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# QUANTITATIVE ESTIMATES OF SAMPLING CONSTANTS IN MODEL SPACES

A. HARTMANN, P. JAMING & K. KELLAY

ABSTRACT. We establish quantitative estimates for sampling (dominating) sets in model spaces associated with meromorphic inner functions, *i.e.* those corresponding to de Branges spaces. Our results encompass the Logvinenko-Sereda-Panejah (LSP) Theorem including Kovrijkine's optimal sampling constants for Paley-Wiener spaces. It also extends Dyakonov's LSP theorem for model spaces associated with bounded derivative inner functions. Considering meromorphic inner functions allows us to introduce a new geometric density condition, in terms of which the sampling sets are completely characterized. This, in comparison to Volberg's characterization of sampling measures in terms of harmonic measure, enables us to obtain explicit estimates on the sampling constants. The methods combine Baranov-Bernstein inequalities, reverse Carleson measures and Remez inequalities.

## 1. INTRODUCTION

An important question in signal theory is to know how much information of a signal is needed in order to be recovered exactly. This information can be given in discrete form (points giving rise to so-called sampling sequences) or more generally by subsets of the sets on which the signal is defined. A prominent class of signals is given by the Paley-Wiener space  $PW_\sigma^2$  of entire functions of type  $\sigma > 0$  and which are square integrable on the real line (finite energy). By the Paley-Wiener theorem it is known that this is exactly the space of Fourier transforms of functions in  $L^2(-\sigma, \sigma)$  (finite energy signals on  $(-\sigma, \sigma)$ ). For the Paley-Wiener space, it is known (Shannon-Whittaker-Kotelnikov Theorem) that  $\frac{\pi}{\sigma}\mathbb{Z}$  forms a sampling sequence for  $PW_\sigma^2$ , which means that each function can be exactly reconstructed from its values on  $\frac{\pi}{\sigma}\mathbb{Z}$  with an appropriate control of norm (actually every sequence  $\gamma\mathbb{Z}$  with  $\gamma \leq \pi/\sigma$  is sampling for  $PW_\sigma^2$ ). More general sequences

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can be considered; we refer to the seminal paper [13] which provides criteria, yet difficult to check, involving the famous Muckenhoupt condition.

Panejah [20, 21], Kac'nelson [15] and Logvinenko-Sereda [17] were interested in a characterization of subsets  $\Gamma \subset \mathbb{R}$  allowing to recover Paley-Wiener functions, or more precisely their norm, from their restriction to  $\Gamma$ . In other words, they were looking for sets  $\Gamma$  for which there exists a constant  $C = C(\Gamma)$  such that, for every  $f \in PW_\sigma^2$ ,

$$(1.1) \quad \int_{\mathbb{R}} |f(x)|^2 dx \leq C \int_{\Gamma} |f(x)|^2 dx.$$

Such sets  $\Gamma$  are said to be *dominating* and the least constant appearing in (1.1) is called the *sampling constant* of  $\Gamma$ . In view of the result on sampling sequences, it is not too surprising that in order to be dominating in the Paley-Wiener space,  $\Gamma$  has to satisfy a so-called relative density condition. This condition means that each interval  $I$  of a given length  $a$  ( $a$  large enough) contains a minimal proportion of  $\Gamma$ ,

$$|\Gamma \cap I| \geq \gamma |I|$$

for some  $\gamma > 0$  independent of  $I$ . A very natural question is to establish a link between  $\gamma$  and  $C(\Gamma)$ . This is particularly interesting for applications where one wants to measure  $f$  on a set  $\Gamma$  as small as possible while aiming at an estimate closest possible to the norm of  $f$ . This requires a knowledge of the sampling constants depending on the size of  $\Gamma$ . For the Paley-Wiener space, essentially optimal quantitative estimates of these constants were given by Kovrijkine [16], see also [22].

The Paley-Wiener space is a special occurrence of so-called model spaces (see below for precise definitions), which do not only occur in the setting of signal theory, but also in control problems as well as in the context of second order differential operators (e.g. Schrödinger operators, Sturm-Liouville problems, which naturally connect the theory to that of de Branges spaces of entire functions which are unitarily equivalent to a subclass of model spaces), see [18] for an interesting account of such connections. As it turns out the problem of dominating sets in model spaces was completely solved by Volberg who provided a description for general inner functions in terms of harmonic measure [24] (see also [12] for  $p = 1$ ). Dyakonov was more interested in a geometric description for dominating sets [7]. He revealed that the notion of relative density indeed generalizes to a much broader class than just Paley-Wiener spaces. His achievement is that relatively dense sets are dominating in a model space  $K_\Theta^p$  precisely when the inner function  $\Theta$  defining the model space has bounded derivative.

The aim and novelty of this paper is to consider an appropriate density condition and quantitative estimates of the sampling constants in the broader setting of model spaces associated to general meromorphic inner functions (these correspond exactly to the situation of de Branges spaces of entire functions). Our framework is larger than Dyakonov's in that we allow  $\Theta$  to have an unbounded derivative, and this immediately leads to the question on how to measure the size of the set  $\Gamma$ . Indeed, according to Dyakonov's results, relative density is intimately related to the boundedness of  $\Theta'$ , and so, in our more general setting, we have to replace uniform intervals of length  $a$ , which were already considered in the

Paley-Wiener space in [20, 21, 15, 17] or more generally in [7] when  $\Theta'$  is bounded, by a suitable family of intervals. The main tool in this direction is a Whitney type covering of  $\mathbb{R}$  introduced by Baranov [2] in which the length of the test intervals is given in terms of the distance to some level set of  $\Theta$  (as a matter of fact, in the Paley-Wiener space this distance is constant). Once we obtain a geometric characterization of dominating sets, we determine the dependence of the sampling constant  $C(\Gamma)$  on the parameters arising in the characterization.

Let us discuss some of the main ingredients used in this paper. A central tool is a Remez-type inequality which allows to estimate mean values of holomorphic functions on an interval in terms of their means on a measurable subset of the interval. This involves some uniform estimates of the function which we will explain a little bit more below. It is essentially this Remez-type inequality which determines the dependence of the sampling constants on the density. The Remez-inequality requires also some relative smallness condition of the intervals we have to consider. For this reason we first need to reduce the problem to intervals which satisfy this smallness condition. For this reduction we introduce a class of test sets which have a fixed size with respect to the Baranov intervals (independently of  $\Gamma$ ). The sampling constants for this class of test sets turn out to be uniformly bounded from below by a reverse Carleson measure result discussed in [5]. In the next step, following the line of proof performed in [16] (*see also* [14]), we will show that there are sufficiently many intervals in a test set containing points where the behavior of the functions and their derivatives are controlled by a generalized Bernstein type inequality originally due to Baranov. With this control, we can estimate the Taylor coefficients of the function whose norm we are interested in, and which allows us to obtain the uniform control required in the Remez-type inequality we alluded to above. Finally, as in Kovrijkine's work ([16]; similar estimates have also been used in [19]), we are then able to apply the Remez type inequality on the small intervals involving the proportion of  $\Gamma$  contained in the test sets to obtain the quantitative control we are interested in.

We will also recall the proof of Baranov's Bernstein result with the necessary details since we will need a slightly more precise form than that given in [3].

The paper is organized as follows. In the next section we introduce the necessary notation and our main result. In the succeeding section we will prove the necessity of our density condition. Bernstein type inequalities as considered by Baranov are one main ingredient in the proof of the sufficiency and will be discussed in Section 4, as well as a reverse Carleson measure result in model spaces. In the last section we prove the sufficiency of our density condition.

Throughout the paper, we use the notation  $C(x_1, \dots, x_n)$  to denote constants that depend only on some parameters  $x_1, \dots, x_n$  that may be numbers, functions or sets. Constants may change from line to line.

We occasionally write  $A \simeq B$  to say that there exists a constant  $C$  independent of  $A, B$  such that  $C^{-1}A \leq B \leq CA$ .

## 2. NOTATION AND STATEMENT OF THE MAIN RESULT

In order to state our main result, we first need to introduce the necessary notation. Let  $\Theta$  be an inner function in the upper half-plane  $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$ , *i.e.* a bounded analytic function with non tangential limits of modulus 1 almost everywhere on  $\mathbb{R}$ . For  $1 \leq p \leq +\infty$ , we denote by  $K_\Theta^p$  the shift co-invariant subspace (with respect to the adjoint of the multiplication semi-group  $e^{isx}$ ,  $s > 0$ ) in the Hardy class  $H^p = H^p(\mathbb{C}^+)$ , which is given (on  $\mathbb{R}$ ) by

$$K_\Theta^p = H^p \cap \overline{\Theta H^p}.$$

If  $\Theta(z) = \Theta_\tau(z) = \exp(2i\tau z)$  for some  $\tau > 0$ , then  $K_{\Theta_\tau}^p$  is up to the entire factor  $e^{-i\tau z}$  equal to the Paley–Wiener space  $PW_\tau^p$ , which is the space of entire functions on  $\mathbb{C}$  of exponential type at most  $\tau$ , whose restriction to the real line belong to  $L^p(\mathbb{R})$ .

Before considering more general model spaces, let us briefly recall the situation in the Paley-Wiener space.

**Definition 2.1.** A measurable set  $\Gamma \subset \mathbb{R}$  is called

— *relatively dense* if there exists  $\gamma, \ell > 0$  such that, for every  $x \in \mathbb{R}$ ,

$$(2.2) \quad |\Gamma \cap [x, x + \ell]| \geq \gamma \ell.$$

— *dominating* for  $K_\Theta^p$  if there exists a constant  $C(\Theta, p, \Gamma)$  such that, for every  $f \in K_\Theta^p$ ,

$$(2.3) \quad \int_{\mathbb{R}} |f|^p dx \leq C(\Theta, p, \Gamma) \int_{\Gamma} |f|^p dx.$$

The smallest constant  $C(\Theta, p, \Gamma)$  appearing in (2.3) will be called the sampling constant of  $\Gamma$  in  $K_\Theta^p$ .

Fixing  $\tau$  and  $p$ , the Logvinenko-Sereda theorem [17, 12] on equivalence of norms asserts that  $\Gamma$  is relatively dense if and only if it is a dominating set for  $PW_\tau^p$ .

