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EXISTENCE OF STATIONARY SOLUTIONS FOR SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH SUPERDIFFUSION

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ABSTRACT. In this article, we establish the existence of solutions of a system of integro-differential equations arising in population dynamics in the case of anomalous diffusion. The proof of the existence of solutions is based on a fixed point technique. Solvability conditions for elliptic operators without the Fredholm property in unbounded domains are used.

1. Introduction. In the present work, we address the existence of stationary solutions of the system of integro-differential equations

$$\frac{\partial u_s}{\partial t} = -D_s\sqrt{-\Delta}u_s + \int_{\mathbb{R}^d} K_s(x-y)g_s(u(y,t))\,dy + f_s(x), \quad 1 \le s \le N, \tag{1.1}$$

appearing in cell population dynamics. We believe that such a model is relatively new. The single equation analogous to (1.1) with the standard Laplacian in the diffusion term was studied in [**29**]. Herein, the space variable x corresponds to the cell genotype, $u_s(x,t)$ are densities for different groups of cells as functions of their genotype and time, and $u(x,t) = 3D(u_1(x,t), u_2(x,t), \dots, u_N(x,t))^T$. The right side of system (1.1) describes the evolution of cell densities by means of cell proliferation, mutations and cell influx. The anomalous diffusion terms here correspond to the change of genotype via small random mutations, and the nonlocal terms describe large mutations. In general, the anomalous diffusion comprises the case of the subdiffusion when the problem involves a fractional derivative with respect to time, and the superdiffusion when the negative Laplacian raised to a fractional power as in (1.1) is involved. Functions $g_s(u)$ are the rates of cell

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birth dependent upon u (density dependent proliferation), and the functions $K_s(x-y)$ show the proportion of newly born cells changing their genotype from y to x. Let us assume here that they depend on the distance between the genotypes. The last term on the right side of (1.1) describes cell influxes for different genotypes.

The square root of minus Laplacian in system (1.1) represents a particular example of *superdiffusion* actively studied in relation to different applications in plasma physics and turbulence **[7, 15]**, surface diffusion **[12, 13]**, semiconductors **[14]**, and so on. The physical meaning of superdiffusion is that the random process occurs with longer jumps in comparison with normal diffusion. The moments of jump length distribution are finite in the case of normal diffusion, but this is not the case for superdiffusion. The operator $\sqrt{-\Delta}$ is defined via spectral calculus. A similar problem in the presence of the standard Laplacian in the diffusion term was recently treated **[29, 30, 31]**. A single equation analogous to system (1.1) is addressed in **[30]**.

Let us set all $D_s = 1$ and study the existence of solutions of the system of equations

$$-\sqrt{-\Delta}u_s + \int_{\mathbb{R}^d} K_s(x-y)g_s(u(y))\,dy + f_s(x) = 0, \quad 1 \le s \le N. \tag{1.2}$$

We consider the case where the linear part of operator (1.2) does not satisfy the Fredholm property such that conventional methods of nonlinear analysis may not be applicable. Solvability conditions for non Fredholm operators, which is the novelty of our approach, along with the method of contraction mappings, will be used. Possible applications of our results are studies of nonlinear elliptic problems in unbounded domains involving operators with the essential spectrum containing the origin.

Let us consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either vanishing in the whole space or converging to 0 at ∞. For $a \ge 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.3) contains the origin. As a consequence, this operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$, the dimension of its kernel and the codimension of its image, are not finite.

In this paper, we study some properties of operators of this kind. Recall that elliptic equations with non Fredholm operators were extensively treated in recent years. Approaches in weighted Sobolev and Hölder spaces were developed [**2, 3, 4, 5, 6**]. Schrödinger-type operators without the Fredholm property were studied via the methods of spectral and scattering theory [**16, 17, 21, 22, 23**]. The Laplacian operator with drift, from the point of view of non Fredholm operators, was treated in [**26**] and linearized Cahn-Hilliard problems in [**24, 27**]. Articles [**25, 28**] were devoted to the studies of nonlinear, non Fredholm, elliptic problems. Significant applications to the theory of reaction-diffusion equations were developed [**9, 10**]. Non Fredholm operators also appear when studying wave systems with an infinite number of localized traveling waves, see [**1**]. In particular, in the case of $a = 0$, the operator A is Fredholm in some properly chosen weighted spaces, see [**2, 3, 4, 5, 6**]. However, the situation where $a \neq 0$ is significantly different, and the approach developed in these articles cannot be applied. Front propagation problems with superdiffusion were extensively treated in recent years, see e.g., [**18, 19**].

