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1 **Computation and characterization of local sub-filter-scale energy transfers**  
2 **in atmospheric flows**

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15 \*

## ABSTRACT

16 Atmospheric motions are governed by turbulent motions associated to non-  
17 trivial energy transfers at small scales (direct cascade) and/or at large scales  
18 (inverse cascade). Although it is known that the two cascades coexist, energy  
19 fluxes have been previously investigated from the spectral point of view but  
20 not on their instantaneous spatial and local structure. Here, we compute local  
21 and instantaneous sub-filter scale energy transfers in two sets of reanalyses  
22 (NCEP-NCAR and ERA-Interim) in the troposphere and the lower strato-  
23 sphere for the year 2005. The fluxes are mostly positive (towards subgrid  
24 scales) in the troposphere and negative in the stratosphere reflecting the baro-  
25 clinic and barotropic nature of the motions respectively. The most intense  
26 positive energy fluxes are found in the troposphere and are associated with  
27 baroclinic eddies or tropical cyclones. The computation of such fluxes can  
28 be used to characterize the amount of energy lost or missing at the smallest  
29 scales in climate and weather models.

## 30 1. Introduction

31 In the classical turbulence phenomenology, valid in homogeneous flows, energy is injected at  
 32 large scales, transferred downscale at a constant averaged rate  $\varepsilon$  (Kolmogorov (1941) cascade)  
 33 and dissipated at small scales by viscous effects (Frisch 1995). This phenomenology is based on  
 34 the so-called Karman-Howarth-Monin (KHM equation), derived directly from the Navier-Stokes  
 35 equation, that reads:

$$\frac{1}{2}\partial_t E + \varepsilon = -\frac{1}{4}\vec{\nabla}_\ell \cdot \langle \delta \vec{u} (\delta u)^2 \rangle + \nu \nabla_\ell^2 E, \quad (1)$$

36 where  $u$  is the velocity field,  $\nu$  the molecular viscosity,  $\langle \rangle$  means statistical average,  $\varepsilon$  is the mean  
 37 non-dimensional energy injection rate,  $\delta \vec{u} = \vec{u}(\vec{x} + \vec{\ell}) - \vec{u}(\vec{x})$  is the velocity increments over a  
 38 distance  $\ell$  and  $E(\ell) = \langle (\delta u)^2 \rangle / 2$  is a measure of the kinetic energy at scale  $\ell$ . KHM equation  
 39 can be seen as the counter-part in the physical space of traditional spectral energy budgets. In  
 40 stationary situations,  $\frac{1}{2}\partial_t E = 0$ , and the KHM equation describes how the injected energy  $\varepsilon$  is  
 41 split, at each given scale, into the two terms of the r.h.s: the "inertial term",  $-\frac{1}{4}\vec{\nabla}_\ell \cdot \langle \delta \vec{u} (\delta u)^2 \rangle$ , that  
 42 comes from the non-linear interactions, and the viscous term  $\nu \nabla_\ell^2 E$ , that comes from the viscosity.  
 43 This energy budget is valid at any scale, and depending on the value of the scale considered,  
 44 either the inertial or the viscous term dominates: for scales larger than the Kolmogorov scale  
 45  $\eta = (\nu^3/\varepsilon)$ , the inertial terms dominates, while the viscous term dominates for scales smaller  
 46 than  $\eta$ . Overall, the picture is that of an energy cascade: as the scale is decreased, the energy  
 47 injected at large scale  $\varepsilon$ , is gradually transferred at a constant rate to the scales  $\ell < \eta$ , where it is  
 48 dissipated into heat. The term  $-\frac{1}{4}\vec{\nabla}_\ell \cdot \langle \delta \vec{u} (\delta u)^2 \rangle$  therefore describes an energy flux at the scale  
 49  $\ell$ , characterizing the cascade. It must be positive in order to describe a forward energy cascade,  
 50 from large to small scale. This condition actually depends very much on the type of turbulence  
 51 we are considering: for example in 3D turbulence, the energy cascade is forward, while in 2D

52 incompressible turbulence, it is backward, with formation of larger and larger structures. In any  
53 case, based on this equation, the observation of the existence of a constant energy flux and on a  
54 hypothesis of self-similarity, Kolmogorov was able to find out that the energy spectrum obeys a  
55  $k^{-5/3}$  law, corresponding to a forward energy cascade.

56 In the atmosphere, turbulence is much more complex than the homogeneous and isotropic  
57 one because of the influence of density stratification and rotation (Holton and Hakim 2012).  
58 Turbulence in such condition is known to develop a complex dynamics, with power law energy  
59 spectra, as revealed by accurate numerical simulations and laboratory experiments (Levich and  
60 Tzvetkov 1985; Schertzer et al. 1997; Falkovich 1992; Pouquet and Marino 2013). Depending  
61 on the scale of the flow, energy transfers can be directed either towards smaller scales (direct  
62 cascade) or towards larger scales (inverse cascade) (Bartello 1995). To date, there is in fact  
63 no general consensus about the direction of cascades in the atmosphere. Observed energy  
64 spectra in the troposphere and in the lower stratosphere (Nastrom and Gage 1985) exhibit  $k^{-5/3}$   
65 law, generally connected to direct cascades, and/or  $k^{-3}$  power laws, associated to an inverse  
66 energy cascade. The inverse cascade has been historically associated to the quasi-geostrophic two  
67 dimensional dynamics induced by rotation (Charney 1971), and fed by baroclinic instability. Tung  
68 and Orlando (2003) simulated the Nastrom-Gage energy spectrum of atmospheric turbulence  
69 as a function of wavelength with a two-level quasi-geostrophic model, and were able to obtain  
70 both spectral behaviours with this simple dynamics. Kitamura and Matsuda (2006) analysed the  
71 role of stratification and rotation in the generation of the cascades, observing that in experiments  
72 without planetary rotation, the obtained spectral slope was steeper and energy transfer to larger  
73 vertical wave-numbers was increased. Some theories for a mesoscale inverse cascade for stratified  
74 (not quasi-geostrophic) turbulence were proposed by Gage (1979) and Lilly (1983) but these  
75 are no longer considered viable. According to Lindborg (2005), atmospheric mesoscale  $-5/3$

76 energy spectra can be explained by the existence of a direct cascade arising in the limit of strong  
77 stratification while the role of planetary rotation is to inhibit the cascade process at large scales  
78 leading to an accumulation of kinetic energy and steepening of the kinetic energy spectrum at  
79 small wave numbers. Evidence of the existence of a direct energy cascade comes from high  
80 resolution direct numerical simulations of stratified flows (Lindborg 2006). They also suggest  
81 that the direction of the cascade may be crucially dependent on the ratio of the Brunt-Väisälä  
82 frequency to the rotation frequency.

83

84 A way to clarify the situation is to compute directly the energy fluxes. In the classical picture  
85 of turbulence, such energy transfers are related directly to the skewness of velocity increments  
86  $\langle \delta \mathbf{u} (\delta u)^2 \rangle$ , and the direction of the cascade is provided by the sign of this quantity (negative  
87 for direct cascade, positive for inverse cascade). This quantity which is global, since it relies on  
88 space-average, has thus been used in the past to quantify the direction of the energy transfer. From  
89 the observed stratospheric third-order structure function, Lindborg and Cho (2001) argued that  
90 there is a forward energy cascade in the mesoscale range of atmospheric motions. In that study  
91 the authors pointed out that for scales smaller than 100 km the statistical inhomogeneities can be  
92 neglected while this assumption is not valid for larger scales.