As mentioned in the introduction, the Logvinenko-Sereda theorem has been extended to model spaces by Volberg [24] and Havin-Jörnicke [12]. Dyakonov [7, Theorem 3] proved that the classes of dominating sets for  $K_\Theta^p$  and relatively dense sets coincide if and only if  $\Theta$  has a bounded derivative:  $\Theta' \in L^\infty(\mathbb{R})$ . Though Dyakonov was not interested in explicit constants  $C(\Theta, p, \Gamma)$  a precise analysis of his method allows to obtain some estimates in this more restrictive situation. In order to state this estimate, let us recall the definition of harmonic measure. For  $z = x + iy \in \mathbb{C}^+$  and  $t \in \mathbb{R}$ , let

$$P_z(t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}$$

be the usual Poisson kernel in the upper half plane. For a measurable set  $\Gamma \subset \mathbb{R}$ , we denote by  $\omega_z(\Gamma)$  its harmonic measure at  $z$  defined by:

$$\omega_z(\Gamma) = \int_{\Gamma} P_z(t) dt.$$

It is easily shown that  $\Gamma$  is relatively dense if and only if  $\delta_y := \inf\{\omega_z(\Gamma) : \text{Im } z = y\} > 0$  for some (all)  $y > 0$ .

Dyakonov proved that, when  $\Theta' \in L^\infty(\mathbb{R})$  and  $\delta_y > 0$  then  $\Gamma$  is dominating for  $K_\Theta^p$ . An estimate of the sampling constants is not given in [7] but may be deduced from the proof (see Remark 2.5).

The aim of this work is to improve the estimates of Dyakonov's theorem as well as to establish a link between an appropriate density and the sampling constant for general meromorphic inner functions. Meromorphic inner functions are inner functions the zeros of which only accumulate at infinity, and whose singular inner part is reduced to  $e^{i\tau z}$ ,  $\tau \geq 0$ , *i.e.*,

$$\Theta(z) = e^{i\tau z} \prod_{\lambda \in \Lambda} b_\lambda, \quad z \in \mathbb{C}_+,$$

where  $\Lambda = \{\lambda\} \subset \mathbb{C}_+$  is a Blaschke sequence in the upper half plane,

$$\sum_{\lambda \in \Lambda} \frac{\text{Im } \lambda}{1 + |\lambda|^2} < +\infty$$

only accumulating at  $\infty$ . Recall that the Blaschke factor in the upper half plane is given by

$$b_\lambda(z) = \frac{|\lambda^2 + 1|}{\lambda^2 + 1} \frac{z - \lambda}{z - \bar{\lambda}}, \quad z \in \mathbb{C}_+.$$

In the more general situation when  $\Theta$  has not necessarily bounded derivative, we have to adapt the concept of relative density. More precisely the size of the testing intervals, which was constant in the setting of classical relative density, has to take into account the distribution of zeros of  $\Theta$ . To this end, we will need the notion of sublevel set: given  $\varepsilon \in (0, 1)$ , this is defined by

$$L(\Theta, \varepsilon) = \{z \in \mathbb{C}_+ : |\Theta(z)| < \varepsilon\}.$$

With this definition in mind, we can introduce one key tool in our setting. In [3, Lemma 3.3], Baranov constructed a Whitney type covering of  $\mathbb{R}$ . For this, define

$$d_\varepsilon(x) = \text{dist}(x, L(\Theta, \varepsilon)).$$

Then Baranov's construction yields a disjoint covering of  $\mathbb{R}$  by intervals  $I_n$  the length of which is comparable to the distance to the sublevel set. More precisely, following Baranov, we have  $I_n = [s_n, s_{n+1})$ , where  $(s_n)_n$  is a strictly increasing sequence  $\lim_{n \rightarrow \pm\infty} s_n = \pm\infty$ , given by

$$(2.4) \quad \int_{s_n}^{s_{n+1}} \frac{1}{d_\varepsilon(x)} dx = c,$$

where  $c > 0$  is some fixed constant. Moreover there exists  $\alpha \geq 1$  such that

$$(2.5) \quad \frac{1}{\alpha} d_\varepsilon(x) \leq |I_n| \leq \alpha d_\varepsilon(x), \quad x \in I_n.$$

Such a sequence will be henceforth called a Baranov sequence. In order to put our work in some more perspective to Dyakonov's work we shall recall an important connection

between  $d_\varepsilon(x)$  and  $|\Theta'(x)|$ . This requires the notion of the spectrum of  $\Theta$  which is defined as

$$\sigma(\Theta) = \{z \in \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\} : \liminf_{\zeta \rightarrow z} |\Theta(\zeta)| = 0\}.$$

For meromorphic inner functions  $\sigma(\Theta)$  consists of the zeros of  $\Theta$  and, provided  $\Theta$  is not a finite Blaschke product, the point  $\infty$ . Setting  $d_0(x) = \text{dist}(x, \sigma(\Theta))$ , Baranov showed in [2, Theorem 4.9] that

$$d_\varepsilon(x) \simeq \min(d_0(x), |\Theta'(x)|^{-1}), \quad x \in \mathbb{R}.$$

In particular, the Baranov intervals will be small when  $\Theta'$  is big.

Recall that Volberg [24] characterized dominating sets in  $K_\Theta^p$  as those sets for which

$$\inf_{z \in \mathbb{C}^+} (|\Theta(z)| + \omega_z(\Gamma)) > 0.$$

This characterization, based on harmonic measure, gives us an intuition that we cannot expect to measure the size of  $\Gamma$  only by looking at how much mass it puts on a Baranov interval. Indeed, harmonic measure of a set is not very sensitive with respect to the exact place where we put the set. For this reason we need to consider *amplified* intervals. For an interval  $I$  and  $a > 0$ , we will denote by  $I^a$  the amplified interval of  $I$  having same center as  $I$  and length  $a|I|$ .

We are now in a position to introduce our new notion of relative density.

**Definition 2.2.** Let  $(I_n)_{n \in \mathbb{Z}}$  be a Baranov sequence,  $\gamma \in (0, 1)$  and  $a \geq 1$ . A Borel set  $\Gamma$  is called  $(\gamma, a)$ -relatively dense with respect to  $(I_n)_{n \in \mathbb{Z}}$  if, for every  $n \in \mathbb{Z}$ ,

$$(2.6) \quad |\Gamma \cap I_n^a| \geq \gamma |I_n^a| = \gamma a |I_n|.$$

In case  $a = 1$  we will simply call the sequence  $\gamma$ -dense with respect to  $(I_n)_{n \in \mathbb{Z}}$ .

We would like to mention that if  $(I_n)_{n \in \mathbb{Z}}$  and  $(\tilde{I}_n)_{n \in \mathbb{Z}}$  are two Baranov sequences then if  $\Gamma$  is  $(\gamma, a)$ -relatively dense with respect to  $(I_n)_{n \in \mathbb{Z}}$  then there is a  $\tilde{\gamma} > 0$  and a  $\tilde{a} > 1$  such that  $\Gamma$  is also  $(\tilde{\gamma}, \tilde{a})$ -relatively dense with respect to  $(\tilde{I}_n)_{n \in \mathbb{Z}}$ . This follows essentially from the fact that the Baranov intervals are defined by  $\int_{I_n} d_\varepsilon^{-1}(x) dx = c$  where  $c$  is a constant (different for  $(I_n)_{n \in \mathbb{Z}}$  and  $(\tilde{I}_n)_{n \in \mathbb{Z}}$ ), see e.g. [3, Lemma 3.3], and that neighboring intervals are of comparable length. Therefore, in the remaining part of the paper,  $(I_n)_{n \in \mathbb{Z}}$  will be a *fixed* Baranov sequence.

The main result of this paper is the following.

**Theorem 2.3.** *Let  $p \in (1, \infty)$ ,  $\Theta$  be a meromorphic inner function, and let  $\Gamma \subset \mathbb{R}$  be a measurable set. Then the following conditions are equivalent*

- (i)  $\Gamma$  is  $(\gamma, a)$ -relatively dense with respect to  $(I_n)_{n \in \mathbb{Z}}$ , for some  $\gamma > 0$  and some  $a \geq 1$ ,
- (ii)  $\inf_{z \in \mathbb{C}^+} (\omega_z(\Gamma) + |\Theta(z)|) > 0$

(iii) there exists  $C > 0$  such that for every  $f \in K_{\Theta}^p$ ,

$$\int_{\mathbb{R}} |f(x)|^p dx \leq C \int_{\Gamma} |f(x)|^p dx.$$

Moreover, if the equivalent conditions (i)-(iii) hold, then

$$C \leq \exp\left(C(\Theta, p, \varepsilon) \frac{a^2}{\gamma} \ln \frac{1}{\gamma}\right).$$

We need to make some remarks here. First, the equivalence between (ii) and (iii) is of course Volberg's result. His result works in a much broader situation since he considers arbitrary inner functions and not only meromorphic inner ones. Also he considers not only harmonic measure of subsets of  $\mathbb{R}$  but harmonic extensions of general  $L^1$  functions (with respect to the measure  $(1 + |x|^2)^{-1} dx$ ). One novelty here is the connection with the new notion of relative density. Next we want to discuss two main differences with Dyakonov's work besides the control of the sampling constant. The obvious difference is that the boundedness assumption of  $\Theta'$  is not required. To handle that situation, the Baranov intervals will be small where  $\Theta'$  is big, and so the  $(\gamma, a)$ -relative density means, loosely speaking, that the more zeros of  $\Theta$  we put somewhere, the more mass of  $\Gamma$  has to be localized there in order to correctly measure functions in  $K_{\Theta}^p$ . Another observation is that even when  $\Theta'$  is bounded, this result gives some new information. More precisely, when  $\Theta$  has very few zeros in certain regions, then the corresponding Baranov intervals will be big, and so we distribute mass of  $\Gamma$  — comparably to the length of the Baranov interval — wherever we want in such an interval (this is perfectly coherent with Volberg's harmonic measure characterization), while classical relative density requires some uniform distribution in such big intervals. Indeed, each subinterval of sufficiently large but fixed length has to contain a fixed portion of  $\Gamma$ .

A special situation occurs when  $a = 1$ , i.e. when the sequence  $\Gamma$  is  $\gamma$ -relatively dense with respect to  $(I_n)_n$ , in other words each Baranov interval (without amplification) contains a least proportion  $\gamma$  of  $\Gamma$ . In this situation we improve significantly the constant. More precisely we have the following result.

**Corollary 2.4.** *Let  $p \in (1, \infty)$ ,  $\Theta$  be a meromorphic inner function, and let  $\Gamma \subset \mathbb{R}$  be a measurable set. If  $\Gamma$  is  $\gamma$ -relatively dense with respect to  $(I_n)_{n \in \mathbb{Z}}$ ,  $\gamma > 0$ , then for every  $f \in K_{\Theta}^p$ ,*

$$\int_{\mathbb{R}} |f(x)|^p dx \leq \left(\frac{1}{\gamma}\right)^{C(\Theta, p, \varepsilon)} \int_{\Gamma} |f(x)|^p dx.$$

We should point out that in this situation we thus obtain a polynomial dependence on  $1/\gamma$  in accordance with Kovrijkine's optimal result in the Paley-Wiener space.