We set $K_s(x) = \varepsilon_s H_s(x)$ with $\varepsilon_s \geq 0$,

$$\varepsilon := \max_{1 \leq s \leq N} \varepsilon_s,$$

and we suppose that the next assumption holds.

Assumption 1.1. Let $1 \leq s \leq N$ be such that $f_s(x) : \mathbb{R}^3 \to \mathbb{R}$, $f_s(x) \in L^1(\mathbb{R}^3)$ and $\nabla f_s(x) \in L^2(\mathbb{R}^3)$. Furthermore, $f_s(x)$ is nonzero for a certain s. Assume also that $H_s(x) : \mathbb{R}^3 \to \mathbb{R}$ is such that $H_s(x) \in L^1(\mathbb{R}^3)$ and $\nabla H_s(x) \in L^2(\mathbb{R}^3)$. Moreover,

$$H^2 := \sum_{s=1}^{N} \|H_s\|^2_{L^1(\mathbb{R}^3)} > 0$$

and

$$Q^2 := \sum_{s=1}^{N} \|\nabla H_s\|^2_{L^2(\mathbb{R}^3)} > 0.$$

We choose space dimension $d = 3$, which is related to the solvability conditions for the linear Poisson equation (3.1) stated in Lemma 3.1. The results obtained below can be generalized to $d > 3$. From the point of view of applications the space dimension is not limited to $d = 3$ because the space variable is correspondent to cell genotype but not to the usual physical space.

By virtue of the standard Sobolev inequality, see e.g., [**11,** page 183], under Assumption 1.1, we have

$$f_s(x) \in L^2(\mathbb{R}^3), \quad 1 \le s \le N.$$

We use the Sobolev space of vector functions

$$H^2(\mathbb{R}^3, \mathbb{R}^N) := \{u(x) : \mathbb{R}^3 \longrightarrow \mathbb{R}^N \mid u_s(x) \in L^2(\mathbb{R}^3), \Delta u_s \in L^2(\mathbb{R}^3),\, 1 \le s \le N\},$$

equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)} := \sum_{s=1}^{N} \|u_s\|^2_{H^2(\mathbb{R}^3)} = \sum_{s=1}^{N} \{\|u_s\|^2_{L^2(\mathbb{R}^3)} + \|\Delta u_s\|^2_{L^2(\mathbb{R}^3)}\}. \tag{1.4}$$

Also,

$$\|u\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)} := \sum_{s=1}^{N} \|u_s\|^2_{L^2(\mathbb{R}^3)}.$$

The Sobolev embedding implies

$$\|\phi\|_{L^\infty(\mathbb{R}^3)} \le c_e \|\phi\|_{H^2(\mathbb{R}^3)}, \tag{1.5}$$

where $c_e > 0$ is the constant of the embedding. The hat symbol will denote the standard Fourier transform such that

$$\widehat{\phi}(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \phi(x) e^{-ipx} dx. \tag{1.6}$$

When all nonnegative parameters $\varepsilon_s = 0$, we obtain the linear Poisson equations

$$\sqrt{-\Delta} u_s = f_s(x), \quad 1 \le s \le N. \tag{1.7}$$

By virtue of Lemma 3.1, along with Assumption 1.1, problem (1.7) has a unique solution $u_{0,s}(x) \in H^1(\mathbb{R}^3)$ such that no orthogonality conditions are required. Lemma 3.1 gives us that, in dimensions $d < 3$, we need specific orthogonality relations for the solvability of (1.7) in $H^1(\mathbb{R}^d)$. (We will not study the problem in dimensions $d > 3$ to avoid extra technicalities because the proof will be based on similar ideas, see Lemma 3.1.) By means of Assumption 1.1, using that

$$\|\Delta u_s\|^2_{L^2(\mathbb{R}^3)} = \|\nabla f_s(x)\|^2_{L^2(\mathbb{R}^3)}$$

we derive for the unique solution $u_{0,s}(x)$ of (1.7) that $u_{0,s}(x) \in H^2(\mathbb{R}^3)$ such that

$$u_0(x) = (u_{0,1}(x), u_{0,2}(x), \dots, u_{0,N}(x))^T \in H^2(\mathbb{R}^3, \mathbb{R}^N).$$

