Time-Stepping Approximation of Rigid-Body Dynamics with Perfect Unilateral Constraints. II: The Partially Elastic Impact Case

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Time-Stepping Approximation of Rigid-Body Dynamics with Perfect Unilateral Constraints. II: The Partially Elastic Impact Case

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As in Paoli (Arch Rational Mech Anal, 2010), we consider a discrete mechanical system with a non-trivial mass matrix subjected to perfect unilateral constraints described by geometrical inequalities $f_\alpha(q) \geqq 0$, $\alpha \in \{1, \dots, \nu\}$ ($\nu \geqq 1$), but we assume now that the transmission of the velocities at impacts is governed by Newton's Law with a coefficient of restitution $e \in (0, 1]$ (so that the impact is partially elastic). We generalize the time-discretization of the second order differential inclusion describing the dynamics proposed in Paoli (Arch Rational Mech Anal, 2010) to this case and, once again, we prove its convergence.

1. Introduction

As in [11], we consider a discrete mechanical system subjected to perfect unilateral constraints. More precisely, we denote by $u \in \mathbb{R}^d$ the generalized coordinates of a typical configuration of the system, and we assume that the set K of admissible configurations is described by $\nu \geqq 1$ geometrical inequalities

$$f_\alpha(u) \geqq 0, \quad \alpha \in \{1, \dots, \nu\}$$

where f_α is a smooth function (at least C^1) such that $\nabla f_\alpha(u)$ does not vanish in a neighbourhood of $\big\{u \in \mathbb{R}^d;\ f_\alpha(u) = 0\big\}$.

Then, the dynamics is described by the following measure differential inclusion (see [7] for instance)

$$M(u)\ddot{u} - g(t, u, \dot{u}) \in -N_K(u) \tag{1}$$

where $M(u)$ is the mass matrix of the system and $N_K(u)$ is the normal cone to K at u, given by

$$N_K(u) = \begin{cases} \{0\} & \text{if } u \in \mathrm{Int}(K), \\ \left\{ \sum_{\alpha \in J(u)} \lambda_\alpha \nabla f_\alpha(u), \lambda_\alpha \leqq 0 \ \ \forall \alpha \in J(u) \right\} & \text{if } u \in \partial K, \\ \emptyset & \text{if } u \notin K, \end{cases}$$

with $J(u) = \left\{ \alpha \in \{1, \ldots, \nu\}; \ f_\alpha(u) \leqq 0 \right\}$ for all $u \in \mathbb{R}^d$. We also define the tangent cone to K at u by

$$T_K(u) = \left\{ w \in \mathbb{R}^d; (\nabla f_\alpha(u), w) \geqq 0 \quad \forall \alpha \in J(u) \right\}$$

where (v, w) denotes the Euclidean scalar product of vectors v and w in $\mathbb{R}^d$. Since $u(s) \in K$ for all s and we infer that

$$\dot{u}(t+0) \in T_K\left(u(t)\right), \quad \dot{u}(t-0) \in -T_K\left(u(t)\right) \quad (t > 0)$$

whenever $\dot{u}(t \pm 0)$ exists. It follows that the velocities are discontinuous at impacts if $\dot{u}(t-0) \notin T_K\left(u(t)\right)$, and (1) implies that

$$M\left(u(t)\right)\left(\dot{u}(t+0) - \dot{u}(t-0)\right) \in -N_K\left(u(t)\right).$$

This relation does not uniquely determine $\dot{u}(t+0)$, so we should add an impact law. Following Moreau [8] and Ballard [1], we consider in this article a more general impact law, as in [11]: we assume that there exists a restitution coefficient $e \in [0, 1]$ such that

$$\dot{u}(t+0) = -e\dot{u}(t-0) + (1+e)\mathrm{Proj}_{M(u(t))}\left(T_K\left(u(t)\right), \dot{u}(t-0)\right) \tag{2}$$

where $\mathrm{Proj}_{M(u)}$ denotes the projection relatively to the Riemannian metric defined by the inertia operator $M(u)$.

It should be observed that this model is energetically consistent: indeed, the kinetic energy of the system, given by $E_c = \|\dot{u}\|^2_{M(u)}/2$, decreases at impacts if $e \in (0, 1]$ and is conserved if $e = 1$. Moreover, when $e = 0$, we recognize the case of "standard inelastic shocks" already studied in [11].

For admissible initial data $(u_0, v_0) \in K \times T_K(u_0)$ we consider the following initial-value problem:

Problem (P) Find $u : [0, \tau] \to \mathbb{R}^d$ $(\tau > 0)$ such that:

(P1) u is an absolutely continuous function from $[0, \tau]$ to K and $\dot{u} \in BV(0, \tau; \mathbb{R}^d)$,

(P2) the differential inclusion

$$M(u)\ddot{u} - g(t, u, \dot{u}) \in -N_K(u)$$

is satisfied in the following sense: there exists a (non-unique) non-negative measure μ such that the Stieltjes measure $\mathrm{d}\dot{u} = \ddot{u}$ and the usual Lebesgue measure $\mathrm{d}t$ admit densities with respect to $\mathrm{d}\mu$, that is, there exist two $\mathrm{d}\mu$-integrable functions v'_μ and t'_μ such that $\ddot{u} = \mathrm{d}\dot{u} = v'_\mu \mathrm{d}\mu$, $\mathrm{d}t = t'_\mu \mathrm{d}\mu$ and

$$M\left(u(t)\right) v'_\mu(t) - g\big(t, u(t), \dot{u}(t)\big) t'_\mu(t) \in -N_K\left(u(t)\right) \quad \mathrm{d}\mu \text{ almost everywhere,} \tag{3}$$

(P3) for all $t \in (0, \tau)$

$$\dot{u}(t+0) = -e\dot{u}(t-0) + (1+e)\mathrm{Proj}_{M(u(t))}\left(T_K\left(u(t)\right), \dot{u}(t-0)\right)$$

(P4) $u(0) = u_0$, $\dot{u}(0+0) = v_0$.

For this model of impact (for any value of the restitution coefficient $e \in [0, 1]$), the existence and uniqueness of a maximal solution for the initial-value problem has been proved by Ballard [1], when the data are analytical. Uniqueness cannot be expected for less regular data (see [2,14] or [1] for counter-examples) but existence results have still been established in the single constraint case (that is, $\nu = 1$): see [5,9,12] for a trivial mass matrix (that is, $M(u) \equiv \mathrm{Id}_{\mathbb{R}^d}$), and [13,15] for a non-trivial mass matrix. All these results rely on the study of a sequence of approximate solutions constructed either by a penalty method [12,15] or by a time-stepping scheme [5,9,13].

As in the inelastic shock case, these techniques encounter a new difficulty for the multi-constraint case since, in general, the motion is not continuous with respect to the data. Nevertheless, in the framework given by the sufficient conditions ensuring continuity on data established in [1,10], we may expect some convergence results. So, in this article we propose a generalization to the partially elastic case of the time-stepping scheme introduced in [11] and we study its convergence.

More precisely, we assume the same kind of regularity for the data as in [10,11], that is,

(H1) g is a continuous function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ $(T > 0)$ to $\mathbb{R}^d$;

(H2) for all $\alpha \in \{1, \ldots, \nu\}$, the function f_α belongs to $C^1(\mathbb{R}^d)$, ∇f_α is locally Lipschitz continuous and does not vanish in a neighbourhood of $\left\{u \in \mathbb{R}^d;\ f_\alpha(u) = 0\right\}$;

(H3) the set K is defined by

$$K = \left\{u \in \mathbb{R}^d;\ f_\alpha(u) \geqq 0,\ \alpha \in \{1, \ldots, \nu\}\right\}$$

and the active constraints along ∂K are functionally independent, that is, for all $u \in \partial K$ the vectors $(\nabla f_\alpha(u))_{J(u)}$ are linearly independent;

(H4) M is a mapping of class C^1 from $\mathbb{R}^d$ to the set of symmetric positive definite $d \times d$ matrices.
With this last assumption we may define $M^{-1}(u)$, $M^{1/2}(u)$ and $M^{-1/2}(u)$ for all $u \in \mathbb{R}^d$ and the corresponding mappings are of class C^1 from $\mathbb{R}^d$ to the set of symmetric positive definite $d \times d$ matrices.
Let F be a function such that

(H5) F is continuous from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times [0, h^*]$ $(h^* > 0)$ to $\mathbb{R}^d$ and is consistent with respect to g, that is,

$$F(t, u, v, 0) = M^{-1}(u)g(t, u, v) \quad \forall (t, u, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

For admissible initial data $(u_0, v_0) \in K \times T_K(u_0)$ we consider the initial-value problem (P). We define a time-stepping scheme as follows:

- the initial positions U^0 and U^1 are given by

$$U^0 = u_0, \quad U^1 \in \mathrm{Argmin}_{Z\in K} \left\| u_0 + hv_0 + hz(h) - Z \right\|_{M(u_0)} \tag{4}$$

with $\lim_{h\to 0} z(h) = 0$,

- for all $n \in \left\{1, \ldots, \left\lfloor \frac{T}{h} \right\rfloor\right\}$, let

$$W^n = \frac{2U^n - (1-e)U^{n-1} + h^2 F^n}{1+e}, \quad F^n = F\left(nh, U^n, \frac{U^n - U^{n-1}}{h}, h\right) \tag{5}$$

and

$$U^{n+1} = -eU^{n-1} + (1+e)Z^n \tag{6}$$

with

$$Z^n \in \mathrm{Argmin}_{Z\in K} \|W^n - Z\|_{M(U^n)} \tag{7}$$

where $\|\cdot\|_{M(U)}$ is the norm associated to the kinetic metric at U defined by $\|Z\|^2_{M(U)} = (Z, Z)_{M(U)}$ with

$$(Z, Z')_{M(U)} = (Z, M(U)Z') = (M(U)Z, Z')$$

for all $(Z, Z', U) \in (\mathbb{R}^d)^3$.

For $e = 0$ we recover the scheme already studied in [11] and, if $M(u) \equiv \mathrm{Id}_{\mathbb{R}^d}$ for all $u \in \mathbb{R}^d$ and K is convex, we recognize the scheme introduced in [9] for the first time.

As in [11], we now define the approximate solutions u_h by

$$u_h(t) = U^n + (t - nh)\frac{U^{n+1} - U^n}{h} \quad \forall t \in [nh, (n+1)h] \cap [0, T]$$

for all $n \in \{0, \ldots, \lfloor T/h \rfloor\}$ and $h \in (0, h^*]$.

Since the impact law (2) leads to some discontinuity with respect to the data, as when the active constraints at impacts are not orthogonal when $e \neq 0$ (see [10]), we cannot expect convergence of the approximate motions unless we add some assumptions on the geometry of active constraints along ∂K.

So, for all $u \in K$ and for all $\alpha \in J(u)$, we define

$$e_\alpha(u) = \frac{M^{-1/2}(u)\nabla f_\alpha(u)}{|M^{-1/2}(u)\nabla f_\alpha(u)|}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$, and we assume that

(H6) for all compact subset $\mathcal{B}$ of $\mathbb{R}^d$, there exist $C_\mathcal{B} > 0$ and $r_\mathcal{B} > 0$ such that for all $(q_1, q_2) \in (K \cap \mathcal{B})^2$ such that $|q_1 - q_2| \leqq r_\mathcal{B}$, and for all $(\alpha, \beta) \in J(q_1) \times J(q_2)$, such that $\alpha \neq \beta$, we have

$$\left|\left(e_\alpha(q_1), e_\beta(q_2)\right)\right| \leqq C_\mathcal{B}|q_1 - q_2|. \tag{8}$$

Let us observe that, for $q_1 = q_2$, (H6) reduces to the "angle condition" given in [10], which ensures continuity on data. In particular, for all $q \in K$

$$\big(e_\alpha(q), e_\beta(q)\big) = 0 \quad \forall(\alpha, \beta) \in J(q)^2,\ \alpha \neq \beta$$

which means that the active constraints are orthogonal for the local momentum metric defined by $M^{-1}(q)$.

Then, under assumptions (H1)–(H6) we prove the convergence of a subsequence of the approximate solutions $(u_h)_{h^* \geqq h > 0}$ to a solution of problem (P). The various steps of the proof are the same as in [11], but the non-trivial restitution coefficient $e \in (0, 1]$ leads to considerably more technicalities in the estimates of the velocity.

The paper is organized as follows. In the next sections we establish a priori estimates for the discrete velocities and accelerations on a non-trivial time interval $[0, \tau]$, with $0 < \tau \leqq T$. Then we pass to the limit when h tends to zero on $[0, \tau]$: using Ascoli's and Helly's theorems we obtain the convergence of a subsequence of $(u_h)_{h^* \geqq h > 0}$ to a limit u which satisfies (P1) and (P2). Next, by a local study in the neighbourhood of any impact time, we prove that the limit u also satisfies (P3) and (P4). Finally we conclude with some global results.

2. A priori estimates for the discrete velocities

As in [11], we begin the study with a priori estimates of the discrete velocities for a more general scheme in which the initialization procedure involves an initial time t_{0h} depending on h. The purpose of this modification is to allow us, in the last section of the paper, to extend the estimates of the discrete velocities, by considering as "new" initial data, the already constructed approximate positions at some time steps t_{0h} and $t_{0h} + h$.

More precisely, let $\mathcal{B}$ be a given convex compact subset of $\mathbb{R}^d$ such that $\mathcal{B} \cap K \neq \emptyset$. Possibly decreasing h^*, we may assume without loss of generality that

$$|z(h)| \leqq 1 \quad \forall h \in (0, h^*]. \tag{9}$$

For all $h \in (0, h^*]$, let $t_{0h} \in [0, T)$, and U^0 and U^1 be given in $\mathcal{B} \cap K$ and K, respectively. For all $n \in \left\{1, \ldots, \left\lfloor \frac{T - t_{0h}}{h} \right\rfloor\right\}$ we define

$$U^{n+1} = -eU^{n-1} + (1 + e)Z^n$$

with

$$Z^n \in \mathrm{Argmin}_{Z \in K} \| W^n - Z \|_{M(U^n)}$$

and

$$W^n = \frac{2U^n - (1 - e)U^{n-1} + h^2 F^n}{1 + e}, \quad F^n = F\left(t_{0h} + nh, U^n, \frac{U^n - U^{n-1}}{h}, h\right).$$

For all $h \in (0, h^*]$ and $n \in \left\{0, \ldots, \left\lfloor \frac{T - t_{0h}}{h} \right\rfloor\right\}$, let

$$V^n = \frac{U^{n+1} - U^n}{h}.$$

Then, we observe that

Lemma 1. *For all $h \in (0, h^*]$ and $n \in \{1, \ldots, \lfloor (T - t_{0h})/h \rfloor\}$, we have*

$$M(U^n)(V^{n-1} - V^n + hF^n) \in N_K(Z^n).$$

Proof. The proof is the same as in Lemma 1 of [11], in which we simply replace U^{n+1} by Z^n. More precisely, let $h \in (0, h^*]$ and $n \in \{1, \ldots, \lfloor (T - t_{0h})/h \rfloor\}$. By definition of Z^n we have

$$\|W^n - Z^n\|^2_{M(U^n)} \leqq \|W^n - Z\|^2_{M(U^n)}$$

for all $Z \in K$, which yields

$$(W^n - Z^n, Z - Z^n)_{M(U^n)} \leqq \frac{1}{2}\|Z^n - Z\|^2_{M(U^n)} \quad \forall Z \in K.$$

But, using formulae (5) and (6), we get

$$\begin{aligned} W^n - Z^n &= \frac{2U^n - (1-e)U^{n-1} + h^2F^n}{1+e} - \frac{U^{n+1} + eU^{n-1}}{1+e} \\ &= h\frac{V^{n-1} - V^n + hF^n}{1+e}. \end{aligned} \tag{10}$$

Thus

$$\frac{2h}{1+e}(V^{n-1} - V^n + hF^n, Z - Z^n)_{M(U^n)} \leqq \|Z^n - Z\|^2_{M(U^n)} \quad \forall Z \in K.$$

and the rest of the proof remains identical. □

Let us introduce the same notation as in [11]. We define

$$\lambda_{\min}(u) = \|M(u)\|, \quad \lambda_{\min}(u) = \frac{1}{\|M^{-1}(u)\|} \quad \forall u \in \mathbb{R}^d.$$

Since $u \mapsto M(u)$ is continuous with values in the set of symmetric positive definite matrices, the mappings $u \mapsto \lambda_{\max}(u)$ and $u \mapsto \lambda_{\min}(u)$ are well defined and continuous from $\mathbb{R}^d$ to $\mathbb{R}^*_+$. Moreover

$$\lambda_{\min}(u)|w|^2 \leqq \|w\|^2_{M(u)} \leqq \lambda_{\max}(u)|w|^2 \quad \forall w \in \mathbb{R}^d, \quad \forall u \in \mathbb{R}^d.$$

Since $\mathcal{B}$ is compact, there exists $\delta > 0$ such that, for all $(q, q') \in \mathcal{B} \times \mathbb{R}^d$ such that $|q - q'| \leqq \delta$, we have:

$$\begin{aligned} \left|\lambda_{\min}(q) - \lambda_{\min}(q')\right| &\leqq \frac{1}{2} \inf_{u \in \mathcal{B}} \lambda_{\min}(u), \\ \left|\lambda_{\max}(q) - \lambda_{\max}(q')\right| &\leqq \frac{1}{2} \sup_{u \in \mathcal{B}} \lambda_{\max}(u). \end{aligned}$$

We define

$$B_0 = \left\{ u \in \mathbb{R}^d ; \, \mathrm{dist}(u, \mathcal{B}) \leqq \delta \right\}. \tag{11}$$

Then B_0 is also a convex compact subset of $\mathbb{R}^d$, and we have

$$\frac{1}{2} \inf_{u \in \mathcal{B}} \lambda_{\min}(u) \leqq \inf_{u \in B_0} \lambda_{\min}(u), \;\; \sup_{u \in B_0} \lambda_{\max}(u) \leqq \frac{3}{2} \sup_{u \in \mathcal{B}} \lambda_{\max}(u).$$

We let

$$\lambda_{\min} = \frac{1}{2} \inf_{u \in \mathcal{B}} \lambda_{\min}(u), \;\; \lambda_{\max} = \frac{3}{2} \sup_{u \in \mathcal{B}} \lambda_{\max}(u). \tag{12}$$

Of course, we have

$$0 < \lambda_{\min} |w|^2 \leqq \|w\|^2_{M(u)} \leqq \lambda_{\max} |w|^2 \quad \forall w \in \mathbb{R}^d \backslash \{0\}, \quad \forall u \in B_0.$$

Let $C_0 > 0$ and C_F be given by

$$C_F = \sup \left\{ |F(t, u, v, h)| \, ; \;\; t \in [0, T], \;\; u \in B_0 \cup B_1, \;\; |v| \leqq C_0, \;\; h \in [0, h^*] \right\} \tag{13}$$

with $B_1 = \overline{B}(u_0, C_0 T + 1)$. Since the mappings M, M^{-1}, $M^{1/2}$ and $M^{-1/2}$ are of class C^1 on $\mathbb{R}^d$, they are Lipschitz continuous on $B_0 \cup B_1$ and we denote by $L_M, L_{M^{-1}}, L_{M^{1/2}}$ and $L_{M^{-1/2}}$ the corresponding Lipschitz constants. Moreover the functions ∇f_α, $1 \leqq \alpha \leqq \nu$, are locally Lipschitzian. There exists also a positive real number L_f such that

$$\left| \nabla f_\alpha(Z) - \nabla f_\alpha(Z') \right| \leqq L_f |Z - Z'| \;\; \forall (Z, Z') \in (B_0 \cup B_1)^2, \quad \forall \alpha \in \{1, \ldots, \nu\}.$$

We can obtain, as in [11], some rough estimates on the discrete velocities V^n. More precisely:

Proposition 1. *Let $C_0 > 0$ and $h_0^* \in (0, h^*]$ such that*

$$h_0^* \leqq \min \left(\frac{C_0}{2C_F}, \frac{\delta}{8C_0} \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}} \right)$$

where C_F is defined by (13) and $\lambda_{\min}$, $\lambda_{\max}$ are given by (12). Let $h \in (0, h_0^*]$, $\tau_h = \min(\delta/(2C_0), T - t_{0h})$ and assume that there exists $n \in \{1, \ldots, \lfloor \tau_h / h \rfloor\}$ such that

$$|V^l| \leqq C_0 \quad \forall l \in \{0, \ldots, n-1\}.$$

Then

$$|V^n| \leqq 4 \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0. \tag{14}$$

Moreover, for all $l \in \{2, \ldots, n\}$ such that $J(Z^l) \neq \emptyset$,

$$\left(\nabla f_\alpha(Z^l), V^l + e V^{l-2} \right) \leqq \frac{L_f h}{2(1+e)} |V^l + e V^{l-2}|^2 \quad \forall \alpha \in J(Z^l).$$

Proof. The proof is almost identical to the proof of Proposition 1 in [11]. More precisely, for all $l \in \{0, \dots, n\}$ we have

$$|U^l - U^0| \leqq \sum_{k=0}^{l-1} h|V^k| \leqq lhC_0 \leqq C_0\tau_h \leqq \delta \quad \forall\, l \in \{0, \dots, n\},$$

and thus $U^l \in B_0$. By definition of Z^n we have

$$\|W^n - Z^n\|_{M(U^n)} \leqq \|W^n - Z^{n-1}\|_{M(U^n)}$$

since $Z^{n-1} \in K$, with the convention $Z^0 = U^0$. Recalling (10) we have

$$W^n - Z^n = \frac{h}{1+e}(V^{n-1} - V^n + hF^n),$$

and with formulae (5) and (6) we have also

$$W^n - Z^{n-1} = \begin{cases} \frac{h}{1+e}(V^{n-1} + eV^{n-2} + hF^n) & \text{if } n \geqq 2, \\ \frac{h}{1+e}(2V^0 + hF^1) & \text{if } n = 1. \end{cases}$$

Hence

$$\|V^n\|_{M(U^n)} \leqq \begin{cases} 2\|V^{n-1}\|_{M(U^n)} + e\|V^{n-2}\|_{M(U^n)} + 2h\|F^n\|_{M(U^n)} & \text{if } n \geqq 2, \\ 3\|V^0\|_{M(U^1)} + 2h\|F^1\|_{M(U^1)} & \text{if } n = 1, \end{cases} \tag{15}$$

and

$$|V^n| \leqq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\,(3C_0 + 2hC_F) \leqq 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}C_0.$$

It follows that $Z^l \in B_0$ for all $l \in \{1, \dots, n\}$. Indeed,

$$\begin{aligned} |Z^l - U^0| &= \left|\frac{U^{l+1} + eU^{l-1}}{1+e} - U^0\right| \leqq \frac{1}{1+e}\left(|U^{l+1} - U^0| + e|U^{l-1} - U^0|\right) \\ &\leqq C_0\tau_h \leqq \frac{\delta}{2} \quad \text{if } l \in \{1, \dots, n-1\}, \end{aligned}$$

and

$$\begin{aligned} |Z^l - U^0| &= \left|\frac{U^{l+1} + eU^{l-1}}{1+e} - U^0\right| \leqq \frac{h}{1+e}|V^l| \\ &\quad + \frac{1}{1+e}\left(|U^l - U^0| + e|U^{l-1} - U^0|\right) \\ &\leqq 4h\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}C_0 + C_0\tau_h \leqq \frac{\delta}{2} + C_0\tau_h \leqq \delta \quad \text{if } l = n. \end{aligned}$$

Assume now that $n \geqq 2$, and let $l \in \{2, \ldots, n\}$ such that $J(Z^l) \neq \emptyset$. For all $\alpha \in J(Z^l)$, we have

$$0 \leqq f_\alpha(Z^{l-1}) - f_\alpha(Z^l) = \int_0^1 \left(\nabla f_\alpha \left(Z^l + t(Z^{l-1} - Z^l) \right), Z^{l-1} - Z^l \right) \mathrm{d}t.$$

But

$$Z^{l-1} - Z^l = -\frac{h}{1+e} \left(V^l + eV^{l-2} \right)$$

and thus

$$\begin{aligned} \left(\nabla f_\alpha(Z^l), V^l + eV^{l-2} \right) &\leqq - \int_0^1 \left(\nabla f_\alpha \left(Z^l + t(Z^{l-1} - Z^l) \right) \right. \\ &\quad \left. - \nabla f_\alpha(Z^l), V^l + eV^{l-2} \right) \mathrm{d}t \\ &\leqq \int_0^1 \left| \nabla f_\alpha \left(Z^l + t(Z^{l-1} - Z^l) \right) - \nabla f_\alpha(Z^l) \right| \\ &\quad \times \left| V^l + eV^{l-2} \right| \mathrm{d}t. \end{aligned}$$

It follows that

$$\left(\nabla f_\alpha(Z^l), V^l + eV^{l-2} \right) \leqq \frac{L_f h}{2(1+e)} |V^l + eV^{l-2}|^2.$$

□

Now we prove a more precise estimate for the discrete velocities.

Proposition 2. *Let $C_0 > 0$ and assume that there exist $C_0^* > 0$ and $h_0^* \in (0, h^*]$ such that*

$$h_0^* \leqq \min \left(\frac{C_0^*}{2C_F}, \frac{\delta}{8C_0} \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}}, \frac{\min(r_{B_0}, r'_{B_0})}{C_0 \left(\nu + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \right)}, \frac{1}{72\nu^{3/2} C_e C_0 \left(\nu + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \right)} \right)$$

and

$$|V^0| = \left| \frac{U^1 - U^0}{h} \right| \leqq C_0^* < \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \right)^{-(\nu+1)} C_0 \quad \forall h \in (0, h_0^*]$$

where C_F is defined by (13), $\lambda_{\min}$ and $\lambda_{\max}$ are given by (12) and $C_e = \max(C_{B_0}, C'_{B_0})$ with C'_{B_0} and r'_{B_0} defined at Lemma 15 (see Appendix).