*Remark 2.5.* Let us also compare this result to what may be obtained from Dyakonov's proof when  $\Theta' \in L^\infty(\mathbb{R})$ . In this situation, there is a suitable  $y > 0$  such that  $L(\Theta, \varepsilon) \subset \{z \in \mathbb{C} : \text{Im } z > y\}$  (see e.g. [7, p. 2222]). Fix such an  $y > 0$  and let  $m_y = \inf\{\Theta(z) : 0 < \text{Im } z < y\} > 0$  and  $\delta_y = \inf\{\omega_z(\Gamma) : \text{Im } z = y\}$ . Recall that Dyakonov proved that if



$\delta_y > 0$ , then  $\Gamma$  is dominating. A careful inspection of his proof further leads to the estimate

$$\|f\|_{L^p(\mathbb{R})} \leq \frac{2^{\frac{1}{p\delta_y}}}{m_y} \|f\|_{L^p(\Gamma)}, \quad f \in K_{\Theta}^p, \quad 0 < p < \infty.$$

This estimate gives an exponential control of the sampling constant depending on  $\delta_y$  (and hence on  $\gamma$ ), while our estimate is polynomial in  $\gamma$ .

*Remark 2.6.* We shall discuss here another natural guess for a necessary density condition. As mentioned above, the Baranov intervals are given by (2.4) where the constant  $c$  is fixed arbitrarily. Then one could think that if  $\Gamma$  is dominating then there exists a suitable  $c$  such that the associated intervals satisfy  $|I_n \cap \Gamma| \geq \gamma |I_n|$ . As it turns out, this does not work.

Here is an example : let  $\Lambda = (2^n i)_{n \geq 0}$ . Then  $L(\Theta, \varepsilon)$  is like a Stolz type angle

$$\Gamma_{\alpha}(0) = \{z = x + iy : y \geq \alpha(1 + |x|)\},$$

and hence  $d_{\varepsilon}(x) \simeq 1 + |x|$ .

From Volberg's characterization, it is clear that  $\Gamma = \mathbb{R}_-$  is dominating (in the Stolz angle  $\Gamma_{\alpha}(0)$ , the harmonic measure of  $\mathbb{R}_-$  is bounded from below by a strictly positive constant). However it is not dominating. Indeed from  $d_{\varepsilon}(x) \simeq 1 + |x|$  it can be deduced that one can choose  $I_n = [q^n, q^{n+1})$  for  $n \in \mathbb{N}$ , and any fixed  $q > 1$ ,  $I_{-n} = -I_n$ . Clearly for  $n$  big enough we have  $|I_n \cap \mathbb{R}_-| = \emptyset$ .

Observe that for  $a > q$  there is  $\gamma > 0$  such that  $|I_n^a \cap \mathbb{R}_-| \geq \gamma |I_n^a|$ , so that  $\mathbb{R}_-$  is  $(\gamma, a)$ -dense, while it is never  $\gamma$ -dense with respect to any Baranov sequence.

*Remark 2.7.* Meromorphic inner functions are those appearing in the context of de Branges spaces of entire functions [6]. Note that for an entire function  $E$  satisfying  $|E(z)| \geq |E(\bar{z})|$ ,  $\text{Im } z > 0$ , and having zeros only in the open lower half plane, the de Branges space is defined by  $\mathcal{H}(E) = \{F \in \text{Hol}(\mathbb{C}) : F/E, F^*/E \in H^2\}$  and  $\|F\|_{\mathcal{H}(E)} = \|F/E\|_{L^2(\mathbb{R})}$  (here  $F^*(z) = \overline{F(\bar{z})}$ ). Since  $F \mapsto F/E$  maps unitarily  $\mathcal{H}(E)$  onto the model space  $K_{\Theta}$ , where  $\Theta(z) = E^*(z)/E(z)$ , an immediate consequence of our results is a characterization of those measurable  $\Gamma$  for which  $\|F\|_{\mathcal{H}(E)}^2 \leq C \int_{\Gamma} |f(x)|^2 / |E(x)|^2 dx$  (with the same control of constants as in the corresponding model spaces).

### 3. PROOF OF (III) IMPLIES (I) IN THEOREM 2.3.

Our aim is to test the sampling inequality on normalized reproducing kernels. Recall that the reproducing kernel for  $K_{\Theta}^p$  at  $\lambda \in \mathbb{C}_+$  is defined by

$$(3.7) \quad k_{\lambda}(z) = \frac{i}{2\pi} \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{z - \bar{\lambda}}, \quad z \in \mathbb{C}_+.$$

This means that for every  $f \in K_{\Theta}^p$  and every  $\lambda \in \mathbb{C}_+$ , we have  $f(\lambda) = \langle f, k_{\lambda} \rangle = \int_{\mathbb{R}} f(x) \overline{k_{\lambda}(x)} dx$ . Then, classical estimates give for  $\lambda = x + iy \in L(\Theta, \varepsilon)$ ,

$$\|k_{\lambda}\|_p^p = \frac{1}{(2\pi)^p} \int_{\mathbb{R}} \left| \frac{1 - \overline{\Theta(\lambda)}\Theta(t)}{t - \bar{\lambda}} \right|^p dt \geq \frac{(1 - \varepsilon)^p}{(2\pi)^p} \int_{\mathbb{R}} \frac{1}{|t - \lambda|^p} dt \simeq \frac{1}{y^{p-1}} \int_{\mathbb{R}} \frac{1}{1 + |t|^p} dt$$

Since  $p > 1$ , the integral appearing in the last expression converges. Let  $c_p$  be the constant such that  $\|k_\lambda\|_p^p \geq c_p/y^{p-1}$ . For the discussions to come we set

$$C_1 = \left( \frac{1 + \varepsilon}{2\pi c_p} \right)^p,$$

where  $\varepsilon$  defines the sublevel set  $L(\Theta, \varepsilon)$ .

Now we are now in a position to prove the above mentioned implication. Suppose that we have (iii) of Theorem 2.3

$$\int_{\mathbb{R}} |f(x)|^p dx \leq C \int_{\Gamma} |f(x)|^p dx,$$

which we will now apply to normalized reproducing kernels. This means that

$$(3.8) \quad \int_{\Gamma} \left| \frac{k_\lambda(x)}{\|k_\lambda\|_p} \right|^p dx \geq \frac{1}{C}, \quad \lambda \in \mathbb{C}_+.$$

Recall from (2.5) that there is  $\alpha$  such that

$$\frac{1}{\alpha} d_\varepsilon(x) \leq |I_n| \leq \alpha d_\varepsilon(x), \quad x \in I_n, \quad n \in \mathbb{N}.$$

This is in particular true when  $x = t_n$  is the center of  $I_n$ . We deduce that there exists  $\lambda_n = x_n + iy_n \in L(\Theta, \varepsilon)$  with  $|t_n - \lambda_n|/\alpha \leq |I_n| \leq \alpha|t_n - \lambda_n|$ . It should be noted that  $\operatorname{Re} \lambda_n$  does not need to be in  $I_n$ . Still we know that  $|x_n - t_n| \leq \alpha|I_n|$ . Then

$$\tilde{I}_n := [x_n - \frac{b}{\alpha}|I_n|, x_n + \frac{b}{\alpha}|I_n|) \subset [x_n - b|I_n|, x_n + b|I_n|) \subset I_n^a$$

for  $b = a/2 - \alpha$  (which, in order for  $b > 0$  requires  $a > 2\alpha$ ).

Then given any  $a \geq 1$ ,

$$(3.9) \quad \begin{aligned} \int_{\Gamma} \left| \frac{k_{\lambda_n}(x)}{\|k_{\lambda_n}\|_p} \right|^p dx &\leq \left( \frac{1 + \varepsilon}{2\pi c_p} \right)^p \int_{\Gamma} \frac{(\operatorname{Im} \lambda_n)^{p-1}}{|x - \lambda_n|^p} dx \\ &\leq C_1 \int_{\Gamma \cap I_n^a} \frac{y_n^{p-1}}{|x - \lambda_n|^p} dx + C_1 \int_{\mathbb{R} \setminus I_n^a} \frac{y_n^{p-1}}{|x - \lambda_n|^p} dx \\ &\leq C_1 \int_{\Gamma \cap I_n^a} \frac{y_n^{p-1}}{|x - \lambda_n|^p} dx + C_1 \int_{\mathbb{R} \setminus \tilde{I}_n} \frac{y_n^{p-1}}{|x - \lambda_n|^p} dx. \end{aligned}$$

We start estimating the second integral in (3.9). In order to do so, we make the change of variable  $u = (|x - x_n|/y_n)^p$ , so that

$$du = \frac{p|x - x_n|^{p-1}}{y_n^p} dx.$$

With this change of variable, and equivalence of  $\ell^p$  and  $\ell^2$ -norms in a two-dimensional vector space, we get

$$\begin{aligned} \int_{\mathbb{R} \setminus \tilde{I}_n} \frac{y_n^{p-1}}{|x - \lambda_n|^p} dx &\leq 2 \int_{\mathbb{R} \setminus \tilde{I}_n} \frac{y_n^{p-1}}{|x - x_n|^p + y_n^p} dx = \frac{2}{p} \int_{|u| \geq (b/\alpha)^p} \frac{1}{1 + |u|} \frac{du}{|u|^{1-1/p}} \\ &\leq \frac{4}{p} \int_{u \geq (b/\alpha)^p} \frac{du}{u^{2-1/p}} \\ &= \frac{4}{p-1} \left( \frac{a-2\alpha}{2\alpha} \right)^{p-1}. \end{aligned}$$

In particular, we may choose

$$a \geq 2\alpha \left[ \left( \frac{p-1}{8CC_1} \right)^{1/(p-1)} + 1 \right],$$

so that

$$(3.10) \quad C_1 \int_{\mathbb{R} \setminus \tilde{I}_n} \frac{\operatorname{Im} \lambda_n^{p-1}}{|x - \lambda_n|^p} dx \leq \frac{1}{2C}.$$

Without loss of generality we can assume that  $a$  is an integer. From (3.9) and (3.8) we deduce that

$$(3.11) \quad \int_{\Gamma \cap I_n^a} \frac{y_n^{p-1}}{|x - \lambda_n|^p} dx \geq \frac{1}{2CC_1}.$$

We will show that there exists  $\gamma_2 > 0$  such that for every  $n$ , there is  $\ell \leq a$ , with

$$(3.12) \quad |\Gamma \cap (I_n^{\ell+1} \setminus I_n^\ell)| \geq \gamma_2 |I_n|.$$

Once this is established we can deduce condition (2.6). Indeed, this follows from the following estimate:

$$|\Gamma \cap I_n^a| \geq |\Gamma \cap I_n^{\ell+1} \setminus I_n^\ell| \geq \gamma_2 |I_n| = \frac{\gamma_2}{a} a |I_n|,$$

which yields (2.6) with  $\gamma = \gamma_2/a$ .