93

94 Another approach relies on the spectral kinetic energy budgets (see e.g. Augier and Lindborg  
95 (2013) and Peng et al. (2015)). Alternatively, one may compute energy budget in the physical  
96 space, by considering appropriate generalization of the Karman-Howarth-Monin equation to in-  
97 clude the influence of rotation and stratification. To this aim, Augier et al. (2012) recently con-  
98 sidered a set of primitive equations for incompressible, non-diffusive and inviscid stably stratified  
99 fluid in the Boussinesq approximation, in order to account for both the kinetic energy (KE) and

100 available potential energy (APE) in a modified version of the KHM equation. The primitive equa-  
 101 tions for the rotating stratified Boussinesq fluid were written as:

$$\begin{aligned}
 \vec{\nabla} \cdot \vec{u} &= 0, \\
 \partial_t \vec{u} + (\vec{u} \cdot \vec{\nabla}) \vec{u} + 2\vec{\Omega} \times \vec{u} &= -\vec{\nabla} p + b\vec{e}_z + \nu \Delta \vec{u}, \\
 \partial_t b + (\vec{u} \cdot \vec{\nabla}) b &= -N^2 u_z + \kappa \Delta b,
 \end{aligned} \tag{2}$$

102 where  $u$  is the velocity field,  $u_z$  its vertical component,  $\Omega$  the rotation rate,  $p$  the rescaled pres-  
 103 sure,  $\nu$  and  $\kappa$  the viscosity and diffusivity,  $N = \sqrt{-(g/\rho_0)(d\bar{\rho}/dz)}$  the constant Brunt-Väisälä  
 104 frequency,  $\mathbf{e}_z$  the vertical unit vector, and  $b = -\rho'g/\rho_0$  the buoyancy perturbation,  $g$  the acceler-  
 105 ation due to gravity,  $\rho_0$  a reference density,  $\bar{\rho}(z)$  the mean profile and  $\rho'$  a density perturbation.  
 106 The resulting generalized KHM equation was written as:

$$\begin{aligned}
 \frac{1}{2} \partial_t E + \varepsilon &= -\frac{1}{4} \vec{\nabla}_\ell \cdot \langle \delta \vec{u} \left[ (\delta u)^2 + \frac{(\delta b)^2}{N^2} \right] \rangle, \\
 &\equiv \vec{\nabla} \cdot \vec{J},
 \end{aligned} \tag{3}$$

107 where  $E$  is now

$$E = \frac{1}{2} \langle (\delta u)^2 \rangle + \langle ((\delta b)^2 / N^2) \rangle, \tag{4}$$

108 and we have omitted contributions due to viscosity and diffusivity. Note that the rotation does  
 109 not enter explicitly into this energy budget because the Coriolis force is perpendicular to the flow.  
 110 It enters implicitly into the energy budget through third-order correlations, that can be shown  
 111 to depend explicitly on the rotation rate (Campagne 2015). This shows that the energy flux  $\vec{J}$  is  
 112 made of a KE flux and an APE flux, and formalizes the Lorenz Energy Cycle (LEC) description  
 113 of the atmospheric energy budget (Lorenz 1955) via the conversion of APE into KE and then into  
 114 dissipative heating. The KE and APE fluxes can have different direction, so that the resulting  
 115 energy flux can be positive, or negative, depending on the scale, isotropy or stratification, and

116 the corresponding direction of the energy cascade is hard to be predicted (Lovejoy and Schertzer  
117 2010). These approaches nevertheless only provide a global in space estimate of the energy  
118 transfers, so that one cannot connect them with observed coherent structures observed in the  
119 atmosphere. One improvement of our understanding of energy transfers would therefore require  
120 their local in space and time estimates, at any given scale.

121

122 This is now possible through an important breakthrough made by Duchon and Robert (2000),  
123 who reformulated the energy budget of the Navier-Stokes equations into a form allowing for the  
124 definition of energy transfers local in space and time and valid for any geometry including when  
125 strong inhomogeneity and anisotropy are present. Its ability to provide interesting information  
126 about energy transfers at a given scale  $\ell$  has been so far exploited in the experimental set-up of the  
127 Von Karman swirling flow to measure the scale to scale energy transfers and non viscous energy  
128 disipation (Kuzzay et al. 2015; Saw et al. 2016). The Duchon and Robert indicator requires only  
129 the 3D velocity fields and provides, for each instant, 3D maps of the sub-filter energy transfers at  
130 a scale  $\ell$ . The interest of this formulation is that it is devoid of any adjustable parameters unlike,  
131 for exemple, local estimates of energy budgets based on LES methods (Kuzzay et al. 2015).

132 In this work we adapt the definition of such an indicator to the atmospheric dynamics providing  
133 the first local maps of sub-filter-scale energy transfers without any adjustable parameter. The goal  
134 of this work is i) to identify and characterize the atmospheric motions responsible for large energy  
135 transfers and ii) to compute global time and spatial average and assess whether the reanalyses  
136 over(under)-represent energy fluxes. The paper is structured as follows. After presenting the  
137 indicator, we will study these transfers in the NCEP-NCAR and ERA-Interim reanalyses - to  
138 study the sensitivity of the results to the resolution - for the year 2005. This year is ideal as it does  
139 not correspond to major ENSO events or volcanic eruptions. We investigate: i) the vertical and



140 horizontal global averages, ii) the distribution of energy transfers at different scales. Results are  
 141 displayed in arbitrary units but in the same scale for NCEP-NCAR and ERA-Interim reanalyses.  
 142 We finally discuss the implications of our results on a theoretical and practical level.

## 143 2. Methods

144 For any solutions of the Navier-Stokes equations, Duchon and Robert (2000) defined energy  
 145 transfers in a fluid at an arbitrary scale  $\ell$  using a local energy balance equation

$$\partial_t E^\ell + \partial_j \left( u_j E^\ell + \frac{1}{2} (u_j \hat{p} + \hat{u}_j p) + \frac{1}{4} (\widehat{u^2 u_j} - \widehat{u^2} \hat{u}_j) - \nu \partial_j E^\ell \right) = -\nu \partial_j u_i \partial_j \hat{u}_i - \mathcal{D}_\ell, \quad (5)$$

146 where  $u_i$  are the components of the velocity field and  $p$  the pressure,  $\hat{u}$  and  $\hat{p}$  their coarse-grained  
 147 component at scale  $\ell$ ,  $E^\ell = \frac{\hat{u}_i \hat{u}_i}{2}$  is the kinetic energy per unit mass at scale  $\ell$  (such that  $\lim_{\ell \rightarrow 0} E^\ell =$   
 148  $u^2/2$ ),  $\mathcal{D}_\ell$  is expressed in terms of velocity increments  $\delta \vec{u}(\vec{r}, \vec{x}) \stackrel{def}{=} \vec{u}(\vec{x} + \vec{r}) - \vec{u}(\vec{x}) \equiv \delta \vec{u}(\vec{r})$  (the  
 149 dependence on  $\ell$  and  $\vec{x}$  is kept implicit) as:

$$\mathcal{D}_\ell(\vec{u}) = \frac{1}{4\ell} \int_{\mathcal{V}} d\vec{r} (\vec{\nabla} G_\ell)(\vec{r}) \cdot \delta \vec{u}(\vec{r}) |\delta \vec{u}(\vec{r})|^2, \quad (6)$$

150 where  $G$  is a smooth filtering function, non-negative, spatially localized and such that  
 151  $\int d\vec{r} G(\vec{r}) = 1$ , and  $\int d\vec{r} |\vec{r}|^2 G(\vec{r}) \approx 1$ . The function  $G_\ell$  is rescaled with  $\ell$  as  $G_\ell(\vec{r}) = \ell^{-3} G(\vec{r}/\ell)$ .  
 152 The choice of  $G$  slightly determines the local energy budget, in the sense that different choices of  
 153  $G$  may result in different level of kinetic or potential energy at a given scale  $\ell$ , as well as a differ-  
 154 ent repartition of injected energy between the kinetic and potential part. In that respect, the local  
 155 energy budget is  $G$ -sensitive and one should choose the filter that is more appropriate to boundary  
 156 conditions, and symmetry of the equations. However, we have checked on numerical simulations  
 157 of incompressible Navier-Stokes equation that the spatial average of the different terms is not sen-  
 158 sitive to the choice of  $G$ . Moreover, as shown in Duchon and Robert (2000), the choice of  $G$  has

159 no impact on the value of  $\mathcal{D}_\ell$ , in the limit  $\ell \rightarrow 0$ , as long as it satisfies the properties specified  
 160 previously. So we expect the  $G$ -sensitivity of the analysis to decrease with scale.