Then there exists $\tau_0 > 0$, depending only on $\mathcal{B}$, C_0, C_0^* and the data, such that

$$|V^n| = \left| \frac{U^{n+1} - U^n}{h} \right| \leqq C_0 \quad \forall nh \in [0, \min(\tau_0, T - t_{0h})], \quad \forall h \in (0, h_0^*].$$

Proof. As in Proposition 1, let us define $\tau_h = \min(\delta/(2C_0), T - t_{0h})$. We begin the proof with the study of the $(\nu + 1)$ first velocities. More precisely, we prove that they satisfy the announced estimate:

Lemma 2. *For all $h \in (0, h_0^*]$ and for all $n \in \{0, \dots, \min(\nu, \lfloor \tau_h/h \rfloor)\}$ we have*

$$|V^n| \leqq C_0^* \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)^n < C_0.$$

Proof. The result is almost a direct consequence of Proposition 1. Indeed, let $h \in (0, h_0^*]$. By definition of C_0^* we have

$$|V^0| = \left|\frac{U^1 - U^0}{h}\right| \leqq C_0^* \leqq C_0,$$

and with (15), we already know that $U^1 \in B_0$ and

$$\|V^1\|_{M(U^1)} \leqq 3\|V^0\|_{M(U^1)} + 2h\|F^1\|_{M(U^1)}.$$

Hence

$$|V^1| \leqq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\left(3|V^0| + 2h|F^1|\right) \leqq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}(3C_0^* + 2hC_F) \leqq 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}C_0^*.$$

Now, let $n \in \{2, \dots, \nu - 1\}$ and assume that

$$|V^l| \leqq C_0^* \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)^l \quad \forall l \in \{0, \dots, n-1\}.$$

Then $|V^l| \leqq C_0$ for all $l \in \{0, \dots, n-1\}$ and, once again using (15), we infer that $U^n \in B_0$ and

$$\|V^n\|_{M(U^n)} \leqq 2\|V^{n-1}\|_{M(U^n)} + e\|V^{n-2}\|_{M(U^n)} + 2h\|F^n\|_{M(U^n)}.$$

Hence

$$\begin{aligned}|V^n| &\leqq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\left(2C_0^*\left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)^{n-1} + eC_0^*\left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)^{n-2} + 2hC_F\right)\\ &\leqq C_0^*\left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)^n.\end{aligned}$$

□

Let us assume now that $h \in (0, h_0^*]$, $\nu + 1 \leqq n \leqq \lfloor \tau_h / h \rfloor$ and $|V^l| \leqq C_0$ for all $l \in \{0, \dots, n-1\}$. We already know from Proposition 1 that

$$U^l \in B_0, \quad Z^l \in B_0 \quad \forall l \in \{0, \dots, n\},$$

and

$$|V^n| \leqq 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} C_0.$$

But we may obtain a more precise estimate on V^n: using Proposition 3 (see below) there exists a constant $C_{\mathcal{B},C_0}$, depending only on $\mathcal{B}$, C_0 and the data, such that

$$\left|\tilde{V}^n\right|^2 \leqq \max\left(\left|\tilde{V}^{n-1}\right|^2, \dots, \left|\tilde{V}^{n-\nu}\right|^2, \left|\tilde{V}^{n-\nu-1}\right|^2\right) + C_{\mathcal{B},C_0} h$$

where we define

$$\tilde{V}^l = M^{1/2}(Z^l) V^l \quad \forall l \in \left\{0, \dots, \left\lfloor \frac{T - t_{0h}}{h} \right\rfloor\right\}.$$

By an immediate induction, we infer that

$$|V^n|^2 \leqq \frac{\lambda_{\max}}{\lambda_{\min}} \max\left(\left|V^0\right|^2, \dots, \left|V^\nu\right|^2\right) + \frac{C_{\mathcal{B},C_0}}{\lambda_{\min}} (n - \nu) h,$$

and by defining

$$\tau_0 = \min\left(\frac{\lambda_{\min}}{C_{\mathcal{B},C_0}} \frac{15 C_0^2}{16}, \frac{\delta}{2C_0}\right)$$

we obtain

$$|V^n| \leqq C_0 \quad \forall nh \in [0, \min(\tau_0, T - t_{0h})], \quad \forall h \in (0, h_0^*].$$

□

Proposition 3. *Let us assume now that $h \in (0, h_0^*]$, $\nu + 1 \leqq n \leqq \lfloor \tau_h / h \rfloor$ and*

$$|V^l| \leqq C_0 \quad \forall l \in \{0, \dots, n-1\}. \tag{16}$$

There exists a constant $C_{\mathcal{B},C_0}$, depending only on $\mathcal{B}$, C_0 and the data, such that

$$\left|\tilde{V}^n\right|^2 \leqq \max\left(\left|\tilde{V}^{n-1}\right|^2, \dots, \left|\tilde{V}^{n-\nu}\right|^2, \left|\tilde{V}^{n-\nu-1}\right|^2\right) + C_{\mathcal{B},C_0} h \tag{17}$$

with

$$\tilde{V}^l = M^{1/2}(Z^l) V^l \quad \forall l \in \left\{0, \dots, \left\lfloor \frac{T - t_{0h}}{h} \right\rfloor\right\}. \tag{18}$$

The proof of this result is rather technical and will be divided into several lemmas. Let us recall here some useful relations about the Z^n's. By definition of the scheme, we have

$$Z^n = \frac{U^{n+1} + eU^{n-1}}{1+e} \quad \forall n \in \left\{1, \ldots, \left\lfloor \frac{T - t_{0h}}{h} \right\rfloor\right\} \tag{19}$$

and by convention $Z^0 = U^0$. It follows that

$$U^n - Z^n = \frac{h}{1+e}(eV^{n-1} - V^n) \quad \forall n \in \left\{1, \ldots, \left\lfloor \frac{T - t_{0h}}{h} \right\rfloor\right\} \tag{20}$$

and

$$Z^n - Z^{n-1} = \frac{h}{1+e}(V^n + eV^{n-2}) \quad \forall n \in \left\{2, \ldots, \left\lfloor \frac{T - t_{0h}}{h} \right\rfloor\right\}. \tag{21}$$

Proof. First, we observe that (17) is obvious if $J(Z^n) = \emptyset$. Indeed, with Lemma 1, we know that

$$M(U^n)(V^{n-1} - V^n + hF^n) \in N_K(Z^n)$$

and thus

$$V^n = V^{n-1} + hF^n \quad \text{if } J(Z^n) = \emptyset.$$

With (21) we infer that

$$\begin{aligned} |\tilde{V}^n|^2 = |V^n|^2_{M(Z^n)} &= |V^{n-1}|^2_{M(Z^n)} + 2h(V^{n-1}, F^n)_{M(Z^n)} + h^2|F^n|^2_{M(Z^n)} \\ &\leqq |\tilde{V}^{n-1}|^2 + L_M|Z^n - Z^{n-1}||V^{n-1}|^2 + 2h\lambda_{\max}|F^n||V^{n-1}| \\ &\quad + h^2\lambda_{\max}|F^n|^2 \\ &\leqq |\tilde{V}^{n-1}|^2 + hL_M C_0^3 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} + 3h\lambda_{\max}C_F C_0. \end{aligned} \tag{22}$$

From now on, let us denote simply $J_l = J(Z^l)$ for all $l \in \{1, \ldots, n\}$ and assume that $J_n \neq \emptyset$. We observe that there exists $k \in \{1, \ldots, \nu\}$ such that

$$J_{n-k} \subset \bigcup_{p=0}^{k-1} J_{n-p}.$$

Indeed, for $k \geqq 1$, Card $\left(\bigcup_{p=0}^{k} J_{n-p}\right) \geqq$ Card $\left(\bigcup_{p=0}^{k-1} J_{n-p}\right) + 1$ if $J_{n-k} \not\subset \left(\bigcup_{p=0}^{k-1} J_{n-p}\right)$ and we have $J_n \neq \emptyset$, $\left(\bigcup_{p=0}^{k} J_{n-p}\right) \subset \{1, \ldots, \nu\}$. We define $k \in \{1, \ldots, \nu\}$ such that

$$J_{n-k} \subset \left(\bigcup_{p=0}^{k-1} J_{n-p}\right) \quad \text{and} \quad J_{n-m} \not\subset \left(\bigcup_{p=0}^{m-1} J_{n-p}\right), \quad \forall m \in \{1, \ldots, k-1\}.$$

With (16) and (14) we have that, for all $m \in \{n-k, \dots, n\}$

$$|Z^m - Z^n| \leqq \sum_{p=m+1}^{n} \frac{h}{1+e} |V^p + eV^{p-2}| \leqq (n-m)C_0 h + \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right) C_0 h$$
$$\leqq \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} + \nu\right) C_0 h \leqq \min(r_{B_0}, r'_{B_0}) \quad \forall h \in (0, h_0^*]. \tag{23}$$

Thus, using Lemma 15, we may define

$$e_\alpha^n = \frac{M^{-1/2}(Z^n)\nabla f_\alpha(Z^n)}{\left|M^{-1/2}(Z^n)\nabla f_\alpha(Z^n)\right|} \quad \forall \alpha \in \bigcup_{p=0}^{k} J_{n-p}.$$

We introduce the following notation:

$$\tilde{J}_n = \left(\bigcup_{p=0}^{k} J_{n-p}\right) = \left(\bigcup_{p=0}^{k-1} J_{n-p}\right), \quad E_n = \operatorname{span}\left\{e_\alpha^n, \alpha \in \tilde{J}_n\right\},$$

and

$$\tilde{F}^m = M^{1/2}(Z^m)F^m \quad \forall m \in \{n-k, \dots, n\},$$
$$\tilde{V}_N^m = \operatorname{Proj}(E_n, \tilde{V}^m), \ \tilde{V}_T^m = \operatorname{Proj}(E_n^\perp, \tilde{V}^m) \quad \forall m \in \{n-k-1, \dots, n\},$$

where the projections are defined with respect to the Euclidean metric on $\mathbb{R}^d$.

Now let us compare $\tilde{V}^n$ and $\tilde{V}^{n-k-1}$. More precisely, we first compare $\tilde{V}_T^n$ and $\tilde{V}_T^{n-k-1}$.

Lemma 3. *There exists $C_1 > 0$ such that*

$$\left|\tilde{V}_T^n - \tilde{V}_T^{n-k-1}\right| = \left|\operatorname{Proj}\left(E_n^\perp, \tilde{V}^n - \tilde{V}^{n-k-1}\right)\right| \leqq C_1 h.$$

Proof. We know from Lemma 1 that, for all $m \in \{n-k, \dots, n\}$, we have

$$M(U^m)(V^{m-1} - V^m + hF^m) \in N_K(Z^m).$$

Thus, there exist non-positive real numbers $(\mu_\alpha^m)_{\alpha \in J_m}$ such that

$$M(U^m)(V^{m-1} - V^m + hF^m) = \sum_{\alpha \in J_m} \mu_\alpha^m M^{1/2}(Z^m) e_\alpha^m,$$

with $e_\alpha^m = e_\alpha(Z^m)$ for all $\alpha \in J_m$. Moreover, if $J_m \neq \emptyset$, assumption (H6) implies that $(e_\alpha^m)_{\alpha \in J_m}$ is orthonormal, thus

$$|\mu_\beta^m| = \left|\left(e_\beta^m, M^{-1/2}(Z^m)M(U^m)(V^{m-1} - V^m + hF^m)\right)\right|$$
$$\leqq \left\|M^{-1/2}(Z^m)\right\| \left\|M(U^m)\right\| |V^{m-1} - V^m + hF^m|$$

and with Proposition 1, we infer that

$$|\mu_\beta^m| \leqq C_2 = \frac{\lambda_{\max}}{\sqrt{\lambda_{\min}}}\left(2 + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right) C_0 \tag{24}$$

for all $\beta \in J_m$ and for all $m \in \{n-k, \ldots, n\}$. It follows that

$$\begin{aligned}\tilde{V}^{m-1} - \tilde{V}^m + h\tilde{F}^m = &\left(M^{1/2}(Z^{m-1}) - M^{1/2}(Z^m)\right) V^{m-1} + \sum_{\alpha \in J_m} \mu_\alpha^m e_\alpha^n \\ &+ \sum_{\alpha \in J_m} \mu_\alpha^m \left(M^{1/2}(Z^m) M^{-1}(U^m) M^{1/2}(Z^m) e_\alpha^m - e_\alpha^n\right).\end{aligned} \tag{25}$$

By summation for $m = n-k$ to $m = n$, we infer that

$$\begin{aligned}\tilde{V}^n - \tilde{V}^{n-k-1} = &\sum_{m=n-k}^{n} \sum_{\alpha \in J_m} (-\mu_\alpha^m) M^{1/2}(Z^m) M^{-1}(U^m) M^{1/2}(Z^m) e_\alpha^m \\ &+ \sum_{m=n-k}^{n} \left(M^{1/2}(Z^m) - M^{1/2}(Z^{m-1})\right) V^{m-1} + h\tilde{F}^m.\end{aligned} \tag{26}$$

But $\sum_{\alpha \in J_m} \mu_\alpha^m e_\alpha^n \in E_n$ since $J_m \subset \tilde{J}_n$, so

$$\begin{aligned}&\mathrm{Proj}\left(E_n^\perp, \tilde{V}^n - \tilde{V}^{n-k-1}\right) \\ &= \mathrm{Proj}\left(E_n^\perp, \sum_{m=n-k}^{n} \sum_{\alpha \in J_m} \mu_\alpha^m \left(e_\alpha^n - M^{1/2}(Z^m) M^{-1}(U^m) M^{1/2}(Z^m) e_\alpha^m\right)\right) \\ &\quad + \mathrm{Proj}\left(E_n^\perp, \sum_{m=n-k}^{n} \left(M^{1/2}(Z^m) - M^{1/2}(Z^{m-1})\right) V^{m-1} + h\tilde{F}^m\right)\end{aligned}$$

which yields

$$\begin{aligned}&\left|\mathrm{Proj}\left(E_n^\perp, \tilde{V}^n - \tilde{V}^{n-k-1}\right)\right| \\ &\leqq \sum_{m=n-k}^{n} \sum_{\alpha \in J_m} |\mu_\alpha^m| \left(|e_\alpha^n - e_\alpha^m| + \left\|M^{1/2}(Z^m)\right\|^2 \left\|M^{-1}(Z^m) - M^{-1}(U^m)\right\|\right) \\ &\quad + \sum_{m=n-k}^{n} \left\|M^{1/2}(Z^m) - M^{1/2}(Z^{m-1})\right\| \left|V^{m-1}\right| + h \left\|M^{1/2}(Z^m)\right\| |F^m|.\end{aligned}$$

Using the Lipschitz property of $M^{1/2}$ and M^{-1} on B_0 we get

$$\begin{aligned}\left|\mathrm{Proj}\left(E_n^\perp, \tilde{V}^n - \tilde{V}^{n-k-1}\right)\right| \leqq &\sum_{m=n-k}^{n} \sum_{\alpha \in J_m} |\mu_\alpha^m| \left(|e_\alpha^n - e_\alpha^m| + \lambda_{\max} L_{M^{-1}} |U^m - Z^m|\right) \\ &+ \sum_{m=n-k}^{n} L_{M^{1/2}} C_0 \left|Z^m - Z^{m-1}\right| + \sqrt{\lambda_{\max}} h C_F.\end{aligned}$$

But $m \geqq n-k \geqq n-\nu \geqq 1$, thus with (16), (14), (19) and (21)

$$\begin{aligned}\left|Z^m - Z^{m-1}\right| &= \frac{h}{1+e}\left|V^m + eV^{m-2}\right| \\ &\leqq \begin{cases} hC_0\left(1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right) & \text{if } m = n, \\ hC_0 & \text{if } 2 \leqq m \leqq n-1, \end{cases} \\ \left|Z^1 - Z^0\right| &= \frac{h}{1+e}\left|V^1 + V^0\right| \leqq 2hC_0, \end{aligned} \tag{27}$$

and with (20)

$$\begin{aligned}\left|U^m - Z^m\right| &= \frac{h}{1+e}\left|V^m - eV^{m-1}\right| \\ &\leqq \begin{cases} hC_0\left(1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right) & \text{if } m = n, \\ hC_0 & \text{if } 1 \leqq m \leqq n-1. \end{cases} \end{aligned} \tag{28}$$

It follows that

$$\begin{aligned}\left|\mathrm{Proj}\left(E_n^{\perp}, \tilde{V}^n - \tilde{V}^{n-k-1}\right)\right| \leqq \sum_{m=n-k}^{n} h\left(L_{M^{1/2}}C_0^2\left(1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)+\sqrt{\lambda_{\max}}C_F\right) \\ + \sum_{m=n-k}^{n}\sum_{\alpha\in J_m}\left|\mu_\alpha^m\right|\left(\left|e_\alpha^n - e_\alpha^m\right| + \lambda_{\max}L_{M^{-1}}C_0 h \right. \\ \left. \times\left(1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)\right).\end{aligned}$$

Since

$$|Z^m - Z^n| \leqq (n-m)C_0 h + \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right)C_0 h \leqq \min(r_{B_0}, r'_{B_0})$$

for all $m \in \{n-k, \ldots, n\}$ (see (23)), we infer from Lemma 15 that

$$|e_\alpha^m - e_\alpha^n| \leqq C'_{B_0}|Z^m - Z^n| \leqq hC'_{B_0}C_0\left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} + k - 1\right).$$

Finally, using (24) and recalling that $k \in \{1, \ldots, \nu\}$, we may conclude with

$$\begin{aligned}C_1 = (\nu+1)\left(L_{M^{1/2}}C_0^2\left(1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)+\sqrt{\lambda_{\max}}C_F\right) \\ +(\nu+1)\nu C_2 C_0\left(C'_{B_0}\left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}+\nu-1\right)+\lambda_{\max}L_{M^{-1}}\left(1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)\right).\end{aligned}$$

□

In order to compare $\tilde{V}_N^n$ and $\tilde{V}_N^{n-k-1}$, we first prove that

Lemma 4. *There exists $C_3 > 0$ such that, for all $\beta \in \tilde{J}_n$:*

$$\left(\tilde{V}^{n-k-1}, e_\beta^n\right) - C_3 h \leqq \left(\tilde{V}^n, e_\beta^n\right) \leqq -e\left(\tilde{V}^{n-k-1}, e_\beta^n\right) + C_3 h.$$

Proof. With (26)

$$\begin{aligned}
&\left(e_\beta^n, \tilde{V}^n - \tilde{V}^{n-k-1} - h\sum_{m=n-k}^{n} \tilde{F}^m\right) \\
&\quad = \sum_{m=n-k}^{n} \sum_{\alpha \in J_m} \left(-\mu_\alpha^m\right)\left(e_\beta^n, M^{1/2}(Z^m)M^{-1}(U^m)M^{1/2}(Z^m)e_\alpha^m\right) \\
&\qquad + \sum_{m=n-k}^{n} \left(e_\beta^n, \left(M^{1/2}(Z^m) - M^{1/2}(Z^{m-1})\right)V^{m-1}\right) \quad \forall \beta \in \tilde{J}_n.
\end{aligned}$$

But, for all $\beta \in \tilde{J}_n$, there exists $l \in \{n-k+1, \ldots, n\}$ such that $\beta \in J_l$. Thus, if $\alpha \in J_m$ and $\alpha \neq \beta$, we get

$$\begin{aligned}
\left|\left(e_\alpha^m, e_\beta^n\right)\right| &\leqq \left|\left(e_\alpha^m, e_\beta^l\right)\right| + \left|\left(e_\alpha^m, e_\beta^n - e_\beta^l\right)\right| \\
&\leqq C_{B_0}\left|Z^m - Z^l\right| + C'_{B_0}\left|Z^l - Z^n\right| \\
&\leqq C_{B_0}\left(|l-m| + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right)hC_0 \\
&\quad + C'_{B_0}\left((n-l) + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right)hC_0 \\
&\leqq 2C_e\left(k + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right)hC_0, \quad \text{with } C_e = \max(C_{B_0}, C'_{B_0}).
\end{aligned}$$

If $\beta = \alpha$, we have

$$\begin{aligned}
\left(e_\alpha^m, e_\alpha^n\right) &= \left|e_\alpha^n\right|^2 + \left(e_\alpha^m - e_\alpha^n, e_\alpha^n\right) \\
&\geqq 1 - C'_{B_0}\left|Z^m - Z^n\right| \geqq 1 - C_e\left(k + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right)hC_0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left(e_\beta^n, \tilde{V}^n - \tilde{V}^{n-k-1} - h\sum_{m=n-k}^{n} \tilde{F}^m\right) \\
&\quad \geqq -\sum_{m=n-k}^{n}\sum_{\alpha\in J_m} \left|\mu_\alpha^m\right| 2C_e\left(k + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right)hC_0 \\
&\qquad -\sum_{m=n-k}^{n}\sum_{\alpha\in J_m} \left|\mu_\alpha^m\right| \left\|M^{1/2}(Z^m)\right\|^2 \left\|M^{-1}(Z^m) - M^{-1}(U^m)\right\| \\
&\qquad -\sum_{m=n-k}^{n} \left\|M^{1/2}(Z^m) - M^{1/2}(Z^{m-1})\right\| \left|V^{m-1}\right|
\end{aligned}$$

and using estimates (24), (27), (28) and the Lipschitz property of M^{-1} and $M^{-1/2}$ we get

$$\begin{aligned}
\left(e_\beta^n, \tilde{V}^n\right) \geqq \left(e_\beta^n, \tilde{V}^{n-k-1}\right) &- (\nu+1)L_{M^{1/2}}C_0^2 h\left(1 + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right) \\
&- (\nu+1)\sqrt{\lambda_{\max}}C_F h - \hat{C}_3 h
\end{aligned}$$

with

$$\hat{C}_3 = (\nu+1)\nu C_2 C_0 h\left(2C_e\left(k+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}-1\right) + \lambda_{\max}L_{M^{-1}}\left(1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)\right).$$

This is the left side of Lemma 4. For the right-hand side we consider $l \in \{n-k+1, \ldots, n\}$ such that $\beta \in J_l$ and $\beta \notin \bigcup_{m=l+1}^{n} J_m$. Then,

$$\begin{aligned}
&\left(e_\beta^n, \tilde{V}^n - \tilde{V}^l - h\sum_{m=l+1}^{n} \tilde{F}^m\right) \\
&\quad = \sum_{m=l+1}^{n}\sum_{\alpha\in J_m} \left(-\mu_\alpha^m\right)\left(e_\beta^n, M^{1/2}(Z^m)M^{-1}(U^m)M^{1/2}(Z^m)e_\alpha^m\right) \\
&\qquad + \sum_{m=l+1}^{n}\left(e_\beta^n, \left(M^{1/2}(Z^m) - M^{1/2}(Z^{m-1})\right)V^{m-1}\right).
\end{aligned}$$

Since $\beta \notin \bigcup_{m=l+1}^{n} J_m$, we have

$$\begin{aligned}
\left|\left(e_\beta^n, e_\alpha^m\right)\right| &\leqq \left|\left(e_\beta^n - e_\beta^l, e_\alpha^m\right)\right| + \left|\left(e_\beta^l, e_\alpha^m\right)\right| \\
&\leqq C_e\left((n-l) + (m-l) + 8\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 2\right)C_0 h \\
&\leqq 2C_e\left((n-l) + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right)C_0 h \qquad (29)
\end{aligned}$$

for all $m \in \{l+1, \dots, n\}$, for all $\alpha \in J_m$ and thus

$$\begin{aligned}\left(\tilde{V}^n, e_\beta^n\right) \leqq \left(\tilde{V}^l, e_\beta^n\right) + h \sum_{m=l+1}^{n} \left(\tilde{F}^m, e_\beta^n\right) \\ + (n-l)\nu C_2 \left(2C_e \left((n-l) + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right)\right. \\ \left. + L_{M^{-1}} \lambda_{\max} \left(1 + \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)\right) C_0 h \\ + (n-l) L_{M^{1/2}} C_0^2 h \left(1 + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right).\end{aligned}$$

Moreover, $\beta \in J_l$ and $l \geqq n-k+1 \geqq 2$ thus, with Proposition 1,

$$\left(\nabla f_\beta(Z^l), V^l + eV^{l-2}\right) \leqq \frac{L_f h}{2(1+e)} \left|V^l + eV^{l-2}\right|^2$$

and

$$\left(e_\beta^l, M^{1/2}(Z^l)(V^l + eV^{l-2})\right) \leqq \frac{L_f h}{2m_{B_0}} \left|V^l + eV^{l-2}\right|^2.$$

With (16), (14) and (21) it follows that

$$\begin{aligned}\left(\tilde{V}^l + e\tilde{V}^{l-2}, e_\beta^l\right) \leqq \frac{L_f h}{2m_{B_0}} |V^l + eV^{l-2}|^2 \\ + e\left(\left(M^{1/2}(Z^{l-2}) - M^{1/2}(Z^l)\right) V^{l-2}, e_\beta^l\right) \leqq C_4 h\end{aligned}$$

with

$$C_4 = \frac{L_f}{2m_{B_0}} C_0^2 \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} + e\right)^2 + eL_{M^{1/2}} C_0^2 \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} + 3\right).$$

Recalling that $|e_\beta^l - e_\beta^n| \leqq C_e |Z^l - Z^n|$, we infer that

$$\left(\tilde{V}^l + e\tilde{V}^{l-2}, e_\beta^n\right) \leqq C_5 h$$

with

$$C_5 = C_4 + C_e \left((n-l) + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right) C_0^2 \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} + e\right) \sqrt{\lambda_{\max}}.$$

Finally, again using (25), we get

$$\begin{aligned}\tilde{V}^{l-2} - \tilde{V}^{n-k-1} = \sum_{m=n-k}^{l-2} h\tilde{F}^m - \left(M^{1/2}(Z^{m-1}) - M^{1/2}(Z^m)\right) V^{m-1} \\ - \sum_{m=n-k}^{l-2} \sum_{\alpha \in J_m} \mu_\alpha^m \left(M^{1/2}(Z^m) M^{-1}(U^m) M^{1/2}(Z^m) e_\alpha^m\right)\end{aligned}$$

and with (H6) (see (8)), we obtain

$$\begin{aligned}&\left(\tilde{V}^{l-2}-\tilde{V}^{n-k-1},e_\beta^{l-2}\right)\\&\geqq -(k'+1)\sqrt{\lambda_{\max}}C_F h-(k'+1)\nu C_2\\&\quad\times\left(2C_e\left(k'+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}-1\right)+\lambda_{\max}L_{M^{-1}}\right)C_0 h\\&\quad-2(k'+1)L_{M^{1/2}}C_0^2 h\end{aligned}$$

with $k'=l-(n-k)-2$. Hence, with (29)

$$\begin{aligned}\left(\tilde{V}^{l-2}-\tilde{V}^{n-k-1},e_\beta^n\right)\geqq&-(k'+1)h\left(\nu C_2C_0\left(2C_e\left(k'+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}-1\right)\right.\right.\\&\left.+\lambda_{\max}L_{M^{-1}}\right)+2L_{M^{1/2}}C_0^2+\sqrt{\lambda_{\max}}C_F\Bigg)\\&-2C_0^2C_e h\left(n-l+1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)\sqrt{\lambda_{\max}}.\end{aligned}$$

Observing that $n-\nu\leqq n-k<l\leqq n$, the conclusion follows with

$$C_3=\max(\hat{C}_3,\tilde{C}_3)+(\nu+1)\sqrt{\lambda_{\max}}C_F+(\nu+1)L_{M_{1/2}}C_0^2\left(1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)$$

and

$$\begin{aligned}\tilde{C}_3=&(\nu-1)\nu C_2C_0\left(2C_e\left(\nu+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}-2\right)+\lambda_{\max}L_{M^{-1}}\left(1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)\right)\\&+C_4+2C_0^2C_e\left(\nu+1+4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)\left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}+3e\right)\sqrt{\lambda_{\max}}.\end{aligned}$$

□

Now, we may apply the following lemma.