It thus remains to prove (3.12). Suppose to the contrary that for every  $\delta > 0$ , there is  $n_\delta$  such that for every  $\ell \leq a-1$  we have

$$(3.13) \quad |\Gamma \cap (I_{n_\delta}^{\ell+1} \setminus I_{n_\delta}^\ell)| \leq \delta |I_{n_\delta}|.$$

Recall that  $|\lambda_{n_\delta} - t_{n_\delta}| \leq \alpha |I_{n_\delta}|$ .

Putting  $I_n^0 = \emptyset$ , we will now discuss the first integral in (3.9).

$$\begin{aligned} \int_{\Gamma \cap I_{n_\delta}^a} \frac{y_{n_\delta}^{p-1}}{|x - \lambda_{n_\delta}|^p} dx &= \sum_{\ell=0}^{a-1} \int_{\Gamma \cap (I_{n_\delta}^{\ell+1} \setminus I_{n_\delta}^\ell)} \frac{y_{n_\delta}^{p-1}}{|x - \lambda_{n_\delta}|^p} dx \\ &\leq 2 \sum_{\ell=0}^{a-1} \int_{\Gamma \cap (I_{n_\delta}^{\ell+1} \setminus I_{n_\delta}^\ell)} \frac{y_{n_\delta}^{p-1}}{(x - x_{n_\delta})^p + y_{n_\delta}^p} dx. \end{aligned}$$

Let now  $u_\ell$  be the closest point of  $I_{n_\delta}^{\ell+1} \setminus I_{n_\delta}^\ell$  to  $x_{n_\delta}$ . Then

$$\begin{aligned} \int_{\Gamma \cap I_{n_\delta}^a} \frac{y_{n_\delta}^{p-1}}{|x - \lambda_{n_\delta}|^p} dx &\leq 2 \sum_{\ell=0}^{a-1} \frac{y_{n_\delta}^{p-1}}{(u_\ell - x_{n_\delta})^p + y_{n_\delta}^p} \times |\Gamma \cap (I_{n_\delta}^{\ell+1} \setminus I_{n_\delta}^\ell)| \\ &\leq 2\delta |I_{n_\delta}| \sum_{\ell=0}^{a-1} \frac{y_{n_\delta}^{p-1}}{(u_\ell - x_{n_\delta})^p + y_{n_\delta}^p}, \end{aligned}$$

where we have used (3.13). Now,  $|u_\ell - x_{n_\delta}|$  essentially behaves as  $|\ell - \ell_0| \times |I_n|/2$  where  $\ell_0$  is such that  $x_{n_\delta} \in I_{n_\delta}^{\ell_0+1} \setminus I_{n_\delta}^{\ell_0}$ . Then we can control the sum by twice the sum starting in 0. Also, keeping in mind that  $|I_n|/\alpha \leq y_n \leq \alpha|I_n|$ ,

$$\begin{aligned} \int_{\Gamma \cap I_{n_\delta}^a} \frac{y_{n_\delta}^{p-1}}{|x - \lambda_{n_\delta}|^p} dx &\leq 4\delta |I_{n_\delta}| \sum_{\ell=0}^{a-1} \frac{y_{n_\delta}^{p-1}}{(|I_{n_\delta}| \ell/2)^p + y_{n_\delta}^p} \\ &\leq 4\delta |I_{n_\delta}| \sum_{\ell=0}^{a-1} \frac{y_{n_\delta}^{p-1}}{|I_{n_\delta}|^p (\ell/2)^p + (y_{n_\delta}/|I_{n_\delta}|)^p} \\ &\leq 4\delta \sum_{\ell=0}^{\infty} \alpha^{p-1} \frac{1}{(\ell/2)^p + \alpha^{-p}} \end{aligned}$$

since the last sum converges. Choosing  $\delta$  small enough, we get

$$\int_{\Gamma \cap I_{n_\delta}^a} \frac{y_{n_\delta}^{p-1}}{|x - \lambda_{n_\delta}|^p} dx < \frac{1}{2CC_1}$$

contradicting (3.11). □

#### 4. BACKGROUND ON THE BARANOV-BERNSTEIN INEQUALITY AND REVERSE CARLESON MEASURES FOR MODEL SPACES

Recall that we consider meromorphic inner functions  $\Theta$ . Associated with  $\Theta$  we will need two constants. The first one comes from Baranov's result on Bernstein inequalities, and the second one from a reverse Carleson measure result in  $K_\Theta^p$ .

**4.1. Baranov-Bernstein inequalities for model spaces.** In order to state Baranov's result we need some more notation. Given  $\varepsilon \in (0, 1)$  we have already introduced the sub-level set  $L(\Theta, \varepsilon) = \{z \in \mathbb{C}_+ : |\Theta(z)| < \varepsilon\}$ . Also, recall that for  $x \in \mathbb{R}$ , we had  $d_\varepsilon(x) = \text{dist}(x, L(\Theta, \varepsilon))$ .

The reproducing kernel was defined in (3.7). We now need a generalization of this. Indeed, there is a formula for the  $n$ -th derivative (see [3, Formula (2.2)] for general  $n$  or [2, Formula (7)] for  $n = 1$ ): for  $f \in K_\Theta^p$ ,

$$f^{(n)}(z) = n! \int_{\mathbb{R}} f(t) \overline{k_z(t)}^{n+1} dt,$$

(observe that in this formula,  $\overline{k_z(t)}^{n+1}$  is the  $(n+1)$ -th power of  $\overline{k_z(t)}$  and not the  $(n+1)$ -th derivative).

**Theorem 4.1** (Baranov). *Let  $\Theta$  be a meromorphic inner function. Suppose that  $\varepsilon \in (0, 1)$ ,  $1 < p < \infty$ . Then for every  $f \in K_\Theta^p$  and every  $n \in \mathbb{N}$ ,*

$$(4.14) \quad \|f^{(n)} d_\varepsilon^n\|_p \leq C(\Theta, p, \varepsilon) n! \left(\frac{4}{\varepsilon}\right)^n \|f\|_p.$$

The statement given here is a slightly more precise quantitative version of Baranov's Bernstein inequality (see [3, Theorem 1.5]) and its proof is largely similar to that of [2, 3]. We shall reproduce Baranov's argument below in order to get the right dependence on  $n$  of the constant.

*Proof.* Let  $f \in K_\Theta^p$ ,  $x \in \mathbb{R}$ , and write

$$\begin{aligned} \frac{1}{n!} d_\varepsilon^n(x) f^{(n)}(x) &= d_\varepsilon^n(x) \int_{\mathbb{R}} f(t) \overline{k_x(t)}^{n+1} dt \\ &= I_1 f(x) + I_2 f(x), \end{aligned}$$

where

$$\begin{aligned} I_1 f(x) &= d_\varepsilon^n(x) \int_{|t-x| \geq d_\varepsilon(x)/2} f(t) \overline{k_x(t)}^{n+1} dt, \\ I_2 f(x) &= d_\varepsilon^n(x) \int_{|t-x| < d_\varepsilon(x)/2} f(t) \overline{k_x(t)}^{n+1} dt. \end{aligned}$$

Put  $h(x) = d_\varepsilon(x)/2$ . We have

$$|I_1 f(x)| \leq 2^{n+1} d_\varepsilon^n(x) \int_{|t-x| \geq d_\varepsilon(x)/2} \frac{|f(t)|}{|t-x|^{n+1}} dt \leq 2 \cdot 4^n \times \tilde{I}_1 f(x)$$

where

$$\tilde{I}_1 f(x) = h(x) \int_{|t-x| \geq h(x)} \frac{|f(t)|}{|t-x|^2} dt.$$

According to [2, Theorem 3.1]  $\tilde{I}_1$  is a bounded operator from  $L^p$  to  $L^p$  for  $p > 1$ . Therefore, there exists  $c_1(\Theta, \varepsilon, p)$  such that

$$(4.15) \quad \|I_1 f\|_{L^p} \leq c_1(\Theta, \varepsilon, p) 4^n \|f\|_{L^p}.$$

Let us now estimate  $I_2 f$ . Since  $\Theta$  is a meromorphic inner function,  $\Theta$  admits an analytic continuation across  $\mathbb{R}$ .

By the Schwarz Reflection Principle,

$$\tilde{\Theta}(\zeta) = \begin{cases} \Theta(\zeta) & \text{if } \operatorname{Im} \zeta \geq 0, \\ 1/\overline{\Theta(\bar{\zeta})} & \text{if } \operatorname{Im} \zeta \leq 0 \end{cases}$$

(in the latter case  $\text{Im } \zeta$  has to be sufficiently close to 0 to avoid the poles of  $\Theta$ ). Let  $D_x = D(x, d_\varepsilon(x)/2)$  the disc of radius  $d_\varepsilon(x)/2$  centered at  $x$ . We put

$$h_x(\zeta) := \begin{cases} \frac{\tilde{\Theta}(\zeta) - \Theta(x)}{\zeta - x} & \text{if } \zeta \in D_x \setminus \{x\}, \\ \Theta'(x) & \text{if } \zeta = x. \end{cases}$$

The function  $h_x$  is well defined and analytic. From the Maximum Principle, we deduce that

$$|h_x(\zeta)| \leq \sup_{\zeta \in \partial D_x} |h_x(\zeta)| \leq 2 \frac{1 + 1/\varepsilon}{d_\varepsilon(x)} \leq \frac{4}{\varepsilon d_\varepsilon(x)}.$$

It follows that

$$\begin{aligned} I_2 f(x) &\leq d_\varepsilon^n(x) \int_{|t-x| < d_\varepsilon(x)/2} |f(t)| |h_x(t)|^{n+1} dt \\ &\leq \frac{2}{\varepsilon} \left(\frac{4}{\varepsilon}\right)^n \frac{2}{d_\varepsilon(x)} \int_{|t-x| < d_\varepsilon(x)/2} |f(t)| dt = \frac{2}{\varepsilon} \left(\frac{4}{\varepsilon}\right)^n Mf(x) \end{aligned}$$

where  $M\varphi(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \varphi(t) dt$  is the Hardy-Littlewood maximal operator (recall that  $x$  is real). Since  $M$  is bounded from  $L^p$  to  $L^p$ , we obtain a constant  $C(p)$  such that

$$(4.16) \quad \|I_2 f\|_{L^p} \leq \frac{C(p)}{\varepsilon} \left(\frac{4}{\varepsilon}\right)^n \|f\|_{L^p}.$$

As a result  $\|f^{(n)} d_\varepsilon^n\|_p \leq \frac{n!}{2\pi} (\|I_1 f\|_{L^p} + \|I_2 f\|_{L^p})$ , and (4.15)-(4.16) imply (4.14).  $\square$

**4.2. Carleson and sampling measures for model spaces.** The second result which will be important in this paper concerns reverse Carleson measures for model spaces. Recall that the Carleson window of an interval  $I$  is given by

$$S(I) = \{z = x + iy \in \mathbb{C}_+ : x \in I, 0 < y < |I|\}.$$

We need the following result about reverse Carleson measures for  $K_\Theta^p$ , see [5] for the case  $p = 2$  and [10] for the general case  $p > 1$  (and which works without requiring the Carleson measure condition).

