, \tau_1 - \varepsilon]. \end{aligned}$$

Now we prove that:

**Lemma 6.** *We have*

$$\begin{aligned} \limsup_{h_i \rightarrow 0^+} \sup \left\{ \|V^n\|_{M(U^n)}, 0 \leq nh_i \leq \tau_1 - \varepsilon \right\} \\ \leq \text{ess sup} \left\{ \|\dot{u}(t)\|_{M(u(t))}, 0 \leq t \leq \tau_1 - \varepsilon \right\}. \end{aligned}$$

**Proof.** Let us prove this result by contradiction. Assume that

$$\limsup_{h_i \rightarrow 0^+} \sup \left\{ \|V^n\|_{M(U^n)}, 0 \leq nh_i \leq \tau_1 - \varepsilon \right\} > S$$

with  $S = \text{ess sup} \left\{ \|\dot{u}(t)\|_{M(u(t))}, 0 \leq t \leq \tau_1 - \varepsilon \right\}$ . Then, there exist  $\gamma > 0$ ,  $\tilde{h}_\varepsilon^* \in (0, h_\varepsilon^*]$  and a subsequence  $(h_{\varphi(i)})_{i \in \mathbb{N}}$  decreasing to zero such that

$$\sup \left\{ \|V^n\|_{M(U^n)}, 0 \leq nh_{\varphi(i)} \leq \tau_1 - \varepsilon \right\} \geq S + \gamma \quad \forall h_{\varphi(i)} \in (0, \tilde{h}_\varepsilon^*].$$

It follows that there exists  $n_{\varphi(i)} \in \{0, \dots, \lfloor (\tau_1 - \varepsilon)/h_{\varphi(i)} \rfloor\}$  such that

$$\|V^{n_{\varphi(i)}}\|_{M(U^{n_{\varphi(i)}})} \geq S + \gamma \quad \forall h_{\varphi(i)} \in (0, \tilde{h}_\varepsilon^*].$$

Possibly extracting another subsequence, still denoted  $(h_{\varphi(i)})_{i \in \mathbb{N}}$ , we may assume without loss of generality that the sequence  $(n_{\varphi(i)}h_{\varphi(i)})_{i \in \mathbb{N}}$  converges to a limit  $\tilde{\tau} \in [0, \tau_1 - \varepsilon]$ .

First, we observe that  $\tilde{\tau} > 0$ . Indeed, with the same computations as in Proposition 2 (see (13)), we obtain that, for all  $h_i \in (0, h_\varepsilon^*]$  and for all  $nh_i \in [0, \tau_1 - \varepsilon]$

$$\|V^n\|_{M(U^n)} \leq \|V^{n-1}\|_{M(U^{n-1})} + C_2 h_i \leq \|V^0\|_{M(U^0)} + C_2 n h_i \quad (51)$$

with

$$C_2 = \sqrt{\lambda_{\max, B_1}} C_F + L_{M^{1/2}} C_0^2 + \frac{3L_f}{2m_{B_1}} \frac{\lambda_{\max, B_1}}{\lambda_{\min, B_1}} \nu C_{*, B_1} C_0^2,$$

$B_1 = \overline{B}(u_0, C_0 T + 1)$  and  $\lambda_{\max, B_1}, \lambda_{\min, B_1}$  given by (15).

Thus, for all  $h_{\varphi(i)} \in (0, \tilde{h}_\varepsilon^*]$

$$S + \gamma \leq \|V^{n_{\varphi(i)}}\|_{M(U^{n_{\varphi(i)}})} \leq \|v_{h_{\varphi(i)}}(0)\|_{M(u_0)} + C_2 n_{\varphi(i)} h_{\varphi(i)}$$

and at the limit when  $i$  tends to  $+\infty$ , we get

$$S + \gamma \leq \|v(0)\|_{M(u_0)} + C_2 \tilde{\tau} = \|v_0\|_{M(u_0)} + C_2 \tilde{\tau}.$$

On the other hand, the right continuity of  $\|\dot{u}(\cdot + 0)\|_{M(u)}$  implies that, for all  $\rho > 0$ , there exists  $\tau_\rho \in (0, \tau_1 - \varepsilon]$  such that, for all  $t \in [0, \tau_\rho]$

$$\left| \|\dot{u}(t + 0)\|_{M(u(t))} - \|\dot{u}(0 + 0)\|_{M(u_0)} \right| \leq \rho.$$