161 In the sequel, we choose a spherically symmetric function of  $x$  which has a Gaussian shape, be-  
 162 cause this filter function occurs naturally in any observational or experimental flow measurement,  
 163 or in LES simulations. The corresponding energy budget will then have a straightforward physical  
 164 meaning. This filter given by:

$$G_\ell(r) = \frac{1}{N} \exp(-1/(1 - (r/(2\ell))^2)), \quad (7)$$

165 where  $N$  is a normalization constant such that  $\int d^3r G_\ell(r) = 1$ . As we show later, results with  
 166 this filter for two different data reanalysis are consistent, at different resolutions. This makes us  
 167 confident that our results are robust.

168 As noticed by Duchon and Robert, the average of  $\mathcal{D}_\ell(\vec{u})$  can be viewed as a weak form of the  
 169 transfer term  $-\frac{1}{4}\vec{\nabla}_\ell \cdot \langle \delta\vec{u}(\delta u)^2 \rangle$  in the anisotropic version of the KHM equation Eq. (1), the  
 170 divergence being taken not on the term itself, but instead on the test function  $G_\ell$ . Therefore,  
 171  $\mathcal{D}_\ell(\vec{u})$  is a local version (no average is taken) of the energy transfer term of the KHM equation. By  
 172 construction,  $\mathcal{D}_\ell(\vec{u})$  represents the amount of energy transferred at the scale  $\ell$  by the inertial term.  
 173 For scales larger than the Kolmogorov scale (which is very small in atmospheric flow), this term  
 174 provides most of the energy transfer, since the viscous contribution is negligible. Its sign provides  
 175 the direction of the fluxes in the scale space: a positive sign implies transfer towards the scales  
 176 smaller than  $\ell$ .

177 By construction, the intrinsic weak formulation of  $\mathcal{D}_\ell(\vec{u})$  makes it less sensitive to noise than  
 178 classical gradients, or even than the usual KHM relation: indeed, the derivative in scale is not  
 179 applied directly to the velocity increments, but rather on the smoothing function, followed by a  
 180 local angle averaging. This guarantees that no additional noise is introduced by the procedure.

181 Even more, the noise coming from the estimate of the velocity is naturally averaged out by the  
182 angle smoothing as shown in Kuzzay et al. (2015). In the same study, the authors argued that the  
183 Duchon and Robert approach was a better alternative to the widespread large eddies simulation  
184 based method for the computation of energy fluxes, since it relies on very few arbitrary hypotheses.  
185 Experimentally, in the von Karman set-up, the DR formula provided a better estimate of the energy  
186 dissipation than a LES method: in particular, estimates of the injected and dissipated powers were  
187 within 20% of the measured value using the LES-PIV method, whereas reached 98% of the actual  
188 dissipation rate of energy with the DR formula (Kuzzay et al. 2015).

189 This approach for atmospheric dynamics requires taking into account density stratification,  
190 and considering Boussinesq equations instead of Navier-Stokes equations. We have adapted the  
191 Duchon-Robert formalism to the Boussinesq equations. The equation for the kinetic energy is  
192 simply restated as

$$\partial_t E^\ell + \vec{\nabla} \cdot \vec{J}_K^\ell = -\nu \partial_j u_i \partial_j \hat{u}_i - \mathcal{D}_\ell + \frac{1}{2} (b \hat{u}_z + \hat{b} u_z). \quad (8)$$

193 Using the point-split buoyancy perturbation as fundamental variable, we can then obtain an equa-  
194 tion related to the local variance of the buoyancy perturbation (details are given in the appendix)

$$\partial_t E_T^\ell + \vec{\nabla} \cdot \vec{J}_T^\ell = -\mathcal{D}_\ell^T - \frac{1}{2} (b \hat{u}_z + \hat{b} u_z) - \kappa \partial_j \hat{b} \partial_j b / N^2, \quad (9)$$

195 where  $E_T^\ell = \frac{\hat{b}b}{2N^2}$  is the available potential energy at scale  $\ell$ ,  $\mathcal{D}_\ell^T$  is expressed in terms of the  
196 increments  $\delta b(\vec{r}, \vec{x}) \stackrel{def}{=} b(\vec{x} + \vec{r}) - b(\vec{x}) \equiv \delta b(\vec{r})$  (the dependence on  $\ell$  and  $\vec{x}$  is kept implicit in the  
197 equations) as

$$\mathcal{D}_\ell^T = \frac{1}{4\ell} \int_{\mathcal{V}} d\vec{r} (\vec{\nabla} G_\ell)(\vec{r}) \cdot \delta \vec{u}(\vec{r}) |(\delta b)|^2 / N^2. \quad (10)$$

198 Considering now that the energy for stratified flows is given by expression (4), we can sum equa-  
199 tion (8) and (9), to get the total local energy balance

$$\partial_t E^\ell + \vec{\nabla} \cdot \vec{J}^\ell = -\mathcal{D}_\ell(\vec{u}, b) - \nu \partial_j \hat{u}_i \partial_j u_i - \kappa \partial_j \hat{b} \partial_j b / N^2, \quad (11)$$

200 where

$$\vec{J}^\ell = \vec{J}_K^\ell + \vec{J}_T^\ell, \quad (12)$$

201 is the spatial energy flux, and

$$\mathcal{D}_\ell(\vec{u}, b) = \frac{1}{4\ell} \int_{\mathcal{V}} d\vec{r} (\vec{\nabla} G_\ell)(\vec{r}) \cdot \delta\vec{u} \left[ (\delta u)^2 + \frac{(\delta b)^2}{N^2} \right], \quad (13)$$

202 is the total local scale to scale energy flux. It is easy to see that the average of  $\mathcal{D}_\ell(\vec{u}, b)$  is a  
203 weak formulation of the energy transfer terms of the generalized KHM equation of Augier et al  
204 Eq. (3). The DR indicator  $\mathcal{D}_\ell(\vec{u}, b)$  is thus a local energy transfer term, that can be split into a  
205 kinetic (dynamical) part  $\mathcal{D}_\ell(\vec{u})$  (the original DR indicator) and a potential (thermodynamic) part  
206 (the remaining part, implying the field  $b$ ). In order to easily implement the expression of  $\mathcal{D}_\ell(\vec{u}, b)$   
207 in climate models, the buoyancy parameter has been rewritten as a function of temperature  $T$  using  
208 the equation of state for dry air:  $\delta b = -\delta p / \rho_0 R \cdot 1 / \delta T$ , where  $\rho_0$  is a reference density at surface  
209 pressure and  $\delta p$  is a pressure horizontal perturbation, which is set to be about 1 hPa each 100 km.  
210 Furthermore, in Eq. 13, we set a constant Brunt-Väisälä frequency, amounting to  $1.2 \times 10^{-2} \text{ s}^{-1}$   
211 (Holton and Hakim 2012). In this way, the computation of  $\mathcal{D}_\ell(\vec{u}, b)$  only requires the numerical  
212 3D velocity  $u$  and  $T$  fields.