**Theorem 4.2** (Blandignères *et al*). *Let  $\Theta$  be an inner function and  $\varepsilon > 0$ . Let  $\mu \in M_+(\mathbb{R} \cup \mathbb{C}^+)$ . Then there exists an  $N_0 = N_0(\Theta, \varepsilon) > 1$  such that if*

$$(4.17) \quad \inf_I \frac{\mu(S(I))}{|I|} > 0,$$

where the infimum is taken over all intervals  $I \subset \mathbb{R}$  with

$$S(I^{N_0}) \cap L(\Theta, \varepsilon) \neq \emptyset,$$

then, for every  $f \in K_\Theta^p$ ,

$$(4.18) \quad \int_{\mathbb{R}} |f|^p dx \leq C(\Theta, \mu, \varepsilon) \int_{\mathbb{R} \cup C^+} |f|^p d\mu.$$

The smallest possible constant in (4.18) will be called *reverse Carleson constant* (it correspond exactly to the sampling constant when  $d\mu = \chi_{\Gamma} dm$ , where  $\chi_{\Gamma}$  is the characteristic function of  $\Gamma$ ).

The theorem above will allow us to make the first reduction we mentioned in the introduction. Indeed, it will imply that the sampling constants of certain reference sets denoted by  $F_0^{a,\sigma}$  in the theorem below are uniform.

From now on, we will fix an integer

$$(4.19) \quad N \geq \max \left( (1 + \alpha) \sqrt{2} N_0, \frac{40 \times 8^{1/p} \alpha}{\varepsilon} \right),$$

where  $\alpha$  is the Baranov constant from (2.5) and  $N_0$  is given in Theorem 4.2.

As discussed earlier, in order to use the Remez-type inequality, we need intervals which are sufficiently small. For this reason we need to subdivide (uniformly) the Baranov intervals (or their amplified companions  $I_n^a$ ). This will be done now. Let us start from a fixed Baranov sequence  $(I_n)_n$ , and partition

$$I_n^a = \bigcup_{k=1}^{aN} I_{n,k}^a$$

where the  $I_{n,k}^a$ 's are intervals of length  $|I_{n,k}^a| = |I_n^a|/(aN) = |I_n|/N$ . (With no loss of generality, we may increase  $a$  in such a way that  $aN$  is an integer). For an application  $\sigma : \mathbb{N} \rightarrow \{1, 2, \dots, aN\}$ , set  $I_n^{a,\sigma} = I_{n,\sigma(n)}^a$ . Our aim is to compare these intervals with the unamplified intervals  $I_{k,l} := I_{k,l}^1$ . To this end, for a given  $\sigma$ , we define

$$A_n^\sigma = \{(k, l) : I_{k,l} \cap I_n^{a,\sigma} \neq \emptyset\},$$

so that  $A_n^\sigma$  is the smallest index set for which

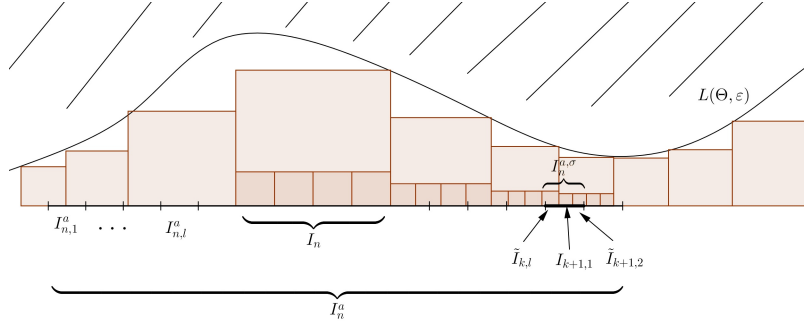
$$I_n^{a,\sigma} \subset \bigcup_{(k,l) \in A_n^\sigma} I_{k,l}.$$

We will also use the following notation for  $(k, l) \in A_n^\sigma$ :

$$\tilde{I}_{k,l} = I_{k,l} \cap I_n^{a,\sigma}$$

so that  $I_n^{a,\sigma} = \bigcup_{(k,l) \in A_n^\sigma} \tilde{I}_{k,l}$ . Then consider the set

$$F^{a,\sigma} = \bigcup_n \left( \bigcup_{(k,l) \in A_n^\sigma} \tilde{I}_{k,l} \right).$$


 FIGURE 1. Intervals  $I_n^a$ ,  $I_{n,k}^a$ ,  $I_n^{a,\sigma}$ ,  $\tilde{I}_{k,l}$ 

**Theorem 4.3.** *Let  $\Theta$  be an inner function and  $1 < p < \infty$ . Let  $\varepsilon > 0$ ,  $N_0 = N_0(\Theta, \varepsilon)$  as in Theorem 4.2, and let  $N$  be as in (4.19). Suppose  $\sigma : \mathbb{N} \rightarrow \{1, \dots, aN\}$ .*

*Given  $0 < \eta < 1$ . If for every  $n$  we choose  $A_n^0 \subset A_n^\sigma$  in such a way that*

$$(4.20) \quad \left| \bigcup_{(k,l) \in A_n^0} \tilde{I}_{k,l} \right| \geq \eta |I_n^{a,\sigma}|$$

then

$$F_0^{a,\sigma} = \bigcup_n \left( \bigcup_{(k,l) \in A_n^0} \tilde{I}_{k,l} \right)$$

is uniformly dominating meaning that there exists a constant  $C = C(\Theta, \alpha, p, \varepsilon)$ , independent of  $\sigma$ ,  $\eta$ ,  $a$  and the choice of  $A_n^0$  with (4.20), such that for every  $f \in K_\Theta^p$ , we have

$$(4.21) \quad \int_{F_0^{a,\sigma}} |f(t)|^p dt \leq \|f\|_{L^p(\mathbb{R})}^p \leq e^{C \frac{a^2}{\eta}} \int_{F_0^{a,\sigma}} |f(t)|^p dt.$$

Before proving this theorem, we discuss the special case  $a = 1$  that we need for Corollary 2.4. In this case, given  $\sigma$ , we have  $I_n^{a,\sigma} = I_{n,\sigma(n)}$ , and hence  $A_n^\sigma = \{(n, \sigma(n))\}$ . We also choose  $A_n^0 = A_n^\sigma$ , so that in (4.20) we have  $\eta = 1$ . Also  $F_0^\sigma := F_0^{1,\sigma} = \bigcup_n I_{n,\sigma(n)}$  and we get (4.21) with  $e^{C \frac{a^2}{\eta}}$  replaced by  $e^C$ . Let us state this as a separated result.

**Corollary 4.4.** *Let  $\Theta$  be an inner function and  $1 < p < \infty$ . Let  $\varepsilon > 0$ ,  $N_0 = N_0(\Theta, \varepsilon)$  as in Theorem 4.2, and let  $N$  be as in (4.19). Suppose  $\sigma : \mathbb{N} \rightarrow \{1, \dots, N\}$ .*

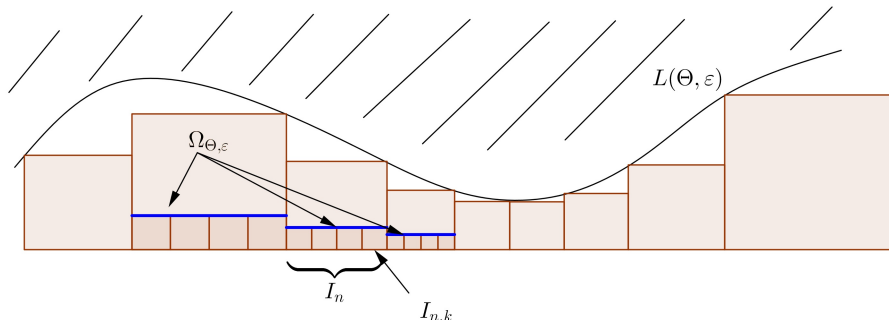
*Then the set*

$$F^\sigma = \bigcup_n I_{n,\sigma(n)}$$

is uniformly dominating meaning that there exists a constant  $C = C(\Theta, \alpha, p, \varepsilon)$ , independent of  $\sigma$ , such that for every  $f \in K_\Theta^p$ , we have

$$(4.22) \quad \int_{F^\sigma} |f(t)|^p dt \leq \|f\|_{L^p(\mathbb{R})}^p \leq e^C \int_{F^\sigma} |f(t)|^p dt.$$



FIGURE 2. Level sets, Baranov intervals, and  $\Omega_{\Theta, \varepsilon}$ 

Let us introduce the following notations (see Figure 2). Let

$$\Omega_{\Theta, \varepsilon} := \bigcup_{n \in \mathbb{Z}} \{z = x + iy \in \mathbb{C}_+ : x \in I_n, y = |I_n|/N, n \in \mathbb{N}\}.$$

Consider the measure

$$d\mu = \sum_{n \in \mathbb{Z}} \mathbf{1}_{I_n} dx \otimes \delta_{|I_n|/N}(y),$$

which defines usual arc length measure on the upper edges of

$$\bigcup_{k \in \mathbb{Z}} \bigcup_{l=1, \dots, N} S(I_{k,l}).$$

For the proof of Theorem 4.3 we need the following lemmas.

**Lemma 4.5.** *In the notation above,  $\mu$  is a Carleson measure and a reverse Carleson measure.*

*Proof.* Since  $\mu$  is the Lebesgue line measure supported on horizontal segments which, projected along the imaginary axis onto the real line, have intersection of Lebesgue measure zero, it is clearly a Carleson measure.