It follows that

$$\|\dot{u}(0 + 0)\|_{M(u_0)} - \rho = \|v_0\|_{M(u_0)} - \rho \leq \|\dot{u}(t)\|_{M(u(t))} \quad \text{almost everywhere on } [0, \tau_\rho],$$

and thus

$$\|v_0\|_{M(u_0)} - \rho \leq S \quad \forall \rho > 0.$$

Hence  $\|v_0\|_{M(u_0)} \leq S$  and  $\tilde{\tau} > 0$ .

Then, once again using the estimate (51), we obtain

$$S + \gamma \leq \|V^{n_{\varphi(i)}}\|_{M(U^{n_{\varphi(i)}})} \leq \|V^{n_{\varphi(i)}-p}\|_{M(U^{n_{\varphi(i)}-p})} + C_2 p h_{\varphi(i)}$$

for all  $p \in \{0, \dots, n_{\varphi(i)}\}$ , for all  $h_{\varphi(i)} \in (0, \tilde{h}_\varepsilon^*]$ . It follows that

$$S + \frac{\gamma}{2} \leq \|V^k\|_{M(U^k)}$$

for all  $kh_{\varphi(i)} \in [\max(0, n_{\varphi(i)}h_{\varphi(i)} - \frac{\gamma}{2C_2}), n_{\varphi(i)}h_{\varphi(i)}]$ . Moreover, for all  $t \in [kh_{\varphi(i)}, (k+1)h_{\varphi(i)}]$ ,

$$\begin{aligned} \left| \|v_{h_{\varphi(i)}}(t)\|_{M(U^k)} - \|v_{h_{\varphi(i)}}(t)\|_{M(u_{h_{\varphi(i)}}(t))} \right| &\leq L_{M^{1/2}} \left| U^k - u_{h_{\varphi(i)}}(t) \right| C_0 \\ &\leq L_{M^{1/2}} C_0^2 h_{\varphi(i)}. \end{aligned}$$

Since  $(n_{\varphi(i)}h_{\varphi(i)})_{i \in \mathbb{N}}$  converges to  $\tilde{\tau} > 0$ , we infer that there exists an interval  $I \subset [0, \tau_1 - \varepsilon]$  with a non-empty interior, such that

$$I \subset \left[ \max \left( 0, (n_{\varphi(i)} + 1)h_{\varphi(i)} - \frac{\gamma}{2C_2} \right), n_{\varphi(i)}h_{\varphi(i)} \right]$$

and

$$S + \frac{\gamma}{4} \leq \|v_{h_{\varphi(i)}}(t)\|_{M(u_{h_{\varphi(i)}}(t))} \quad \forall t \in I$$

for all  $h_{\varphi(i)}$  small enough.

Then, passing to the limit as  $i$  tends to  $+\infty$ , we obtain

$$S + \frac{\gamma}{4} \leq \|v(t)\|_{M(u(t))} \quad \forall t \in I$$

But  $v(t) = \dot{u}(t)$  almost everywhere on  $I$ , and

$$S \geq \text{ess sup} \{ \|\dot{u}(t)\|_{M(u(t))}; t \in I \}$$

which yields a contradiction.  $\square$

With the previous lemma, possibly decreasing  $h_\varepsilon^*$  we get

$$U^n \in \overline{B}(u_0, C + 1), \quad \|V^n\|_{M(U^n)} \leq C + 1 \quad \forall nh_i \in [0, \tau_1 - \varepsilon], \quad \forall h_i \in (0, h_\varepsilon^*].$$

It follows that

$$|V^n| \leq (C + 1) \sup \left\{ \|M^{-1/2}(q)\|; q \in \overline{B}(u_0, C + 1) \right\} = C' \leq C_0^* < C_0.$$

We choose now  $l(h_i) \in \{0, \dots, \lfloor T/h \rfloor\}$  such that

$$l(h_i)h_i \in \left[ \tau_1 - \frac{\tau'}{2}, \tau_1 - \frac{\tau'}{4} \right] \quad \forall h_i \in (0, h_\varepsilon^*]$$

and let

$$\hat{U}^0 = U^{l(h_i)}, \quad \hat{U}^1 = U^{l(h_i)+1}, \quad t_{0h_i} = l(h_i)h_i.$$

We have

$$|\hat{U}^1 - \hat{U}^0| = h_i |V^{l(h_i)}| \leq C_0^* h_i$$

and  $\hat{U}^0, \hat{U}^1$  belong to  $\mathcal{B} \cap K$  for all  $h_i \in (0, h_\varepsilon^*]$ .