213 The sign and geometry of the zones associated with high and low values of  $\mathcal{D}_\ell(\vec{u}, b)$  will then  
214 provide interesting information about the dynamics of the energy exchange in the atmosphere. For  
215 example, a study of the occurrence of high and low values of  $\mathcal{D}_\ell(\vec{u})$  in the von Kármán swirling  
216 flow has revealed that such events are associated with well defined, characteristic geometry of the  
217 velocity field (Saw et al. 2016). For the kinetic (dynamical) part, positive values of  $\mathcal{D}_\ell(\vec{u})$  are  
218 measured whenever there is a strong convergence of the flow. Divergent flows are instead associ-  
219 ated to negative values of  $\mathcal{D}_\ell(\vec{u})$ , and they point to injection of energy from the sub-filter scales.  
220 This simple description is not valid anymore when we also consider the potential (thermodynamic)

221 component. For all these reasons, we cannot reduce the computation of  $\mathcal{D}_\ell(\vec{u})$  to only that of the  
222 divergence/vorticity.

### 223 3. Analysis

224 For this study, outputs of the ERA-Interim and NCEP-NCAR Reanalysis 1 have been used.  
225 ERA-Interim is a modern generation reanalysis with a much higher resolution model. NCEP-  
226 NCAR was pioneering when it was developed, but is run at a comparatively low resolution and  
227 does not take advantage of as many observations.

228  
229 ERA-Interim is the currently operational Reanalysis product at the European Center for  
230 Medium-Range Weather Forecasting (ECMWF) (Dee et al. 2011). Released in 2007, it provides  
231 reanalyzed data from 1979 to nowadays, stored at an original T255 spectral resolution (about  
232 80 km horizontal resolution), with 60 vertical hybrid model levels. A 12h four-dimensional  
233 variational data assimilation (4D-Var) is adopted. As a forecast model, the Integrated Forecast  
234 Model (IFS), Cy31r2 release, is used, fully coupling modules for the atmosphere, ocean waves  
235 and land surface. Sea-surface temperatures (SST) and sea-ice concentration (SIC) are ingested  
236 as boundary conditions and interpolated on a reduced-Gaussian grid as needed. In our case  
237 zonal, meridional and vertical wind components are considered at a  $0.75^\circ \times 0.75^\circ$  horizontal  
238 resolution over 12 pressure levels between 1000 and 100 hPa. A 12h time-step is considered.  
239 Known problems concerning these datasets are the lack of dry mass conservation (Berrisford et al.  
240 2011) and the slight asymmetry between evaporation and precipitation (Dee et al. 2011). The  
241 turbulent fluxes are based on the tiled ECMWF scheme for surface exchanges over land (Viterbo  
242 and Beljaars 1995; Viterbo and Betts 1999). Each gridbox is divided into up to six fractions  
243 (over land) depending on the type of surface, having different transfer coefficients based on a

244 Monin-Obukhov formulation. Similarly, over oceans, two different coefficients are used for stable  
245 and unstable conditions (Beljaars 1995).

246  
247 NCEP-NCAR Reanalysis 1 has been developed in a joint effort by the National Center for  
248 Environmental Prediction (NCEP) and the National Center for Atmospheric Research (NCAR)  
249 (Kalnay et al. 1996). The simulation is operational since January 1995, covering a period from  
250 1948 to nowadays. Data assimilation is performed via a 3D variational scheme (Parrish and  
251 Derber 1992). It features a T62 spectral resolution, corresponding to a  $2.5^\circ \times 2.5^\circ$  horizontal grid  
252 (about 200 km horizontal resolution), with 28 sigma levels. Most of the major physical processes  
253 involving the climate system are parametrized. SST, SIC, snow cover, albedo, soil wetness  
254 and roughness length are ingested as boundary conditions. Data are archived at an original 6h  
255 time-step, and such a temporal resolution is retained for our analysis. The atmospheric model  
256 which provides the NCEP/NCAR reanalysis data, uses bulk aerodynamic formulas to estimate  
257 the turbulent fluxes, with exchange coefficients depending on empirical profiles extending the  
258 Monin-Obukhov similarity relationship (Miyakoda and Sirutis 1986). For more details on the  
259 comparison between different subgrid parametrization of surface fluxes, one might refer to Brunke  
260 et al. (2011).

261

## 262 *a. Analysis of local energy transfers*

### 263 1) YEAR AND SEASONS' AVERAGES OF LOCAL ENERGY TRANSFERS

264 We begin the analysis by studying the latitudinal averages and the spatial features of the DR  
265 indicator for both the ERA-Interim reanalysis and the coarser NCEP-NCAR reanalysis. To enable  
266 comparison between the two datasets, one has to choose the analysis length larger than the reso-

267 lution scale of NCEP-NCAR (200km) since going below the resolution size introduces spurious  
 268 effects dependent on the filter design. On the other hand, since we want to have as much details as  
 269 possible, we have to choose the smallest scale consistent with those requirements. Here, we thus  
 270 adopt a scale of  $\ell = 220$  km, this scale being the smallest that provide reliable estimates of DR  
 271 indicator. A further discussion of the dependence of the results with scale is done in section 3.a.3.

272 Results obtained for both reanalysis are consistent with each other, as can be checked from  
 273 Figure 1 (ERA) and Fig. 2 (NCEP). The gross features do not depend on whether one undertakes a  
 274 year average (a,d), or seasonal (b,c,e,f): in the panels (a,b,c), which show height dependence of the  
 275 longitudinally averaged  $\langle \mathcal{D}_\ell(\vec{u}, b) \rangle_{long}$ , one observes the the total local energy transfers  $\mathcal{D}_\ell(\vec{u}, b)$   
 276 are mostly positive in the troposphere, about zero at the tropopause and negative in the lower  
 277 stratosphere. By looking at cuts at different pressure levels, one can look more precisely about  
 278 the spatial distribution of the yearly and seasonal averages of  $\langle \mathcal{D}_\ell(\vec{u}, b) \rangle_{time}$ . Close to the ground  
 279 ( $P = 1000$  hPa), the DR indicator is approximately zero except in proximity of sharp elevation  
 280 gradients (Antarctica costs, Himalaya, Greenland and Andes mountain ranges). By splitting the  
 281 local energy transfers is their kinetic  $\langle \mathcal{D}_\ell(\vec{u}) \rangle_{time}$  and thermodynamic part  $\langle \mathcal{D}_\ell^T \rangle_{time}$  (Fig. S1-S4 of  
 282 Supplementary Material), one sees that this effect is mostly due to the the density fluctuations (i.e.  
 283 the thermodynamic component of the DR indicator) that produce these negative fluxes.

284 In the middle troposphere ( $P = 500$  hPa), the behavior of  $\mathcal{D}_\ell(\vec{u}, b)$  is associated to that of the jet  
 285 stream, since the most intense positive patterns are observed in winter for the northern hemisphere  
 286 and summer for the southern hemisphere. In the lower stratosphere ( $P = 100$  hPa),  $\mathcal{D}_\ell(\vec{u}, b)$  is  
 287 negative at the middle latitudes, and become slightly positive in polar regions and in the intertrop-  
 288 ical convergence zone. Overall, the splitting between the kinetic and thermodynamic component  
 289 detailed in the Supplementary material suggests that the dynamical component dominates with

290 respect to the thermodynamic one, although the DR thermodynamic contributions are significant  
291 especially in the proximity of the ground.