Let us now consider the reverse Carleson measure condition. Suppose  $I$  is a real interval with  $S(I^{N_0}) \cap L(\Theta, \varepsilon) \neq \emptyset$ , then for every  $x \in I$ , and with (4.19) in mind,

$$d_\varepsilon(x) \leq \sqrt{2}N_0|I| \leq \frac{N|I|}{1 + \alpha},$$

and hence,

$$|I| \geq \frac{(1 + \alpha)d_\varepsilon(x)}{N},$$

where  $x$  is arbitrary in  $I$ . On the other hand, there is  $n$  such that  $x \in I_n$ , so that when  $z = x + iy \in \Omega(\Theta, \varepsilon)$  we have  $y = |I_n|/N$  and from (2.5) we know that since  $x \in I_n$  we thus get

$$y = \frac{|I_n|}{N} \leq \frac{\alpha d_\varepsilon(x)}{N} \leq |I|.$$

As a result, for every  $x \in I$ , the corresponding  $z = x + iy \in \Omega_{\Theta, \varepsilon}$  is in  $S(I)$  so that  $\mu(S(I)) = |I|$ . Whence (4.17) is fulfilled and we conclude from Theorem 4.2 that  $\mu$  is a reverse Carleson measure.  $\square$

**Lemma 4.6.** *In the notation above we have*

$$\inf\{\omega_z(F_0^{a, \sigma}) : z \in \Omega_{\Theta, \varepsilon}\} \geq \delta = \frac{4\eta}{(2 + N(a + 1))^2 \pi} > 0,$$

independently of the choice of  $\sigma$  and  $A_n^0 \subset A_n^\sigma$  satisfying (4.20).

*Proof.* Observe that if  $z = x + iy \in \Omega_{\Theta, \varepsilon}$  then there exists  $n \in \mathbb{Z}$ , such that  $x \in I_n$  and  $y = |I_n|/N = |I_n^{a, \sigma}|$ .

Now, for  $t \in I_n^{a, \sigma} \subset I_n^a$ , the distance from  $t$  to  $x$  is bounded by the distance of one edge of  $I_n$  to the opposite edge of  $I_n^a$ , that is  $|x - t| \leq |I_n|/2 + |I_n^a|/2$ . Therefore

$$|z - t| \leq |y| + |x - t| \leq |I_n^{a, \sigma}| + \frac{1}{2}|I_n| + \frac{aN}{2}|I_n^{a, \sigma}| = |I_n^{a, \sigma}| \left(1 + \frac{N(a + 1)}{2}\right).$$

Hence

$$\begin{aligned} \omega_z(F_0^{a, \sigma}) &\geq \omega_z\left(\bigcup_{(k, l) \in A_n^0} \tilde{I}_{k, l}\right) = \frac{1}{\pi} \int_{\bigcup_{(k, l) \in A_n^0} \tilde{I}_{k, l}} \frac{|I_n^{a, \sigma}|}{|z - t|^2} dt \\ &\geq \frac{4 \times \left|\bigcup_{(k, l) \in A_n^0} \tilde{I}_{k, l}\right| \times |I_n^{a, \sigma}|}{(2 + N(a + 1))^2 \pi |I_n^{a, \sigma}|^2} \\ &\geq \frac{4\eta}{(2 + N(a + 1))^2 \pi} \end{aligned}$$

which proves the lemma.  $\square$

*Proof of Theorem 4.3.* The left hand inequality (Carleson embedding) is immediate.

Let us consider the right hand embedding (reverse Carleson inequality). From an idea of Havin-Jörjicke [11] and Dyakonov [7], we know that for every  $1 < q < +\infty$ , and for every  $f \in H^p$ , we have the Jensen inequality

$$\begin{aligned} |f(z)|^q &\leq 2 \left( \int_{F_0^{a, \sigma}} |f(t)|^q P_z(t) dt \right)^{\omega_z(F_0^{a, \sigma})} \left( \int_{\mathbb{R}} |f(t)|^q P_z(t) dt \right)^{1 - \omega_z(F_0^{a, \sigma})} \\ (4.23) \quad &= 2 \int_{\mathbb{R}} |f(t)|^q P_z(t) dt \left( \frac{\int_{F_0^{a, \sigma}} |f(t)|^q P_z(t) dt}{\int_{\mathbb{R}} |f(t)|^q P_z(t) dt} \right)^{\omega_z(F_0^{a, \sigma})} \end{aligned}$$

where  $P_z$  is the Poisson kernel in the upper half plane. Recall that

$$\Omega_{\Theta, \varepsilon} = \bigcup_{n \in \mathbb{Z}} \{z = x + iy \in \mathbb{C}_+ : x \in I_n, y = |I_n|/N, n \in \mathbb{N}\}.$$

It follows from Lemma 4.6 and (4.23) that, for  $z \in \Omega_{\Theta, \varepsilon}$ ,

$$\begin{aligned}
|f(z)|^q &\leq 2 \int_{\mathbb{R}} |f(t)|^q P_z(t) dt \left( \frac{\int_{F_0^{a, \sigma}} |f(t)|^q P_z(t) dt}{\int_{\mathbb{R}} |f(t)|^q P_z(t) dt} \right)^\delta \\
(4.24) \qquad &= 2 \left( \int_{F_0^{a, \sigma}} |f(t)|^q P_z(t) dt \right)^\delta \left( \int_{\mathbb{R}} |f(t)|^q P_z(t) dt \right)^{1-\delta}
\end{aligned}$$

Now pick a function  $f \in L^p(\mathbb{R})$  and let  $s = \frac{1}{2} \left(1 - \frac{1}{p}\right)$  so that  $(1-s)p = \frac{1+p}{2}$ . Note that  $0 < s < 1$  and that  $1 < (1-s)p < p$ .

Write  $q = (1-s)p$  and define the two harmonic functions  $u(z) = \int_{\mathbb{R}} |f(t)|^q P_z(t) dt$  and  $u_\sigma(z) = \int_{\mathbb{R}} \chi_{F_0^{a, \sigma}}(t) |f(t)|^q P_z(t) dt$ . Then (4.24) reads as

$$(4.25) \qquad |f(z)|^q \leq 2u_\sigma(z)^\delta u(z)^{1-\delta}.$$

Since by Lemma 4.5  $\mu$  is a Carleson measure, and so, in view of [9, Theorem I.5.6], there exists a constant  $C(p)$  such that, for every  $\varphi \in L^{1/(1-s)}(\mathbb{R}) = L^{\frac{1+p}{2p}}(\mathbb{R})$ ,

$$\int_{\Omega_{\Theta, \varepsilon}} \left[ \int_{\mathbb{R}} |\varphi(t)| P_z(t) dt \right]^{\frac{1}{1-s}} d\mu(z) \leq C(p) \int_{\mathbb{R}} |\varphi(t)|^{\frac{1}{1-s}} dt.$$

Applying this to  $\varphi = \chi_{F_0^{a, \sigma}} |f|^{(1-s)p}$  and to  $\varphi = |f|^{(1-s)p}$ , which are both in  $L^{1/(1-s)}$ , we get

$$(4.26) \qquad \int_{\Omega_{\Theta, \varepsilon}} u_\sigma(z)^{\frac{1}{1-s}} d\mu(z) \leq C(p) \int_{F_0^{a, \sigma}} |f(t)|^p dt$$

$$(4.27) \qquad \int_{\Omega_{\Theta, \varepsilon}} u(z)^{\frac{1}{1-s}} d\mu(z) \leq C(p) \int_{\mathbb{R}} |f(t)|^p dt.$$

Now, integrating (4.25) with respect to  $\mu$  we get with (4.24)

$$\begin{aligned}
\int_{\Omega_{\Theta, \varepsilon}} |f(z)|^p d\mu(z) &= \int_{\Omega_{\Theta, \varepsilon}} |f(z)|^{sp} |f(z)|^q d\mu(z) \\
&\leq 2 \int_{\Omega_{\Theta, \varepsilon}} |f(z)|^{sp} u_\sigma(z)^\delta u(z)^{1-\delta} d\mu(z) \\
&\leq 2 \left( \int_{\Omega_{\Theta, \varepsilon}} |f(z)|^p d\mu(z) \right)^s \left( \int_{\Omega_{\Theta, \varepsilon}} u_\sigma(z)^{\frac{\delta}{1-s}} u(z)^{\frac{1-\delta}{1-s}} d\mu(z) \right)^{1-s}
\end{aligned}$$

where we have applied Hölder's inequality with exponents  $1/s$ ,  $1/(1-s)$ . It follows that

$$\begin{aligned} \int_{\Omega_{\Theta,\varepsilon}} |f(z)|^p d\mu(z) &\leq 2^{\frac{1}{1-s}} \int_{\Omega_{\Theta,\varepsilon}} u_\sigma(z)^{\frac{\delta}{1-s}} u(z)^{\frac{1-\delta}{1-s}} d\mu(z) \\ &\leq 2^{\frac{2p}{1+p}} \left( \int_{\Omega_{\Theta,\varepsilon}} u_\sigma(z)^{\frac{1}{1-s}} d\mu(z) \right)^\delta \left( \int_{\Omega_{\Theta,\varepsilon}} u(z)^{\frac{1}{1-s}} d\mu(z) \right)^{1-\delta} \end{aligned}$$

where we have again used Hölder's inequality, now with exponents  $1/\delta$ ,  $1/(1-\delta)$  ( $\delta$  can be assumed in  $(0, 1)$ ). Using (4.26)-(4.27) this gives

$$\int_{\Omega_{\Theta,\varepsilon}} |f(z)|^p d\mu(z) \leq 2^{\frac{2p}{1+p}} C(p) \left( \int_{F_0^{a,\sigma}} |f(t)|^p dt \right)^\delta \left( \int_{\mathbb{R}} |f(t)|^p dt \right)^{1-\delta}.$$

On the other hand, from Lemma 4.5, we know that  $\mu$  is reverse Carleson with constant  $C_R$ , so

$$\int_{\mathbb{R}} |f(t)|^p dt \leq C_R^p \int_{\Omega_{\Theta,\varepsilon}} |f|^p d\mu \leq C_R 2^{\frac{2p}{1+p}} C(p) \left( \int_{F_0^{a,\sigma}} |f(t)|^p dt \right)^\delta \left( \int_{\mathbb{R}} |f(t)|^p dt \right)^{1-\delta},$$

which yields

$$\int_{\mathbb{R}} |f(t)|^p dt \leq \left( C_R 2^{\frac{2p}{1+p}} C(p) \right)^{1/\delta} \int_{F_0^{a,\sigma}} |f(t)|^p dt.$$

It remains to remember the form of  $\delta$  as given in Lemma (4.6) to conclude.  $\square$

## 5. PROOF OF (I) IMPLIES (III) IN THEOREM 2.3 AND ESTIMATE OF THE CONSTANTS.

In view of the construction of  $F_0^{a,\sigma}$  the main idea is to switch to the sets  $\tilde{I}_{k,l}$ ,  $(k, l) \in A_m^0$  ( $k, l, m$  appropriate).