Lemma 5. *There exists $C_5>0$ such that, for all $w\in E_n$, we have*

$$\left||w|^2-\sum_{\alpha\in\tilde{J}_n}(w,e_\alpha^n)^2\right|\leqq C_5 h\sum_{\alpha\in\tilde{J}_n}(w,e_\alpha^n)^2.$$

Proof. Let $w\in E_n=\operatorname{span}\left\{e_\alpha^n,\alpha\in\tilde{J}_n\right\}$. There exists $(\mu_\beta)_{\beta\in\tilde{J}_n}\in\mathbb{R}^{\operatorname{Card}\tilde{J}_n}$ such that

$$w=\sum_{\beta\in\tilde{J}_n}\mu_\beta e_\beta^n,$$

and for all $\alpha \in \tilde{J}_n$

$$(w, e_\alpha^n) = \sum_{\beta \in \tilde{J}_n} \mu_\beta (e_\alpha^n, e_\beta^n) = \mu_\alpha + \sum_{\beta \in \tilde{J}_n \setminus \{\alpha\}} \mu_\beta (e_\alpha^n, e_\beta^n).$$

But $\tilde{J}_n = \bigcup_{p=0}^{k-1} J_{n-p} = \bigcup_{p=0}^{k} J_{n-p}$, and there exists $(p, q) \in \{0, \ldots, k-1\}^2$ such that $\alpha \in J_{n-p}$ and $\beta \in J_{n-q}$. Hence,

$$\begin{aligned}\left|(e_\alpha^n, e_\beta^n)\right| &\leqq \left|(e_\alpha^n - e_\alpha^{n-p}, e_\beta^n)\right| + \left|(e_\alpha^{n-p}, e_\beta^{n-q})\right| + \left|(e_\alpha^{n-p}, e_\beta^n - e_\beta^{n-q})\right| \\ &\leqq \left|e_\alpha^n - e_\alpha^{n-p}\right| + \left|(e_\alpha^{n-p}, e_\beta^{n-q})\right| + \left|e_\beta^n - e_\beta^{n-q}\right|.\end{aligned}$$

We recall that, for all $(m, l) \in \{0, \ldots, k-1\}^2$ we have

$$|Z^{n-m} - Z^{n-l}| \leqq \left(\nu + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 2\right) C_0 h \leqq \min(r_{B_0}, r'_{B_0}).$$

Thus, with (8) and Lemma 15 we get for all $\beta \in \tilde{J}_n \setminus \{\alpha\}$

$$\begin{aligned}\left|e_\alpha^n - e_\alpha^{n-p}\right| &\leqq \left(p + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right) C'_{B_0} C_0 h, \\ \left|e_\beta^n - e_\beta^{n-q}\right| &\leqq \left(q + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right) C'_{B_0} C_0 h, \\ \left|(e_\alpha^{n-p}, e_\beta^{n-q})\right| &\leqq \left(|p-q| + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 1\right) C_{B_0} C_0 h,\end{aligned}$$

and

$$\left|(e_\alpha^n, e_\beta^n)\right| \leqq 3\left(k + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 2\right) C_e C_0 h.$$

Let us denote

$$C_6 = 3 C_e C_0 \left(\nu + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} - 2\right).$$

We infer that

$$\left|(w, e_\alpha^n) - \mu_\alpha\right| \leqq \sum_{\beta \in \tilde{J}_n \setminus \{\alpha\}} |\mu_\beta| |(e_\alpha^n, e_\beta^n)| \leqq C_6 h \sum_{\beta \in \tilde{J}_n \setminus \{\alpha\}} |\mu_\beta|$$

and since $\mathrm{Card}(\tilde{J}_n) \leqq \nu$ and $k \leqq \nu$ we get

$$\left|(w, e_\alpha^n) - \mu_\alpha\right| \leqq C_6 h \sqrt{\nu - 1} \left(\sum_{\beta \in \tilde{J}_n \setminus \{\alpha\}} |\mu_\beta|^2\right)^{1/2}.$$

On the other hand, we also have

$$|w|^2 = \sum_{(\alpha,\beta)\in \tilde{J}_n{}^2} \mu_\alpha \mu_\beta (e_\alpha^n, e_\beta^n) = \sum_{\alpha\in\tilde{J}_n} \mu_\alpha^2 + \sum_{(\alpha,\beta)\in \tilde{J}_n{}^2, \alpha\neq\beta} \mu_\alpha \mu_\beta (e_\alpha^n, e_\beta^n)$$

and, with the same arguments as above, we get

$$\begin{aligned}\left| |w|^2 - \sum_{\alpha\in\tilde{J}_n} \mu_\alpha^2 \right| &\leqq \sum_{(\alpha,\beta)\in \tilde{J}_n{}^2, \alpha\neq\beta} |\mu_\alpha \mu_\beta| \left| (e_\alpha^n, e_\beta^n) \right| \leqq C_6 h \sum_{(\alpha,\beta)\in \tilde{J}_n{}^2, \alpha\neq\beta} |\mu_\alpha \mu_\beta| \\ &\leqq C_6 h(\nu - 1) \sum_{\alpha\in\tilde{J}_n} \mu_\alpha^2.\end{aligned}$$

It follows that

$$\begin{aligned}\left| |w|^2 - \sum_{\alpha\in\tilde{J}_n} (w, e_\alpha^n)^2 \right| &\leqq \left| |w|^2 - \sum_{\alpha\in\tilde{J}_n} \mu_\alpha^2 \right| + \left| \sum_{\alpha\in\tilde{J}_n} \mu_\alpha^2 - (w, e_\alpha^n)^2 \right| \\ &\leqq C_6 h(\nu - 1) \sum_{\alpha\in\tilde{J}_n} \mu_\alpha^2 + \sum_{\alpha\in\tilde{J}_n} \left| \mu_\alpha^2 - (w, e_\alpha^n)^2 \right|\end{aligned}$$

and for all $\alpha \in \tilde{J}_n$

$$\begin{aligned}\left| \mu_\alpha^2 - (w, e_\alpha^n)^2 \right| &= \left| \mu_\alpha - (w, e_\alpha^n) \right| \left| \mu_\alpha + (w, e_\alpha^n) \right| \leqq \left| \mu_\alpha - (w, e_\alpha^n) \right| \\ &\quad \times \left(2|\mu_\alpha| + \left| \mu_\alpha - (w, e_\alpha^n) \right| \right) \\ &\leqq \left| \mu_\alpha - (w, e_\alpha^n) \right| \left(2 \left(\sum_{\beta\in\tilde{J}_n} |\mu_\beta|^2 \right)^{1/2} + \left| \mu_\alpha - (w, e_\alpha^n) \right| \right) \\ &\leqq C_6 h \sqrt{\nu - 1} \left(2 + C_6 h \sqrt{\nu - 1} \right) \sum_{\beta\in\tilde{J}_n} \mu_\beta^2.\end{aligned}$$

Since

$$h_0^* \leqq \frac{1}{72\nu^{3/2} C_e C_0 \left(\nu + 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \right)}$$

we get

$$\left| \mu_\alpha^2 - (w, e_\alpha^n)^2 \right| \leqq 3C_6 h \sqrt{\nu - 1} \sum_{\beta\in\tilde{J}_n} \mu_\beta^2$$

for all $\alpha \in \tilde{J}_n$. Thus

$$\sum_{\alpha\in\tilde{J}_n} (w, e_\alpha^n)^2 \geqq \left(1 - 3C_6 h\nu \sqrt{\nu - 1} \right) \sum_{\alpha\in\tilde{J}_n} \mu_\alpha^2 \geqq \frac{1}{2} \sum_{\alpha\in\tilde{J}_n} \mu_\alpha^2$$

which yields finally

$$\left| |w|^2 - \sum_{\alpha \in \tilde{J}_n} (w, e_\alpha^n)^2 \right| \leqq C_6 h \left((\nu - 1) + 3\nu\sqrt{\nu - 1} \right) \sum_{\alpha \in \tilde{J}_n} \mu_\alpha^2$$
$$\leqq 4\nu^{3/2} C_6 h \sum_{\alpha \in \tilde{J}_n} (w, e_\alpha^n)^2.$$

The conclusion follows with $C_7 = 4\nu^{3/2} C_6$. □

Now we can prove that

$$|\tilde{V}^n|^2 \leqq \max \left(|\tilde{V}^{n-1}|^2, \ldots, |\tilde{V}^{n-\nu}|^2, \left| \tilde{V}^{n-\nu-1} \right|^2 \right) + C_{\mathcal{B}, C_0} h$$

where $C_{\mathcal{B}, C_0}$ is a constant which depends only on $\mathcal{B}$ and C_0.

Indeed, if $J_n = \emptyset$, we have (see (22))

$$|\tilde{V}^n|^2 \leqq |\tilde{V}^{n-1}|^2 + h L_M C_0^3 4 \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} + 3h\lambda_{\max} C_F C_0.$$

If $J_n \neq \emptyset$, we have with Lemma 3 and Lemma 4

$$\left| \tilde{V}_T^n \right| \leqq \left| \tilde{V}_T^{n-k-1} \right| + \left| \text{Proj} \left(E_n^\perp, \tilde{V}^n - \tilde{V}^{n-k-1} \right) \right| \leqq \left| \tilde{V}_T^{n-k-1} \right| + C_1 h$$

and, for all $\beta \in \tilde{J}_n$

$$\left| \left(\tilde{V}^n, e_\beta^n \right) \right| = \left| \left(\tilde{V}_N^n, e_\beta^n \right) \right| \leqq \left| \left(\tilde{V}^{n-k-1}, e_\beta^n \right) \right| + C_3 h = \left| \left(\tilde{V}_N^{n-k-1}, e_\beta^n \right) \right| + C_3 h.$$

With Lemma 5 we get

$$\left| \tilde{V}^n \right|^2 = \left| \tilde{V}_T^n \right|^2 + \left| \tilde{V}_N^n \right|^2$$
$$\leqq \left| \tilde{V}_T^{n-k-1} \right|^2 + 2C_1 h \left| \tilde{V}_T^{n-k-1} \right| + (C_1 h)^2 + (1 + C_7 h) \sum_{\beta \in \tilde{J}_n} \left(\tilde{V}_N^n, e_\beta^n \right)^2$$

But

$$\left| \tilde{V}_T^{n-k-1} \right| \leqq \left| \tilde{V}^{n-k-1} \right| \leqq C_0 \sqrt{\lambda_{\max}}$$

and

$$\left| \left(\tilde{V}_N^{n-k-1}, e_\beta^n \right) \right| \leqq \left| \tilde{V}^{n-k-1} \right| \leqq C_0 \sqrt{\lambda_{\max}}.$$

Thus

$$\left| \tilde{V}^n \right|^2 \leqq \left| \tilde{V}_T^{n-k-1} \right|^2 + 2C_1 h C_0 \sqrt{\lambda_{\max}} + (C_1 h)^2$$
$$+ (1 + C_7 h) \sum_{\beta \in \tilde{J}_n} \left(\left(\tilde{V}_N^{n-k-1}, e_\beta^n \right)^2 + 2C_3 C_0 \sqrt{\lambda_{\max}} h + (C_3 h)^2 \right).$$

Since $h_0^* \leqq \frac{1}{3C_7}$, we have

$$1 - C_7 h > 0 \quad \text{and} \quad \frac{1 + C_7 h}{1 - C_7 h} \leqq 1 + 3C_7 h.$$

Thus

$$\left|\tilde{V}^n\right|^2 \leqq \left|\tilde{V}_T^{n-k-1}\right|^2 + \frac{1 + C_7 h}{1 - C_7 h}\left|\tilde{V}_N^{n-k-1}\right|^2 + C_0\sqrt{\lambda_{\max}} h\,(2C_1 + 2\nu C_3(1 + C_7 h))$$
$$+ (C_1 h)^2 + (C_3 h)^2(1 + C_7 h)\nu.$$

Finally, in both cases ($J_n = \emptyset$ or $J_n \neq \emptyset$), we have

$$\left|\tilde{V}^n\right|^2 \leqq \max\left(\left|\tilde{V}^{n-1}\right|^2, \ldots, \left|\tilde{V}^{n-\nu-1}\right|^2\right) + C_{\mathcal{B},C_0} h$$

with

$$C_{\mathcal{B},C_0} = \max\Bigg(L_M C_0^3 4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} + 3\lambda_{\max} C_F C_0,\, C_0\sqrt{\lambda_{\max}}\left(2C_1 + 2\nu C_3(1 + C_7 h^*)\right)$$
$$+\; C_1^2 h^* + C_3^2 h^*(1 + C_7 h^*)\nu + 3C_7\lambda_{\max} C_0^2\Bigg)$$

which allows us to conclude the proof. □

Let us now consider the initialization procedure given by formula (4), that is, let $t_{0h} = 0$ and

$$U^0 = u_0, \quad U^1 \in \operatorname{Argmin}_{Z \in K} \|u_0 + h v_0 + h z(h) - Z\|_{M(u_0)}, \ \lim_{h \to 0} z(h) = 0,$$

for all $h \in (0, h^*]$. We choose $\mathcal{B} = \overline{B}(u_0, C + 1)$ with $C \geqq 0$. By applying the preceding results, we get a uniform estimate of the discrete velocities on a non-trivial time interval:

Theorem 1. *For all $C_0^* \geqq 2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}(|v_0| + 1)$ and for all $C_0 > \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)^{\nu+1} C_0^*$, there exist $h_0^* \in (0, h^*]$ and $\tau_0 > 0$, depending only on $\mathcal{B}$, C_0 and C_0^* and the data, such that*

$$|V^n| = \left|\frac{U^{n+1} - U^n}{h}\right| \leqq C_0 \quad \forall nh \in [0, \min(\tau_0, T)], \quad \forall h \in (0, h_0^*].$$

Proof. Let $C_0^* \geqq 2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}(|v_0|+1)$, and let C_0 and h_0^* be defined as in Proposition 2. With (9) we have

$$|z(h)| \leqq 1 \quad \forall h \in (0, h^*]$$

and by definition of U^1, we have

$$\left\|u_0 + h v_0 + h z(h) - U^1\right\|_{M(u_0)} \leqq \|u_0 + h v_0 + h z(h) - Z\|_{M(u_0)} \quad \forall Z \in K.$$

By choosing $Z = U^0 = u_0$, we get

$$\|V^0\|_{M(u_0)} \leqq 2\left(\|v^0\|_{M(u_0)} + \|z(h)\|_{M(u_0)}\right).$$

Since $u_0 \in \mathcal{B} \subset B_0$, we infer that

$$|V^0| \leqq 2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\,(|v_0| + 1) \leqq C_0^* \quad \forall h \in (0, h^*].$$

It follows that

$$|V^0| \leqq C_0^* < \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)^{-(\nu+1)} C_0 \quad \forall h \in (0, h_0^*]$$

and we may apply Proposition 2, which yields the announced result. □

3. Convergence of the approximate solutions $(u_h)_{h^* \geqq h > 0}$

Before passing to the limit as h tends to zero in the sequence $(u_h)_{h^* \geqq h > 0}$, we prove an estimate for the discrete accelerations.

Proposition 4. *Let us assume that there exist $C_0 > 0$, $\tau_0 > 0$, $h_0^* \in (0, h^*]$ and a subsequence $(h_i)_{i\in\mathbb{N}}$, decreasing to zero, such that*

$$|V^n| \leqq C_0 \quad \forall n h_i \in [0, \min(\tau_0, T)], \quad \forall h_i \in (0, h_0^*]. \tag{30}$$

Then there exist $h_1^* \in (0, h_0^*]$ and $C_0' > 0$ such that, for all $h_i \in (0, h_1^*]$

$$\sum_{n=1}^{N} \left|V^n - V^{n-1}\right| \leqq C_0', \quad \text{with } N = \left\lfloor \frac{\min(\tau_0, T)}{h_i} \right\rfloor.$$

Proof. We begin the proof as in [11] Proposition 3. More precisely, let $\mathcal{B} = \overline{B}(u_0, C + 1)$ with $C \geqq 0$, B_0 be defined by (11) and C_F be defined by (13). Without loss of generality, possibly decreasing h_0^*, we assume that $C_0 h_0^* \leqq 1$ and $C_F h_0^* \leqq C_0$. We denote $K_1 = K \cap B_1 = K \cap \overline{B}(u_0, C_0 T + 1)$ and

$$\begin{aligned}
\lambda_{\min, B_1} &= \inf_{u\in B_1} \lambda_{\min}(u) = \frac{1}{\sup_{u\in B_1} \|M^{-1}(u)\|}, \\
\lambda_{\max, B_1} &= \sup_{u\in B_1} \lambda_{\max}(u) = \sup_{u\in B_1} \|M(u)\|.
\end{aligned}$$

Let $h_i \in (0, h_0^*]$. By definition of the scheme, we have $Z^n \in K$ for all $n \in \{0, \ldots, \lfloor T/h_i \rfloor\}$ and assumption (30) implies that

$$\left|U^n - U^0\right| \leqq h_i \sum_{k=0}^{n-1} \left|V^k\right| \leqq n h_i C_0 \leqq C_0 T \quad \forall n \in \{0, \ldots, N+1\}$$

and with (20)

$$\begin{aligned}\left|Z^n - U^0\right| &\leqq \left|U^n - U^0\right| + \frac{h_i}{1+e}\left|V^n - eV^{n-1}\right| \\ &\leqq C_0 T + C_0 h_i \leqq C_0 T + 1 \qquad \forall n \in \{1, \ldots, N\},\end{aligned}$$

thus $Z^n \in K_1$ for all $n \in \{1, \ldots, N\}$.

The continuity of the mappings $f_\alpha, \alpha \in \{1, \ldots, \nu\}$, implies that, for all $q \in \mathbb{R}^d$, there exists $r_q > 0$ such that

$$f_\alpha(q') \geqq \frac{f_\alpha(q)}{2} > 0 \quad \forall q' \in \overline{B}(q, r_q), \quad \forall \alpha \notin J(q). \tag{31}$$

Without loss of generality, we may assume that $r_q \leqq \min\left(r_{B_1}, r'_{B_1}, 1/(4\nu^2 C_{e,B_1})\right)$ where $C_{e,B_1} = \max(C_{B_1}, C'_{B_1})$ and $C_{B_1}, r_{B_1}, C'_{B_1}$ and r'_{B_1} are the constants defined at assumption (H6) and Lemma 15 for the compact set B_1. Then, the compactness of K_1 implies that there exists $(q_j)_{1 \leqq j \leqq l}$ such that $q_j \in K_1$ for all $j \in \{1, \ldots, l\}$ and

$$K_1 \subset \bigcup_{i=1}^{l} B\left(q_i, \frac{r_{q_i}}{2}\right).$$

We define

$$r = \min_{1 \leqq i \leqq l} \frac{r_{q_i}}{2}.$$

Let $h_1^* \in \left(0, \min\left(h_0^*, r/(2C_0)\right)\right]$ and $h_i \in (0, h_1^*]$. Using (21), we infer that for all $(n, m) \in \{1, \ldots, N\}^2$ such that $n \leqq m - 1$ we have

$$|Z^m - Z^n| \leqq \sum_{k=n+1}^{m} \frac{h_i}{1+e}\left|V^k + eV^{k-2}\right| \leqq (m-n) h_i C_0.$$

Let $p = \lfloor r/(C_0 h_i) \rfloor$ and let $n \in \{1, \ldots, N-1\}$. Then, there exists $i \in \{1, \ldots, l\}$ such that $Z^n \in B\left(q_i, r_{q_i}/2\right)$ and $Z^m \in B(q_i, r_{q_i})$ for all $m \in \left\{n, \ldots, \min(N, n+p)\right\}$. With (31) we infer that

$$f_\alpha(Z^m) > 0 \quad \forall \alpha \notin J(q_i) \quad \forall m \in \{n, \ldots, \min(N, n+p)\}$$

that is, $J(Z^m) \subset J(q_i)$ for all $m \in \{n, \ldots, \min(N, n+p)\}$.

Let $m \in \{n+1, \ldots, \min(N, n+p)\}$. Since $|Z^m - q_i| \leqq r_{q_i} \leqq \min(r_{B_1}, r'_{B_1})$, we may define

$$e_\alpha^m = e_\alpha(Z^m) = \frac{M^{-1/2}(Z^m)\nabla f_\alpha(Z^m)}{\left|M^{-1/2}(Z^m)\nabla f_\alpha(Z^m)\right|} \quad \forall \alpha \in J(q_i)$$

and we let $\mu_\alpha^m = 0$ if $\alpha \in J(q_i) \backslash J(Z^m)$. It follows (with Lemma 1) that

$$M(U^m)(V^{m-1} - V^m + h_i F^m) = \sum_{\beta \in J(q_i)} \mu_\beta^m M^{1/2}(Z^m) e_\beta^m,$$

with $\mu_\beta^m \leqq 0$ for all $\beta \in J(q_i)$. From assumption (H6) we know that, for all $(\alpha, \beta) \in J(Z^m)^2$

$$(e_\alpha^m, e_\beta^m) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Thus for all $\beta \in J(Z^m)$

$$\left(M^{-1/2}(Z^m)M(U^m)(V^{m-1} - V^m + h_i F^m), e_\beta^m\right) = \mu_\beta^m \leqq 0$$

and

$$\begin{aligned} |\mu_\beta^m| &= \left(M^{-1/2}(Z^m)M(U^m)(V^m - V^{m-1} - h_i F^m), e_\beta^m\right) \\ &\leqq \left|M^{-1/2}(Z^m)M(U^m)(V^m - V^{m-1} - h_i F^m)\right| \\ &\leqq \left\|M^{-1/2}(Z^m)\right\| \left\|M(U^m)\right\| |V^m - V^{m-1} - h_i F^m|. \end{aligned}$$

On the other hand, for all $\alpha \in J(q_i)\backslash J(Z^m)$

$$\left(M^{-1/2}(Z^m)M(U^m)(V^{m-1} - V^m + h_i F^m), e_\alpha^m\right) = \sum_{\beta \in J(Z^m)} \mu_\beta^m (e_\beta^m, e_\alpha^m)$$

and since $(\alpha, \beta) \in J(q_i)^2$ with $\alpha \neq \beta$, we get with assumption (H6) (see (8)) and Lemma 15

$$\begin{aligned} \left|(e_\beta^m, e_\alpha^m)\right| &\leqq \left|\left(e_\beta^m, e_\alpha^m - e_\alpha(q_i)\right)\right| + \left|\left(e_\alpha(q_i), e_\beta^m\right)\right| \\ &\leqq (C'_{B_1} + C_{B_1})|Z^m - q_i| \leqq 2C_{e,B_1} r_{q_i}. \end{aligned}$$

It follows that, for all $\alpha \in J(q_i)\backslash J(Z^m)$

$$\begin{aligned} &\left|\left(M^{-1/2}(Z^m)M(U^m)(V^{m-1} - V^m + h_i F^m), e_\alpha^m\right)\right| \\ &\leqq \sum_{\beta \in J(Z^m)} |\mu_\beta^m| \left|(e_\beta^m, e_\alpha^m)\right| \\ &\leqq 2\nu C_{e,B_1} r_{q_i} \left|M^{-1/2}(Z^m)M(U^m)(V^m - V^{m-1} - h_i F^m)\right|. \end{aligned}$$

Then we infer that

$$\begin{aligned} &\left|M^{-1/2}(Z^m)M(U^m)(V^m - V^{m-1} - h_i F^m)\right| \\ &= \left|\sum_{\beta \in J(Z^m)} -\mu_\beta^m e_\beta^m\right| \leqq \sum_{\beta \in J(Z^m)} |\mu_\beta^m| \\ &= \sum_{\beta \in J(Z^m)} \left(M^{-1/2}(Z^m)M(U^m)(V^m - V^{m-1} - h_i F^m), e_\beta^m\right) \\ &\leqq \sum_{\beta \in J(q_i)} \left(M^{-1/2}(Z^m)M(U^m)(V^m - V^{m-1} - h_i F^m), e_\beta^m\right) \\ &\quad + 2\nu^2 C_{e,B_1} r_{q_i} \left|M^{-1/2}(Z^m)M(U^m)(V^m - V^{m-1} - h_i F^m)\right|. \end{aligned}$$

Thus, for all $m \in \{n+1, \ldots, \min(N, n+p)\}$

$$
\begin{aligned}
&(1 - 2\nu^2 C_{e,B_1} r_{q_i}) \left| M^{-1/2}(Z^m) M(U^m)(V^m - V^{m-1} - h_i F^m) \right| \\
&\quad \leqq \sum_{\beta \in J(q_i)} \left(M^{-1/2}(Z^m) M(U^m)(V^m - V^{m-1} - h_i F^m), e_\beta^m \right).
\end{aligned}
$$

But $r_{q_i} \leqq 1/(4\nu^2 C_{e,B_1})$, so we get

$$
\begin{aligned}
&\frac{1}{2} \left| M^{-1/2}(Z^m) M(U^m)(V^m - V^{m-1} - h_i F^m) \right| \\
&\quad \leqq \sum_{\beta \in J(q_i)} \left(M^{-1/2}(Z^m) M(U^m)(V^m - V^{m-1} - h_i F^m), e_\beta^m \right)
\end{aligned}
$$

for all $m \in \{n+1, \ldots, \min(N, n+p)\}$.

Now, we rewrite the right-hand side in order to obtain a telescopic sum. More precisely

$$
\begin{aligned}
&\left(M^{-1/2}(Z^m) M(U^m)(V^m - V^{m-1} - h_i F^m), e_\beta^m \right) \\
&\quad \leqq - \left(M^{-1/2}(Z^{m-1}) M(U^{m-1}) V^{m-1}, e_\beta^{m-1} \right) \\
&\qquad + \left(M^{-1/2}(Z^m) M(U^m) V^m, e_\beta^m \right) \\
&\qquad - \left((M^{-1/2}(Z^m) M(U^m) - M^{-1/2}(Z^{m-1}) M(U^{m-1})) V^{m-1}, e_\beta^{m-1} \right) \\
&\qquad - \left(M^{-1/2}(Z^m) M(U^m) V^{m-1}, e_\beta^m - e_\beta^{m-1} \right) \\
&\qquad - h_i \left(M^{-1/2}(Z^m) M(U^m) F^m, e_\beta^m \right).
\end{aligned}
$$

Using the Lipschitz properties of $M^{-1/2}$ and M on B_1 we get

$$
\begin{aligned}
&\left| \left((M^{-1/2}(Z^m) M(U^m) - M^{-1/2}(Z^{m-1}) M(U^{m-1})) V^{m-1}, e_\beta^{m-1} \right) \right| \\
&\quad \leqq L_{M^{-1/2}} |Z^m - Z^{m-1}| \left\| M(U^m) \right\| |V^{m-1}| \\
&\qquad + L_M |U^m - U^{m-1}| \left\| M^{-1/2}(Z^{m-1}) \right\| |V^{m-1}| \\
&\quad \leqq \left(L_{M^{-1/2}} \lambda_{\max, B_1} + \frac{L_M}{\sqrt{\lambda_{\min, B_1}}} \right) C_0^2 h_i
\end{aligned}
$$

and with Lemma 15

$$
\begin{aligned}
&\left| \left(M^{-1/2}(Z^m) M(U^m) V^{m-1}, e_\beta^m - e_\beta^{m-1} \right) \right| \\
&\quad \leqq \left\| M^{-1/2}(Z^m) \right\| \| M(U^m) \| |V^{m-1}| C_{e,B_1} |Z^m - Z^{m-1}| \leqq \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} C_{e,B_1} C_0^2 h_i.
\end{aligned}
$$

Finally we obtain

$$\begin{aligned}
&\frac{1}{2}\left|M^{-1/2}(Z^m)M(U^m)(V^m - V^{m-1} - h_i F^m)\right| \\
&\quad \leqq \sum_{\beta \in J(q_i)} \left(-\left(M^{-1/2}(Z^{m-1})M(U^{m-1})V^{m-1}, e_\beta^{m-1}\right)\right. \\
&\qquad \left. + \left(M^{-1/2}(Z^m)M(U^m)V^m, e_\beta^m\right)\right) \\
&\qquad + h_i \nu \left(C_0^2 L_{M^{-1/2}} \lambda_{\max, B_1} + \frac{L_M C_0^2}{\sqrt{\lambda_{\min, B_1}}} + (C_{e, B_1} C_0^2 + C_F) \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} \right)
\end{aligned}$$

for all $m \in \{n+1, \ldots, \min(N, n+p)\}$. It follows that

$$\begin{aligned}
&\sum_{m=n+1}^{\min(N,n+p)} \left|M^{-1/2}(Z^m)M(U^m)(V^m - V^{m-1} - h_i F^m)\right| \leqq 4 \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} C_0 \\
&\quad + 2p' \nu h_i \left(C_0^2 L_{M^{-1/2}} \lambda_{\max, B_1} + \frac{L_M C_0^2}{\sqrt{\lambda_{\min, B_1}}} + (C_{e, B_1} C_0^2 + C_F) \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} \right)
\end{aligned}$$

with $p' = \min(N, n+p) - n$. Let

$$\begin{aligned}
\tilde{C}_0 = 2\nu \frac{\sqrt{\lambda_{\max, B_1}}}{\lambda_{\min, B_1}} &\left(C_0^2 L_{M^{-1/2}} \lambda_{\max, B_1} + \frac{L_M C_0^2}{\sqrt{\lambda_{\min, B_1}}} \right. \\
&\left. + (C_{e, B_1} C_0^2 + C_F) \frac{\lambda_{\max, B_1}}{\sqrt{\lambda_{\min, B_1}}} \right).
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{m=n+1}^{\min(N,n+p)} |V^m - V^{m-1}| &\leqq p' h_i C_F + \sum_{m=n+1}^{\min(N,n+p)} \left\| M^{-1}(U^m) \right\| \left\| M^{1/2}(Z^m) \right\| \\
&\quad \times \left| M^{-1/2}(Z^m) M(U^m)(V^m - V^{m-1} - h_i F^m) \right| \\
&\leqq 4 \left(\frac{\lambda_{\max, B_1}}{\lambda_{\min, B_1}} \right)^{3/2} C_0 + p' h_i (C_F + \tilde{C}_0).
\end{aligned}$$

Finally,

$$\begin{aligned}
\sum_{m=1}^{N} \left|V^m - V^{m-1}\right| &\leqq \sum_{k=0}^{\lfloor N/p \rfloor - 1} \sum_{m=kp+1}^{(k+1)p} \left|V^m - V^{m-1}\right| + \sum_{m=\lfloor N/p \rfloor p + 1}^{N} \left|V^m - V^{m-1}\right| \\
&\leqq \left(\left\lfloor \frac{N}{p} \right\rfloor + 1 \right) 4 \left(\frac{\lambda_{\max, B_1}}{\lambda_{\min, B_1}} \right)^{3/2} C_0 + N h_i (C_F + \tilde{C}_0).
\end{aligned}$$

Since $p = \left\lfloor \frac{r}{C_0 h_i} \right\rfloor$ and $N = \left\lfloor \frac{\min(\tau_0, T)}{h_i} \right\rfloor$ we get $\left\lfloor \frac{N}{p} \right\rfloor \leqq \frac{C_0 T}{r - C_0 h_1^*}$ for all $h_i \in (0, h_1^*]$ and the conclusion follows with

$$C_0' = \left(\frac{C_0 T}{r - C_0 h_1^*} + 1 \right) 4 \left(\frac{\lambda_{\max, B_1}}{\lambda_{\min, B_1}} \right)^{3/2} C_0 + T(C_F + \tilde{C}_0).$$

□

Now we can pass to the limit as h tends to zero. The steps of the proof are quite similar to [11]. For the sake of self-sufficiency, we state the main results here.