## 292 2) CORRELATION WITH ENERGY SPECTRUM

293 The above result shows that the kinetic energy flux are globally positive in the troposphere, indi-  
294 cating a *direct* kinetic energy cascade, while they are negative in the lower stratosphere indicating  
295 a *inverse* kinetic energy cascade. Our results are therefore consistent with those found by Peng  
296 et al. (2015) who also found upscale transfer in the lower stratosphere at outer mesoscale length  
297 scales and downscale transfers at scales smaller than 360 km (KE) or 200 km (APE).

298 To get some insight on these cascades, we have further computed the kinetic horizontal energy  
299 spectra where  $k$  is the inverse of the wavelength from the horizontal velocity fields at different  
300 pressure levels in the two reanalysis. They are reported in Figure 3. One sees that for  $P \leq 500$   
301 hPa (corresponding to the stratosphere), the energy spectrum is mostly scaling like  $k^{-3}$ , while for  
302  $P \geq 500$  hPa (middle troposphere), the energy spectrum scales like  $k^{-5/3}$ , at least for scales larger  
303 than  $\ell = 220$  km-in agreement with the Nastrom-Gage spectrum in the lower stratosphere at scales  
304 between  $10^3$  and  $10^2$  km. In the ERA Interim data, the spectrum steepens below this scale and is  
305 closer to  $k^{-2}$ . These values are however to be taken with caution, since our resolution does not able  
306 one to distinguish clearly between a slope of  $-5/3$  and  $-7/5$  or  $-11/5$  and  $-3$ , which are classical  
307 spectral slope that appear in rotating stratified or quasi 2D turbulence. Moreover, the spectra are  
308 computed on a Cartesian grid which weight oddly higher latitudes, so that it is hard to trust results  
309 NCEP-NCAR reanalysis below  $\ell = 220$  km. The difference in spectra between the troposphere  
310 and stratosphere is a well-observed property of the kinetic energy spectra in the atmosphere, and  
311 several explanations have been proposed for that, some of them complementary to each other.  
312 The existence of an individual range spanning from planetary wavelengths to the edge of the sub-



313 inertial range) in the troposphere has been observed, e.g. by Koshyk et al. (1999). They found  
 314 that above 250 hPa there is a transition to a two-ranges spectrum at synoptic scales, so that the  
 315 sub-inertial range makes his way deep until about 2000 km wavelength and the spectrum exhibits  
 316 a slope of  $-5/3$ . At larger wavelengths the slope is still equal to  $-3$ . They explain this transition  
 317 with the rotational component of the kinetic energy decaying with height at a higher rate than  
 318 the divergent component. Accordingly, Burgess et al. (2013) attribute this transition to the zonal  
 319 mean-eddy interaction overcoming the eddy-eddy interaction in setting up the transient kinetic  
 320 energy peak. Žagar et al. (2017) propose also that this might be attributed to the predominant role  
 321 of non-linear inertial-gravity waves. In accurate numerical simulations, Pouquet et al. (2017) have  
 322 also shown that by changing the value of  $f/N$ , one also change the value of the crossover between  
 323 the large scales involving an inverse energy cascade, and the small scale, involving a direct energy  
 324 cascade. In the terrestrial atmosphere the value of  $f$  is practically constant, whereas  $N$  can vary up  
 325 to a factor 2 between troposphere and stratosphere, thus changing the direction of the cascade.

### 326 3) PROBABILITY DISTRIBUTION FUNCTIONS OF INSTANTANEOUS LOCAL ENERGY TRANS- 327 FERS

328 In addition to time average, it is also interesting to study the probability distribution function of  
 329 instantaneous local energy transfers,  $\mathcal{D}_\ell(\vec{u}, b)$  at a different height (pressure level), and see how  
 330 it varies with scale and height. This is provided in Figure 4 for ERA-Interim and Figure 5 for  
 331 NCAR reanalyses. Panels (b,d,f) show the distributions at each level for  $\ell = 220$  km. Panels (a,b)  
 332 show the kinetic component  $\mathcal{D}_\ell(\vec{u}, 0)$ , panels (c,d) the thermodynamic component  $\mathcal{D}_\ell^T$  and (e,f) the  
 333 total  $\mathcal{D}_\ell(\vec{u}, b)$ . Tables S1 and S2 of the supplemental material report the values of mean, standard  
 334 deviation, skewness and kurtosis as a function of the height for the total DR indicator.

335 Overall, all distributions are skewed, and exhibit fat tails. The sign of the skewness depends  
336 on the height: for both the total and kinetic component, it is positive in the lower troposphere,  
337 and negative for  $P < 500$  hPa, in agreement with the time averages. For the thermodynamic part,  
338 the behaviour is opposite, with a negative skewness at low altitude ( $P > 700hPa$ ) and positive  
339 skewness at large altitude. In such case, the distribution is totally asymmetric, and includes only  
340 positive transfer, indicating that in the high part of the atmosphere, the density fluctuations only  
341 contribute to a downscale energy transfer. Although there is agreement between the ERA interim  
342 and the NCEP-NCAR data, the latter shows fatter tails. This might be due either to the different  
343 resolution of the datasets and/or on the different physical parametrizations.

344 Looking now at the dependence with scale at fixed height, we see that both the kinetic and total  
345 local energy transfer display similar behaviour, with a tendency to have fatter tails with decreasing  
346 scales. This means that the energy imbalance of the reanalysis is reduced when we look at motions  
347 whose characteristic scales are larger. This type of behaviour, also observed in local energy trans-  
348 fers measured in a laboratory turbulent von Karman flow (Saw et al. 2016), might be due to the  
349 fact that, at larger scales, the atmosphere becomes more wave like (and so less turbulent) (Rhines  
350 1979). Regarding the thermodynamical part of the transfer, the scale dependence is much more  
351 mild on the positive side of the distribution, and even absent in the negative part of the distribution.

#### 352 4) POSSIBLE INTERPRETATION

353 A possible way to explain the sign of the DR indicators is to invoke the relation between baro-  
354 clinic and barotropic flows and direct and inverse cascades. In Tung and Orlando (2003), it is  
355 argued that the baroclinic motions responsible for the genesis and decay of extratropical cyclones  
356 are mostly associated to direct cascades (corresponding to positive  $\mathcal{D}_\ell(\vec{u}, b)$ ), while the essentially  
357 barotropic motions governing the lower stratosphere dynamics (Salby (1996), Chapter 17), are

358 associated to an inverse energy cascade (that would correspond to negative values of  $\mathcal{D}_\ell(\vec{u}, b)$ ). To  
359 check such interpretation, we have analyzed the maps of  $\mathcal{D}_\ell(\vec{u})$  collected each 6 or 12h depending  
360 on the datasets. They are collected for NCEP-NCAR in the supplementary video. Large positive  
361 and negative values of the DR indicator are found as dipoles in baroclinic eddies. When increasing  
362 the scale  $\ell$  of the analysis, the tails become lighter as the local positive and negative contributions  
363 get averaged out.