Suppose the set  $\Gamma$  is  $(\gamma, a)$ -relatively dense with respect to the Baranov sequence  $(I_n)_n$ :

$$|\Gamma \cap I_n^a| \geq \gamma a |I_n| = \gamma |I_n^a|.$$

Since the  $I_{n,k}^a$ 's partition  $I_n^a$ , this implies that for every  $n$  there exists at least one  $k$ , denoted by  $k = \sigma(n)$ , such that

$$|\Gamma \cap I_{n,k}^a| \geq \gamma |I_{n,k}^a|.$$

By our previously introduced notation  $I_n^{a,\sigma} = I_{n,\sigma(n)}^a$ . Recall that  $A_n^\sigma = \{(k, l) : I_{k,l} \cap I_n^{a,\sigma} \neq \emptyset\}$  and  $I_n^{a,\sigma} = \bigcup_{(k,l) \in A_n^\sigma} \tilde{I}_{k,l}$ . Then the relative density condition translates to

$$(5.28) \quad \sum_{(k,l) \in A_n^\sigma} |\Gamma \cap \tilde{I}_{k,l}| = |\Gamma \cap I_n^{a,\sigma}| \geq \gamma |I_n^{a,\sigma}|.$$

Set

$$A_n^0 = \{(k, l) \in A_n^\sigma : |\Gamma \cap \tilde{I}_{k,l}| \geq \frac{\gamma}{2} |\tilde{I}_{k,l}|\}.$$

then decomposing  $A_n^\sigma = A_n^0 \cup A_n^\sigma \setminus A_n^0$ , we deduce from (5.28)

$$\sum_{(k,l) \in A_n^0} |\Gamma \cap \tilde{I}_{k,l}| \geq \gamma |I_n^{a,\sigma}| - \frac{\gamma}{2} \sum_{(k,l) \in A_n^\sigma \setminus A_n^0} |\tilde{I}_{k,l}| \geq \frac{\gamma}{2} |I_n^{a,\sigma}|,$$

which in particular yields condition (4.20) with

$$\eta = \gamma/2.$$

From now on we will use the notation

$$F := F_0^{a,\sigma} = \bigcup_n J_n,$$

where, as above,  $J_n = \tilde{I}_{k,l}$  for an appropriate  $(k,l) \in A_n^0$ . In particular, we deduce from the very definition of  $A_n^0$  that for every  $n$

$$|\Gamma \cap J_n| \geq \frac{\gamma}{2} |J_n|.$$

We should also recall that since  $J_n = \tilde{I}_{k,l} \subset I_{k,l}$  we have for every  $x \in J_n$ ,  $|J_n| \leq \frac{\alpha}{N} \text{dist}(x, L(\Theta, \varepsilon))$ .

A main ingredient of our proof is a Remez-type inequality which requires the control of the uniform norm on a bigger set depending on that of a smaller set. Our method only works when the sets are sufficiently small. For that reason we have to reduce the situation to sufficiently small sets which will be achieved using Theorem 4.3.

### 5.1. Step 1 — Reduction to the dominating set $F$ .

**Corollary 5.1.** *With the notation of Theorems 4.1 and 4.3, there exists a constant  $\tilde{C} = C(\Theta, \alpha, p, \varepsilon)$ , depending only on  $\Theta, \alpha, p$  and  $\varepsilon$  such that, for every  $f \in K_\Theta^p$  and every  $k \in \mathbb{N}$ , we have*

$$(5.29) \quad \int_F (|f^{(k)}(x)| d_\varepsilon(x)^k)^p dx \leq e^{\tilde{C} \frac{\alpha^2}{\gamma}} \left( \frac{4^k k!}{\varepsilon^k} \right)^p \int_F |f(x)|^p dx.$$

*Proof.* Indeed, using successively a trivial estimate, Theorem 4.1 and Theorem 4.3, we get for every  $f \in K_\Theta^p$ ,

$$\begin{aligned} \int_F (|f^{(k)}(x)| d_\varepsilon(x)^k)^p dx &\leq \int_{\mathbb{R}} (|f^{(k)}(x)| d_\varepsilon(x)^k)^p dx \\ &\leq C^p(\Theta, p, \varepsilon) \left( \frac{4^k k!}{\varepsilon^k} \right)^p \int_{\mathbb{R}} |f(x)|^p dx \\ &\leq C^p(\Theta, p, \varepsilon) \left( \frac{4^k k!}{\varepsilon^k} \right)^p e^{C \frac{\alpha^2}{\gamma}} \int_F |f(x)|^p dx \end{aligned}$$

as claimed.  $\square$

Inequality (5.29) means that we have a Bernstein inequality with respect to  $F$  so that we can replace  $\mathbb{R}$  by  $F$  in Theorem 2.3.

**5.2. Step 2 — Good intervals.** Assume that (5.29) holds. An integer  $n$  and the corresponding interval will be called *bad* if there exists an integer  $m_n$  such that

$$\int_{J_n} (|f^{(m_n)}(x)|d_\varepsilon(x)^{m_n})^p dx \geq e^{\tilde{C}\frac{\alpha^2}{\gamma}} 4^{m_n} \left(\frac{4^{m_n} m_n!}{\varepsilon^{m_n}}\right)^p \int_{J_n} |f(x)|^p dx.$$

Observe that  $m_n \geq 1$ . We will say that  $n$  and  $J_n$  are *good* if they are not bad.

**Claim 1.** *The good intervals contain most of the mass of  $f$  in the sense that*

$$\int_{\cup_{n \text{ is good}} J_n} |f(x)|^p dx \geq \frac{2}{3} \int_F |f(x)|^p dx.$$

*Proof of Claim 1.* By definition of bad intervals

$$\begin{aligned} \int_{\cup_{n \text{ is bad}} J_n} |f(x)|^p dx &= \sum_{n \text{ is bad}} \int_{J_n} |f(x)|^p dx \\ &\leq \sum_{n \text{ is bad}} \frac{e^{-\tilde{C}\frac{\alpha^2}{\gamma}}}{4^{m_n}} \left(\frac{\varepsilon^{m_n}}{4^{m_n} m_n!}\right)^p \int_{J_n} (|f^{(m_n)}(x)|d_\varepsilon(x)^{m_n})^p dx. \end{aligned}$$

Now

$$\begin{aligned} \frac{e^{-\tilde{C}\frac{\alpha^2}{\gamma}}}{4^{m_n}} \left(\frac{\varepsilon^{m_n}}{4^{m_n} m_n!}\right)^p \int_{J_n} (|f^{(m_n)}(x)|d_\varepsilon(x)^{m_n})^p dx \\ \leq \sum_{k \geq 1} \frac{e^{-\tilde{C}\frac{\alpha^2}{\gamma}}}{4^k} \left(\frac{\varepsilon^k}{4^k k!}\right)^p \int_{J_n} (|f^{(k)}(x)|d_\varepsilon(x)^k)^p dx. \end{aligned}$$

By Fubini's theorem we get

$$\begin{aligned} \int_{\cup_{n \text{ is bad}} J_n} |f(x)|^p dx &\leq \sum_{k \geq 1} \frac{e^{-\tilde{C}\frac{\alpha^2}{\gamma}}}{4^k} \left(\frac{\varepsilon^k}{4^k k!}\right)^p \int_{\cup_{n \text{ is bad}} J_n} (|f^{(k)}(x)|d_\varepsilon(x)^k)^p dx \\ &\leq \sum_{k \geq 1} \frac{e^{-\tilde{C}\frac{\alpha^2}{\gamma}}}{4^k} \left(\frac{\varepsilon^k}{4^k k!}\right)^p \int_F (|f^{(k)}(x)|d_\varepsilon(x)^k)^p dx. \end{aligned}$$

Bernstein's Inequality (5.29) then implies

$$\int_{\cup_{n \text{ is bad}} J_n} |f(x)|^p dx \leq \sum_{k \geq 1} \frac{1}{4^k} \int_F |f(x)|^p dx = \frac{1}{3} \int_F |f(x)|^p dx,$$

from which Claim 1 follows.  $\square$

**5.3. Step 3 — Good points.** Let  $\kappa > 1$ . For each *good*  $n$ , we will say that a point  $x \in J_n$  is  $\kappa$ -*good* if, for every  $k \in \mathbb{N}$ ,

$$|f^{(k)}(x)|^p \leq 2\kappa e^{\tilde{C}\frac{\alpha^2}{\gamma}} \times 8^k \left( \frac{4^k k!}{\varepsilon^k d_\varepsilon(x)^k} \right)^p \frac{1}{|J_n|} \int_{J_n} |f(x)|^p dx.$$

**Claim 2.** Let  $G_n$  be the set of  $\kappa$ -good points in  $J_n$ . Then  $|G_n| \geq \left(1 - \frac{1}{\kappa}\right) |J_n|$ .

*Remark 5.2.* Let  $\kappa = \frac{4}{\gamma}$ . As  $|J_n \cap \Gamma| \geq \frac{\gamma}{2} |J_n|$ , we get

$$|G_n \cap \Gamma| \geq |J_n \cap \Gamma| - |J_n \setminus G_n| \geq \frac{\gamma}{2} |J_n| - \frac{\gamma}{4} |J_n| = \frac{\gamma}{4} |J_n| \geq \frac{\gamma}{4} |G_n|.$$

This means that there are many good points in  $\Gamma$ , but we shall not use this fact.

*Proof of Claim 2.* Let  $B_n = J_n \setminus G_n$  be the set of bad points. Then for every  $x \in B_n$ , there exists  $k_x \geq 0$  such that

$$\frac{1}{|J_n|} \int_{J_n} |f(y)|^p dy \leq \frac{e^{-\tilde{C}\frac{\alpha^2}{\gamma}}}{2\kappa \times 8^{k_x}} \left( \frac{\varepsilon^{k_x} d_\varepsilon(x)^{k_x}}{4^{k_x} k_x!} \right)^p |f^{(k_x)}(x)|^p.$$

Therefore

$$\frac{1}{|J_n|} \int_{J_n} |f(y)|^p dy \leq \sum_{k \geq 0} \frac{e^{-\tilde{C}\frac{\alpha^2}{\gamma}}}{2\kappa \times 8^k} \left( \frac{\varepsilon^k d_\varepsilon(x)^k}{4^k k!} \right)^p |f^{(k)}(x)|^p.$$

Integrating both sides over  $B_n$ , we obtain

$$\begin{aligned} \frac{|B_n|}{|J_n|} \int_{J_n} |f(y)|^p dy &\leq \sum_{k \geq 0} \frac{e^{-\tilde{C}\frac{\alpha^2}{\gamma}}}{2\kappa \times 8^k} \left( \frac{\varepsilon^k}{4^k k!} \right)^p \int_{B_n} \left( |f^{(k)}(x)| d_\varepsilon(x)^k \right)^p dx \\ &\leq \sum_{k \geq 0} \frac{e^{-\tilde{C}\frac{\alpha^2}{\gamma}}}{2\kappa \times 8^k} \left( \frac{\varepsilon^k}{4^k k!} \right)^p \int_{J_n} \left( |f^{(k)}(x)| d_\varepsilon(x)^k \right)^p dx. \end{aligned}$$

Now, since  $n$  is good, we get

$$\frac{|B_n|}{|J_n|} \int_{J_n} |f(y)|^p dy \leq \sum_{k \geq 0} \frac{4^k}{2\kappa \times 8^k} \int_{J_n} |f(x)|^p dx = \frac{1}{\kappa} \int_{J_n} |f(x)|^p dx.$$

Since  $f$  cannot be 0 almost everywhere on  $J_n$ ,  $\int_{J_n} |f(x)|^p dx \neq 0$  and we get  $|B_n| \leq \frac{1}{\kappa} |J_n|$  which yields Claim 2.  $\square$

In the next step, we will need a Remez type inequality. There exist different versions of such inequalities, *e.g.* [19, Lemma B]. The one that seems most suitable for our needs is the following straightforward adaptation of a result of O. Kovrijkine [16, Corollary, p 3041] *see also* [14, Theorem 4.3].