Let us recall the definition of the approximate solutions $(u_h)_{h^* \geqq h > 0}$

$$u_h(t) = U^n + (t - nh) \frac{U^{n+1} - U^n}{h} \quad \forall t \in [nh, (n+1)h] \cap [0, T] \tag{32}$$

and let us define

$$v_h(t) = V^n = \frac{U^{n+1} - U^n}{h} \quad \forall t \in [nh, (n+1)h) \cap [0, T] \tag{33}$$

for all $n \in \{0, \ldots, \lfloor T/h \rfloor\}$ and $h \in (0, h^*]$.

Let us assume from now on that

(H7) there exist $C_0 > 0$, $\tau_0 > 0$, $h_0^* \in (0, h^*]$ and a subsequence $(h_i)_{i \in \mathbb{N}}$, decreasing to zero, such that

$$|V^n| \leqq C_0 \quad \forall n h_i \in [0, \min(\tau_0, T)], \quad \forall h_i \in (0, h_0^*].$$

We define $\mathcal{B} = \overline{B}(u_0, C + 1)$ with $C \geqq 0$. Let B_0 and C_F be defined by (11) and (13), respectively. We assume (without loss of generality) that $C_0 h_0^* \leqq 1$ and $C_F h_0^* \leqq C_0$. Let us denote $\tau = \min(\tau_0, T)$.

From assumption (H7) and Proposition 4 we know that $(u_{h_i})_{h_1^* \geqq h_i > 0}$ is uniformly C_0-Lipschitz continuous on $[0, \tau]$ and $(v_{h_i})_{h_1^* \geqq h_i > 0}$ is uniformly bounded in $L^\infty(0, \tau; \mathbb{R}^d) \cap BV(0, \tau; \mathbb{R}^d)$. Hence, $(u_{h_i})_{h_1^* \geqq h_i > 0}$ is equicontinuous and, using Ascoli's and Helly's theorems, we infer that there exist a subsequence, still denoted $(h_i)_{i \in \mathbb{N}}$, $u \in C^0\left([0, \tau]; \mathbb{R}^d\right)$ and $v \in BV(0, \tau; \mathbb{R}^d)$ such that

$$u_{h_i} \to u \quad \text{strongly in } C^0\left([0, \tau]; \mathbb{R}^d\right), \tag{34}$$

and

$$v_{h_i} \to v \quad \text{pointwise in } [0, \tau]. \tag{35}$$

Moreover we have

$$u_{h_i}(t) = u_0 + \int_0^t v_{h_i}(s)\, \mathrm{d}s \quad \forall t \in [0, T], \quad \forall h_i \in (0, h^*].$$

Thus, with Lebesgue's theorem, we get

$$u(t) = \lim_{h_i \to 0} \left(u_0 + \int_0^t v_{h_i}(s)\, \mathrm{d}s \right) = u_0 + \int_0^t v(s)\, \mathrm{d}s \quad \forall t \in [0, \tau]. \tag{36}$$

We infer that u is C_0-Lipschitz continuous and

$$u_{h_i}(t), u(t) \in \overline{B}(u_0, C_0\tau) \subset B_1 = \overline{B}(u_0, C_0T + 1) \quad \forall t \in [0, \tau], \ \forall h_i \in (0, h_1^*].$$

Moreover, u is absolutely continuous on $[0, \tau]$, thus u admits a derivative (in the classical sense) almost everywhere on $[0, \tau]$ and $\dot{u} \in L^1(0, \tau; \mathbb{R}^d)$. From (36) we infer that $\dot{u}(t) = v(t)$ for all $t \in [0, \tau]$ such that v is continuous at t. Possibly modifying $\dot{u}$ on a countable subset of $[0, \tau]$, we may assume without loss of generality that $\dot{u} = v$.

As usual, we adopt the convention

$$\begin{aligned} \dot{u}(0-0) &= v(0-0) = v(0) = \dot{u}(0), \\ \dot{u}(\tau+0) &= v(\tau+0) = v(\tau) = \dot{u}(\tau). \end{aligned} \tag{37}$$

Then we observe that

Lemma 6. *For all $t \in [0, \tau]$, $u(t) \in K$.*

Proof. Let $t \in (0, \tau)$. For all $h_i \in (0, \min(t, \tau - t))$ there exists $n \in \left\{1, \ldots, \left\lfloor \frac{\tau}{h_i} \right\rfloor\right\}$ such that $t \in [nh_i, (n+1)h_i) \subset (0, \tau)$. Then, observing that $Z^n \in K$ we get

$$\operatorname{dist}(u(t), K) \leqq \left|u(t) - Z^n\right| \leqq \left|u(t) - u_{h_i}(t)\right| + \left|u_{h_i}(t) - U^n\right| + \left|U^n - Z^n\right|.$$

But

$$\begin{aligned} \left|u_{h_i}(t) - U^n\right| &= \left|u_{h_i}(t) - u_{h_i}(nh_i)\right| \leqq |t - nh_i||V^n| \leqq C_0 h_i, \\ \left|U^n - Z^n\right| &= \frac{h_i}{1+e}|V^n - eV^{n-1}| \leqq C_0 h_i, \end{aligned}$$

thus

$$\operatorname{dist}(u(t), K) \leqq \left\|u - u_{h_i}\right\|_{C^0([0,\tau];\mathbb{R}^d)} + 2C_0 h_i.$$

By passing to the limit as h_i tends to zero, we obtain $\operatorname{dist}(u(t), K) \leqq 0$, that is, $u(t) \in K$ for all $t \in (0, \tau)$.

Since K is a closed subset of $\mathbb{R}^d$ and u is continuous at $t = 0$ and $t = \tau$, the same property holds on $[0, \tau]$. □

3.1. Study of property (P2)

Now, let us prove that u satisfies property (P2), that is, the differential inclusion (3). First, we observe (as in [11]) that there exists at least one non-negative measure μ such that both the Stieltjes measure $\ddot{u} = \mathrm{d}\dot{u} = dv$ and the Lebesgue measure $\mathrm{d}t$ admit densities with respect to μ. Indeed, let μ be defined by $\mathrm{d}\mu = |\mathrm{d}\dot{u}| + \mathrm{d}t$: μ is non-negative and the measures $\ddot{u} = \mathrm{d}\dot{u}$ and $\mathrm{d}t$ are absolutely continuous with respect to μ.

Let now $\mu = |\mathrm{d}\dot{u}| + \mathrm{d}t$. We denote by v'_μ and t'_μ the densities of $\mathrm{d}\dot{u} = dv$ and $\mathrm{d}t$ with respect to $\mathrm{d}\mu$. We have to prove that

$$M(u(t))\, v'_\mu(t) - g(t, u(t), \dot{u}(t))\, t'_\mu(t) \in -N_K(u(t)) \quad d\mu \text{ almost everywhere.}$$

By Jeffery's theorem (see [4] or [6]) we know that there exists a $\mathrm{d}\mu$-negligible set $N \subset [0,\tau]$ such that, for all $t \in [0,\tau]\backslash N$:

$$v'_\mu(t) = \lim_{\varepsilon\to 0^+} \frac{\mathrm{d}\dot{u}\,(I_\varepsilon)}{\mathrm{d}\mu\,(I_\varepsilon)}, \quad t'_\mu(t) = \lim_{\varepsilon\to 0^+} \frac{\mathrm{d}t\,(I_\varepsilon)}{\mathrm{d}\mu\,(I_\varepsilon)}$$

with $I_\varepsilon = [t, t+\varepsilon] \cap [0,\tau]$.

We define

$$N' = \{t \in [0,\tau];\, \dot{u}(t+0) = \dot{u}(t-0) \neq \dot{u}(t)\}$$

(we may observe that the convention (37) implies that $0 \notin N'$ and $\tau \notin N'$). Since $\dot{u} = v$ belongs to $BV(0,\tau;\mathbb{R}^d)$, N' is at most a countable subset of $[0,\tau]$ and we infer that N' is negligible with respect to $|\mathrm{d}\dot{u}|$.

Finally, let $N_0 = \{t \in \{0\} \cup \{\tau\};\, \dot{u} \text{ is continuous at } t\}$. The set N_0 is finite (it contains at most the two points $t = 0$ and $t = \tau$), so it is negligible with respect to $|\mathrm{d}\dot{u}|$, and it follows that $N \cup N' \cup N_0$ is also negligible with respect to $\mathrm{d}\mu$. We have:

Proposition 5. *Let $t \in [0,\tau]\backslash(N \cup N' \cup N_0)$ such that $\dot{u}$ is continuous at t. Then*

$$M\,(u(t))\,v'_\mu(t) - g\,(t, u(t), \dot{u}(t))\,t'_\mu(t) \in -N_K\,(u(t))\,. \tag{38}$$

Proof. We begin the proof in the same way as in the proof of Proposition 4 in [11]. More precisely, let $t \in [0,\tau]\backslash(N \cup N' \cup N_0)$ such that $\dot{u}$ is continuous at t. Then $t \in (0,\tau)$ and, for simplicity, we will denote $\bar{u} = u(t)$ in the remainder of the proof. By definition of $N_K(\bar{u})$, (38) is equivalent to

$$\left(g\,(t, \bar{u}, \dot{u}(t))\,t'_\mu(t) - M(\bar{u})v'_\mu(t),\, w\right) \leqq 0 \tag{39}$$

for all $w \in T_K(\bar{u}) = \left\{w \in \mathbb{R}^d;\, (\nabla f_\alpha(\bar{u}), w) \geqq 0,\ \forall \alpha \in J(\bar{u})\right\}$.

Let us consider $\tilde{T}_K(\bar{u})$ defined by

$$\tilde{T}_K(\bar{u}) = \begin{cases} \left\{w \in \mathbb{R}^d;\, (\nabla f_\alpha(\bar{u}), w) > 0\ \forall \alpha \in J(\bar{u})\right\} & \text{if } J(\bar{u}) \neq \emptyset, \\ \mathbb{R}^d & \text{otherwise.} \end{cases}$$

Since $\tilde{T}_K(\bar{u})$ is dense in $T_K(\bar{u})$, we only need to prove that (39) holds for all $w \in \tilde{T}_K(\bar{u})$. As in the proof of Proposition 4 in [11], we define $r_{\bar{u}} > 0$ such that

$$J(q) \subset J(\bar{u}) \quad \forall q \in \overline{B}(\bar{u}, r_{\bar{u}}),$$

and, for all $w \in \tilde{T}_K(\bar{u})$, we define $r_w \in (0, r_{\bar{u}}]$ such that $w \in T_K(q)$ for all $q \in \overline{B}(\bar{u}, r_w)$.

Using the continuity of u and the uniform convergence of $(u_{h_i})_{i\in\mathbb{N}}$ to u on $[0,\tau]$, we also infer that, for all $w \in \tilde{T}_K(\bar{u})$, there exists $\tilde{\varepsilon}_w \in \left(0, \min\left(t, \frac{\tau - t}{2}\right)\right)$, such that for all $\varepsilon \in (0, \tilde{\varepsilon}_w]$ there exists $h_\varepsilon \in (0, h_1^*]$, such that

$$u(s) \in \overline{B}\left(\bar{u}, \frac{r_w}{3}\right) \quad \forall s \in [t, t+\varepsilon],$$

$$h_\varepsilon \leqq \min\left(\frac{r_w}{3C_0}, \frac{\varepsilon}{3}\right), \quad \|u - u_{h_i}\|_{C^0([0,\tau];\mathbb{R}^d)} \leqq \frac{r_w}{3} \quad \forall h_i \in (0, h_\varepsilon].$$

It follows that for all $\varepsilon \in (0, \tilde{\varepsilon}_w]$ and for all $h_i \in (0, h_\varepsilon]$

$$u_{h_i}(s) \in \overline{B}\left(\bar{u}, \frac{2r_w}{3}\right) \quad \forall s \in [t, t+\varepsilon],$$

and

$$Z^n \in \overline{B}(\bar{u}, r_w) \quad \forall n h_i \in [t, t+\varepsilon],$$

since

$$\left|Z^n - u_{h_i}(nh_i)\right| = |Z^n - U^n| \leqq C_0 h_i \leqq C_0 h_\varepsilon \leqq \frac{r_w}{3}.$$

Now let $w \in \tilde{T}_K(\bar{u})$, $\varepsilon \in (0, \tilde{\varepsilon}_w]$ and $h_i \in (0, h_\varepsilon]$. We define j and k by

$$j = \left\lfloor \frac{t}{h_i} \right\rfloor, \quad k = \left\lfloor \frac{t+\varepsilon}{h_i} \right\rfloor.$$

We have

$$0 < t_j = jh_i \leqq t < t_{j+1} < \cdots < t_k = kh_i \leqq t+\varepsilon < t_{k+1} < \tau.$$

From Lemma 1 we know that, for all $n \in \{j+1, \ldots, k\}$, we have

$$(V^{n-1} - V^n + h_i F^n, w)_{M(U^n)} \leqq 0$$

since $w \in T_K(Z^n)$. Next we continue the proof exactly as in [11]. □

Let us now consider the case of discontinuities of the velocity. Let $t \in [0, \tau] \backslash (N \cup N' \cup N_0)$ such that $\dot{u}$ is discontinuous at t. Then, $\dot{u}(t+0) \neq \dot{u}(t-0)$ and $\mathrm{d}\dot{u}$ possesses a Dirac mass at $\{t\}$. Thus $\{t\}$ is not negligible anymore with respect to $\mathrm{d}\mu$ and (3) is equivalent to

$$M\left(u(t)\right)\left(\dot{u}(t+0) - \dot{u}(t-0)\right) \in -N_K\left(u(t)\right).$$

This property is a direct consequence of the following result:

Proposition 6. *For all $t \in [0, \tau]$ we have*

$$M\left(u(t)\right)\left(\dot{u}(t+0) - \dot{u}(t-0)\right) \in -N_K\left(u(t)\right).$$

Proof. Once again, the proof is similar to the proof of the analogous result in [11]. More precisely, let $t \in [0, \tau]$ and for simplicity denote $\bar{u} = u(t)$. Thanks to the density of $\tilde{T}_K(\bar{u})$ in $T_K(\bar{u})$, we only need to prove that

$$(M(\bar{u})\left(\dot{u}(t-0) - \dot{u}(t+0)\right), w) \leqq 0 \quad \forall w \in \tilde{T}_K(\bar{u}).$$

Let $w \in \tilde{T}_K(\bar{u})$. As in the proof of the previous proposition, we define $r_w > 0$ such that

$$J(q) \subset J(\bar{u}) \quad \text{and} \quad w \in T_K(q) \text{ for all } q \in \overline{B}(\bar{u}, r_w).$$

We also define $\tilde{\varepsilon}_w \in (0, \tau/2)$ such that for all $\varepsilon \in (0, \tilde{\varepsilon}_w]$ we have

$$u(s) \in \overline{B}\left(\bar{u}, \frac{r_w}{3}\right) \quad \forall s \in [t-\varepsilon, t+\varepsilon] \cap [0, \tau],$$

and there exists $h_\varepsilon \in \left(0, \min(h_1^*, r_w/3C_0, \varepsilon/3)\right]$ such that

$$u_{h_i}(s) \in \overline{B}\left(\bar{u}, \frac{2r_w}{3}\right) \quad \forall s \in [t-\varepsilon, t+\varepsilon] \cap [0, \tau], \quad \forall h_i \in (0, h_\varepsilon],$$

and

$$Z^n \in \overline{B}(\bar{u}, r_w) \quad \forall n h_i \in [t-\varepsilon, t+\varepsilon] \cap [0, \tau], \quad \forall h_i \in (0, h_\varepsilon],$$

(we recall that $Z^0 = U^0$ by convention). Now we consider $\varepsilon \in (0, \tilde{\varepsilon}_w]$ and $h_i \in (0, h_\varepsilon]$. We define $t_\varepsilon^- = \max(t-\varepsilon, 0)$, $t_\varepsilon^+ = \min(t+\varepsilon, \tau)$ and

$$j = \left\lfloor \frac{t_\varepsilon^-}{h_i} \right\rfloor, \quad k = \left\lfloor \frac{t_\varepsilon^+}{h_i} \right\rfloor$$

that is, we have

$$0 \leqq t_j = j h_i \leqq t_\varepsilon^- < t_{j+1} < \cdots < t_k = k h_i \leqq t_\varepsilon^+ \leqq \tau.$$

It follows that

$$v_{h_i}(t_\varepsilon^-) = V^j, \quad v_{h_i}(t_\varepsilon^+) = V^k$$

and we have $w \in T_K(Z^n)$ for all $n \in \{j+1, \ldots, k\}$. Then, we continue the proof exactly as in Proposition 5 of [11]. □

3.2. Transmission of the velocity at impacts

With the previous proposition, we observe that $\dot{u}(t-0) = \dot{u}(t+0)$ for all $t \in [0, \tau]$ such that $J(u(t)) = \emptyset$. That is, $\dot{u}$ is continuous at t if $u(t) \in \mathrm{Int}(K)$ and in this case the impact law (2) is satisfied. Thus it remains only to prove that

$$\dot{u}(\bar{t}+0) = -e\dot{u}(\bar{t}-0) + (1+e)\mathrm{Proj}_{M(u(\bar{t}))}\left(T_K\left(u(\bar{t})\right), \dot{u}(\bar{t}-0)\right) \tag{40}$$

for all $\bar{t} \in (0, \tau)$ such that $J\left(u(\bar{t})\right) \neq \emptyset$.

Let $\bar{t} \in (0, \tau)$ be such that $J\left(u(\bar{t})\right) \neq \emptyset$. For simplicity, let us denote $\bar{u} = u(\bar{t})$ and $\dot{u}^+ = \dot{u}(\bar{t}+0)$, $\dot{u}^- = \dot{u}(\bar{t}-0)$. With Proposition 6 we already know that $M(\bar{u})(\dot{u}^- - \dot{u}^+) \in N_K(\bar{u})$, that is, there exist non-positive real numbers $(\mu_\alpha)_{\alpha \in J(\bar{u})}$ such that

$$M^{1/2}(\bar{u})\left(\dot{u}^- - \dot{u}^+\right) = \sum_{\alpha \in J(\bar{u})} \mu_\alpha e_\alpha(\bar{u}), \tag{41}$$

where we recall that

$$e_\alpha(\bar{u}) = \frac{M^{-1/2}(\bar{u})\nabla f_\alpha(\bar{u})}{\left|M^{-1/2}(\bar{u})\nabla f_\alpha(\bar{u})\right|} \quad \forall \alpha \in J(\bar{u}).$$

Then (40) reduces to

$$\dot{u}^+ + e\dot{u}^- \in T_K(\bar{u}), \quad \left(\dot{u}^- - \dot{u}^+, \dot{u}^+ + e\dot{u}^-\right)_{M(\bar{u})} = 0,$$

that is,

$$\left(\nabla f_\alpha(\bar{u}), \dot{u}^+ + e\dot{u}^-\right) \geqq 0 \quad \forall \alpha \in J(\bar{u})$$

and

$$\left(M^{1/2}(\bar{u})(\dot{u}^- - \dot{u}^+), M^{1/2}(\bar{u})(\dot{u}^+ + e\dot{u}^-)\right) = 0.$$

From (41) we infer that (40) is equivalent to

$$\left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})(\dot{u}^+ + e\dot{u}^-)\right) \geqq 0 \quad \forall \alpha \in J(\bar{u})$$

and

$$\sum_{\alpha \in J(\bar{u})} \mu_\alpha \left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})(\dot{u}^+ + e\dot{u}^-)\right) = 0.$$

But $\mu_\alpha \leqq 0$ for all $\alpha \in J(\bar{u})$, so (40) is satisfied if and only if

$$\left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})(\dot{u}^+ + e\dot{u}^-)\right) \geqq 0 \quad \text{for all } \alpha \in J(\bar{u}) \text{ such that } \mu_\alpha = 0,$$
$$\left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})(\dot{u}^+ + e\dot{u}^-)\right) = 0 \quad \text{for all } \alpha \in J(\bar{u}) \text{ such that } \mu_\alpha \neq 0.$$

First we observe that, for all $\alpha \in J(\bar{u})$, we have with (41) and assumption (H6)

$$\left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})\dot{u}^-\right) = \left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})\dot{u}^+\right) + \mu_\alpha,$$

and, since $\dot{u}^+ \in T_K(\bar{u})$ and $\dot{u}^- \in -T_K(\bar{u})$,

$$\left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})\dot{u}^+\right) \geqq 0, \quad \left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})\dot{u}^-\right) \leqq 0.$$

It follows that

$$\left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})\dot{u}^+\right) = \left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})\dot{u}^-\right) = 0$$

if $\alpha \in J(\bar{u})$ such that $\mu_\alpha = 0$ and it remains only to prove that $\left(e_\alpha(\bar{u}), M^{1/2}(\bar{u})(\dot{u}^+ + e\dot{u}^-)\right) = 0$ for all $\alpha \in J(\bar{u})$ such that $\mu_\alpha \neq 0$.

As in [11], we construct a neighbourhood of $\bar{u}$, $\overline{B}(\bar{u}, r_{\bar{u}})$ with $r_{\bar{u}} \in \left(0, \min(r_{B_1}, r'_{B_1})/2\right)$, such that $J(q) \subset J(\bar{u})$ for all $q \in \overline{B}(\bar{u}, r_{\bar{u}})$, and d Lipschitz continuous mappings v_α, $\alpha \in \{1, \ldots, d\}$, such that, for all $q \in \overline{B}(\bar{u}, r_{\bar{u}})$ $(v_\alpha(q))_{\alpha \in \{1,\ldots,d\}}$ is a basis of $\mathbb{R}^d$, $|v_\alpha(q)| = 1$ for all $\alpha \in \{1, \ldots, d\}$ and $v_\alpha(q) = e_\alpha(q)$ for all $\alpha \in J(q)$, for all $q \in \overline{B}(\bar{u}, r_{\bar{u}}) \cap K$.

We define the dual basis $(w_\alpha(q))_{1 \leqq \alpha \leqq d}$ for all $q \in \overline{B}(\bar{u}, r_{\bar{u}})$. The mappings w_α $(1 \leqq \alpha \leqq d)$ are also Lipschitz continuous on $\overline{B}(\bar{u}, r_{\bar{u}})$: we let

$$C_{\bar{u}} = \sup\left\{|v_\alpha(q)|, |w_\alpha(q)|; q \in \overline{B}(\bar{u}, r_{\bar{u}}), \alpha \in \{1, \ldots, d\}\right\}$$

and $L_{\bar{u}} \in \mathbb{R}_*^+$ be such that, for all $\alpha \in \{1, \ldots, d\}$ and for all $(q, q') \in \overline{B}(\bar{u}, r_{\bar{u}})^2$

$$\left|v_\alpha(q) - v_\alpha(q')\right| \leqq L_{\bar{u}}|q - q'|, \quad \left|w_\alpha(q) - w_\alpha(q')\right| \leqq L_{\bar{u}}|q - q'|.$$

Since the mappings ∇f_α ($\alpha \in \{1, \ldots, \nu\}$) are locally Lipschitz continuous, possibly modifying $C_{\bar{u}}$, we may assume without loss of generality that

$$|\nabla f_\alpha(q)| \leqq C_{\bar{u}},$$

for all $\alpha \in \{1, \ldots, \nu\}$ and for all $(q, q') \in \overline{B}(\bar{u}, r_{\bar{u}})^2$.