#### 364 **4. Discussion**

365 Weather and climate models do not resolve the viscous scales, which for the atmospheric mo-  
366 tions are order of 0.1 mm (Priestley 1959). Up to date, their resolution ranges from  $\simeq 2$  km of  
367 regional weather models to  $\simeq 100$  km of global climate models. To correctly represent dissipation  
368 effects at a scale  $\ell$ , the turbulent cascade needs to be parametrized at each grid point depending on  
369 the type of motion and the geographical constraints. Despite the importance of such energy trans-  
370 fers, their distribution and their time and spatial behavior is known only partially through field  
371 campaigns (Lübken 1997) or by global averages (Sellers 1969; Seinfeld and Pandis 2016). This  
372 does not ensure a global coverage and does not tell the direction of the energy transfers in the free  
373 troposphere. In this paper, we have used Duchon and Robert (2000) to compute and characterize  
374 the distribution of instantaneous and local sub-filter energy transfers in the atmosphere using 3D  
375 velocity fields obtained in NCEP-NCAR and ERA-Interim reanalysis. Those energy transfers are  
376 highly correlated with the baroclinic eddies occurring at mid-latitudes and with severe tropical  
377 cyclones. Our computation of local energy transfer provides the direction of the local energy cas-  
378 cade at a certain scale  $\ell$  in physical space. At the grid resolution  $\Delta x$ , the value of  $\mathcal{D}_{\Delta x}(\vec{u}, b)$  is an  
379 exact measure of the amount of energy *that must be transferred to subgrid scales* (positive DR  
380 contributions) or *that must be injected from the subgrid scales* (negative contributions) in order

381 to equilibrate energy budgets. If the simulation is perfectly resolved, so that  $\Delta x = \eta$ , this equi-  
382 libration is of course guaranteed by the contribution due to viscosity. In most cases, however,  
383 the Kolmogorov scale is not resolved, and one needs to artificially increase the viscosity, so as  
384 to absorb or produce this energy flux. The information about  $\mathcal{D}_{\Delta x}(\vec{u}, b)$  could then be used to  
385 interactively adjust the viscosity to account for the energy conservation laws in the atmosphere  
386 (Lucarini and Ragone 2011). Furthermore, the expression of  $\mathcal{D}_\ell(\vec{u})$  is separable in a dynamical  
387 and a thermodynamic contributions. Although most of the total  $\mathcal{D}_\ell(\vec{u}, b)$  contribution is due to  
388 the dynamical component, negative fluxes are found at the ground in presence of mountain ranges  
389 and sharp temperature/pressure gradients, positive fluxes in the middle troposphere reinforce the  
390 dynamic contributions. We have also observed that extreme events as tropical and extratropical  
391 storms are associated with large values of  $\mathcal{D}_\ell(\vec{u}, b)$ , even at the ground.

392 The quantity  $\mathcal{D}_\ell(\vec{u}, b)$  could also be a proxy of the flux of energy that can be exploited in wind  
393 turbines (Miller et al. 2011, 2015). Although our analysis is performed for large scale general  
394 circulation models, the Duchon and Robert (2000) formula can be applied to regional climate and  
395 weather prediction models. At smaller scales, it will be extremely interesting to analyze the rela-  
396 tion between  $\mathcal{D}_\ell(\vec{u}, b)$  and the genesis of extreme wind gusts or even tornadoes. At such scales,  
397 one could investigate the distributions of  $\mathcal{D}_\ell(\vec{u}, b)$  to the instantaneous subgrid scales dissipation  
398 obtained by field measurements (Higgins et al. 2003). It will also been worth investigating whether  
399 adaptive asymptotic methods, as those proposed by Klein et al. (2001) or the Lagrangian scale-  
400 dependant models for the subgrid scales in Large Eddy Simulations (Bou-Zeid et al. 2004), afford  
401 better energy balances, i.e. the spatial and temporal average of  $\mathcal{D}_\ell(\vec{u}, b)$  is closer to zero.

402 It is evident that the resolution plays an important role in determining spurious energy fluxes by  
403 looking at the difference in the  $\mathcal{D}_\ell(\vec{u}, b)$  indicator near the ground (NCAR vs ERA-Interim re-  
404 analysis). However, it is positively surprising that the average spatial and vertical structure of the

405 indicators is very similar in both reanalysis.

406 Several waves phenomena in the atmospheric - gravity and Rossby waves - are tied to these hor-  
 407 izontal density variations, and are associated with energy conversion between available potential  
 408 and kinetic energy. One key question is whether this diagnostic may incorrectly assess such en-  
 409 ergy conversion as an energy transfer across scale. For future research directions, it might be worth  
 410 applying the diagnostic to a simple gravity or Rossby wave model.

## 411 APPENDIX

### 412 A1. Derivation of the local Duchon-Robert equation for Boussinesq equations

413 We start from the Boussinesq equation Eqs. (2) for the buoyancy perturbation and write it at two  
 414 different position,  $\vec{x}$  and  $\vec{x}'$  for  $b(\vec{x})$  and  $b' = b(\vec{x}')$  and  $u(\vec{x})$  and  $u' = u(\vec{x}')$ :

$$415 \quad \partial_t b + \partial_j(u_j b) = -N^2 u_z + \kappa \partial_j^2 b, \quad (\text{A1})$$

$$\quad \partial_t b' + \partial_j(u'_j b') = -N^2 u'_z + \kappa \partial_j^2 b', \quad (\text{A2})$$

416 Multiplying the equation (A1) by  $b'$  and equation (A2) by  $b$  and adding the results we obtain

$$\quad \partial_t(bb') + b\partial_j(b'u'_j) + b'\partial_j(bu_j) = -N^2(b'u_z + bu'_z) + \kappa(b'\partial_j^2 b + b\partial_j^2 b'). \quad (\text{A3})$$

417 To simplify the equation, we can write the diffusive term as

$$\quad b'\partial_j^2 b + b\partial_j^2 b' = \partial_j^2 bb' - 2\partial_j b \partial_j b', \quad (\text{A4})$$

418 while the nonlinear can be written as

$$\quad b\partial_j b' u'_j + b'\partial_j b u_j = b\delta u_j \partial_j b' + \partial_j b u_j b', \quad (\text{A5})$$

419 where  $\delta u_j = u'_j - u_j$  as before. Considering the term  $(u'_j - u_j)(b' - b)^2 = \delta u_j(\delta b)^2$ , it reads as

$$\quad \delta u_j(\delta b)^2 = b'^2(\delta u_j) + b^2(\delta u_j) - 2b'(u'_j - u_j)b \quad (\text{A6})$$

420 Using now the identities  $\nabla_r \cdot (\delta \mathbf{u}) = \nabla_r \cdot \mathbf{u}' = 0$ , and after some manipulations we have:

$$b \delta u_j \partial_j b' = \frac{1}{2} \left[ \partial_j (b'^2 \delta u_j) - \partial_j (\delta u_j (\delta b)^2) \right] + \partial_j b u_j b' \quad (\text{A7})$$

421 Substituting the results from the equations (A4) and (A7) and multiplying both the sides by 1/2  
422 and simplifying gives:

$$\begin{aligned} \partial_t \left( \frac{1}{2} b b' \right) + \frac{1}{2} \partial_j \left( (u_j b') b + \frac{1}{2} b'^2 \delta u_j - \kappa \partial_j (b b') \right) &= \frac{1}{4} \nabla_r \cdot \delta \vec{u} (\delta b)^2 - \kappa \partial_j b \partial_j b' \\ &\quad - \frac{N^2}{2} (b u'_z + b' u_z) ; \end{aligned} \quad (\text{A8})$$

423 Applying the filter operator  $G_\ell$ , and noting  $\hat{f} = f * G_\ell$  (\* being the convolution), we get:

$$\begin{aligned} \partial_t \left( \frac{1}{2} \hat{b} \hat{b}' \right) + \vec{\nabla} \cdot \left( \frac{1}{2} (\vec{u} \hat{b}) b + \frac{1}{4} \widehat{(b^2 \vec{u})} - \frac{1}{4} \widehat{(b^2)} \vec{u} \right) - \kappa \vec{\nabla} \cdot \left( \frac{1}{2} \hat{b} \hat{b}' \right) \\ = - \frac{1}{4\ell} \int d\vec{r} (\vec{\nabla} G)_l \cdot \delta \vec{u}(r) (\delta b)^2 - \kappa \vec{\nabla} b \cdot \vec{\nabla} \hat{b} - \frac{N^2}{2} (b \hat{u}_z + \hat{b} u_z). \end{aligned} \quad (\text{A9})$$

424 Introducing  $E_T^\ell = \hat{b} \hat{b}' / 2N^2$ , the available potential energy at scale  $\ell$ , and the terms

$$\vec{J}_T^\ell = \left( \frac{1}{2} (\vec{u} \hat{b}) b + \frac{1}{4} \widehat{(b^2 \vec{u})} - \frac{1}{4} \widehat{(b^2)} \vec{u} - \kappa \vec{\nabla} \cdot \left( \frac{1}{2} \hat{b} \hat{b}' \right) \right) / N^2, \quad (\text{A10})$$

$$\mathcal{D}_\ell^b = \frac{1}{4\ell} \int d^d r (\vec{\nabla} G)_l \cdot \delta \vec{u}(r) (\delta b)^2 / N^2, \quad (\text{A11})$$

425 we get the equation, Eq (9) of section 2.