We look for the resulting solution of the nonlinear system of equations (1.2) as

$$u(x) = u_0(x) + u_p(x), \tag{1.8}$$

with

$$u_p(x) = (u_{p,1}(x), u_{p,2}(x), \dots, u_{p,N}(x))^T.$$

In a straightforward manner, we derive the perturbative system of equations

$$\sqrt{-\Delta} u_{p,s} = \varepsilon_s \int_{\mathbb{R}^3} H_s(x-y) g_s(u_0(y) + u_p(y))\, dy, \quad 1 \le s \le N. \tag{1.9}$$

We introduce a closed ball in our Sobolev space as

$$B_\rho := \{u(x) \in H^2(\mathbb{R}^3, \mathbb{R}^N) \mid \|u\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} \le \rho\}, \quad 0 < \rho \le 1. \tag{1.10}$$

We now look for the solution of (1.9) as the fixed point of the auxiliary nonlinear system

$$\sqrt{-\Delta} u_s = \varepsilon_s \int_{\mathbb{R}^3} H_s(x-y) g_s(u_0(y) + v(y))\, dy, \quad 1 \le s \le N, \tag{1.11}$$

in ball (1.10). For a given vector function $v(y)$, this is a system of equations with respect to $u(x)$. The left side of (1.11) contains the non Fredholm operator $\sqrt{-\Delta} : H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$. Since its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$, this operator has no bounded inverse. The analogous situation appeared in articles **[25, 28]**, but as distinct from the present work; the problems treated there required orthogonality relations. The fixed point technique was applied in **[20]** to estimate the perturbation to the standing solitary wave of the nonlinear Schrödinger (NLS) equation where either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem there had the Fredholm property, see **[8]**, **[20,** Assumption 1]. We define a closed ball in the space of N dimensions

$$I := \{z \in \mathbb{R}^N \mid |z| \le c_e \|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + c_e\}. \tag{1.12}$$

We now introduce the following quantities with $1 \le s,\ j \le N$,

$$a_{2,s,j} := \sup_{z\in I}\left|\nabla \frac{\partial g_s}{\partial z_j}\right|, \qquad a_{2,s} := \sqrt{\sum_{j=1}^{N} a_{2,s,j}^2}, \quad a_2 := \max_{1\le s\le N} a_{2,s}.$$

Also,

$$a_{1,s} := \sup_{z\in I}|\nabla g_s(z)|, \quad a_1 := \max_{1\le s\le N} a_{1,s}.$$

The next assumption follows from the nonlinear part of problem (1.2).

Assumption 1.2. Let $1 \le s \le N$ be such that $g_s(z) : \mathbb{R}^N \to \mathbb{R}$ with $g_s(z) \in C_2(\mathbb{R}^N)$. We also assume that $g_s(0) = 0$, $\nabla g_s(0) = 0$ and $a_2 > 0$.

Here, $C_2(\mathbb{R}^N)$ denotes the space of twice continuously differentiable functions on $\mathbb{R}^N$. It follows that a_1, as defined above, is also positive; otherwise, all functions $g_s(z)$ will be constants in the ball I, and then, $a_2 = 0$. For example, $g_s(z) = z^2$, $z \in \mathbb{R}^N$, obviously satisfies Assumption 1.2.

We introduce operator T_g such that $u = T_g v$, where u is a solution of the system of equations (1.11).

Our main result is as follows.

Theorem 1.3. *Let Assumptions* 1.1 *and* 1.2 *hold. Then, system* (1.11) *defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon*$ for some $\varepsilon* > 0$. The unique fixed point $u_p(x)$ of the map T_g is the only solution of the system of equations* (1.9) *in B_ρ.*

It would be natural to conjecture here that this fixed point continuously depends on parameters of the contracting operator, namely, on functions g_s for the operator T_g. We leave this as an open question.