**Lemma 5.3** (Kovrijkine's Remez Type Inequality). *Let  $p \in [1, \infty)$ . Let  $\Phi$  be an analytic function,  $J$  an interval and  $E \subset J$  a set of positive measure. Let  $M = \max_{D_J} |\Phi(z)|$  where  $D_J = \{z \in \mathbb{C}, \text{dist}(z, J) < 4|J|\}$  and let  $m = \max_J |\Phi(x)|$ , then*

$$(5.30) \quad \int_J |\Phi(s)|^p ds \leq \left( \frac{300|J|}{|E|} \right)^{p \frac{\ln(M/m)}{\ln 2} + 1} \int_E |\Phi(s)|^p ds.$$

**5.4. Step 4 — Conclusion.** It remains to apply Lemma 5.3 with  $\Phi = f$ ,  $J = J_n$  a good interval,  $E = \Gamma \cap J_n$ . We write  $M = \max_{y \in D_{J_n}} |f(y)|$  and  $m = \max_{x \in J_n} |f(x)|$ .

First, note that if  $x \in J_n$  is  $\kappa$ -good then, by assumption (4.19) on  $N$ ,

$$(5.31) \quad \frac{|J_n|}{d_\varepsilon(x)} \leq \frac{\alpha}{N} \leq \frac{\varepsilon}{40 \times 8^{1/p}}.$$

Further, for such an  $x$ , and a  $y$  with  $|x - y| < 10|J_n|$ , we get

$$\begin{aligned} |f(y)| &\leq \sum_{k \geq 0} \frac{|f^{(k)}(x)|}{k!} |x - y|^k \\ &\leq (2\kappa e^{\tilde{C} \frac{\alpha^2}{\gamma}})^{1/p} \sum_{k \geq 0} 8^{k/p} \left( \frac{4|x - y|}{\varepsilon d_\varepsilon(x)} \right)^k \left( \frac{1}{|J_n|} \int_{J_n} |f(x)|^p dx \right)^{1/p} \\ &= (2\kappa e^{\tilde{C} \frac{\alpha^2}{\gamma}})^{1/p} \sum_{k \geq 0} \left( \frac{|x - y|}{10|J_n|} \right)^k \left( \frac{40 \times 8^{1/p} |J_n|}{\varepsilon d_\varepsilon(x)} \right)^k \left( \frac{1}{|J_n|} \int_{J_n} |f(x)|^p dx \right)^{1/p} \\ &\leq (2\kappa e^{\tilde{C} \frac{\alpha^2}{\gamma}})^{1/p} \sum_{k \geq 0} \left( \frac{|x - y|}{10|J_n|} \right)^k \sup_{J_n} |f(x)|^p \end{aligned}$$

with (5.31). Summing this last series, we obtain

$$(5.32) \quad |f(y)| \leq \frac{(2\kappa e^{\tilde{C} \frac{\alpha^2}{\gamma}})^{1/p}}{1 - \frac{|x-y|}{10|J_n|}} \max_{J_n} |f(x)|.$$

But now, from Claim 2, we know that the set  $G_n$  of  $\kappa$ -good points has measure  $|G_n| \geq (1 - \frac{1}{\kappa}) |J_n|$ . Thus, if  $y \in D_{J_n}$ , there exists  $z \in J_n$  such that  $|z - y| \leq 4|J_n|$ , and there exists  $x \in G_n$  such that  $|z - x| \leq \frac{1}{2\kappa} |J_n|$  which yields  $|y - x| \leq \left( 4 + \frac{1}{2\kappa} \right) |J_n|$ . We can

now chose  $\kappa$  to be such that  $\frac{\kappa^{1/p}}{1 - \frac{1}{10} \left( 4 + \frac{1}{2\kappa} \right)} \leq 2$  (which is possible since the left hand side goes to  $20/11 < 2$  when  $\kappa \rightarrow 1$ ). Then, (5.32) implies

$$(5.33) \quad \max_{y \in D_{J_n}} |f(y)| \leq 2 \times (2e^{\tilde{C} \frac{\alpha^2}{\gamma}})^{1/p} \max_{x \in J_n} |f(x)|,$$



from which we obtain  $M/m \leq 2 \times (2e^{\tilde{C}\frac{a^2}{\gamma}})^{1/p}$ . Since  $|\Gamma \cap J_n| \geq \frac{\gamma}{2}|J_n|$ , Kovrijkine's Remez Type Inequality then reads

$$(5.34) \quad \int_{J_n} |f(x)|^p dx \leq \left(\frac{C_1}{\gamma}\right)^{C_2 \frac{a^2}{\gamma}} \int_{J_n \cap \Gamma} |f(x)|^p dx$$

where  $C_1 = C_1(\Theta, \alpha, p, \varepsilon)$ . Summing over all good intervals gives the result.  $\square$

We finish this section commenting on the proof of Corollary 2.4. We first observe that in view of Corollary 4.4 the constant  $e^{\tilde{C}a^2/\gamma}$  appearing in (5.29) turns out to be  $e^{\tilde{C}}$ , with  $\tilde{C} > C$  where  $C$  is the constant in (4.22). With this in mind, and following the lines of the proof above we see that reaching (5.33) the constant does not depend on  $\gamma$ , so that finally the exponent in (5.34) is just a constant as required  $\square$

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#### REFERENCES

- [1] A. ALEKSANDROV *On embedding theorems for coinvariant subspaces of the shift operator II*. Studies on linear operators and function theory. 27, Zap. Nauchn. Sem. POMI, **262**, POMI, St. Petersburg 1999, 5–48; English transl., J. Math. Sci. (New York) **110** (2002), 2907–2929.
- [2] A. BARANOV *Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings*. J. Funct. Anal. **223** (2005), 116–146.
- [3] A. BARANOV *Embeddings of model subspaces of the Hardy class: compactness and Schatten-von Neumann ideals*. Izv. Math. **73** (2009), 1077–1100.
- [4] A. BARANOV, E. FRICAIN, J. MASHREGHI *Weighted norm inequalities for de Branges–Rovnyak spaces and their applications*. Amer. J. Math. **132** (2010), 125–155.
- [5] A. BLANDIGNÈRES, E. FRICAIN, F. GAUNARD, A. HARTMANN & W. ROSS *Reverse Carleson embeddings for model spaces* J. Lond. Math. Soc. (2) **88** (2013), 437–464.
- [6] L. DE BRANGES *Hilbert spaces of entire functions*. Prentice-Hall, Inc., Englewood Clis, N.J., 1968.
- [7] K. DYAKONOV *Entire functions of exponential type and model subspaces in  $H^p$* . J. Math. Sci. **71** (1994), 2222–2233
- [8] K. DYAKONOV *Division and multiplication by inner functions and embedding theorems for star-invariant subspaces*. Amer. J. Math. **115** (1993), 881–902.
- [9] J. B. GARNETT *Bounded analytic functions*. Revised first edition. Graduate Texts in Mathematics, **236**. Springer, New York, 2007. xiv+459 pp.
- [10] E. FRICAIN, A. HARTMANN & W. ROSS *A survey on reverse Carleson measures*. <http://arxiv.org/abs/1601.00226>
- [11] V. HAVIN & B. JÖRICKE *The uncertainty principle in harmonic analysis*. Springer-Verlag, Berlin, 1994.
- [12] V. HAVIN & B. JÖRICKE *Traces of harmonic functions and comparison of  $L_p$ -norms of analytic functions*. Math. Nachr. **123** (1985), 225–254.
- [13] S. V. HRUSCEV, N. K. NIKOL'SKII & B. S. PAVLOV *Unconditional bases of exponentials and of reproducing kernels*. Lecture Notes in Math. **864** (1981), 214–335.
- [14] S. GHOBBER & P. JAMING *The Logvinenko–Sereda theorem for the Fourier-Bessel transform*. Integral Transforms Spec. Funct. **24** (2013), 470–484.
- [15] V. È. KACNEL'SON *Equivalent norms in spaces of entire functions*. Mat. Sb. (N.S.), **92** (1973), 34–54 English Transl. Math. USSR Sb. **21** (1973), 33–53.

- [16] O. KOVRIJKINE *Some results related to the Logvinenko-Sereda theorem*. Proc. Amer. Math. Soc. **129** (2001), 3037–3047.
- [17] V. N. LOGVINENKO & YU. F. SEREDA *Equivalent norms in spaces of entire functions of exponential type*. Teor. Funktsii, Funktsional. Anal. i Prilozhen **19** (1973), 234–246.
- [18] N. MAKAROV & A. POLTORATSKI *Meromorphic inner functions, Toeplitz kernels and the uncertainty principle*. In Perspectives in analysis, volume 27 of Math. Phys. Stud., pages 185–252. Springer, Berlin, 2005.
- [19] F. NAZAROV; M. SODIN & A. VOLBERG *Local dimension-free estimates for volumes of sublevel sets of analytic functions*. Israel J. Math. **133** (2003), 269–283.
- [20] B. P. PANEJAH. *On some problems in harmonic analysis*. Dokl. Akad. Nauk SSSR, **142** (1962), 1026–1029.
- [21] B. P. PANEJAH. *Some inequalities for functions of exponential type and a priori estimates for general differential operators*. Russian Math. Surveys **21** (1966), 75–114.
- [22] A. REZNIKOV *Sharp constants in the Paneyah-Logvinenko-Sereda theorem*. C.R. Acad. Sci. Paris **348** (2010), 141–144.
- [23] A. VOLBERG & S. TREIL *Imbedding theorems for the invariant subspaces of the backward shift operator*, Studies on linear operators and function theory. XV, Zap. Nauchn. Sem. LOMI, vol. 149, Nauka, Leningrad 1986, pp. 38–51; English transl., J. Soviet Math. **42** (1988), 1562–1572.
- [24] A. VOLBERG *Thin and thick families of rational fractions*. Complex analysis and spectral theory (Leningrad, 1979/1980), pp. 440–480, Lecture Notes in Math., 864, Springer, Berlin-New York, 1981.

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