426 Now, to study the inviscid limit  $\nu \rightarrow 0$ , we take the limit  $\ell \rightarrow 0$  and introducing the available  
427 potential energy  $E_T = b^2 / 2N^2$ , the equation finally simplifies to:

$$\partial_t E_T + \vec{\nabla} \cdot \left( \frac{1}{2} \vec{u} E_T \right) - \kappa \nabla^2 E_T = -b u_z - \mathcal{D}^b - \kappa (\nabla b)^2 / N^2, \quad (\text{A12})$$

428 with

$$\mathcal{D}^b = \lim_{\ell \rightarrow 0} \mathcal{D}_\ell^b. \quad (\text{A13})$$

429 The equation for the kinetic energy has been derived in Duchon and Robert (2000), without the  
430 term due to rotation and buoyancy. For the rotation, it is straightforward to see that it only adds

431 a terms  $\vec{u} \cdot (2\vec{\Omega} \times \vec{u}) + \vec{u} \cdot (2\vec{\Omega} \times \vec{u})$  which vanishes, due to the symmetry of the  $\times$  operator. The  
 432 buoyancy adds a new term that can be simply included, so that the equation for the kinetic energy  
 433 writes as:

$$\begin{aligned} \partial_t E^\ell + \partial_j \left( \hat{u}_j E^\ell + \frac{1}{2} (u_j \hat{p} + \hat{u}_j p) + \frac{1}{4} \left( \widehat{u^2 u_j} - \frac{1}{4} \widehat{u^2} u_j \right) - v \partial_j E^\ell \right) = \\ - v \partial_j u_i \partial_j \hat{u}_i - \mathcal{D}_\ell + \frac{1}{2} (b \hat{u}_z + \hat{b} u_z), \end{aligned} \quad (\text{A14})$$

434 with  $\mathcal{D}_\ell$  being given by Eq. 6. Introducing the KE spatial flux:

$$\vec{J}_K^\ell = \hat{u} E^\ell + \frac{1}{2} (\vec{u} \hat{p} + \hat{u} p) + \frac{1}{4} \left( \widehat{(u^2 \vec{u})} - \frac{1}{4} \widehat{(u^2)} \vec{u} \right) - v \vec{\nabla} E^\ell, \quad (\text{A15})$$

435 we get Eq. (8) of section 2. The fact that the rotation does not enter explicitly into the kinetic energy  
 436 budget is well-known, and due to the fact that the Coriolis force does not produce energy. However,  
 437 it influences the energy cascade through the energy redistribution by nonlinear mechanisms such  
 438 as resonant wave interactions (Campagne et al. 2014). This process is taken into account in term  
 439  $\mathcal{D}_\ell$  via the the third-order moment.

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563 **LIST OF FIGURES**

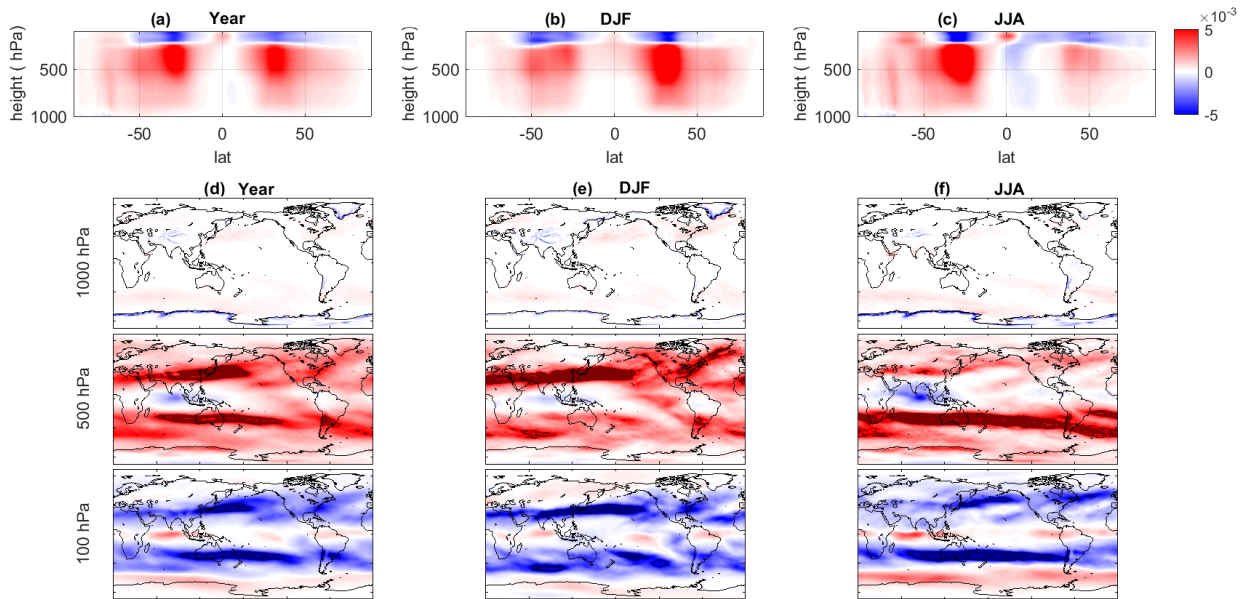
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566 yearly averages, (b,e) winter averages, (c,f) summer averages. . . . . 30