Obviously, the resulting solution of system (1.2) given by (1.8) will be nontrivial due to the fact that the source terms $f_s(x)$ are nontrivial for a certain $s = 1, \dots, N$, and all $g_s(0) = 0$, as assumed.

The next, trivial, lemma will be needed in our applications.

Lemma 1.4. *Consider the function* $\varphi(R) := \alpha R + \beta/R^2$ *for* $R \in (0, +\infty)$ *with the constants* $\alpha, \beta > 0$. *It achieves the minimal value at* $R^* = (2\beta/\alpha)^{1/3}$, *which is given by* $\varphi(R^*) = (3/2^{2/3})\alpha^{2/3}\beta^{1/3}$.

We now proceed to the proof of our main statement.

2. The existence of the perturbed solution.

Proof of Theorem 1.3. We arbitrarily choose $v(x) \in B_\rho$ and denote the terms involved in the integral expressions on the right side of system (1.11) as

$$G_s(x) := g_s(u_0(x) + v(x)), \quad 1 \le s \le N.$$

Applying standard Fourier transform (1.6) to both sides of system (1.11) yields

$$\widehat{u}_s(p) = \varepsilon_s (2\pi)^{3/2} \frac{\widehat{H}_s(p)\widehat{G}_s(p)}{|p|}, \quad 1 \le s \le N.$$

Thus, for the norm, we obtain

$$\|u_s\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^3 \varepsilon_s^2 \int_{\mathbb{R}^3} \frac{|\widehat{H}_s(p)|^2 |\widehat{G}_s(p)|^2}{p^2} \, dp. \tag{2.1}$$

Obviously, for any $\phi(x) \in L^1(\mathbb{R}^3)$,

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{(2\pi)^{3/2}} \|\phi(x)\|_{L^1(\mathbb{R}^3)}. \tag{2.2}$$

In departure from articles **[25, 28]** involving the standard Laplacian operator in the diffusion term, here we do not try to control the norms

$$\left\| \frac{\widehat{H}_s(p)}{|p|} \right\|_{L^\infty(\mathbb{R}^3)}.$$

We estimate the right side of (2.1), applying (2.2) with $R>0$ as

$$(2\pi)^3\varepsilon_s^2\int_{|p|\le R}\frac{|\widehat{H}_s(p)|^2|\widehat{G}_s(p)|^2}{p^2}\,dp + (2\pi)^3\varepsilon_s^2\int_{|p|>R}\frac{|\widehat{H}_s(p)|^2|\widehat{G}_s(p)|^2}{p^2}\,dp \le \varepsilon_s^2\|H_s\|^2_{L^1(\mathbb{R}^3)}\left\{\frac{1}{2\pi^2}\,\|G_s(x)\|^2_{L^1(\mathbb{R}^3)}R+\frac{1}{R^2}\,\|G_s(x)\|^2_{L^2(\mathbb{R}^3)}\right\}. \tag{2.3}$$

Since $v(x)\in B_\rho$, we obtain

$$\|u_0+v\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)}\le\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1,$$

and, by means of Sobolev embedding (1.5),

$$|u_0+v|\le c_e\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+c_e.$$

We use the identity formula

$$G_s(x)=\int_0^1\nabla g_s(t(u_0(x)+v(x)))\cdot(u_0(x)+v(x))\,dt,\quad 1\le s\le N.$$

Throughout this paper, $\cdot$ stands for the scalar product of two vectors in $\mathbb{R}^N$. With ball I defined in (1.12) we derive

$$|G_s(x)|\le\sup_{z\in I}|\nabla g_s(z)||u_0(x)+v(x)|\le a_1|u_0(x)+v(x)|.$$

Hence,

$$\|G_s(x)\|_{L^2(\mathbb{R}^3)}\le a_1\|u_0+v\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)}\le a_1(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1).$$

In addition, for $t\in[0,1]$ and $1\le j\le N$,

$$\frac{\partial g_s}{\partial z_j}\,(t(u_0(x)+v(x)))=\int_0^t\nabla\frac{\partial g_s}{\partial z_j}\,(\tau(u_0(x)+v(x)))\cdot(u_0(x)+v(x))\,d\tau.$$