From the continuity of u and the uniform convergence of $(u_{h_i})_{i\in\mathbb{N}}$ to u on $[0, \tau]$, we infer that there exist $\bar{\varepsilon} \in \left(0, \min(\bar{t}, \tau - \bar{t})/2\right)$ and $h_2^* \in \left(0, \min\left(h_1^*, \frac{\bar{\varepsilon}}{3}, \frac{r_{\bar{u}}}{6C_0}\right)\right]$, where h_1^* is given by Proposition 4, such that

$$u(t) \in \overline{B}\left(\bar{u}, \frac{r_{\bar{u}}}{3}\right) \quad \forall t \in [\bar{t} - \bar{\varepsilon}, \bar{t} + \bar{\varepsilon}],$$
$$\|u - u_{h_i}\|_{C^0([0,\tau];\mathbb{R}^d)} \leqq \frac{r_{\bar{u}}}{3} \quad \forall h_i \in (0, h_2^*].$$

It follows that

$$U^{n+1}, U^n, Z^n, Z^{n\pm1} \in \overline{B}(\bar{u}, r_{\bar{u}}) \quad \forall n h_i \in [\bar{t} - \bar{\varepsilon}, \bar{t} + \bar{\varepsilon}], \quad \forall h_i \in (0, h_2^*]. \tag{42}$$

Furthermore, we infer from Lemma 1 that, for all $nh_i \in [\bar{t} - \bar{\varepsilon}, \bar{t} + \bar{\varepsilon}]$, for all $h_i \in (0, h_2^*]$

$$M(U^n)(V^{n-1} - V^n + h_i F^n) \in N_K(Z^n).$$

If $J(Z^n) \neq \emptyset$, there exist non-positive real numbers $(\mu_\beta^n)_{\beta\in J(Z^n)}$ such that

$$M(U^n)\left(V^{n-1} - V^n + h_i F^n\right) = \sum_{\beta\in J(Z^n)} \mu_\beta^n M^{1/2}(Z^n) e_\beta(Z^n) \tag{43}$$

and observing that $e_\beta(Z^n) = v_\beta(Z^n)$ for all $\beta \in J(Z^n)$, we get with assumption (H6) (see (8))

$$\begin{aligned}\left|\mu_\beta^n\right| &\leqq \left|\left(M(U^n)(V^{n-1} - V^n + h_i F^n), M^{-1/2}(Z^n) v_\beta(Z^n)\right)\right| \\ &\leqq \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}}\left(|V^{n-1} - V^n| + h_i C_F\right) \leqq 3C_0 \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}}.\end{aligned} \tag{44}$$

From now on, let us denote

$$C_2' = 3C_0 \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}}.$$

We begin with the following lemma.

Lemma 7. *Let $\alpha \in J(\bar{u})$ such that $\mu_\alpha \neq 0$. Then, for all $\varepsilon_1 \in (0, \bar{\varepsilon}]$ there exists $h_{\varepsilon_1} \in \left(0, \min(h_2^*, \varepsilon_1/3)\right]$ such that for all $h_i \in (0, h_{\varepsilon_1}]$, there exists $nh_i \in [\bar{t} - \varepsilon_1, \bar{t} + \varepsilon_1]$ such that $f_\alpha(Z^n) \leqq 0$.*

This result is analogous to Lemma 3 in [11]: it means that on any neighbourhood $[\bar{t}-\varepsilon_1, \bar{t}+\varepsilon_1]$ of $\bar{t}$ (with $\varepsilon_1 \in (0, \bar{\varepsilon}]$), the constraint numbered α is saturated by at least one discrete average position Z^n, if h_i is small enough.

Proof. As in the proof of Lemma 3 in [11], we assume that the announced result does not hold, that is, assume that there exists $\varepsilon_1 \in (0, \bar{\varepsilon}]$ such that, for all $h_{\varepsilon_1} \in \left(0, \min(h_2^*, \varepsilon_1/3)\right]$ there exists $h_i \in (0, h_{\varepsilon_1}]$ such that $f_\alpha(Z^n) > 0$ for all $nh_i \in [\bar{t}-\varepsilon_1, \bar{t}+\varepsilon_1]$.

Hence we can extract from $(h_i)_{i\in\mathbb{N}}$ a subsequence denoted $(h_{\varphi(i)})_{i\in\mathbb{N}}$ decreasing to zero, such that $h_{\varphi(i)} \in \left(0, \min(h_2^*, \varepsilon_1/3)\right]$ and

$$f_\alpha(Z^n) > 0 \quad \forall n h_{\varphi(i)} \in [\bar{t}-\varepsilon_1, \bar{t}+\varepsilon_1] \tag{45}$$

for all $i \in \mathbb{N}$.

For all $\varepsilon \in (0, \varepsilon_1]$, let us establish the following estimate:

$$\begin{aligned}&\left|\left(M^{1/2}\left(u_{h_{\varphi(i)}}(t_\varepsilon^-)\right) v_{h_{\varphi(i)}}(t_\varepsilon^-) - M^{1/2}\left(u_{h_{\varphi(i)}}(t_\varepsilon^+)\right) v_{h_{\varphi(i)}}(t_\varepsilon^+), w_\alpha(\bar{u})\right)\right| \\ &\quad \leqq \mathcal{O}\left(\varepsilon + h_{\varphi(i)} + \|u - u_{h_{\varphi(i)}}\|_{C^0([0,\tau];\mathbb{R}^d)}\right)\end{aligned}$$

where $t_\varepsilon^- = \max(\bar{t}-\varepsilon, 0) = \bar{t}-\varepsilon$ and $t_\varepsilon^+ = \min(\bar{t}+\varepsilon, \tau) = \bar{t}+\varepsilon$. Then, by passing to the limit when i tends to $+\infty$, we will infer with (34)–(35) that

$$\left|\left(M^{1/2}\left(u(t_\varepsilon^-)\right) v(t_\varepsilon^-) - M^{1/2}\left(u(t_\varepsilon^+)\right) v(t_\varepsilon^+), w_\alpha(\bar{u})\right)\right| \leqq \mathcal{O}(\varepsilon)$$

and, when ε tends to zero, we will obtain

$$\left|\left(M^{1/2}(\bar{u})\left(\dot{u}(\bar{t}-0) - \dot{u}(\bar{t}+0)\right), w_\alpha(\bar{u})\right)\right| = |\mu_\alpha| \leqq 0$$

which gives a contradiction.

Let $\varepsilon \in (0, \varepsilon_1]$. There exists $i_\varepsilon \in \mathbb{N}$ such that, for all $i \geqq i_\varepsilon$ we have $h_{\varphi(i)} \in (0, \varepsilon/2)$ and we define

$$n_i = \left\lfloor \frac{t_\varepsilon^-}{h_{\varphi(i)}} \right\rfloor, \quad p_i = \left\lfloor \frac{t_\varepsilon^+}{h_{\varphi(i)}} \right\rfloor.$$

Then, $n_i + 1 < p_i$ and for all $n \in \{n_i+1, \ldots, p_i\}$ we have $nh_{\varphi(i)} \in [t_\varepsilon^-, t_\varepsilon^+]$. With (20) we infer that

$$|U^n - Z^n| \leqq C_0 h_{\varphi(i)} \tag{46}$$

and with (45) we know that $\alpha \notin J(Z^n)$, thus with (43) and (44)

$$\begin{aligned}&\left|\left(M^{1/2}(Z^n)\left(V^{n-1} - V^n + h_{\varphi(i)}F^n\right), w_\alpha(Z^n)\right)\right| \\ &= \left|\sum_{\beta\in J(Z^n)} \mu_\beta^n \left(M^{1/2}(Z^n)M^{-1}(U^n)M^{1/2}(Z^n)v_\beta(Z^n), w_\alpha(Z^n)\right)\right| \\ &\leqq \sum_{\beta\in J(Z^n)} \left|\mu_\beta^n\right| \left\|M^{1/2}(Z^n)\right\|^2 \left\|M^{-1}(Z^n) - M^{-1}(U^n)\right\| \left|v_\beta(Z^n)\right| \left|w_\alpha(Z^n)\right| \\ &\leqq \nu C_2' C_{\bar{u}} \lambda_{\max,B_1} L_{M^{-1}} |Z^n - U^n| \leqq \nu C_2' C_{\bar{u}} C_0 \lambda_{\max,B_1} L_{M^{-1}} h_{\varphi(i)}\end{aligned}$$

for all $n \in \{n_i+1, \dots, p_i\}$, if $J(Z^n) \neq \emptyset$. If $J(Z^n) = \emptyset$, this inequality remains true, since $V^{n-1} - V^n + h_{\varphi(i)} F^n = 0$ in this case.

It follows that, for all $n \in \{n_i+1, \dots, p_i\}$

$$\begin{aligned}
&\left(M^{1/2}(Z^{n_i+1})V^{n_i} - M^{1/2}(Z^{p_i+1})V^{p_i}, w_\alpha(\bar{u})\right) \\
&= \sum_{n=n_i+1}^{p_i} \left(M^{1/2}(Z^n)V^{n-1} - M^{1/2}(Z^{n+1})V^n, w_\alpha(\bar{u})\right) \\
&= \sum_{n=n_i+1}^{p_i} \left(M^{1/2}(Z^n)(V^{n-1} - V^n + h_{\varphi(i)}F^n), w_\alpha(Z^n)\right) \\
&\quad + \sum_{n=n_i+1}^{p_i} \left(\left(M^{1/2}(Z^n) - M^{1/2}(Z^{n+1})\right)V^n, w_\alpha(\bar{u})\right) \\
&\quad - \sum_{n=n_i+1}^{p_i} h_{\varphi(i)} \left(M^{1/2}(Z^n)F^n, w_\alpha(Z^n)\right) \\
&\quad + \sum_{n=n_i+1}^{p_i} \left(M^{1/2}(Z^n)(V^{n-1} - V^n), w_\alpha(\bar{u}) - w_\alpha(Z^n)\right).
\end{aligned}$$

Then, using (20), we obtain

$$\begin{aligned}
|\bar{u} - Z^n| &\leqq \left|u(\bar{t}) - u_{h_{\varphi(i)}}(\bar{t})\right| + \left|u_{h_{\varphi(i)}}(\bar{t}) - u_{h_{\varphi(i)}}(nh_{\varphi(i)})\right| + |U^n - Z^n| \\
&\leqq \left\|u - u_{h_{\varphi(i)}}\right\|_{C^0([0,\tau];\mathbb{R}^d)} + C_0|\bar{t} - nh_{\varphi(i)}| + \frac{h_{\varphi(i)}}{1+e}|V^n - eV^{n-1}| \\
&\leqq \left\|u - u_{h_{\varphi(i)}}\right\|_{C^0([0,\tau];\mathbb{R}^d)} + C_0(\varepsilon + h_{\varphi(i)}),
\end{aligned}$$

for all $n \in \{n_i+1, \dots, p_i\}$ and we estimate $|w_\alpha(\bar{u}) - w_\alpha(Z^n)|$ as

$$L_{\bar{u}} \left(\left\|u - u_{h_{\varphi(i)}}\right\|_{C^0([0,\tau];\mathbb{R}^d)} + C_0(\varepsilon + h_{\varphi(i)})\right).$$

Hence, with the estimate of the discrete accelerations obtained at Proposition 4 and (21), we get

$$\begin{aligned}
&\left|\left(M^{1/2}(Z^{n_i+1})V^{n_i} - M^{1/2}(Z^{p_i+1})V^{p_i}, w_\alpha(\bar{u})\right)\right| \\
&\leqq \sum_{n=n_i+1}^{p_i} \nu C_2' C_{\bar{u}} C_0 \lambda_{\max,B_1} L_{M^{-1}} h_{\varphi(i)} \\
&\quad + \sum_{n=n_i+1}^{p_i} C_0 C_{\bar{u}} L_{M^{1/2}} |Z^n - Z^{n+1}| + \sum_{n=n_i+1}^{p_i} h_{\varphi(i)} C_{\bar{u}} C_F \sqrt{\lambda_{\max,B_1}} \\
&\quad + \sqrt{\lambda_{\max,B_1}} L_{\bar{u}} \left(\left\|u - u_{h_{\varphi(i)}}\right\|_{C^0([0,\tau];\mathbb{R}^d)} + C_0(\varepsilon + h_{\varphi(i)})\right) \sum_{n=n_i+1}^{p_i} |V^n - V^{n-1}|
\end{aligned}$$

$$\leqq (2\varepsilon + h_{\varphi(i)})C_{\bar{u}} \left(\nu C_2' C_0 \lambda_{\max,B_1} L_{M^{-1}} + C_0^2 L_{M^{1/2}} + C_F \sqrt{\lambda_{\max,B_1}}\right)$$
$$+ C_0' \sqrt{\lambda_{\max,B_1}} L_{\bar{u}} \left(\left\| u - u_{h_{\varphi(i)}} \right\|_{C^0([0,\tau];\mathbb{R}^d)} + C_0(\varepsilon + h_{\varphi(i)})\right). \tag{47}$$

But $V^{n_i} = v_{h_{\varphi(i)}}(t_\varepsilon^-)$ and $V^{p_i} = v_{h_{\varphi(i)}}(t_\varepsilon^+)$ and

$$\begin{aligned}
&\left|\left(M^{1/2}\left(u_{h_{\varphi(i)}}(t_\varepsilon^-)\right) v_{h_{\varphi(i)}}(t_\varepsilon^-) - M^{1/2}\left(u_{h_{\varphi(i)}}(t_\varepsilon^+)\right) v_{h_{\varphi(i)}}(t_\varepsilon^+), w_\alpha(\bar{u})\right)\right. \\
&\quad \left. - \left(M^{1/2}(Z^{n_i+1})V^{n_i} - M^{1/2}(Z^{p_i+1})V^{p_i}, w_\alpha(\bar{u})\right)\right| \\
&\leqq \left(\left|M^{1/2}\left(u_{h_{\varphi(i)}}(t_\varepsilon^-)\right) - M^{1/2}(Z^{n_i+1})\right|\right. \\
&\quad \left. + \left|M^{1/2}\left(u_{h_{\varphi(i)}}(t_\varepsilon^+)\right) - M^{1/2}(Z^{p_i+1})\right|\right) C_0 \left|w_\alpha(\bar{u})\right| \\
&\leqq C_0 C_{\bar{u}} L_{M^{1/2}} \left(\left|u_{h_{\varphi(i)}}(t_\varepsilon^-) - u_{h_{\varphi(i)}}((n_i+1)h_{\varphi(i)})\right| + \left|U^{n_i+1} - Z^{n_i+1}\right|\right. \\
&\quad \left. + \left|u_{h_{\varphi(i)}}(t_\varepsilon^+) - u_{h_{\varphi(i)}}((p_i+1)h_{\varphi(i)})\right| + \left|U^{p_i+1} - Z^{p_i+1}\right|\right) \\
&\leqq 4C_0^2 C_{\bar{u}} L_{M^{1/2}} h_{\varphi(i)}.
\end{aligned} \tag{48}$$

Finally, from (47)–(48) we obtain

$$\begin{aligned}
&\left|\left(M^{1/2}\left(u_{h_i}(t_\varepsilon^-)\right) v_{h_{\varphi(i)}}(t_\varepsilon^-) - M^{1/2}\left(u_{h_i}(t_\varepsilon^+)\right) v_{h_{\varphi(i)}}(t_\varepsilon^+), w_\alpha(\bar{u})\right)\right| \\
&\quad \leqq \mathcal{O}\left(h_{\varphi(i)} + \varepsilon + \left\| u - u_{h_{\varphi(i)}} \right\|_{C^0([0,\tau];\mathbb{R}^d)}\right)
\end{aligned}$$

which enables us to conclude. □

Let us prove now that

Lemma 8. *For all $\alpha \in J(\bar{u})$ such that $\mu_\alpha \neq 0$, we have*

$$\left(M^{1/2}(\bar{u})\dot{u}^+, v_\alpha(\bar{u})\right) \leqq -e\left(M^{1/2}(\bar{u})\dot{u}^-, v_\alpha(\bar{u})\right).$$

Proof. Let $\varepsilon \in (0, \bar{\varepsilon}]$, thus $\bar{t} \pm \varepsilon \in (0, \tau)$. We apply the previous Lemma, with $\varepsilon_1 = \varepsilon/2$: there exists $h_{\varepsilon/2} \in \left(0, \min(h_2^*, \varepsilon/6)\right]$ such that, for all $h_i \in (0, h_{\varepsilon/2}]$, there exists $nh_i \in \left[\bar{t} - \frac{\varepsilon}{2}, \bar{t} + \frac{\varepsilon}{2}\right]$ such that $f_\alpha(Z^n) \leqq 0$, that is, $\alpha \in J(Z^n)$.

Let $h_i \in (0, h_{\varepsilon/2}]$. We define P_i, p_i and n_i by

$$P_i = \left\lfloor \frac{\bar{t} + \varepsilon/2}{h_i} \right\rfloor, \quad p_i = \left\lfloor \frac{\bar{t} + \varepsilon}{h_i} \right\rfloor, \quad n_i = \left\lfloor \frac{\bar{t} - \varepsilon}{h_i} \right\rfloor.$$

Let us recall that $\bar{\varepsilon} \in \left(0, \min(\bar{t}, \tau - \bar{t})/2\right)$ so $\bar{t} - \varepsilon \geqq \bar{\varepsilon}$ and $\tau - \bar{t} - \varepsilon \geqq \bar{\varepsilon}$. Thus $h_i \in (0, \varepsilon/6] \subset (0, \bar{\varepsilon}/6] \subset \left(0, \min(\bar{t} - \varepsilon, \tau - \bar{t} - \varepsilon)/2\right]$. It follows that

$$\begin{aligned}
&0 < n_i h_i \leqq \bar{t} - \varepsilon < (n_i + 1)h_i < \bar{t} < P_i h_i \leqq \bar{t} + \frac{\varepsilon}{2} < (P_i + 1)h_i < p_i h_i \\
&\quad \leqq \bar{t} + \varepsilon < (p_i + 1)h_i < \tau
\end{aligned}$$

and $v_{h_i}(\bar{t} - \varepsilon) = V^{n_i}$, $v_{h_i}\left(\bar{t} + \frac{\varepsilon}{2}\right) = V^{P_i}$.

Let N_i be the last time step in $\left[\bar{t}-\frac{\varepsilon}{2}, \bar{t}+\frac{\varepsilon}{2}\right]$ such that $f_\alpha(Z^n) \leqq 0$, that is,

$$N_i = \max\left\{n \in \mathbb{N};\, nh_i \in \left[\bar{t}-\frac{\varepsilon}{2}, \bar{t}+\frac{\varepsilon}{2}\right] \text{ and } f_\alpha(Z^n) \leqq 0\right\}.$$

Then, recalling that $h_i \in (0, \varepsilon/6]$ we get $N_i \geqq n_i + \frac{\varepsilon}{2h_i} \geqq n_i + 3$ and with the same computations as in Proposition 1, we obtain that

$$\left(\nabla f_\alpha(Z^{N_i}),\, V^{N_i} + eV^{N_i-2}\right) \leqq \frac{L_f h_i}{2(1+e)}\left|V^{N_i} + eV^{N_i-2}\right|^2 \leqq L_f h_i C_0^2$$

and with Lemma 15

$$\left(v_\alpha(Z^{N_i}),\, M^{1/2}(Z^{N_i})(V^{N_i} + eV^{N_i-2})\right) \leqq \frac{L_f C_0^2}{m_{B_1}} h_i.$$

It follows that

$$\begin{aligned}\left(v_\alpha(Z^{N_i}),\, M^{1/2}(Z^{N_i})V^{N_i}\right) \leqq\ & -e\left(v_\alpha(Z^{N_i-2}),\, M^{1/2}(Z^{N_i-2})V^{N_i-2}\right)\\ & + \frac{L_f C_0^2}{m_{B_1}} h_i + e\left|M^{1/2}(Z^{N_i-2})v_\alpha(Z^{N_i-2})\right.\\ & \left. -\ M^{1/2}(Z^{N_i})v_\alpha(Z^{N_i})\right| C_0.\end{aligned}$$

Thus, using the estimate (21) twice and the Lipschitz property of $M^{1/2}$ on B_1 and v_α on $\overline{B}(\bar{u}, r_{\bar{u}})$, we obtain

$$\left(v_\alpha(Z^{N_i}),\, M^{1/2}(Z^{N_i})V^{N_i}\right) \leqq -e\left(v_\alpha(Z^{N_i-2}),\, M^{1/2}(Z^{N_i-2})V^{N_i-2}\right) + C_3' h_i$$

with

$$C_3' = \frac{L_f C_0^2}{m_{B_1}} + 2C_0^2(L_{M^{1/2}} + L_{\bar{u}}\sqrt{\lambda_{\max,B_1}}).$$

By definition of N_i, we have $f_\alpha(Z^n) > 0$ for all $n \in \{N_i+1, \ldots, P_i\}$, thus

$$\begin{aligned}&\left|\left(M^{1/2}(Z^{P_i})V^{P_i}, v_\alpha(Z^{P_i})\right) - \left(M^{1/2}(Z^{N_i})V^{N_i}, v_\alpha(Z^{N_i})\right)\right|\\ &\leqq \left|\left(M^{1/2}(Z^{N_i})V^{N_i}, v_\alpha(Z^{P_i}) - v_\alpha(Z^{N_i})\right)\right|\\ &\quad + \sum_{n=N_i+1}^{P_i}\left|\left(M^{1/2}(Z^n)V^n - M^{1/2}(Z^{n-1})V^{n-1}, v_\alpha(Z^{P_i})\right)\right|\\ &\leqq \left|\left(M^{1/2}(Z^{N_i})V^{N_i}, v_\alpha(Z^{P_i}) - v_\alpha(Z^{N_i})\right)\right|\\ &\quad + \sum_{n=N_i+1}^{P_i}\left|\left(\left(M^{1/2}(Z^n) - M^{1/2}(Z^{n-1})\right)V^{n-1}, v_\alpha(Z^{P_i})\right)\right|\\ &\quad + \sum_{n=N_i+1}^{P_i} h_i\left|\left(M^{1/2}(Z^n)F^n, v_\alpha(Z^{P_i})\right)\right|\\ &\quad + \sum_{n=N_i+1}^{P_i}\left|\left(M^{1/2}(Z^n)(V^n - V^{n-1} - h_i F^n), v_\alpha(Z^{P_i})\right)\right|.\end{aligned}$$

With the definition of P_i and N_i, we have immediately $P_i h_i \leqq \bar{t} + \varepsilon/2$ and $N_i h_i \geqq \bar{t} - \varepsilon/2$, so $(P_i - N_i)h_i \leqq \varepsilon$ and with (21), we get $|Z^{P_i} - Z^{N_i}| \leqq C_0 \varepsilon$. Furthermore, using the Lipschitz property of $M^{1/2}$ on B_1 and recalling that the mappings v_α ($\alpha \in \{1, \ldots, \nu\}$) are $L_{\bar{u}}$-Lipschitz continuous on $\overline{B}(\bar{u}, r_{\bar{u}})$, we get

$$\begin{aligned}
&\left|\left(M^{1/2}(Z^{P_i})V^{P_i}, v_\alpha(Z^{P_i})\right) - \left(M^{1/2}(Z^{N_i})V^{N_i}, v_\alpha(Z^{N_i})\right)\right| \\
&\quad \leqq C_0^2 \sqrt{\lambda_{\max,B_1}} L_{\bar{u}} \varepsilon + \varepsilon (L_{M^{1/2}} C_0^2 + \sqrt{\lambda_{\max,B_1}} C_F) \\
&\qquad + \sum_{n=N_i+1}^{P_i} \left|\left(M^{1/2}(Z^n)(V^n - V^{n-1} - h_i F^n), v_\alpha(Z^{P_i})\right)\right|. \qquad (49)
\end{aligned}$$

Moreover, for all $n \in \{n_i + 1, \ldots, p_i\}$ we have

$$M(U^n)(V^{n-1} - V^n + h_i F^n) = \sum_{\beta \in J(Z^n)} \mu_\beta^n M^{1/2}(Z^n) v_\beta(Z^n) \qquad (50)$$

with (see (44))

$$0 \leqq -\mu_\beta^n \leqq \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} \left(|V^n - V^{n-1}| + h_i C_F\right) \leqq C_2' = 3C_0 \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} \qquad (51)$$

and the last term of (49) can be estimated as

$$\begin{aligned}
&\sum_{n=N_i+1}^{P_i} \left|\left(M^{1/2}(Z^n)(V^n - V^{n-1} - h_i F^n), v_\alpha(Z^{P_i})\right)\right| \\
&\leqq \sum_{n=N_i+1}^{P_i} \sum_{\beta \in J(Z^n)} |-\mu_\beta^n| \left|\left(M^{1/2}(Z^n) M^{-1}(U^n) M^{1/2}(Z^n) v_\beta(Z^n), v_\alpha(Z^{P_i})\right)\right| \\
&\leqq \sum_{n=N_i+1}^{P_i} \sum_{\beta \in J(Z^n)} |-\mu_\beta^n| \left|(v_\beta(Z^n), v_\alpha(Z^{P_i}))\right| \\
&\quad + \sum_{n=N_i+1}^{P_i} \sum_{\beta \in J(Z^n)} |-\mu_\beta^n| \left\|M^{1/2}(Z^n)\right\|^2 \left\|M^{-1}(U^n) - M^{-1}(Z^n)\right\| \\
&\leqq \sum_{n=N_i+1}^{P_i} \sum_{\beta \in J(Z^n)} \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} \left(|V^n - V^{n-1}| + h_i C_F\right) \left|(v_\beta(Z^n), v_\alpha(Z^{P_i}))\right| \\
&\quad + \sum_{n=N_i+1}^{P_i} \sum_{\beta \in J(Z^n)} 3C_0 \frac{\lambda_{\max,B_1}^2}{\sqrt{\lambda_{\min,B_1}}} L_{M^{-1}} |U^n - Z^n|.
\end{aligned}$$

But, with (20) we have

$$|U^n - Z^n| \leqq C_0 h_i \qquad (52)$$

and, recalling that $\beta \in J(Z^n)$ and $\alpha \in J(Z^{N_i})$, we infer from (8) and Lemma 15 that

$$\begin{aligned}\left|(v_\beta(Z^n), v_\alpha(Z^{P_i}))\right| &\leqq \left|(v_\beta(Z^n), v_\alpha(Z^{N_i}))\right| + \left|(v_\beta(Z^n), v_\alpha(Z^{P_i}) - v_\alpha(Z^{N_i}))\right| \\ &\leqq C_{B_1}|Z^n - Z^{N_i}| + C'_{B_1}|Z^{P_i} - Z^{N_i}| \leqq 2C_{e,B_1}C_0\varepsilon\end{aligned}$$

with $C_{e,B_1} = \max(C_{B_1}, C'_{B_1})$. Using the estimate of the discrete accelerations obtained at Proposition 4, we obtain

$$\begin{aligned}&\sum_{n=N_i+1}^{P_i} \left|\left(M^{1/2}(Z^n)(V^n - V^{n-1} - h_i F^n), v_\alpha(Z^{P_i})\right)\right| \\ &\leqq 2\nu C_{e,B_1}C_0\varepsilon \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}}(C'_0 + \varepsilon C_F) + 3\nu \frac{\lambda^2_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} C_0^2 L_{M^{-1}}\varepsilon.\end{aligned}$$

Hence

$$\left(M^{1/2}(Z^{P_i})V^{P_i}, v_\alpha(Z^{P_i})\right) \leqq -e\left(M^{1/2}(Z^{N_i-2})V^{N_i-2}, v_\alpha(Z^{N_i-2})\right) + \mathcal{O}(\varepsilon + h_i) \tag{53}$$