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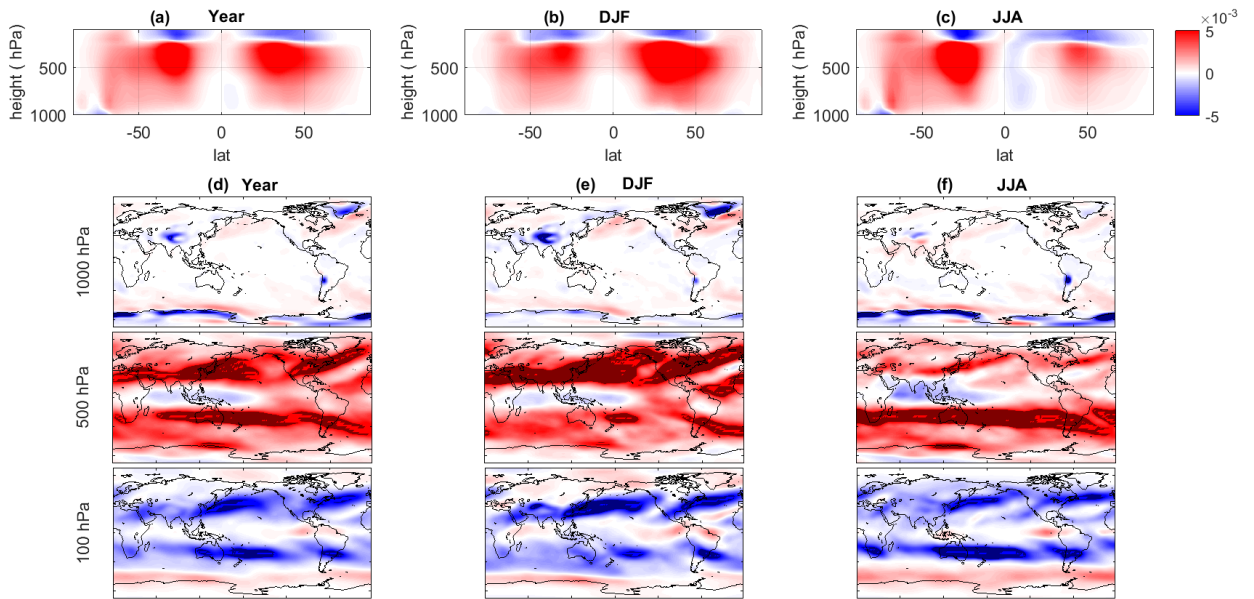
570 **Fig. 3.** Solid lines: spectra  $E(k)$ , where  $k$  is the wavelength) computed, at each pressure level, for  
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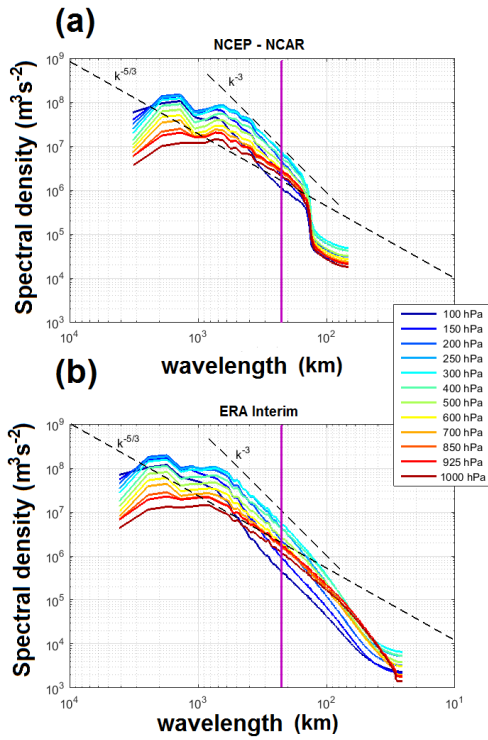


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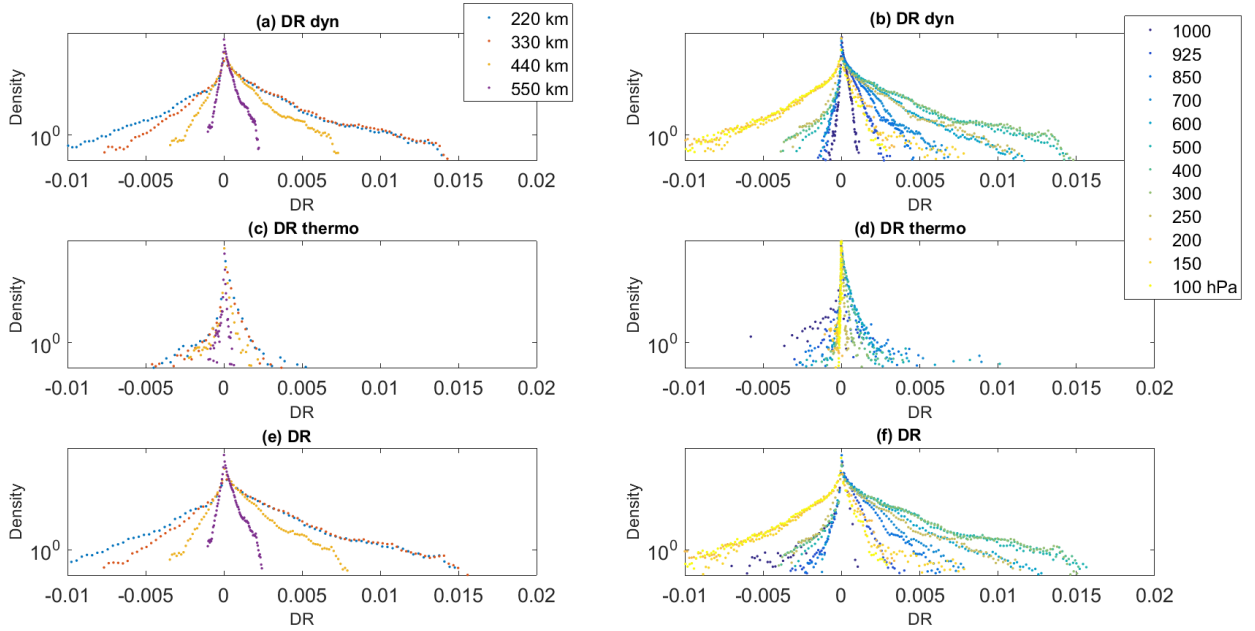


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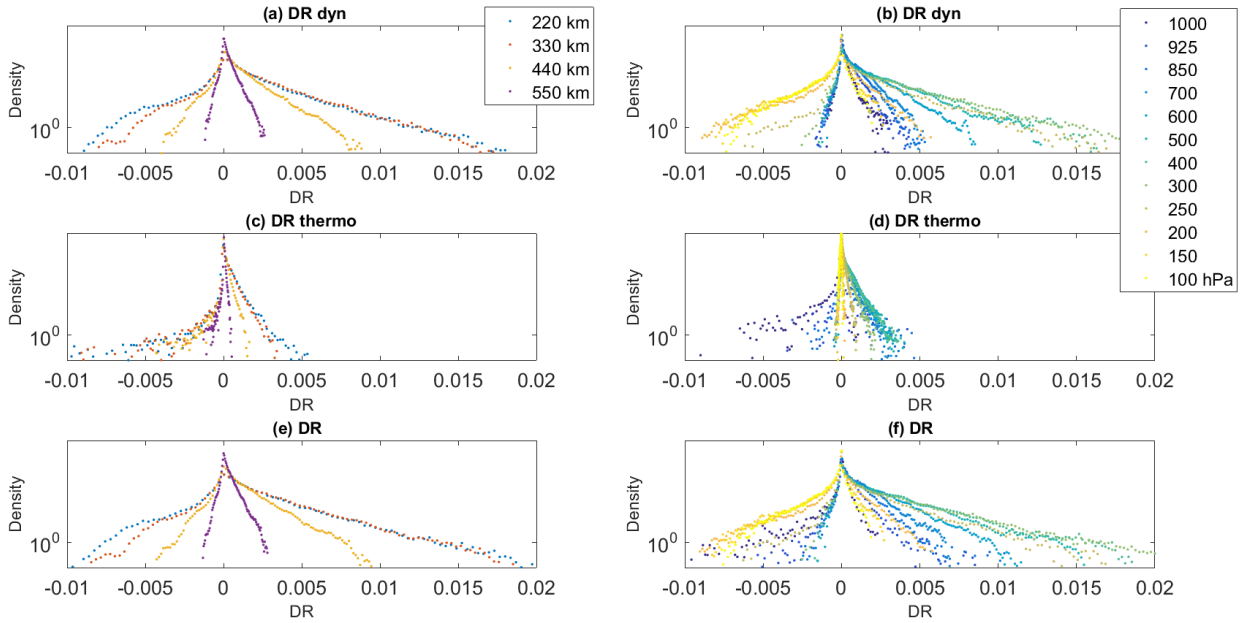




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