This implies

$$\left|\frac{\partial g_s}{\partial z_j}\,(t(u_0(x)+v(x)))\right|\le\sup_{z\in I}\left|\nabla\frac{\partial g_s}{\partial z_j}\right||u_0(x)+v(x)| = a_{2,s,j}|u_0(x)+v(x)|.$$

Hence, by virtue of the Schwarz inequality,

$$|G_s(x)| \leq |u_0(x)+v(x)| \sum_{j=1}^{N} a_{2,s,j} |u_{0,j}(x)+v_j(x)| \leq a_2 |u_0(x)+v(x)|^2,$$

such that

$$\|G_s(x)\|_{L^1(\mathbb{R}^3)} \leq a_2 \|u_0(x)+v(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)} \leq a_2(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^2.$$

Thus, we arrive at the upper bound for the right side of (2.3), given by (2.4)

$$\varepsilon_s^2 \|H_s\|^2_{L^1(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^2 \left\{ \frac{a_2^2}{2\pi^2} (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^2 R + \frac{a_1^2}{R^2} \right\},$$

with $R \in (0,+\infty)$. Lemma 1.4 gives us the minimal value of (2.4). Therefore,

$$\|u_s\|^2_{L^2(\mathbb{R}^3)} \leq \frac{3}{2^{4/3}\pi^{4/3}} \, \varepsilon^2 \|H_s\|^2_{L^1(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^{3(1/3)} a_1^{2/3} a_2^{4/3},$$

such that

$$\|u\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)} \leq \frac{3}{2^{4/3}\pi^{4/3}} \, \varepsilon^2 H^2 (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^{3(1/3)} a_1^{2/3} a_2^{4/3}. \tag{2.5}$$

Obviously, by means of (1.11),

$$-\Delta u_s = \varepsilon_s \sqrt{-\Delta} \int_{\mathbb{R}^3} H_s(x-y) G_s(y)\, dy, \quad 1 \leq s \leq N.$$

Using (2.2), we easily obtain

$$\|\Delta u_s\|^2_{L^2(\mathbb{R}^3)} \leq \varepsilon^2 a_2^2 (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^4 \|\nabla H_s\|^2_{L^2(\mathbb{R}^3)}.$$

Hence,

$$\sum_{s=1}^{N} \|\Delta u_s\|^2_{L^2(\mathbb{R}^3)} \leq \varepsilon^2 a_2^2 (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^4 Q^2. \tag{2.6}$$

The definition of norm (1.4), along with inequalities (2.5) and (2.6), give

$$\|u\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \leq \varepsilon(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^2 a_2^{2/3} \times \sqrt{\frac{3}{2^{4/3}\pi^{4/3}} H^2 a_1^{2/3} + a_2^{2/3} Q^2} \leq \rho,$$

for all $\varepsilon > 0$ sufficiently small; therefore, $u(x) \in B_\rho$ as well. If, for a certain $v(x) \in B_\rho$, there are two solutions $u_{1,2}(x) \in B_\rho$ of system (1.11), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3, \mathbb{R}^N)$ solves

$$\sqrt{-\Delta} w = 0.$$

Due to the fact that the operator $\sqrt{-\Delta}$ does not possess nontrivial square integrable zero modes, $w(x) = 0$ almost everywhere in $\mathbb{R}^3$. Hence, system (1.11) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ small enough.

Our goal is to show that this map is a strict contraction. We arbitrarily choose $v_{1,2}(x) \in B_\rho$. By means of the argument above we have $u_{1,2} = T_g v_{1,2} \in B_\rho$ as well. By virtue of system (1.11),

$$\sqrt{-\Delta} u_{1,s} = \varepsilon_s \int_{\mathbb{R}^3} H_s(x-y) g_s(u_0(y) + v_1(y))\, dy, \quad 1 \le s \le N, \tag{2.7}$$

$$\sqrt{-\Delta} u_{2,s} = \varepsilon_s \int_{\mathbb{R}^3} H_s(x-y) g_s(u_0(y) + v_2(y))\, dy, \quad 1 \le s \le N. \tag{2.8}$$