But $N_i - 2 \geqq n_i + 1$ and

$$\begin{aligned}&\left(M^{1/2}(Z^{N_i-2})V^{N_i-2}, v_\alpha(Z^{N_i-2})\right) - \left(M^{1/2}(Z^{n_i})V^{n_i}, v_\alpha(Z^{n_i})\right) \\ &= \sum_{n=n_i+1}^{N_i-2} \sum_{\beta\in J(Z^n)} -\mu^n_\beta \left(M^{1/2}(Z^n)M^{-1}(U^n)M^{1/2}(Z^n)v_\beta(Z^n), v_\alpha(Z^n)\right) \\ &\quad + \sum_{n=n_i+1}^{N_i-2} \left(\left(M^{1/2}(Z^n) - M^{1/2}(Z^{n-1})\right)V^{n-1}, v_\alpha(Z^n)\right) \\ &\quad + \sum_{n=n_i+1}^{N_i-2} \left(M^{1/2}(Z^{n-1})V^{n-1}, v_\alpha(Z^n) - v_\alpha(Z^{n-1})\right) \\ &\quad + \sum_{n=n_i+1}^{N_i-2} h_i \left(M^{1/2}(Z^n)F^n, v_\alpha(Z^n)\right) \\ &\geqq \sum_{n=n_i+1}^{N_i-2} \sum_{\beta\in J(Z^n)} -\mu^n_\beta \left(M^{1/2}(Z^n)M^{-1}(U^n)M^{1/2}(Z^n)v_\beta(Z^n), v_\alpha(Z^n)\right) \\ &\quad - \left(L_{M^{1/2}}C_0^2 + \sqrt{\lambda_{\max,B_1}}L_{\tilde{u}}C_0^2 + \sqrt{\lambda_{\max,B_1}}C_F\right)\frac{3}{2}\varepsilon.\end{aligned} \tag{54}$$

We rewrite the first term of the right-hand side as follows

$$\sum_{n=n_i+1}^{N_i-2} \sum_{\beta\in J(Z^n)} -\mu_\beta^n \left(M^{1/2}(Z^n)M^{-1}(U^n)M^{1/2}(Z^n)v_\beta(Z^n), v_\alpha(Z^n)\right)$$

$$= \sum_{n=n_i+1}^{N_i-2} \sum_{\beta\in J(Z^n)} -\mu_\beta^n \left(v_\beta(Z^n), v_\alpha(Z^n)\right)$$

$$+ \sum_{n=n_i+1}^{N_i-2} \sum_{\beta\in J(Z^n)} -\mu_\beta^n \left(M^{1/2}(Z^n)\left(M^{-1}(U^n)\right.\right.$$

$$\left.\left. - M^{-1}(Z^n)\right) M^{1/2}(Z^n)v_\beta(Z^n), v_\alpha(Z^n)\right).$$

Then assumption (H6) [see (8)] implies that, for all $n \in \{n_i + 1, \dots, N_i - 2\}$

$$\left(v_\beta(Z^n), v_\alpha(Z^n)\right) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise} \end{cases}$$

if $\alpha \in J(Z^n)$, and with (8) and Lemma 15

$$\begin{aligned} \left|\left(v_\beta(Z^n), v_\alpha(Z^n)\right)\right| &\leqq \left|\left(v_\beta(Z^n), v_\alpha(Z^{N_i})\right)\right| + \left|\left(v_\beta(Z^n), v_\alpha(Z^n) - v_\alpha(Z^{N_i})\right)\right| \\ &\leqq C_{B_1}|Z^n - Z^{N_i}| + C'_{B_1}|Z^n - Z^{N_i}| \leqq 2C_{e,B_1}|Z^n - Z^{N_i}| \\ &\leqq 3C_{e,B_1}C_0\varepsilon \end{aligned}$$

if $\alpha \notin J(Z^n)$. From (52) it follows that

$$\sum_{n=n_i+1}^{N_i-2} \sum_{\beta\in J(Z^n)} -\mu_\beta^n \left(M^{1/2}(Z^n)M^{-1}(U^n)M^{1/2}(Z^n)v_\beta(Z^n), v_\alpha(Z^n)\right)$$

$$\geqq -3\nu \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} \left(C'_0 + \frac{3}{2}\varepsilon C_F\right) C_{e,B_1}C_0\varepsilon - \nu C'_2\lambda_{\max,B_1}L_{M^{-1}}C_0\frac{3}{2}\varepsilon. \quad (55)$$

Thus, combining (53), (54) and (55)

$$\left(M^{1/2}(Z^{P_i})V^{P_i}, v_\alpha(Z^{P_i})\right) \leqq -e\left(M^{1/2}(Z^{n_i})V^{n_i}, v_\alpha(Z^{n_i})\right) + \mathcal{O}(\varepsilon + h_i).$$

Recalling that

$$V^{P_i} = v_{h_i}\left(\bar{t} + \frac{\varepsilon}{2}\right), \quad V^{n_i} = v_{h_i}(\bar{t} - \varepsilon)$$

and

$$\left|Z^{P_i} - u_{h_i}\left(\bar{t} + \frac{\varepsilon}{2}\right)\right| \leqq 2C_0h_i, \quad \left|Z^{n_i} - u_{h_i}(\bar{t} - \varepsilon)\right| \leqq 2C_0h_i,$$

we can pass to the limit as h_i tends to zero, then as ε tends to zero, we get

$$\left(M^{1/2}(\bar{u})\dot{u}^+, v_\alpha(\bar{u})\right) \leqq -e\left(M^{1/2}(\bar{u})\dot{u}^-, v_\alpha(\bar{u})\right).$$

□

It remains for us to prove that, for all $\alpha \in J(\bar{u})$ such that $\mu_\alpha \neq 0$, we have

$$\left(M^{1/2}(\bar{u})\dot{u}^+, v_\alpha(\bar{u})\right) \geqq -e\left(M^{1/2}(\bar{u})\dot{u}^-, v_\alpha(\bar{u})\right).$$

Let us observe that the result is immediate if $\left(M^{1/2}(\bar{u})\dot{u}^-, v_\alpha(\bar{u})\right) = 0$ since we have, already, $\left(M^{1/2}(\bar{u})\dot{u}^+, v_\alpha(\bar{u})\right) \geqq 0$.

So we assume now that $\alpha \in J(\bar{u})$ such that $\mu_\alpha \neq 0$, $\left(M^{1/2}(\bar{u})\dot{u}^-, v_\alpha(\bar{u})\right) < 0$ and we adopt the same notations as in the previous lemma. Since $f_\alpha(\bar{u}) = 0$, $\left(\nabla f_\alpha(\bar{u}), \dot{u}^-\right) < 0$ and $\lim_{t\to\bar{t}^-} \dot{u}(t) = \dot{u}^-$, we infer that, possibly decreasing $\bar{\varepsilon}$, we have

$$f_\alpha(u(t)) \geqq \frac{\left|\left(\nabla f_\alpha(\bar{u}), \dot{u}^-\right)\right|}{2}(\bar{t} - t) \quad \forall t \in [\bar{t} - \bar{\varepsilon}, \bar{t}).$$

Then, the Lipschitz property of f_α on B_1 implies that

$$f_\alpha\left(u_{h_i}(t)\right) \geqq \frac{\left|\left(\nabla f_\alpha(\bar{u}), \dot{u}^-\right)\right|}{2}(\bar{t} - t) - L_f \|u - u_{h_i}\|_{C^0([0,\tau];\mathbb{R}^d)}$$

for all $t \in [\bar{t} - \bar{\varepsilon}, \bar{t})$. As a consequence, for all $\varepsilon \in (0, \bar{\varepsilon}]$, for all $t \in [\bar{t} - \varepsilon, \bar{t} - \varepsilon/2]$ we have

$$f_\alpha\left(u_{h_i}(t)\right) \geqq \left|\left(\nabla f_\alpha(\bar{u}), \dot{u}^-\right)\right| \frac{\varepsilon}{4} - L_f \|u - u_{h_i}\|_{C^0([0,\tau];\mathbb{R}^d)}$$

and, for all $nh_i \in [\bar{t} - \varepsilon, \bar{t} - \varepsilon/2]$

$$f_\alpha(Z^n) \geqq \left|\left(\nabla f_\alpha(\bar{u}), \dot{u}^-\right)\right| \frac{\varepsilon}{4} - L_f \|u - u_{h_i}\|_{C^0([0,\tau];\mathbb{R}^d)} - L_f C_0 h_i.$$

It follows that, for all $\varepsilon \in (0, \bar{\varepsilon}]$ there exists $\tilde{h}_\varepsilon \in (0, h_2^*]$ such that

$$f_\alpha(Z^n) > 0 \quad \forall nh_i \in [\bar{t} - \varepsilon, \bar{t} - \varepsilon/2], \quad \forall h_i \in (0, \tilde{h}_\varepsilon].$$

Now, we consider $\varepsilon \in (0, \bar{\varepsilon}]$. Let $h'_\varepsilon = \min(\tilde{h}_\varepsilon, h_{\varepsilon/2})$ and let $h_i \in (0, h'_\varepsilon]$. We define p_i, n_i and m_i by

$$p_i = \left\lfloor \frac{\bar{t} + \varepsilon}{h_i} \right\rfloor, \quad n_i = \left\lfloor \frac{\bar{t} - \varepsilon}{h_i} \right\rfloor, \quad m_i = \left\lfloor \frac{\bar{t} - \varepsilon/2}{h_i} \right\rfloor.$$

Thus

$$\begin{aligned} 0 < n_i h_i \leqq \bar{t} - \varepsilon < (n_i + 1)h_i < m_i h_i \leqq \bar{t} - \frac{\varepsilon}{2} < (m_i + 1)h_i < p_i h_i \\ \leqq \bar{t} + \varepsilon < (p_i + 1)h_i < \tau. \end{aligned}$$

Now let N'_i be the first time step in $\left[\bar{t} - \frac{\varepsilon}{2}, \bar{t} + \frac{\varepsilon}{2}\right]$ such that $f_\alpha(Z^n) \leqq 0$, that is,

$$N'_i = \min\left\{n \in \mathbb{N};\, nh_i \in \left[\bar{t} - \frac{\varepsilon}{2}, \bar{t} + \frac{\varepsilon}{2}\right] \text{ and } f_\alpha(Z^n) \leqq 0\right\}.$$

Then we have $f_\alpha(Z^n) > 0$ for all $n \in \{n_i + 1, \ldots, N'_i - 1\}$. Moreover,

Lemma 9. *There exists a constant $C_4' > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}]$ and for all $h_i \in (0, h_\varepsilon']$, we have :*

$$\left|\left(M^{1/2}(Z^{N_i'})(V^{n_i} - V^{N_i'-1}), v_\alpha(Z^{N_i'})\right)\right| \leqq C_4'\varepsilon.$$

Proof. With (43) we have

$$\begin{aligned}
&\left(M^{1/2}(Z^{N_i'})(V^{n_i} - V^{N_i'-1}), v_\alpha(Z^{N_i'})\right) \\
&= \sum_{n=n_i+1}^{N_i'-1} \left(M^{1/2}(Z^{N_i'})(V^{n-1} - V^n), v_\alpha(Z^{N_i'})\right) \\
&= -h_i \sum_{n=n_i+1}^{N_i'-1} \left(M^{1/2}(Z^{N_i'})F^n, v_\alpha(Z^{N_i'})\right) \\
&\quad + \sum_{n=n_i+1}^{N_i'-1} \sum_{\beta \in J(Z^n)} \mu_\beta^n \left(M^{1/2}(Z^{N_i'})M^{-1}(U^n)M^{1/2}(Z^n)v_\beta(Z^n), v_\alpha(Z^{N_i'})\right).
\end{aligned}$$

The first term of the right-hand side can be easily estimated:

$$\begin{aligned}
\left|h_i \sum_{n=n_i+1}^{N_i'-1} \left(M^{1/2}(Z^{N_i'})F^n, v_\alpha(Z^{N_i'})\right)\right| &\leqq h_i(N_i' - n_i - 1)C_F\sqrt{\lambda_{\max,B_1}} \\
&\leqq \frac{3}{2}C_F\sqrt{\lambda_{\max,B_1}}\varepsilon. \qquad (56)
\end{aligned}$$

We rewrite the second term of the right-hand side as follows:

$$\begin{aligned}
&\left|\sum_{n=n_i+1}^{N_i'-1} \sum_{\beta \in J(Z^n)} \mu_\beta^n \left(M^{1/2}(Z^{N_i'})M^{-1}(U^n)M^{1/2}(Z^n)v_\beta(Z^n), v_\alpha(Z^{N_i'})\right)\right| \\
&\leqq \sum_{n=n_i+1}^{N_i'-1} \sum_{\beta \in J(Z^n)} \left|\mu_\beta^n\right| \left\|M^{1/2}(Z^{N_i'})M^{-1}(U^n)M^{1/2}(Z^n) - \mathrm{Id}\right\| \\
&\quad + \sum_{n=n_i+1}^{N_i'-1} \sum_{\beta \in J(Z^n)} \left|\mu_\beta^n\right| \left|\left(v_\beta(Z^n), v_\alpha(Z^{N_i'})\right)\right|. \qquad (57)
\end{aligned}$$

Since $\alpha \notin J(Z^n)$ for all $n \in \{n_i + 1, \ldots, N_i' - 1\}$, assumption (H6) and (21) imply that, for all $\beta \in J(Z^n)$

$$\begin{aligned}
\left|\left(v_\beta(Z^n), v_\alpha(Z^{N_i'})\right)\right| &\leqq C_{e,B_1}|Z^n - Z^{N_i'}| \leqq C_{e,B_1} \sum_{k=n+1}^{N_i'} \frac{h}{1+e}|V^k + eV^{k-2}| \\
&\leqq \frac{3}{2}C_{e,B_1}C_0\varepsilon.
\end{aligned}$$

Using estimate (51) for $|\mu_\beta^n|$ and the estimate of the discrete accelerations obtained in Proposition 4, we get

$$\begin{aligned}
&\sum_{n=n_i+1}^{N_i'-1} \sum_{\beta\in J(Z^n)} \left|\mu_\beta^n\right| \left|\left(v_\beta(Z^n), v_\alpha(Z^{N_i'})\right)\right| \\
&\leqq \sum_{n=n_i+1}^{N_i'-1} \frac{3}{2}\nu C_{e,B_1} C_0 \varepsilon \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} \left(|V^n - V^{n-1}| + h_i C_F\right) \\
&\leqq \frac{3}{2}\nu C_{e,B_1} C_0 \varepsilon \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} \left(C_0' + \frac{3\varepsilon}{2} C_F\right). \qquad (58)
\end{aligned}$$

It remains to estimate the first term of the right-hand side of (57). Using the Lipschitz property of $M^{1/2}$ and M^{-1}, we get

$$\begin{aligned}
&\sum_{n=n_i+1}^{N_i'-1} \sum_{\beta\in J(Z^n)} \left|\mu_\beta^n\right| \left\|M^{1/2}(Z^{N_i'}) M^{-1}(U^n) M^{1/2}(Z^n) - \mathrm{Id}\right\| \\
&\leqq \sum_{n=n_i+1}^{N_i'-1} \sum_{\beta\in J(Z^n)} \left|\mu_\beta^n\right| \left(\left\|M^{1/2}(Z^{N_i'}) - M^{1/2}(Z^n)\right\| \left\|M^{-1}(U^n) M^{1/2}(Z^n)\right\|\right. \\
&\quad \left. + \left\|M^{-1}(U^n) - M^{-1}(Z^n)\right\| \left\|M^{1/2}(Z^n)\right\|^2\right) \\
&\leqq \sum_{n=n_i+1}^{N_i'-1} \sum_{\beta\in J(Z^n)} \left|\mu_\beta^n\right| \left(L_{M^{1/2}} C_0 h_i (N_i' - n) \frac{\sqrt{\lambda_{\max,B_1}}}{\lambda_{\min,B_1}} + \lambda_{\max,B_1} L_{M^{-1}} C_0 h_i\right) \\
&\leqq \sum_{n=n_i+1}^{N_i'-1} \sum_{\beta\in J(Z^n)} \frac{3}{2}\varepsilon L_{M^{1/2}} C_0 \left(\frac{\lambda_{\max,B_1}}{\lambda_{\min,B_1}}\right)^{3/2} \left(|V^n - V^{n-1}| + h_i C_F\right) \\
&\quad + \frac{3}{2}\varepsilon\nu C_2' \lambda_{\max,B_1} L_{M^{-1}} C_0 \\
&\leqq \frac{3}{2}\varepsilon L_{M^{1/2}} C_0 \nu \left(\frac{\lambda_{\max,B_1}}{\lambda_{\min,B_1}}\right)^{3/2} \left(C_0' + \frac{3\varepsilon}{2} C_F\right) + \frac{3}{2}\varepsilon\nu C_2' \lambda_{\max,B_1} L_{M^{-1}} C_0.
\end{aligned}$$

Combining this last estimate with (56), (57) and (58), we may conclude. □

Now let us prove that

Lemma 10. *There exists a constant $C_5' > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}]$ and for all $h_i \in (0, h_\varepsilon']$, we have:*

$$f_\alpha(Z^{N_i'+1}) \leqq C_5' h_i^2.$$

Proof. If $\alpha \in J(Z^{N_i'+1})$ the result is obvious. Otherwise,

$$\begin{aligned}
&\left|\left(V^{N_i'} - V^{N_i'+1} + h_i F^{N_i'+1}, M^{1/2}(Z^{N_i'+1})v_\alpha(Z^{N_i'+1})\right)\right| \\
&\leqq \sum_{\beta \in J(Z^{N_i'+1})} |\mu_\beta^{N_i'+1}| \left|\left(M^{-1}(U^{N_i'+1})M^{1/2}(Z^{N_i'+1})v_\beta(Z^{N_i'+1}),\right.\right. \\
&\qquad\qquad \left.\left. M^{1/2}(Z^{N_i'+1})v_\alpha(Z^{N_i'+1})\right)\right| \\
&\leqq \sum_{\beta \in J(Z^{N_i'+1})} |\mu_\beta^{N_i'+1}| \left\|M^{-1}(U^{N_i'+1}) - M^{-1}(Z^{N_i'+1})\right\| \left\|M^{1/2}(Z^{N_i'+1})\right\|^2 \\
&\quad + \sum_{\beta \in J(Z^{N_i'+1})} |\mu_\beta^{N_i'+1}| \left|\left(v_\beta(Z^{N_i'+1}), v_\alpha(Z^{N_i'+1})\right)\right|.
\end{aligned}$$

Since $\alpha \in J(Z^{N_i'})$ and $\alpha \notin J(Z^{N_i'+1})$, assumption (H6) and Lemma 15 imply that

$$\begin{aligned}
\left|\left(v_\beta(Z^{N_i'+1}), v_\alpha(Z^{N_i'+1})\right)\right| &\leqq \left|\left(v_\beta(Z^{N_i'+1}), v_\alpha(Z^{N_i'})\right)\right| \\
&\quad + \left|v_\alpha(Z^{N_i'}) - v_\alpha(Z^{N_i'+1})\right| \\
&\leqq 2C_{e,B_1}C_0 h_i.
\end{aligned}$$

Using the Lipschitz property of M^{-1}, we get finally

$$\begin{aligned}
&\left|\left(V^{N_i'} - V^{N_i'+1}, M^{1/2}(Z^{N_i'+1})v_\alpha(Z^{N_i'+1})\right)\right| \\
&\leqq h_i\sqrt{\lambda_{\max,B_1}}C_F + h_i \nu C_2' C_0 \left(\lambda_{\max,B_1} L_{M^{-1}} + 2C_{e,B_1}\right).
\end{aligned} \tag{59}$$

With exactly the same computations, we also obtain

$$\begin{aligned}
&\left|\left(V^{N_i'-2} - V^{N_i'-1}, M^{1/2}(Z^{N_i'-1})v_\alpha(Z^{N_i'-1})\right)\right| \\
&\leqq h_i\sqrt{\lambda_{\max,B_1}}C_F + h_i \nu C_2' C_0 \left(\lambda_{\max,B_1} L_{M^{-1}} + 2C_{e,B_1}\right)
\end{aligned} \tag{60}$$

since (recall that $\alpha \in J(Z^{N_i'})$ and $\alpha \notin J(Z^{N_i'-1})$)

$$\begin{aligned}
\left|\left(v_\beta(Z^{N_i'-1}), v_\alpha(Z^{N_i'-1})\right)\right| &\leqq \left|\left(v_\beta(Z^{N_i'-1}), v_\alpha(Z^{N_i'})\right)\right| \\
&\quad + \left|\left(v_\alpha(Z^{N_i'}) - v_\alpha(Z^{N_i'-1})\right)\right| \\
&\leqq 2C_{e,B_1}C_0 h_i.
\end{aligned}$$

We define

$$C_6' = \sqrt{\lambda_{\max,B_1}}C_F + \nu C_2' C_0 \left(\lambda_{\max,B_1} L_{M^{-1}} + 2C_{e,B_1}\right).$$

Then, using a Taylor's formula, we get

$$f_\alpha(Z^{N_i'+1}) = f_\alpha(Z^{N_i'}) + \left(\nabla f_\alpha(Z^{N_i'}), Z^{N_i'+1} - Z^{N_i'}\right) + \int_0^1 \left(\nabla f_\alpha\left(Z^{N_i'} + t(Z^{N_i'+1} - Z^{N_i'})\right) - \nabla f_\alpha(Z^{N_i'}), Z^{N_i'+1} - Z^{N_i'}\right) \mathrm{d}t.$$

But $f_\alpha(Z^{N_i'}) = 0$ since $\alpha \in J(Z^{N_i'})$ and

$$Z^{N_i'+1} - Z^{N_i'} = \frac{h_i}{1+e}\left(V^{N_i'+1} + eV^{N_i'-1}\right).$$

Moreover,

$$\left(\nabla f_\alpha(Z^{N_i'}), V^{N_i'+1} + eV^{N_i'-1}\right) = \left(\nabla f_\alpha(Z^{N_i'}), V^{N_i'+1} - V^{N_i'}\right) + \left(\nabla f_\alpha(Z^{N_i'}), V^{N_i'} + eV^{N_i'-2}\right) + e\left(\nabla f_\alpha(Z^{N_i'}), V^{N_i'-1} - V^{N_i'-2}\right). \quad (61)$$

The second term of the right-hand side is estimated with Proposition 1:

$$\left(\nabla f_\alpha(Z^{N_i'}), V^{N_i'} + eV^{N_i'-2}\right) \leqq \frac{L_f h_i}{2(1+e)}|V^{N_i'} + eV^{N_i'-2}|^2 \leqq L_f h_i C_0^2.$$

The first and third terms of the right-hand side of (61) can be estimated by using the Lipschitz property of ∇f_α on B_1 and (59)–(60). Indeed,

$$\begin{aligned}\left(\nabla f_\alpha(Z^{N_i'}), V^{N_i'+1} - V^{N_i'}\right) &= \left(\nabla f_\alpha(Z^{N_i'}) - \nabla f_\alpha(Z^{N_i'+1}), V^{N_i'+1} - V^{N_i'}\right) \\ &\quad + \left|M^{-1/2}(Z^{N_i'+1})\nabla f_\alpha(Z^{N_i'+1})\right| \\ &\quad \times \left(M^{1/2}(Z^{N_i'+1})v_\alpha(Z^{N_i'+1}), V^{N_i'+1} - V^{N_i'}\right) \\ &\leqq 2L_f C_0^2 h_i + \frac{C_{\bar{u}}}{\sqrt{\lambda_{\min,B_1}}} C_6' h_i\end{aligned}$$

and similarly

$$\begin{aligned}\left(\nabla f_\alpha(Z^{N_i'}), V^{N_i'-1} - V^{N_i'-2}\right) &= \left(\nabla f_\alpha(Z^{N_i'}) - \nabla f_\alpha(Z^{N_i'-1}), V^{N_i'-1} - V^{N_i'-2}\right) \\ &\quad + \left|M^{-1/2}(Z^{N_i'-1})\nabla f_\alpha(Z^{N_i'-1})\right| \\ &\quad \times \left(M^{1/2}(Z^{N_i'-1})v_\alpha(Z^{N_i'-1}), V^{N_i'-1} - V^{N_i'-2}\right) \\ &\leqq 2L_f C_0^2 h_i + \frac{C_{\bar{u}}}{\sqrt{\lambda_{\min,B_1}}} C_6' h_i.\end{aligned}$$

Moreover,

$$\left|\int_0^1 \left(\nabla f_\alpha\left(Z^{N_i'} + t(Z^{N_i'+1} - Z^{N_i'})\right) - \nabla f_\alpha(Z^{N_i'}), Z^{N_i'+1} - Z^{N_i'}\right) \mathrm{d}t\right|$$
$$\leqq \int_0^1 L_f |Z^{N_i'+1} - Z^{N_i'}|^2 t \,\mathrm{d}t \leqq \frac{C_0^2 L_f}{2} h_i^2.$$

Thus

$$f_\alpha(Z^{N_i'+1}) \leqq C_5' h_i^2 \quad \forall h_i \in (0, h_2^*]$$

with

$$C_5' = \frac{11}{2} C_0^2 L_f + 2\frac{C_{\bar{u}}}{\sqrt{\lambda_{\min, B_1}}} C_6'.$$

□

It follows that, for all $\varepsilon \in (0, \bar{\varepsilon}]$ and for all $h_i \in (0, h_\varepsilon']$, we have

$$0 \leqq f_\alpha(Z^{N_i'+1}) - f_\alpha(Z^{N_i'}) = f_\alpha(Z^{N_i'+1}) \leqq C_5' h_i^2$$

since $Z^{N_i'+1} \in K$ by construction. But, with the same computations as in Proposition 1, we also have

$$\begin{aligned} f_\alpha(Z^{N_i'+1}) - f_\alpha(Z^{N_i'}) = {} & \frac{h_i}{1+e}\left(\nabla f_\alpha(Z^{N_i'}), V^{N_i'+1} + eV^{N_i'-1}\right) \\ & + \frac{h_i}{1+e}\int_0^1 \left(\nabla f_\alpha\left(Z^{N_i'} + t(Z^{N_i'+1} - Z^{N_i'})\right)\right. \\ & \left. - \nabla f_\alpha(Z^{N_i'}), V^{N_i'+1} + eV^{N_i'-1}\right) \mathrm{d}t. \end{aligned}$$

We can estimate the last term of the right-hand side by

$$\left|\int_0^1 \left(\nabla f_\alpha\left(Z^{N_i'} + t(Z^{N_i'+1} - Z^{N_i'})\right) - \nabla f_\alpha(Z^{N_i'}), V^{N_i'+1} + eV^{N_i'-1}\right) \mathrm{d}t\right|$$
$$\leqq L_f h_i \frac{\left|V^{N_i'+1} + eV^{N_i'-1}\right|^2}{2(1+e)} \leqq L_f C_0^2 h_i$$

and thus

$$-L_f C_0^2 h_i \leqq \left(\nabla f_\alpha(Z^{N_i'}), V^{N_i'+1} + eV^{N_i'-1}\right) \leqq C_5'(1+e)h_i + L_f C_0^2 h_i.$$

Using Lemma 15, we finally get

$$\begin{aligned} \left|\left(M^{1/2}(Z^{N_i'+1}) v_\alpha(Z^{N_i'+1}), V^{N_i'+1} + eV^{N_i'-1}\right)\right| &\leqq \frac{L_f C_0^2 + C_5'(1+e)}{m_{B_1}} h_i \\ &= C_7' h_i. \end{aligned} \tag{62}$$

Now we will estimate $\left(M^{1/2}(Z^{p_i+1}) v_\alpha(Z^{p_i+1}), V^{N_i'+1} - V^{p_i}\right)$.