Then, for all  $n \in \{l(h_i), \dots, \lfloor \frac{T}{h_i} \rfloor\}$ ,  $U^n = \hat{U}^{n-l(h_i)}$  and with Proposition 2, we obtain

$$|V^n| = \left| \frac{\hat{U}^{n-l(h_i)+1} - \hat{U}^{n-l(h_i)}}{h_i} \right| \leq C_0$$

for all  $(n - l(h_i))h_i \in [0, \min(\tau_0, T - l(h_i)h_i)]$ , for all  $h_i \in (0, h_\varepsilon^*]$ .

Hence

$$m(h_i)h_i > l(h_i)h_i + \min(\tau_0, T - l(h_i)h_i) - h_i = \min(l(h_i)h_i + \tau_0, T) - h_i$$

for all  $h_i \in (0, h_\varepsilon^*]$ . But  $l(h_i)h_i \geq \tau_1 - \frac{\tau'}{2}$  and  $\tau' = \min(\tau_0, T) = \tau_0$ , so

$$m(h_i)h_i \geq \min\left(\tau_1 + \frac{\tau'}{2}, T\right) - h_i \quad \forall h_i \in (0, h_\varepsilon^*]$$

and, at the limit, we get

$$\tau_1 = \lim_{h_i \rightarrow 0} m(h_i)h_i \geq \min\left(\tau_1 + \frac{\tau'}{2}, T\right)$$

which is absurd.

Thus  $\tau_1 = \lim_{h_i \rightarrow 0} m(h_i)h_i > \tau(C)$  and there exists  $\tilde{h}_0^* \in (0, h_0^*]$  such that

$$m(h_i)h_i = \tau(h_i) \geq \tau_1 - \frac{\tau_1 - \tau(C)}{2} = \frac{\tau_1 + \tau(C)}{2} > \tau(C) \quad \forall h_i \in (0, \tilde{h}_0^*].$$

Then we apply Theorem 2 with  $\tau_0$  replaced by  $\tau = \frac{\tau_1 + \tau(C)}{2}$  and  $h_0^*$  by  $\tilde{h}_0^*$ , which yields the announced result.

Case 2:  $\tau(C) \geq T$ .

Since  $m(h)h = \tau(h) \leq T$ , we have  $\tau_1 \leq T$ . We consider once again  $\varepsilon \in (0, \tau'/8)$ , and we define as previously  $h_\varepsilon^*$  and  $l(h_i)$  for all  $h_i \in (0, h_\varepsilon^*]$ . Then, we have again

$$m(h_i)h_i > l(h_i)h_i + \min(\tau_0, T - l(h_i)h_i) - h_i \geq \min\left(\tau_1 + \frac{\tau'}{2}, T\right) - h_i \quad (52)$$

for all  $h_i \in (0, h_\varepsilon^*]$ . Thus, if  $\tau_1 < T$ ,  $\min(\tau_1 + \frac{\tau'}{2}, T) \in (\tau_1, T]$  and (52) yields a contradiction with the definition of  $\tau_1 = \lim_{h_i \rightarrow 0} m(h_i)h_i$ . We infer that  $\tau_1 = T$  and (52) implies that  $m(h_i)h_i > T - h_i$ , that is,  $m(h_i) \geq [T/h_i]$  for all  $h_i \in (0, h_\varepsilon^*]$ . Hence, we may apply Theorem 2 to obtain the convergence of a subsequence of  $(u_{h_i}, v_{h_i})_{i \in \mathbb{N}}$ , still denoted  $(u_{h_i}, v_{h_i})_{i \in \mathbb{N}}$ , to a solution of problem (P) on  $[0, T]$ .  $\square$