We define

$$\begin{aligned} G_{1,s}(x) &:= g_s(u_0(x) + v_1(x)), \\ G_{2,s}(x) &:= g_s(u_0(x) + v_2(x)), \quad 1 \le s \le N, \end{aligned}$$

and apply the standard Fourier transform (1.6) to both sides of systems (2.7) and (2.8). We obtain

$$\widehat{u_{1,s}}(p) = \varepsilon_s (2\pi)^{3/2} \frac{\widehat{H}_s(p)\widehat{G_{1,s}}(p)}{|p|}, \qquad \widehat{u_{2,s}}(p) = \varepsilon_s (2\pi)^{3/2} \frac{\widehat{H}_s(p)\widehat{G_{2,s}}(p)}{|p|}.$$

In addition,

$$\|u_{1,s}(x) - u_{2,s}(x)\|^2_{L^2(\mathbb{R}^3)} = \varepsilon_s^2 (2\pi)^3 \int_{\mathbb{R}^3} \frac{|\widehat{H}_s(p)|^2 |\widehat{G_{1,s}}(p) - \widehat{G_{2,s}}(p)|^2}{|p|^2}\, dp.$$

Furthermore, it can be estimated by using (2.2) from above that

$$\varepsilon_s^2 \|H_s\|^2_{L^1(\mathbb{R}^3)} \Big\{ \frac{1}{2\pi^2} \|G_{1,s}(x) - G_{2,s}(x)\|^2_{L^1(\mathbb{R}^3)} R \\ + \|G_{1,s}(x) - G_{2,s}(x)\|^2_{L^2(\mathbb{R}^3)} \frac{1}{R^2} \Big\},$$

where $R \in (0, +\infty)$. For $1 \le s \le N$, we use the identity

$$G_{1,s}(x) - G_{2,s}(x) = \int_0^1 \nabla g_s(u_0(x) + tv_1(x) + (1-t)v_2(x)) \cdot (v_1(x) - v_2(x))\, dt.$$

Clearly, for $v_{1,2}(x) \in B_\rho$ and $t \in [0, 1]$, we easily obtain the upper bound for $\|v_2(x) + t(v_1(x) - v_2(x))\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}$ as

$$t\|v_1(x)\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + (1-t)\|v_2(x)\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \le \rho.$$

Thus, $v_2(x) + t(v_1(x) - v_2(x)) \in B_\rho$ as well. We derive

$$|G_{1,s}(x) - G_{2,s}(x)| \le \sup_{z\in I} |\nabla g_s(z)||v_1(x) - v_2(x)| = a_{1,s}|v_1(x) - v_2(x)|.$$

Therefore,

$$\|G_{1,s}(x) - G_{2,s}(x)\|_{L^2(\mathbb{R}^3)} \le a_{1,s}\|v_1(x) - v_2(x)\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}.$$

In addition, for $1 \le j \le N$,

$$\frac{\partial g_s}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x)) = \int_0^1 \nabla \frac{\partial g_s}{\partial z_j}(\tau[u_0(x) + tv_1(x) + (1-t)v_2(x)]) \cdot (u_0(x) + tv_1(x) + (1-t)v_2(x))\, d\tau.$$

Hence,

$$\left|\frac{\partial g_s}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x))\right| \le \sup_{z\in I}\left|\nabla \frac{\partial g_s}{\partial z_j}\right|\{|u_0(x)| + t|v_1(x)| + (1-t)|v_2(x)|\},$$

where $t \in [0, 1]$. Clearly, by virtue of the Schwarz inequality, we estimate $|G_{1,s}(x) - G_{2,s}(x)|$ from the above by

$$\sum_{j=1}^N a_{2,s,j}\Big\{|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)|\Big\}|v_{1,j}(x) - v_{2,j}(x)| \le a_{2,s}\Big\{|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)|\Big\}|v_1(x) - v_2(x)|.$$

The Schwarz inequality gives the upper bound for

$$\|G_{1,s}(x)-G_{2,s}(x)\|_{L^1(\mathbb{R}^3)}$$

as

$$a_{2,s}\Big\{\|u_0(x)\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)}+\frac{1}{2}\,\|v_1(x)\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)}+\frac{1}{2}\,\|v_2(x)\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)}\Big\}$$
$$\times\,\|v_1(x)-v_2(x)\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)}$$
$$\le a_2\{\|u_0(x)\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1\}\|v_1(x)-v_2(x)\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}.$$