Lemma 11. *There exists a constant $C_8' > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}]$ and for all $h_i \in (0, h_\varepsilon']$, we have:*

$$\left(M^{1/2}(Z^{p_i+1})v_\alpha(Z^{p_i+1}),\, V^{N_i'+1} - V^{p_i}\right) \leqq C_8'(\varepsilon + h_i).$$

Proof. Let $h_i \in (0, h_\varepsilon']$. First we observe that $N_i'+1 < p_i$ since $h_i \leqq h_\varepsilon' \leqq h_{\varepsilon/2} \leqq \frac{\varepsilon}{6}$. With (50) we have

$$\begin{aligned} V^{N_i'+1} - V^{p_i} &= -h_i \sum_{l=N_i'+1}^{p_i-1} F^{l+1} \\ &\quad + \sum_{l=N_i'+1}^{p_i-1} \sum_{\beta \in J(Z^{l+1})} \mu_\beta^{l+1} M^{-1}(U^{l+1}) M^{1/2}(Z^{l+1}) v_\beta(Z^{l+1}) \end{aligned}$$

and

$$\begin{aligned} &\left(V^{N_i'+1} - V^{p_i},\, M^{1/2}(Z^{p_i+1})v_\alpha(Z^{p_i+1})\right) \\ &= -h_i \sum_{l=N_i'+1}^{p_i-1} \left(F^l,\, M^{1/2}(Z^{p_i+1})v_\alpha(Z^{p_i+1})\right) + \sum_{l=N_i'+1}^{p_i-1} \sum_{\beta \in J(Z^{l+1})} \mu_\beta^{l+1} \\ &\quad \times \left(M^{-1}(U^{l+1}) M^{1/2}(Z^{l+1}) v_\beta(Z^{l+1}),\, M^{1/2}(Z^{p_i+1})v_\alpha(Z^{p_i+1})\right) \\ &\leqq \sum_{l=N_i'+1}^{p_i-1} \sum_{\beta \in J(Z^{l+1})} \left|\mu_\beta^{l+1}\right| \left\| M^{1/2}(Z^{p_i+1}) - M^{1/2}(Z^{l+1}) \right\| \\ &\quad \times \left\| M^{-1}(U^{l+1}) \right\| \left\| M^{1/2}(Z^{l+1}) \right\| \\ &\quad + \sum_{l=N_i'+1}^{p_i-1} \sum_{\beta \in J(Z^{l+1})} \left|\mu_\beta^{l+1}\right| \left\| M^{1/2}(Z^{l+1}) \right\|^2 \\ &\quad \times \left\| M^{-1}(U^{l+1}) - M^{-1}(Z^{l+1}) \right\| \\ &\quad + \sum_{l=N_i'+1}^{p_i-1} \sum_{\beta \in J(Z^{l+1})} \mu_\beta^{l+1} \left(v_\beta(Z^{l+1}),\, v_\alpha(Z^{p_i+1})\right) \\ &\quad + h_i(p_i - N_i' - 1)\sqrt{\lambda_{\max, B_1}} C_F \\ &\leqq \frac{3}{2}\varepsilon \sqrt{\lambda_{\max, B_1}} C_F + \sum_{l=N_i'+1}^{p_i-1} \sum_{\beta \in J(Z^{l+1})} \left|\mu_\beta^{l+1}\right| \frac{\sqrt{\lambda_{\max, B_1}}}{\lambda_{\min, B_1}} L_{M^{1/2}} C_0 \frac{3}{2}\varepsilon \end{aligned}$$

$$+\sum_{l=N_i'+1}^{p_i-1} \nu C_2' \lambda_{\max,B_1} L_{M^{-1}} C_0 h_i$$

$$+\sum_{l=N_i'+1}^{p_i-1} \sum_{\beta\in J(Z^{l+1})} \mu_\beta^{l+1} \left(v_\beta(Z^{l+1}), v_\alpha(Z^{p_i+1})\right). \tag{63}$$

We estimate the second term of the right-hand side by using (51) and Proposition 4, that is,

$$\begin{aligned}
&\sum_{l=N_i'+1}^{p_i-1} \sum_{\beta\in J(Z^{l+1})} \left|\mu_\beta^{l+1}\right| \frac{\sqrt{\lambda_{\max,B_1}}}{\lambda_{\min,B_1}} L_{M^{1/2}} C_0 \frac{3}{2}\varepsilon \\
&\leqq \sum_{l=N_i'+1}^{p_i-1} C_0 \nu \left(\frac{\lambda_{\max,B_1}}{\lambda_{\min,B_1}}\right)^{3/2} L_{M^{1/2}} \frac{3}{2}\varepsilon \left(|V^{l+1}-V^l| + h_i C_F\right) \\
&\leqq \frac{3}{2}\varepsilon C_0 \nu \left(\frac{\lambda_{\max,B_1}}{\lambda_{\min,B_1}}\right)^{3/2} L_{M^{1/2}} \left(C_0' + \frac{3}{2}\varepsilon C_F\right).
\end{aligned} \tag{64}$$

For the last term, we observe that, if $\alpha \in J(Z^{l+1})$, then $\mu_\alpha^{l+1} \leqq 0$ and

$$\begin{aligned}
\mu_\alpha^{l+1} \left(v_\alpha(Z^{l+1}), v_\alpha(Z^{p_i+1})\right) &\leqq \mu_\alpha^{l+1} + |\mu_\alpha^{l+1}| \left|v_\alpha(Z^{p_i+1}) - v_\alpha(Z^{l+1})\right| \\
&\leqq |\mu_\alpha^{l+1}| C_{e,B_1} \left|Z^{p_i+1} - Z^{l+1}\right| \leqq \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} \\
&\quad \times \left(|V^{l+1}-V^l| + h_i C_F\right) C_{e,B_1} C_0 (p_i - l) h_i \\
&\leqq \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} \left(|V^{l+1} - V^l| + h_i C_F\right) C_{e,B_1} C_0 \frac{3}{2}\varepsilon.
\end{aligned}$$

On the other hand, if $\beta \in J(Z^{l+1})\backslash\{\alpha\}$, recalling that $\alpha \in J(Z^{N_i'})$, assumption (H6) implies that

$$\begin{aligned}
\left|\mu_\beta^{l+1} \left(v_\beta(Z^{l+1}), v_\alpha(Z^{p_i+1})\right)\right| &\leqq \left|\mu_\beta^{l+1}\right| \left(\left|v_\alpha(Z^{p_i+1}) - v_\alpha(Z^{N_i'})\right|\right. \\
&\quad \left. + \left|\left(v_\alpha(Z^{N_i'}), v_\beta(Z^{l+1})\right)\right|\right) \\
&\leqq \left|\mu_\beta^{l+1}\right| C_{e,B_1} \left(|Z^{p_i+1} - Z^{N_i'}| + |Z^{N_i'} - Z^{l+1}|\right) \\
&\leqq \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} 2 C_{e,B_1} C_0 \left(\frac{3}{2}\varepsilon + h_i\right) \\
&\quad \times \left(|V^{l+1} - V^l| + h_i C_F\right).
\end{aligned}$$

Finally,

$$\sum_{l=N_i'+1}^{p_i-1} \sum_{\beta\in J(Z^{l+1})} \mu_\beta^{l+1}\left(v_\beta(Z^{l+1}), v_\alpha(Z^{p_i+1})\right)$$
$$\leqq \frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}} 2C_{e,B_1}C_0\nu\left(\frac{3}{2}\varepsilon+h_i\right)\left(C_0'+\frac{3}{2}\varepsilon C_F\right) \tag{65}$$

and, combining (63), (64) and (65) we may conclude with

$$C_8' = \frac{3\sqrt{\lambda_{\max,B_1}}}{2}\left(C_F+\nu C_2'\sqrt{\lambda_{\max,B_1}}L_{M^{-1}}C_0\right)$$
$$+\frac{3}{2}\nu C_0\frac{\lambda_{\max,B_1}}{\sqrt{\lambda_{\min,B_1}}}\left(\frac{\sqrt{\lambda_{\max,B_1}}}{\lambda_{\min,B_1}}L_{M^{1/2}}+2C_{e,B_1}\right)\left(C_0'+\frac{3}{2}\bar{\varepsilon}C_F\right).$$

□

Now, for all $\varepsilon\in(0,\bar{\varepsilon}]$ and for all $h_i\in(0,h_\varepsilon']$, we get, using estimate (62),

$$\left(V^{p_i}, M^{1/2}(Z^{p_i+1})v_\alpha(Z^{p_i+1})\right) \geqq \left(V^{N_i'+1}, M^{1/2}(Z^{p_i+1})v_\alpha(Z^{p_i+1})\right) - C_8'(\varepsilon+h_i)$$
$$\geqq \left(V^{N_i'+1}, M^{1/2}(Z^{N_i'+1})v_\alpha(Z^{N_i'+1})\right)$$
$$+\left(V^{N_i'+1}, \left(M^{1/2}(Z^{p_i+1})-M^{1/2}(Z^{N_i'+1})\right)v_\alpha(Z^{N_i'+1})\right)$$
$$+\left(V^{N_i'+1}, M^{1/2}(Z^{p_i+1})\left(v_\alpha(Z^{p_i+1})-v_\alpha(Z^{N_i'+1})\right)\right) - C_8'(\varepsilon+h_i)$$
$$\geqq -e\left(V^{N_i'-1}, M^{1/2}(Z^{N_i'+1})v_\alpha(Z^{N_i'+1})\right)$$
$$-(L_{M^{1/2}}+\sqrt{\lambda_{\max B_1}}C_{e,B_1})C_0|Z^{p_i+1}-Z^{N_i'+1}|$$
$$-C_8'(\varepsilon+h_i)-C_7'h_i$$

and

$$\left(V^{p_i}, M^{1/2}(Z^{p_i+1})v_\alpha(Z^{p_i+1})\right)$$
$$\geqq -e\left(V^{N_i'-1}, M^{1/2}(Z^{N_i'})v_\alpha(Z^{N_i'})\right) - C_8'(\varepsilon+h_i)-C_7'h_i$$
$$-(L_{M^{1/2}}+\sqrt{\lambda_{\max,B_1}}C_{e,B_1})C_0(e|Z^{N_i'+1}-Z^{N_i'}|+|Z^{p_i+1}-Z^{N_i'+1}|)$$
$$\geqq -e\left(V^{N_i'-1}, M^{1/2}(Z^{N_i'})v_\alpha(Z^{N_i'})\right) - C_8'(\varepsilon+h_i)-C_7'h_i$$
$$-(L_{M^{1/2}}+\sqrt{\lambda_{\max,B_1}}C_{e,B_1})C_0^2(p_i-N_i'+1)h_i.$$

So, with Lemma 9

$$\left(V^{p_i}, M^{1/2}(Z^{p_i+1})v_\alpha(Z^{p_i+1})\right)$$
$$\geqq -e\left(V^{n_i}, M^{1/2}(Z^{N_i'})v_\alpha(Z^{N_i'})\right) - C_8'(\varepsilon+h_i)-C_7'h_i$$
$$-eC_4'\varepsilon-(L_{M^{1/2}}+\sqrt{\lambda_{\max,B_1}}C_{e,B_1})C_0^2\left(\frac{3}{2}\varepsilon+h_i\right), \tag{66}$$

for all $\varepsilon \in (0, \bar{\varepsilon}]$ and for all $h_i \in (0, h_\varepsilon']$. Observing that $V^{p_i} = v_{h_i}(\bar{t} + \varepsilon)$, $V^{n_i} = v_{h_i}(\bar{t} - \varepsilon)$ and

$$\begin{aligned}|Z^{p_i+1} - \bar{u}| &\leqq |Z^{p_i+1} - U^{p_i+1}| + \left|u_{h_i}\left((p_i+1)h_i\right) - u_{h_i}(\bar{t})\right| + \left|u_{h_i}(\bar{t}) - u(\bar{t})\right| \\ &\leqq C_0 h_i + C_0\left|(p_i+1)h_i - \bar{t}\right| + \|u_{h_i} - u\|_{C^0([0,\tau];\mathbb{R}^d)} \\ &\leqq \|u_{h_i} - u\|_{C^0([0,\tau];\mathbb{R}^d)} + C_0(\varepsilon + 2h_i)\end{aligned}$$

and similarly

$$|Z^{N_i'} - \bar{u}| \leqq \|u_{h_i} - u\|_{C^0([0,\tau];\mathbb{R}^d)} + C_0\left(\frac{1}{2}\varepsilon + h_i\right)$$

we can pass to the limit in (66) as h_i tends to zero. Then as ε tends to zero, we get

$$\left(M^{1/2}(\bar{u})\dot{u}^+, v_\alpha(\bar{u})\right) \geqq -e\left(M^{1/2}(\bar{u})\dot{u}^-, v_\alpha(\bar{u})\right).$$

Combining this inequality with the result of Lemma 8 we obtain

$$\left(M^{1/2}(\bar{u})\dot{u}^+, v_\alpha(\bar{u})\right) = -e\left(M^{1/2}(\bar{u})\dot{u}^-, v_\alpha(\bar{u})\right)$$

for all $\alpha \in J(\bar{u})$ such that $\mu_\alpha \neq 0$, which concludes the proof of the impact law.

3.3. Study of the initial conditions

Let us prove now that property (P4) holds. We begin with the following lemma:

Lemma 12. *For all $k \in \{0, \ldots, \nu\}$, we have*

$$\lim_{h_i \to 0} V^k = v_0.$$

Proof. Observing that the definitions of U^0 and U^1 are the same as in the inelastic case, we obtain immediately that (see Lemma 5 in [11])

$$\lim_{h_i \to 0} V^0 = \lim_{h_i \to 0} v_{h_i}(0) = v(0) = v_0.$$

As in the previous subsection, we define $r_{u_0} > 0$ such that $J(q) \subset J(u_0)$ for all $q \in \overline{B}(u_0, r_{u_0})$. Possibly decreasing r_{u_0}, we may assume without loss of generality that $r_{u_0} \leqq \min(r_{B_1}, r'_{B_1})/2$. From the continuity of u and the uniform convergence of $(u_{h_i})_{h_1^* \geqq h_i > 0}$ to u on $[0, \tau]$, we infer that there exist $\bar{\varepsilon} \in (0, \tau/2)$ and $\tilde{h}_1^* \in \left(0, \min\left(h_1^*, \frac{\bar{\varepsilon}}{\nu+1}, \frac{r_{u_0}}{6C_0}\right)\right]$ such that

$$\begin{aligned} u(t) &\in \overline{B}\left(\bar{u}_0, \frac{r_{u_0}}{3}\right) \quad \forall t \in [0, \bar{\varepsilon}], \\ \|u - u_{h_i}\|_{C^0([0,\tau];\mathbb{R}^d)} &\leqq \frac{r_{u_0}}{3} \quad \forall h_i \in (0, \tilde{h}_1^*]. \end{aligned}$$

It follows that

$$U^{n+1}, U^n, Z^n, Z^{n+1} \in \overline{B}(u_0, r_{u_0}) \quad \forall n h_i \in [0, \bar{\varepsilon}], \quad \forall h_i \in (0, \tilde{h}_1^*].$$

Let us now distinguish between the two cases $u_0 \in \mathrm{Int}(K)$ and $u_0 \in \partial K$.

Case 1: $u_0 \in \mathrm{Int}(K)$.

Then, $\overline{B}(u_0, r_{u_0}) \subset \mathrm{Int}(K)$. With Lemma 1 we infer that, for all $nh_i \in (0, \bar{\varepsilon}]$, for all $h_i \in (0, \tilde{h}_1^*]$

$$M(U^n)(V^{n-1} - V^n + h_i F^n) \in N_K(Z^n) = \{0\}$$

and thus

$$V^n = V^{n-1} + h_i F^n.$$

It follows that

$$|V^k - V^0| \leqq \sum_{l=1}^{k} h_i |F^l| \leqq k h_i C_F$$

for all $k \in \{1, \ldots, \nu\}$ and for all $h_i \in (0, \tilde{h}_1^*]$ (let us recall that $\tilde{h}_1^* \leqq \bar{\varepsilon}/(\nu + 1)$), and the conclusion follows.

Case 2: $u_0 \in \partial K$.

Let us prove by induction on k that

$$\lim_{h_i \to 0} V^k = v_0$$

for all $k \in \{0, \ldots, \nu\}$. More precisely, let $k \in \{1, \ldots, \nu\}$ and assume that

$$\lim_{h_i \to 0} V^l = v_0 \quad \forall l \in \{0, \ldots, k-1\}. \tag{67}$$

Since $v_0 \in T_K(u_0)$ and $\tilde{T}_K(u_0)$ is dense in $T_K(u_0)$, we may consider a sequence $(v_p)_{p \in \mathbb{N}^*}$ converging to v_0 and such that

$$v_p \in \tilde{T}_K(u_0) = \left\{ w \in \mathbb{R}^d ; (\nabla f_\alpha(u_0), w) > 0 \ \forall \alpha \in J(u_0) \right\} \quad \forall p \in \mathbb{N}^*.$$

As in Lemma 5 in [11], we infer that there exist $t_p > 0$ such that

$$u_0 + t v_p \in K \cap B_1 \quad \forall t \in (0, t_p].$$

By definition of the scheme

$$\|W^k - Z^k\|_{M(U^k)} \leqq \|W^k - Z\|_{M(U^k)} \quad \forall Z \in K.$$

Let $h_p = \min\left(h^*, \frac{t_p(1+e)}{2+(1+e)(k-1)}\right)$. By choosing $Z = u_0 + \left(\frac{2}{1+e} + (k-1)\right) h_i v_p$, we get for all $h_i \in (0, h_p]$

$$\begin{aligned} W^k - Z &= \frac{2U^k - (1-e)U^{k-1} + h_i^2 F^k}{1+e} - \left(u_0 + \left(\frac{2}{1+e} + (k-1)\right) h_i v_p\right) \\ &= \frac{h_i}{1+e}\left(2V^{k-1} + (1+e)\sum_{l=0}^{k-2} V^l + h_i F^k - (2 + (1+e)(k-1))\, v_p\right) \end{aligned}$$

and with (10)

$$\begin{aligned}&\|V^{k-1} - V^k + hF^k\|_{M(U^k)}\\ &\leqq \left\| 2V^{k-1} + (1+e)\sum_{l=0}^{k-2} V^l + h_i F^k - (2+(1+e)(k-1))\, v_p \right\|_{M(U^k)}\end{aligned}$$

with the convention that $\sum_{l=0}^{k-2} V^l = 0$ if $k = 1$, and thus

$$\begin{aligned}\|v_0 - V^k\|_{M(U^k)} \leqq\ & 2h_i \|F^k\|_{M(U^k)} + \|V^{k-1} - v_0\|_{M(U^k)}\\ &+(2+(1+e)(k-1))\left(\|w_k - v_0\|_{M(U^k)} + \|v_0 - v_p\|_{M(U^k)}\right)\end{aligned}$$

with

$$w_k = \frac{1}{2+(1+e)(k-1)}\left(2V^{k-1} + (1+e)\sum_{l=0}^{k-2} V^l\right).$$

It follows that

$$\begin{aligned}|v_0 - V^k| \leqq \sqrt{\frac{\lambda_{\max,B_1}}{\lambda_{\min,B_1}}}\Big(&2h_i|F^k| + |V^{k-1} - v_0|\\ &+ (2+(1+e)(k-1))\left(|w_k - v_0| + |v_0 - v_p|\right)\Big) \quad \forall h_i \in (0, h_p].\end{aligned}$$

But the assumption (67) implies that $\lim_{h_i \to 0} w_k = v_0$ and we get

$$\limsup_{h_i \to 0} |V^k - v_0| \leqq \sqrt{\frac{\lambda_{\max,B_1}}{\lambda_{\min,B_1}}}\,(2+(1+e)(k-1))\,|v_0 - v_p|.$$

This last inequality holds for all $p \in \mathbb{N}^*$, and passing to the limit as p tends to $+\infty$, we obtain

$$\limsup_{h_i \to 0} |V^k - v_0| \leqq 0$$

which implies $\lim_{h_i \to 0} V^k = v_0$. □

Now we can prove that

Lemma 13. *The initial conditions (u_0, v_0) are satisfied in the following sense*

$$u(0) = u_0, \quad \dot{u}(0+0) = v_0.$$

Proof. The first equality is trivial, since

$$u(0) = \lim_{h_i \to 0} u_{h_i}(0) \quad \text{and} \quad u_{h_i}(0) = U^0 = u_0 \quad \forall i \in \mathbb{N}.$$

In order to establish the second equality, we first observe that we already know with Proposition 6 that

$$M(u_0)\,(\dot{u}(0+0) - \dot{u}(0-0)) \in -N_K(u_0).$$

Moreover, with convention (37) and the previous Lemma, we have

$$\dot{u}(0-0) = \dot{u}(0) = v(0) = v_0$$

and thus

$$M(u_0)\,(\dot{u}(0+0) - v_0) \in -N_K(u_0).$$

If $u_0 \in \mathrm{Int}(K)$, then $N_K(u_0) = \{0\}$ and we may conclude.

Let us assume now that $u_0 \in \partial K$. Since v_0 and $\dot{u}(0+0)$ belong to $T_K(u_0)$ we obtain

$$\begin{aligned}(\dot{u}(0+0) - v_0, \dot{u}(0+0))_{M(u_0)} &\geqq 0,\\ (\dot{u}(0+0) - v_0, v_0)_{M(u_0)} &\geqq 0\end{aligned}$$

and thus

$$\|v_0\|^2_{M(u_0)} \leqq (\dot{u}(0+0), v_0)_{M(u_0)} \leqq \|\dot{u}(0+0)\|^2_{M(u_0)}\,. \tag{68}$$

But with Proposition 3 we obtain

$$\|V^n\|^2_{M(Z^n)} \leqq \max\left(\|V^0\|^2_{M(Z^0)}, \ldots, \|V^\nu\|^2_{M(Z^\nu)}\right) + C_{\mathcal{B},C_0} nh$$

for all $nh \in [0, \tilde{\tau}]$, for all $h \in (0, \tilde{h}_1^*]$, where $\tilde{\tau} = \min(\delta/(2C_0), T)$ is defined at Proposition 2.

Let $t \in (0, \tilde{\tau})$ and $h_i \in \left(0, \tilde{h}_1^*\right]$. We define $n = \lfloor t/h_i \rfloor$ and with the Lipschitz property of M on B_1 we get

$$\begin{aligned}\left\|v_{h_i}(t)\right\|^2_{M(u_0)} = \left\|V^n\right\|^2_{M(u_0)} &\leqq \|V^n\|^2_{M(Z^n)} + L_M|Z^n - u_0|C_0^2\\ &\leqq \max\left(\|V^0\|^2_{M(u_0)}, \ldots, \|V^\nu\|^2_{M(u_0)}\right) + C_{\mathcal{B},C_0} nh\\ &\quad + \max_{1\leqq l\leqq \nu}\left(L_M|Z^l - u_0|C_0^2\right) + L_M|Z^n - u_0|C_0^2.\end{aligned}$$

But, for all $k \in \{1, \ldots, n\}$

$$\begin{aligned}|Z^k - u_0| = \left|\frac{U^{k+1} + eU^{k-1}}{1+e} - U^0\right| &\leqq \frac{1}{1+e}\left(|U^{k+1} - U^0| + e|U^{k-1} - U^0|\right)\\ &\leqq (k+1)C_0 h_i\end{aligned} \tag{69}$$

and $Z^0 = U^0$ by convention. So

$$\begin{aligned}\left\|v_{h_i}(t)\right\|^2_{M(u_0)} &\leqq \max\left(\|V^0\|^2_{M(u_0)}, \ldots, \|V^\nu\|^2_{M(u_0)}\right) + (\nu+1)C_0^3 L_M h_i\\ &\quad + C_{\mathcal{B},C_0} t + L_M C_0^3 (t + h_i).\end{aligned}$$

By passing to the limit as h_i tends to zero, we get

$$\|v(t)\|^2_{M(u_0)} \leqq \|v_0\|^2_{M(u_0)} + C_{\mathcal{B},C_0} t + L_M C_0^3 t \quad \forall t \in (0, \tilde{\tau})$$

and, by passing to the limit as t tends to zero

$$\|\dot{u}(0+0)\|^2_{M(u_0)} = \|v(0+0)\|^2_{M(u_0)} \leqq \|v_0\|^2_{M(u_0)}.$$

With (68) it follows that

$$\|v_0\|^2_{M(u_0)} = (\dot{u}(0+0), v_0)_{M(u_0)} = \|\dot{u}(0+0)\|^2_{M(u_0)}$$

and thus $\dot{u}(0+0) = v_0$. □

Finally, with these results, we can state the following theorem.

Theorem 2. *Let us assume that there exist $C_0 > 0, \tau_0 > 0, h_0^* \in (0, h^*]$ and a subsequence of the approximate positions defined by (4)–(7) such that*

$$|V^n| = \left| \frac{U^{n+1} - U^n}{h_i} \right| \leqq C_0 \quad \forall n h_i \in [0, \min(\tau_0, T)], \quad \forall h_i \in (0, h_0^*],$$

with $(h_i)_{i\in\mathbb{N}}$ decreasing to zero. Let u_h and v_h be defined by (32) and (33). Then, there exist a subsequence, still denoted $(h_i)_{i\in\mathbb{N}}$, and $(u, v) \in C^0([0, \min(\tau_0, T)]; \mathbb{R}^d) \times BV\left(0, \min(\tau_0, T); \mathbb{R}^d\right)$ such that

$$\begin{aligned} u_{h_i} &\to u \quad \text{strongly in } C^0\left([0, \min(\tau_0, T)]; \mathbb{R}^d\right), \\ v_{h_i} &\to v \quad \text{pointwise in } [0, \min(\tau_0, T)], \end{aligned}$$

with

$$u(t) = u_0 + \int_0^t v(s)\, \mathrm{d}s \quad \text{for all } t \in [0, \min(\tau_0, T)],$$

and u is a solution of problem (P) on $[0, \min(\tau_0, T)]$.

Combining Theorem 2 with the a priori estimate of the discrete velocities obtained in Theorem 1, we immediately obtain a local convergence result for the numerical scheme, and thus a local existence result for problem (P).

4. Energy estimates and global results

As in the inelastic case, we observe now that we have some energy estimates for the solutions of problem (P).

Proposition 7. *Let $C > \|v_0\|_{M(u_0)}$. Then there exists $\tau(C) > 0$ such that, for any solution u of problem (P) defined on $[0, \tau]$ (with $\tau \in (0, T]$), we have*

$$\begin{aligned} |u(t) - u_0| &\leqq C \quad \forall t \in [0, \min(\tau(C), \tau)], \\ \|\dot{u}(t)\|_{M(u(t))} &\leqq C \quad \mathrm{d}t \text{ almost everywhere on } [0, \min(\tau(C), \tau)]. \end{aligned}$$

Proof. Since property (P3) implies that

$$\|\dot{u}(t+0)\|_{M(u(t))} \leqq \|\dot{u}(t-0)\|_{M(u(t))} \quad \forall t \in (0, \tau)$$

the proof remains the same as in [11] Proposition 6. □

As a consequence, we can prove a "global" convergence result. More precisely, for any given energy level $C > \|v_0\|_{M(u_0)}$, we will prove the convergence of the time-stepping scheme on a time interval $[0, \tau(C)]$ depending only on C and the data.

Theorem 3. *Let $C > \|v_0\|_{M(u_0)}$ and $\tau(C) > 0$ such that, for any solution u of problem (P) defined on $[0, \tau]$ (with $\tau \in (0, T]$), we have*

$$\begin{aligned} |u(t) - u_0| &\leqq C \quad \forall t \in [0, \min(\tau(C), \tau)], \\ \|\dot{u}(t)\|_{M(u(t))} &\leqq C \quad \mathrm{d}t \text{ almost everywhere on } [0, \min(\tau(C), \tau)]. \end{aligned}$$

Let u_h and v_h be the approximate positions and velocities defined by (32) and (33). Then, there exists a subsequence $(u_{h_i}, v_{h_i})_{i\in\mathbb{N}}$, $\tau \in [\min(\tau(C), T), T]$ and $(u, v) \in C^0\left([0, \tau]; \mathbb{R}^d\right) \times BV(0, \tau; \mathbb{R}^d)$ such that

$$\begin{aligned} u_{h_i} &\to u \quad \text{strongly in } C^0\left([0, \tau]; \mathbb{R}^d\right), \\ v_{h_i} &\to v \quad \text{pointwise in } [0, \tau], \end{aligned}$$

with

$$u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s \quad \forall t \in [0, \tau]$$

and u is a solution of problem (P) on $[0, \tau]$.