## Appendix

**Lemma 7.** For all compact subset  $\mathcal{B}$  of  $\mathbb{R}^d$ , there exist  $m_{\mathcal{B}} > 0$  and  $r_{\mathcal{B}} > 0$  such that for all  $q \in K \cap \mathcal{B}$  and for all  $\alpha \in J(q)$  we have

$$|M^{-1/2}(q')\nabla f_\alpha(q')| \geq m_{\mathcal{B}} \quad \forall q' \in \overline{B}(q, r_{\mathcal{B}}).$$

Furthermore, for all  $q \in K \cap \mathcal{B}$ , the family  $(e_\alpha(q))_{\alpha \in J(q)}$  is linearly independent and can be completed as a basis  $(v_j(q))_{1 \leq j \leq d}$ . Let us denote by  $(w_j(q))_{1 \leq j \leq d}$  the dual basis. Then there exists  $C_{*,\mathcal{B}} > 0$  such that

$$|v_j(q)| = 1, \quad |w_j(q)| \leq C_{*,\mathcal{B}} \quad \forall j \in \{1, \dots, d\}, \quad \forall q \in K \cap \mathcal{B}.$$

**Proof.** Let  $\mathcal{B}$  be a given compact subset of  $\mathbb{R}^d$ . For all  $\alpha \in \{1, \dots, \nu\}$  we define

$$\mathcal{B}_\alpha = \mathcal{B} \cap \{q \in \mathbb{R}^d; f_\alpha(q) \leq 0\} \cap K.$$

Then  $\mathcal{B}_\alpha$  is also a compact subset of  $\mathbb{R}^d$  and, for all  $q \in \mathcal{B}_\alpha$ , we have  $f_\alpha(q) = 0$ . Hence, with (H2)

$$\nabla f_\alpha(q) \neq 0 \quad \forall q \in \mathcal{B}_\alpha, \quad \forall \alpha \in \{1, \dots, \nu\}.$$

It follows that there exists  $m_\alpha > 0$  such that

$$m_\alpha = \inf_{q \in \mathcal{B}_\alpha} \left| M^{-1/2}(q)\nabla f_\alpha(q) \right|.$$

By continuity of the mappings  $M^{-1/2}$  and  $\nabla f_\alpha$ , we infer that

$$\forall q \in \mathcal{B}_\alpha, \quad \exists \rho_q > 0 \quad \left| M^{-1/2}(q')\nabla f_\alpha(q') \right| \geq \frac{m_\alpha}{2} \quad \forall q' \in \overline{B}(q, \rho_q).$$

Since  $\mathcal{B}_\alpha$  is compact and  $\mathcal{B}_\alpha \subset \bigcup_{q \in \mathcal{B}_\alpha} B(q, \frac{\rho_q}{2})$ , there exists a finite set of points  $\{q_1, \dots, q_p\} \in \mathcal{B}_\alpha^p$  such that  $\mathcal{B}_\alpha = \bigcup_{i=1}^p B(q_i, \frac{\rho_{q_i}}{2})$ .

By defining  $\rho_\alpha = \min_{1 \leq i \leq p} \frac{\rho_{qi}}{2}$  we obtain that

$$\forall q \in \mathcal{B}_\alpha, \quad \left| M^{-1/2}(q') \nabla f_\alpha(q') \right| \geq \frac{m_\alpha}{2} \quad \forall q' \in \overline{B}(q, \rho_\alpha).$$

Finally, with

$$m_{\mathcal{B}} = \min_{1 \leq \alpha \leq v} \frac{m_\alpha}{2}, \quad r_{\mathcal{B}} = \min_{1 \leq \alpha \leq v} \rho_\alpha,$$

we get the first part of the announced result.

As a consequence, for all  $q \in K \cap \mathcal{B}$ , we can define

$$v_\alpha(q') = \frac{M^{-1/2}(q') \nabla f_\alpha(q')}{\left| M^{-1/2}(q') \nabla f_\alpha(q') \right|} \quad \forall q' \in \overline{B}(q, r_{\mathcal{B}}), \quad \forall \alpha \in J(q).$$

Let  $q \in K \cap \mathcal{B}$  be given. From assumption (H3) we infer that  $(e_\alpha(q))_{\alpha \in J(q)}$  is linearly independent, and there exists a family of vectors  $(e_\beta)_{\beta \in \{1, \dots, d\} \setminus J(q)}$  such that  $|e_\beta| = 1$  for all  $\beta \in \{1, \dots, d\} \setminus J(q)$  and  $\{e_\alpha(q); \alpha \in J(q)\} \cup \{e_\beta; \beta \in \{1, \dots, d\} \setminus J(q)\}$  is a basis of  $\mathbb{R}^d$ .