This allows us to estimate the norm $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)}$ by

$$\varepsilon^2H^2\|v_1-v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)}\left\{\frac{{a_2}^2}{2\pi^2}(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^2R+\frac{{a_1}^2}{R^2}\right\}. \tag{2.9}$$

By virtue of Lemma 1.4, we minimize (2.9) over $R>0$ in order to obtain that $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)}$ is bounded from above by

$$\varepsilon^2H^2\|v_1-v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)}\frac{3}{2^{4/3}}\frac{a_2^{4/3}}{\pi^{4/3}}(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^{4/3}a_1^{2/3}. \tag{2.10}$$

By means of (2.7) and (2.8) we have

$$-\Delta(u_{1,s}-u_{2,s})=\varepsilon_s\sqrt{-\Delta}\int_{\mathbb{R}^3}H_s(x-y)[G_{1,s}(y)-G_{2,s}(y)]\,dy,\quad 1\le s\le N.$$

Hence, via (2.2), we obtain

$$\|\Delta(u_{1,s}-u_{2,s})\|^2_{L^2(\mathbb{R}^3)}\le\varepsilon^2\|\nabla H_s\|^2_{L^2(\mathbb{R}^3)}\|G_{1,s}(x)-G_{2,s}(x)\|^2_{L^1(\mathbb{R}^3)}$$
$$\le\varepsilon^2\|\nabla H_s\|^2_{L^2(\mathbb{R}^3)}a_2^2(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^2\|v_1-v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)},$$

such that

$$\sum_{s=1}^N\|\Delta(u_{1,s}-u_{2,s})\|^2_{L^2(\mathbb{R}^3)}$$

is estimated from above by

$$\varepsilon^2Q^2a_2^2(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)^2\|v_1-v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)}. \tag{2.11}$$

By virtue of inequalities (2.10) and (2.11) the norm $\|u_1-u_2\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}$ is bounded from above by

$$\varepsilon a_2^{2/3}(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)\left[\frac{3}{2^{4/3}}\,\frac{H^2a_1^{2/3}}{\pi^{4/3}}+Q^2a_2^{2/3}\right]^{1/2}\|v_1-v_2\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}.$$

Hence, the map $T_g : B_\rho \to B_\rho$, defined by system (1.11), is a strict contraction for all values of $\varepsilon > 0$ small enough. Its unique, fixed point $u_p(x)$ is the only solution of system (1.9) in B_ρ. The resulting $u(x) \in H^2(\mathbb{R}^3, \mathbb{R}^N)$, given by (1.8), is a solution of the system of equations (1.2). □

3. Auxiliary results. We recall the solvability conditions for the linear Poisson type equation

$$\sqrt{-\Delta}\phi = f(x), \quad x \in \mathbb{R}^d, \ d \in \mathbb{N}, \tag{3.1}$$

established in **[30]**. We denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\overline{g}(x)\, dx, \tag{3.2}$$

with a slight abuse of notation, when the functions involved in (3.2) are not square integrable, for example, the functions present in orthogonality conditions (3.3) and (3.4). Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x) \in L^\infty(\mathbb{R}^d)$, then the integral on the right side of (3.2) makes sense.

The technical result below was easily proved **[30]** by applying the standard Fourier transform to problem (3.1).

Lemma 3.1. *Let $f(x) \in L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$.*

(i) *When $d = 1$ and $|x|f(x) \in L^1(\mathbb{R})$, equation* (3.1) *admits a unique solution $\phi(x) \in H^1(\mathbb{R})$ if and only if the orthogonality condition*

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{3.3}$$

holds.

(ii) *When $d = 2$ and $|x|f(x) \in L^1(\mathbb{R}^2)$, problem* (3.1) *possesses a unique solution $\phi(x) \in H^1(\mathbb{R}^2)$ if and only if the orthogonality relation*

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{3.4}$$

holds.

(iii) *When $d \geq 3$ and $f(x) \in L^1(\mathbb{R}^d)$, equation* (3.1) *has a unique solution $\phi(x) \in H^1(\mathbb{R}^d)$.*

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Note that, in d ove, no orthogonality relations are needed to solve the linear Poisson-type equation (3.1) in $H^1(\mathbb{R}^d)$.

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