Proof. The proof is analogous to the Proof of Theorem 3 in [11]. More precisely, let $C > \|v_0\|_{M(u_0)}$. We define $\mathcal{B} = \overline{B}(u_0, C + 1)$ and B_0, $\lambda_{\min}$, $\lambda_{\max}$ by (11) and (12), respectively. Let C_F be defined by (13), C_0^* and C' be given by

$$C_0^* = \max\left(2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\,(|v_0| + 1)\,, C'\right) \tag{70}$$

$$C' = (C + 1)\sup\left\{\left\|M^{-1/2}(q)\right\|\,; q \in \overline{B}(u_0, C + 1)\right\} \tag{71}$$

and C_0 be chosen as in Theorem 1, that is,

$$C_0 > \left(4\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)^{\nu+1} C_0^*. \tag{72}$$

Then, with Theorem 1, we know that there exist $h_0^* \in (0, h^*]$ and $\tau_0 > 0$, depending only on $\mathcal{B}$, C_0, C_0^* and the data, such that

$$|V^n| = \left|\frac{U^{n+1} - U^n}{h}\right| \leqq C_0 \quad \forall nh \in [0, \min(\tau_0, T)]\,, \quad \forall h \in (0, h_0^*]. \tag{73}$$

Moreover, with Proposition 2, we also know that for all $t_{0h} \in [0, T)$ and for all $(\hat{U}^0, \hat{U}^1) \in (\mathcal{B} \cap K) \times K$ such that

$$\left|\hat{U}^1 - \hat{U}^0\right| \leqq hC_0^* \quad \forall h \in (0, h_0^*],$$

the approximate positions defined by

$$\hat{U}^{n+1} = -e\hat{U}^{n-1} + (1+e)\hat{Z}^n$$

with

$$\hat{Z}^n \in \mathrm{Argmin}_{Z\in K} \|\hat{W}^n - Z\|_{M(\hat{U}^n)}$$

and

$$\hat{W}^n = \frac{2\hat{U}^n - (1-e)\hat{U}^{n-1} + h^2\hat{F}^n}{1+e}, \quad \hat{F}^n = F\left(t_{0h} + nh, \hat{U}^n, \frac{\hat{U}^n - \hat{U}^{n-1}}{h}, h\right)$$

for all $n \in \left\{1, \ldots, \left\lfloor \frac{T-t_{0h}}{h} \right\rfloor\right\}$ and for all $h \in (0, h_0^*]$, satisfy

$$\left|\frac{\hat{U}^{n+1} - \hat{U}^n}{h}\right| \leqq C_0 \quad \forall nh \in [0, \min(\tau_0, T - t_{0h})]\,, \ \forall h \in (0, h_0^*].$$

Let $\tau(h) = m(h)h$ be the maximal discrete time step such that estimate (73) holds, that is, for all $h \in (0, h_0^*]$

$$m(h) = \max\left\{n \in \{0, \ldots, \lfloor T/h \rfloor\}; |V^k| \leqq C_0 \quad \forall k \in \{0, \ldots, n\}\right\}.$$

We define $\tau_1 = \liminf_{h\to 0} \tau(h) = \liminf_{h\to 0} m(h)h$. Theorem 1 implies that $\tau_1 \geqq \tau' = \min(\tau_0, T)$.

Let $\varepsilon \in \left(0, \tau'/8\right)$. Then, there exists a subsequence $(h_i)_{i\in\mathbb{N}}$, decreasing to zero, such that $(\tau(h_i))_{i\in\mathbb{N}}$ converges to τ_1 and there exists $h_\varepsilon^* \in \left(0, \min(h_0^*, \tau'/8)\right]$ such that $m(h_i)h_i \geqq \tau_1 - \varepsilon$ for all $h_i \in (0, h_\varepsilon^*]$. We may apply Theorem 2 with $h_0^* := h_\varepsilon^*$ and $\tau_0 := \tau_1 - \varepsilon$: we infer that there exists a subsequence, still denoted $(h_i)_{i\in\mathbb{N}}$, such that $(u_{h_i}, v_{h_i})_{i\in\mathbb{N}}$ converges to a solution of problem (P) on $[0, \tau_1 - \varepsilon]$. Thus, with Proposition 7 we get

$$\begin{aligned} |u(t) - u_0| &\leqq C \quad \forall t \in [0, \min\left(\tau(C), \tau_1 - \varepsilon\right)]\,, \\ \|\dot{u}(t)\|_{M(u(t))} &\leqq C \quad \mathrm{d}t \text{ almost everywhere on } [0, \min\left(\tau(C), \tau_1 - \varepsilon\right)]\,. \end{aligned}$$

Now we prove that:

Lemma 14. *We have*

$$\begin{aligned} &\lim_{h_i\to 0^+} \sup \sup \left\{\left\|V^n\right\|^2_{M(Z^n)}\,,\ 0 \leqq nh_i \leqq \tau_1 - \varepsilon\right\} \\ &\leqq \mathrm{ess\ sup}\left\{\|\dot{u}(t)\|^2_{M(u(t))}\,,\ 0 \leqq t \leqq \tau_1 - \varepsilon\right\}. \end{aligned}$$

Proof. The proof is quite similar to the proof of Lemma 6 in [11]. More precisely, we prove this result by contradiction. Assume that

$$\limsup_{h_i \to 0^+} \sup \left\{ \left\| V^n \right\|^2_{M(Z^n)},\ 0 \leqq nh_i \leqq \tau_1 - \varepsilon \right\} > S$$

with $S = \text{ess sup} \left\{ \|\dot{u}(t)\|^2_{M(u(t))},\ 0 \leqq t \leq \tau_1 - \varepsilon \right\}$. Then, there exist $\gamma > 0$, $\tilde{h}^*_\varepsilon \in (0, h^*_\varepsilon]$ and a subsequence $(h_{\varphi(i)})_{i\in\mathbb{N}}$ decreasing to zero such that

$$\sup \left\{ \left\| V^n \right\|^2_{M(Z^n)},\ 0 \leqq nh_{\varphi(i)} \leqq \tau_1 - \varepsilon \right\} \geqq S + \gamma \quad \forall h_{\varphi(i)} \in (0, \tilde{h}^*_\varepsilon].$$

It follows that there exists $n_{\varphi(i)} \in \left\{0, \ldots, \lfloor (\tau_1 - \varepsilon)/h_{\varphi(i)} \rfloor\right\}$ such that

$$\left\| V^{n_{\varphi(i)}} \right\|^2_{M(Z^{n_{\varphi(i)}})} \geqq S + \gamma \quad \forall h_{\varphi(i)} \in (0, \tilde{h}^*_\varepsilon].$$

Possibly extracting another subsequence, still denoted $(h_{\varphi(i)})_{i\in\mathbb{N}}$, we may assume without loss of generality that the sequence $(n_{\varphi(i)}h_{\varphi(i)})_{i\in\mathbb{N}}$ converges to a limit $\tilde{\tau} \in [0, \tau_1 - \varepsilon]$.

First, we observe that $\tilde{\tau} > 0$. Indeed, with the same computations as in Proposition 3, we can prove that there exists a constant $C_{B_1,C_0} > 0$, depending only on B_1 and C_0, such that, for all $h_i \in (0, h^*_\varepsilon]$

$$\|V^n\|^2_{M(Z^n)} \leqq \max\left(\|V^{n-1}\|^2_{M(Z^{n-1})}, \ldots, \|V^{n-\nu-1}\|^2_{M(Z^{n-1-\nu})} \right) + C_{B_1,C_0} h_i \tag{74}$$

for all $n \in \{\nu + 1, \ldots, \lfloor (\tau_1 - \varepsilon)/h_i \rfloor\}$, thus

$$\|V^n\|^2_{M(Z^n)} \leqq \max\left(\|V^0\|^2_{M(Z^0)}, \ldots, \|V^\nu\|^2_{M(Z^\nu)} \right) + C_{B_1,C_0} nh_i$$

for all $nh_i \in [0, \tau_1 - \varepsilon]$.

With the Lipschitz property of M on B_1 and (69), we infer that, for all $h_{\varphi(i)} \in (0, \tilde{h}^*_\varepsilon]$

$$\begin{aligned} S + \gamma &\leqq \|V^{n_{\varphi(i)}}\|^2_{M(Z^{n_{\varphi(i)}})} \\ &\leqq \max\left(\|V^0\|^2_{M(u_0)}, \ldots, \|V^\nu\|^2_{M(u_0)} \right) \\ &\quad + C_{B_1,C_0} n_{\varphi(i)} h_{\varphi(i)} + (\nu + 1) C_0^3 L_M h_{\varphi(i)}. \end{aligned}$$

At the limit when i tends to $+\infty$, we get with Lemma 12

$$S + \gamma \leqq \|v_0\|^2_{M(u_0)} + C_{B_1,C_0} \tilde{\tau}.$$

But the right continuity of $\|\dot{u}(\cdot + 0)\|_{M(u)}$ implies that $\|v_0\|^2_{M(u_0)} = \|\dot{u}(0 + 0)\|^2_{M(u_0)} \leqq S$ (see Lemma 13), and finally $\tilde{\tau} > 0$.

We infer that, possibly reducing $\tilde{h}^*_\varepsilon$, $n_{\varphi(i)}h_{\varphi(i)} \geqq \tilde{\tau}/2 \geqq 2\nu h_{\varphi(i)}$ for all $h_{\varphi(i)} \in (0, \tilde{h}^*_\varepsilon]$. Then, once again using estimate (74), we obtain

$$\begin{aligned} S + \gamma \leqq \max&\left(\|V^{n_{\varphi(i)}-p}\|^2_{M(Z^{n_{\varphi(i)}-p})}, \ldots, \|V^{n_{\varphi(i)}-p-\nu}\|^2_{M(Z^{n_{\varphi(i)}-p-\nu})} \right) \\ &+ C_{B_1,C_0} p h_{\varphi(i)} \end{aligned}$$

for all $p \in \{0, \dots, n_{\varphi(i)} - \nu\}$, for all $h_{\varphi(i)} \in (0, \tilde{h}^*_\varepsilon]$. It follows that

$$S + \frac{\gamma}{2} \leqq \max\left(\|V^k\|^2_{M(Z^k)}, \dots, \|V^{k-\nu}\|^2_{M(Z^{k-\nu})}\right)$$

for all $kh_{\varphi(i)} \in \left[\max\left(n_{\varphi(i)}h_{\varphi(i)} - \frac{\gamma}{2C_{B_1,C_0}}, \nu h_{\varphi(i)}\right), n_{\varphi(i)}h_{\varphi(i)}\right]$. Moreover, using (20), we infer that for all $t \in \left[kh_{\varphi(i)}, (k+1)h_{\varphi(i)}\right)$, for all $l \in \{0, \dots, \nu\}$

$$\begin{aligned}
&\left|\|v_{h_{\varphi(i)}}(t - lh_{\varphi(i)})\|^2_{M(Z^{k-l})} - \|v_{h_{\varphi(i)}}(t - lh_{\varphi(i)})\|^2_{M\left(u_{h_{\varphi(i)}}(t)\right)}\right| \\
&\leqq \left\|M(Z^{k-l}) - M\left(u_{h_{\varphi(i)}}(t)\right)\right\| |V^{k-l}|^2 \leqq L_M \left|Z^{k-l} - u_{h_{\varphi(i)}}(t)\right| C_0^2 \\
&\leqq L_M C_0^2 \left(\left|Z^{k-l} - U^{k-l}\right| + \left|u_{h_{\varphi(i)}}\left((k-l)h_{\varphi(i)}\right) - u_{h_{\varphi(i)}}(t)\right|\right) \\
&\leqq L_M C_0^3 (\nu + 2) h_{\varphi(i)}.
\end{aligned}$$

Since $\left(n_{\varphi(i)}h_{\varphi(i)}\right)_{i\in\mathbb{N}}$ converges to $\tilde{\tau} > 0$, there exists an interval $I \subset [0, \tau_1 - \varepsilon]$, with a non-empty interior, such that

$$I \subset \left[\max\left((n_{\varphi(i)} + 1)h_{\varphi(i)} - \frac{\gamma}{2C_{B_1,C_0}}, \nu h_{\varphi(i)}\right), n_{\varphi(i)}h_{\varphi(i)}\right]$$

and

$$S + \frac{\gamma}{4} \leqq \max\left(\|v_{h_{\varphi(i)}}(t)\|^2_{M(u_{h_{\varphi(i)}}(t))}, \dots, \|v_{h_{\varphi(i)}}(t - \nu h_{\varphi(i)})\|^2_{M(u_{h_{\varphi(i)}}(t))}\right) \quad (75)$$

for all $t \in I$. But the sequence $(v_{h_i})_{i\in\mathbb{N}}$ converges pointwise to v on $[0, \tau_1 - \varepsilon]$ and $v = \dot{u}$ almost everywhere on $[0, \tau_1 - \varepsilon]$. Thus, Lebesgue's theorem implies that $(v_{h_i})_{i\in\mathbb{N}}$ converges strongly to $\dot{u}$ in $L^1(0, \tau_1 - \varepsilon; \mathbb{R}^d)$. We may extend v_{h_i} $(i \in \mathbb{N})$ and $\dot{u}$ by zero outside $[0, \tau_1 - \varepsilon]$ and we denote $\tilde{v}_{h_i}$ and $\tilde{\dot{u}}$ the extended functions. Of course $\|\tilde{v}_{h_i} - \tilde{\dot{u}}\|_{L^1(\mathbb{R},\mathbb{R}^d)} = \|v_{h_i} - \dot{u}\|_{L^1(0,\tau_1-\varepsilon;\mathbb{R}^d)}$ and $(\tilde{v}_{h_i})_{i\in\mathbb{N}}$ converges strongly to $\tilde{\dot{u}}$ in $L^1(\mathbb{R}, \mathbb{R}^d)$. The compactness criteria in $L^1(\mathbb{R}, \mathbb{R}^d)$ (see [3]) yields that

$$\lim_{\theta\to 0} \sup_{i\in\mathbb{N}} \int_{\mathbb{R}} \left|\tilde{v}_{h_i}(t - \theta) - \tilde{v}_{h_i}(t)\right| \, \mathrm{d}t = 0.$$

Thus, $\left(\tilde{v}_{h_i}(\cdot - lh_i)\right)_{i\in\mathbb{N}}$ converges to $\tilde{\dot{u}}$ strongly in $L^1(\mathbb{R}, \mathbb{R}^d)$ for all $l \in \{0, \dots, \nu\}$. It follows that $\left(\tilde{v}_{h_{\varphi(i)}}(\cdot - lh_{\varphi(i)})\right)_{i\in\mathbb{N}}$ also converges to $\tilde{\dot{u}}$ strongly in $L^1(\mathbb{R}, \mathbb{R}^d)$ for all $l \in \{0, \dots, \nu\}$ and, possibly extracting another subsequence, still denoted $(h_{\varphi(i)})_{i\in\mathbb{N}}$, we have for all $l \in \{0, \dots, \nu\}$

$$\tilde{v}_{h_{\varphi(i)}}(t - lh_{\varphi(i)}) \to \tilde{\dot{u}}(t) \quad \text{almost everywhere in } \mathbb{R}.$$

By passing to the limit in (75), we get

$$S + \frac{\gamma}{4} \leqq \|\dot{u}(t)\|^2_{M(u(t))} \quad \text{almost everywhere in } I.$$

Recalling that I is an interval included in $[0, \tau_1 - \varepsilon]$ with a non-empty interior, we get

$$S \geqq \text{ess sup} \left\{ \left\| \dot{u}(t) \right\|^2_{M(u(t))}; t \in I \right\}$$

which yields a contradiction. □

With the previous lemma, possibly decreasing h^*_ε we get

$$U^n \in \overline{B}(u_0, C+1), \quad \|V^n\|_{M(Z^n)} \leqq C+1 \quad \forall n h_i \in [0, \min(\tau_1 - \varepsilon, \tau(C))], \quad \forall h_i \in (0, h^*_\varepsilon].$$

Let us now distinguish two subcases.
Case 1: $\tau(C) < T$.

Let us prove that $\tau_1 > \tau(C) = \min(\tau(C), T)$ by contradiction. Hence, assume that $\tau_1 \leqq \tau(C)$. It follows that $\tau' = \min(\tau_0, T) \leqq \tau_1 \leqq \tau(C) < T$ and $\tau' = \tau_0$. Furthermore, recalling that $Z^n = \frac{U^{n+1} + eU^{n-1}}{1+e}$ if $n \geqq 1$ and $Z^0 = U^0$, we infer that $Z^n \in \overline{B}(u_0, C+1)$ for all $nh_i \in [0, \tau_1 - \varepsilon - h_1]$ and

$$|V^n| \leqq (C+1) \sup \left\{ \left\| M^{-1/2}(q) \right\| ; q \in \overline{B}(u_0, C+1) \right\} = C' \leqq C^*_0 < C_0$$

for all $nh_i \in [0, \tau_1 - \varepsilon - h_i]$, for all $h_i \in (0, h^*_\varepsilon]$, where C', C^*_0 and C_0 are given by (70), (71) and (72). We now choose $l(h_i) \in \{0, \ldots, \lfloor T/h_i \rfloor\}$ such that

$$l(h_i) h_i \in \left[\tau_1 - \frac{\tau'}{2}, \tau_1 - \frac{\tau'}{4} \right] \quad \forall h_i \in (0, h^*_\varepsilon]$$

and let

$$\hat{U}^0 = U^{l(h_i)}, \quad \hat{U}^1 = U^{l(h_i)+1}, \quad t_{0h_i} = l(h_i) h_i.$$

We have

$$|\hat{U}^1 - \hat{U}^0| = h_i \left| V^{l(h_i)} \right| \leqq C^*_0 h_i$$

and $\hat{U}^0$, $\hat{U}^1$ belong to $\mathcal{B} \cap K$, for all $h_i \in (0, h^*_\varepsilon]$.

Then, for all $n \in \left\{ l(h_i), \ldots, \left\lfloor \frac{T}{h_i} \right\rfloor \right\}$, $U^n = \hat{U}^{n - l(h_i)}$ and, with Proposition 2, we obtain

$$|V^n| = \left| \frac{\hat{U}^{n-l(h_i)+1} - \hat{U}^{n-l(h_i)}}{h_i} \right| \leqq C_0$$

for all $(n - l(h_i))\, h_i \in [0, \min(\tau_0, T - l(h_i) h_i)]$, for all $h_i \in (0, h^*_\varepsilon]$. Thus

$$m(h_i) h_i > l(h_i) h_i + \min(\tau_0, T - l(h_i) h_i) - h_i \geqq \min\left(\tau_1 + \frac{\tau'}{2}, T \right) - h_i.$$

It follows that $\lim_{h_i \to 0} m(h_i) h_i = \tau_1 \geqq \min\left(\tau_1 + \frac{\tau'}{2}, T \right)$ which gives a contradiction.

Thus $\tau_1 > \tau(C)$, and there exists $\tilde{h}_0^* \in (0, h_0^*]$ such that

$$m(h_i)h_i = \tau(h_i) \geqq \tau_1 - \frac{\tau_1 - \tau(C)}{2} = \frac{\tau_1 + \tau(C)}{2} > \tau(C) \quad \forall h_i \in (0, \tilde{h}_0^*].$$

Then, we apply Theorem 2 with τ_0 replaced by $\tau = \frac{\tau_1+\tau(C)}{2}$ and h_0^* by $\tilde{h}_0^*$, which yields the announced result.

Case 2: $\tau(C) \geqq T$.

Since $m(h)h = \tau(h) \leqq T$, we have $\tau_1 \leqq T$. We choose as previously $l(h_i) \in \{0, \ldots, \lfloor T/h_i \rfloor\}$ such that

$$l(h_i)h_i \in \left[\tau_1 - \frac{\tau'}{2}, \tau_1 - \frac{\tau'}{4}\right] \quad \forall h_i \in (0, h_\varepsilon^*].$$

Then, we again have

$$m(h_i)h_i > l(h_i)h_i + \min(\tau_0, T - l(h_i)h_i) - h_i \geqq \min\left(\tau_1 + \frac{\tau'}{2}, T\right) - h_i \quad (76)$$

for all $h_i \in (0, h_\varepsilon^*]$. Thus, if $\tau_1 < T$, $\min\left(\tau_1 + \frac{\tau'}{2}, T\right) \in (\tau_1, T]$ and (76) yields a contradiction with the definition of $\tau_1 = \lim_{h_i \to 0} m(h_i)h_i$. We infer that $\tau_1 = T$ and (76) implies that $m(h_i)h_i > T - h_i$, that is, $m(h_i) \geqq \lfloor T/h_i \rfloor$ for all $h_i \in (0, h_\varepsilon^*]$. Hence we may apply Theorem 2 to obtain the convergence of a subsequence of $(u_{h_i}, v_{h_i})_{i\in\mathbb{N}}$, still denoted $(u_{h_i}, v_{h_i})_{i\in\mathbb{N}}$, to a solution of problem (P) on $[0, T]$. □

Appendix

Lemma 15. *For all compact subsets $\mathcal{B}$ of $\mathbb{R}^d$, there exist $m_\mathcal{B} > 0$, $C'_\mathcal{B} > 0$ and $r'_\mathcal{B} > 0$ such that for all $q \in K \cap \mathcal{B}$ and for all $\alpha \in J(q)$ we have*

$$|M^{-1/2}(q')\nabla f_\alpha(q')| \geqq m_\mathcal{B} \quad \forall q' \in \overline{B}(q, r'_\mathcal{B})$$

and the mappings

$$q' \mapsto \frac{M^{-1/2}(q')\nabla f_\alpha(q')}{|M^{-1/2}(q')\nabla f_\alpha(q')|}$$

are defined and $C'_\mathcal{B}$-Lipschitz continuous on $\overline{B}(q, r'_\mathcal{B})$.

Proof. Let $\mathcal{B}$ be a given compact subset of $\mathbb{R}^d$. With the same arguments as in [11] Lemma 7, we first obtain that there exist $m_\mathcal{B} > 0$ and $r'_\mathcal{B} > 0$ such that for all $(q, q') \in (K \cap \mathcal{B}) \times \mathbb{R}^d$ such that $|q - q'| \leqq r'_\mathcal{B}$, and for all $\alpha \in J(q)$ we have

$$|M^{-1/2}(q')\nabla f_\alpha(q')| \geqq m_\mathcal{B}.$$

Hence, for all $q \in K \cap \mathcal{B}$, the mappings

$$q' \mapsto \frac{M^{-1/2}(q')\nabla f_\alpha(q')}{|M^{-1/2}(q')\nabla f_\alpha(q')|}, \quad \alpha \in J(q)$$

are defined on $\overline{B}(q, r'_\mathcal{B})$.

Now let us prove that there exists $C'_{\mathcal{B}} > 0$ such that, for all $(q', q'') \in \overline{B}\left(q, r'_{\mathcal{B}}\right)^2$, and for all $\alpha \in J(q)$, we have

$$\left| \frac{M^{-1/2}(q')\nabla f_\alpha(q')}{|M^{-1/2}(q')\nabla f_\alpha(q')|} - \frac{M^{-1/2}(q'')\nabla f_\alpha(q'')}{|M^{-1/2}(q'')\nabla f_\alpha(q'')|} \right| \leqq C'_{\mathcal{B}}|q' - q''|.$$

We define $\tilde{\mathcal{B}} = \left\{q \in \mathbb{R}^d;\ \operatorname{dist}(q, \mathcal{B}) \leqq r'_{\mathcal{B}}\right\}$. The set $\tilde{\mathcal{B}}$ is also compact. Moreover, from assumptions (H2) and (H4) we know that the mappings ∇f_α $(1 \leqq \alpha \leqq \nu)$ and $M^{-1/2}$ are locally Lipschitz continuous on $\mathbb{R}^d$. Thus, for all $\alpha \in \{1, \ldots, \nu\}$ there exits $C_\alpha > 0$ such that

$$\left|M^{-1/2}(q')\nabla f_\alpha(q') - M^{-1/2}(q'')\nabla f_\alpha(q'')\right| \leqq C_\alpha|q' - q''| \quad \forall (q', q'') \in \tilde{\mathcal{B}}^2.$$

Let $q \in \mathcal{B} \cap K$ and $(q', q'') \in \overline{B}(q, r'_{\mathcal{B}})^2$. Then

$$\begin{aligned}
&\left| \frac{M^{-1/2}(q')\nabla f_\alpha(q')}{\left|M^{-1/2}(q')\nabla f_\alpha(q')\right|} - \frac{M^{-1/2}(q'')\nabla f_\alpha(q'')}{\left|M^{-1/2}(q'')\nabla f_\alpha(q'')\right|} \right| \\
&\quad \leqq \left| \frac{M^{-1/2}(q')\nabla f_\alpha(q') - M^{-1/2}(q'')\nabla f_\alpha(q'')}{\left|M^{-1/2}(q')\nabla f_\alpha(q')\right|} \right. \\
&\qquad \left. + \left(\frac{M^{-1/2}(q'')\nabla f_\alpha(q'')}{\left|M^{-1/2}(q')\nabla f_\alpha(q')\right|} - \frac{M^{-1/2}(q'')\nabla f_\alpha(q'')}{\left|M^{-1/2}(q'')\nabla f_\alpha(q'')\right|} \right) \right| \\
&\quad \leqq \frac{\left|M^{-1/2}(q')\nabla f_\alpha(q') - M^{-1/2}(q'')\nabla f_\alpha(q'')\right|}{\left|M^{-1/2}(q')\nabla f_\alpha(q')\right|} \\
&\qquad + \left|M^{-1/2}(q'')\nabla f_\alpha(q'')\right| \left| \frac{\left|M^{-1/2}(q'')\nabla f_\alpha(q'')\right| - \left|M^{-1/2}(q')\nabla f_\alpha(q')\right|}{\left|M^{-1/2}(q')\nabla f_\alpha(q')\right| \left|M^{-1/2}(q'')\nabla f_\alpha(q'')\right|} \right| \\
&\quad \leqq \frac{C_\alpha}{m_{\mathcal{B}}}|q' - q''| + \frac{C_\alpha}{m_{\mathcal{B}}}|q' - q''|.
\end{aligned}$$

By defining

$$C'_{\mathcal{B}} = \max_{1 \leqq \alpha \leqq \nu} \frac{2C_\alpha}{m_{\mathcal{B}}}$$

we may conclude. □

Lemma 16. *Let us recall the definition of $T_K(q)$:*

$$T_K(q) = \left\{v \in \mathbb{R}^d;\ (\nabla f_\alpha(q), v) \geqq 0\ \forall \alpha \in J(q)\right\} \quad \forall q \in \mathbb{R}^d$$

with

$$J(q) = \left\{\alpha \in \{1, \ldots, \nu\};\ f_\alpha(q) \leqq 0\right\}.$$

Then, for all $q_0 \in K$, there exist $\delta > 0$, $r > 0$ and $a \in \mathbb{R}^d$ such that, for all $q \in \overline{B}(q_0, 2\delta)$:

$$\overline{B}(a, r) \subset T_K(q).$$

Proof. See Lemma 8 in [11]. □

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