Let us now define the mappings  $v_\beta, \beta \in \{1, \dots, d\} \setminus J(q)$ , by

$$v_\beta(q') = e_\beta \quad \forall q' \in \overline{B}(q, r_{\mathcal{B}}).$$

The mappings  $v_j, j \in \{1, \dots, d\}$ , are continuous on  $\overline{B}(q, r_{\mathcal{B}})$  and there exists  $r_q \in (0, r_{\mathcal{B}}]$  such that  $(v_j(q'))_{1 \leq j \leq d}$  is a basis of  $\mathbb{R}^d$  for all  $q' \in \overline{B}(q, r_q)$ . Moreover, using the continuity of the mappings  $f_\alpha, \alpha \in \{1, \dots, v\}$ , and possibly decreasing  $r_q$ , we also have

$$J(q') \subset J(q) \quad \forall q' \in \overline{B}(q, r_q).$$

It follows that

$$v_\alpha(q') = e_\alpha(q') \quad \forall \alpha \in J(q'), \quad \forall q' \in \overline{B}(q, r_q) \cap K.$$

Let us denote by  $(w_j(q'))_{1 \leq j \leq d}$  the dual basis of  $(v_j(q'))_{1 \leq j \leq d}$  for all  $q' \in \overline{B}(q, r_q)$ . Then, the mappings  $w_j, j \in \{1, \dots, d\}$ , are continuous on  $\overline{B}(q, r_q)$ . Indeed, let  $(\delta_j)_{1 \leq j \leq d}$  be the canonical basis of  $\mathbb{R}^d$  and define  $(a_{ij}(q'))_{1 \leq i, j \leq d}$  and  $(b_{ij}(q'))_{1 \leq i, j \leq d}$  as the coordinates of  $v_i(q')$  and  $w_i(q')$ ,  $1 \leq i \leq d$ , in the canonical basis  $(\delta_j)_{1 \leq j \leq d}$ . That is,

$$v_i(q') = \sum_{j=1}^d a_{ij}(q') \delta_j, \quad w_i(q') = \sum_{j=1}^d b_{ij}(q') \delta_j \quad \forall i \in \{1, \dots, d\}.$$

We denote by  $A(q') = (A_{ij}(q') = a_{ij}(q'))_{1 \leq i, j \leq d}$  and  $B(q') = (B_{ij}(q') = b_{ji}(q'))_{1 \leq i, j \leq d}$ . Then, by the definition of dual bases, we have

$$\forall (i, j) \in \{1, \dots, d\}^2 \quad (v_i(q'), w_j(q')) = \sum_{k=1}^d a_{ik}(q') b_{jk}(q') = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and thus  $A(q')B(q') = \text{Id}_{\mathbb{R}^d}$ . We infer that  $B(q') = A^{-1}(q')$ . But, the mapping

$$\mathcal{I} : \begin{cases} GL(\mathbb{R}^d) \rightarrow GL(\mathbb{R}^d) \\ Q \mapsto Q^{-1} \end{cases}$$

is continuous, and the mapping  $q' \mapsto A(q')$  is continuous on  $\overline{B}(q, r_q)$  with values in  $GL(\mathbb{R}^d)$ . It follows that  $q' \mapsto B(q')$  is also continuous on  $\overline{B}(q, r_q)$  and we infer that the mappings  $w_j, j \in \{1, \dots, d\}$ , (which are the columns of  $B$ ) are also continuous on  $\overline{B}(q, r_q)$ .

It follows that we can define

$$C_{*,q} = \max \{|w_j(q')|; q' \in \overline{B}(q, r_q)\}.$$

Now, using the compactness of  $K \cap \mathcal{B}$ , we infer that there exists a finite set of points  $(q_k)_{1 \leq k \leq \ell}$  such that  $q_k \in K \cap \mathcal{B}$  for all  $k \in \{1, \dots, \ell\}$  and

$$K \cap \mathcal{B} \subset \bigcup_{k=1}^{\ell} B(q_k, r_{q_k}).$$

Then, the conclusion follows with  $C_{*,\mathcal{B}} = \max_{1 \leq k \leq \ell} C_{*,q_k}$ .  $\square$

**Lemma 8.** *Let us recall the definition of  $T_K(q)$ :*

$$T_K(q) = \left\{ w \in \mathbb{R}^d; (\nabla f_\alpha(q), w) \geq 0 \quad \forall \alpha \in J(q) \right\} \quad \forall q \in \mathbb{R}^d$$

with

$$J(q) = \{\alpha \in \{1, \dots, v\}; f_\alpha(q) \leq 0\}.$$

Then for all  $q_0 \in K$ , there exist  $\delta > 0, r > 0$  and  $a \in \mathbb{R}^d$  such that, for all  $q \in \overline{B}(q_0, 2\delta)$ :

$$\overline{B}(a, r) \subset T_K(q). \quad (53)$$

**Proof.** Let  $q_0$  be in  $K$ .

Since the functions  $(f_\alpha)_{\alpha=1, \dots, v}$  are continuous, we infer that there exists  $\delta_1 > 0$  such that, for all  $\alpha \notin J(q_0)$ , we have

$$f_\alpha(q) > 0 \quad \text{if } |q - q_0| \leq \delta_1.$$

It follows that  $J(q) \subset J(q_0)$  for all  $q \in \overline{B}(q_0, \delta_1)$ .

Consequently, if  $J(q_0) = \emptyset$ , we have  $J(q) = \emptyset$  for all  $q \in \overline{B}(q_0, \delta_1)$  and (53) is satisfied for  $\delta = \delta_1/2$  and for all  $a \in \mathbb{R}^d$  and  $r > 0$ .

Let us assume now that  $J(q_0) \neq \emptyset$ . For all  $\alpha \in J(q_0)$  we define  $\phi_\alpha : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$\phi_\alpha(q, y) = (\nabla f_\alpha(q), y) \quad \forall (q, y) \in \mathbb{R}^d \times \mathbb{R}^d$$

and  $\phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$\phi(q, y) = \min_{\alpha \in J(q_0)} \phi_\alpha(q, y) \quad \forall (q, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Since  $f_\alpha \in C^1(\mathbb{R}^d)$  for all  $\alpha \in \{1, \dots, \nu\}$ , we obtain that the mappings are continuous. Moreover, since  $(\nabla f_\alpha(q_0))_{\alpha \in J(q_0)}$  is linearly independent, we can define a basis  $(\xi_i)_{1 \leq i \leq d}$  of  $\mathbb{R}^d$  such that

$$\xi_\alpha = \nabla f_\alpha(q_0) \quad \forall \alpha \in J(q_0).$$

Let us denote by  $(\zeta_i)_{1 \leq i \leq d}$  the dual basis of  $(\xi_i)_{1 \leq i \leq d}$  and let

$$a = \sum_{\alpha \in J(q_0)} \zeta_\alpha.$$

Then, for all  $\alpha \in J(q_0)$ , we have

$$\phi_\alpha(q_0, a) = (\nabla f_\alpha(q_0), a) = \left( \xi_\alpha, \sum_{\beta \in J(q_0)} \zeta_\beta \right) = 1$$

and  $\phi(q_0, a) = 1$ . By continuity, it follows that there exist  $r > 0$  and  $\delta_2 > 0$  such that

$$\phi(q, y) > 0 \quad \forall (q, y) \in \overline{B}(q_0, \delta_2) \times \overline{B}(a, r).$$

Let  $\delta = \frac{1}{2} \min(\delta_1, \delta_2)$ . For all  $q \in \overline{B}(q_0, 2\delta)$  we have

$$J(q) \subset J(q_0), \quad \phi(q, y) = \min_{\alpha \in J(q_0)} (\nabla f_\alpha(q), y) > 0 \quad \forall y \in \overline{B}(a, r)$$

which implies that

$$\overline{B}(a, r) \subset T_K(q) = \left\{ y \in \mathbb{R}^d; (\nabla f_\alpha(q), y) \geq 0 \quad \forall \alpha \in J(q) \right\}$$

and (53) is satisfied.  $\square$

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