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# MULTIFRACTAL ANALYSIS OF THE BIRKHOFF SUMS OF SAINT-PETERSBURG POTENTIAL

DONG HAN KIM, LINGMIN LIAO, MICHAŁ RAMS, AND BAO-WEI WANG

ABSTRACT. Let  $((0, 1], T)$  be the doubling map in the unit interval and  $\varphi$  be the Saint-Petersburg potential, defined by  $\varphi(x) = 2^n$  if  $x \in (2^{-n-1}, 2^{-n}]$  for all  $n \geq 0$ . We consider asymptotic properties of the Birkhoff sum  $S_n(x) = \varphi(x) + \dots + \varphi(T^{n-1}(x))$ . With respect to the Lebesgue measure, the Saint-Petersburg potential is not integrable and it is known that  $\frac{1}{n \log n} S_n(x)$  converges to  $\frac{1}{\log 2}$  in probability. We determine the Hausdorff dimension of the level set  $\{x : \lim_{n \rightarrow \infty} S_n(x)/n = \alpha\}$  ( $\alpha > 0$ ), as well as that of the set  $\{x : \lim_{n \rightarrow \infty} S_n(x)/\Psi(n) = \alpha\}$  ( $\alpha > 0$ ), when  $\Psi(n) = n \log n$ ,  $n^a$  or  $2^{n^\gamma}$  for  $a > 1$ ,  $\gamma > 0$ . The fast increasing Birkhoff sum of the potential function  $x \mapsto 1/x$  is also studied.

## 1. INTRODUCTION

Let  $T$  be the doubling map on the unit interval  $(0, 1]$  defined by

$$Tx = 2x - [2x] + 1,$$

where  $[x]$  is the smallest integer larger than or equal to  $x$ . Let  $\epsilon_1$  be the function defined by  $\epsilon_1(x) = [2x] - 1$  and  $\epsilon_n(x) := \epsilon_1(T^{n-1}x)$  for  $n \geq 2$ . Then each real number  $x \in (0, 1]$  can be expanded into an infinite series as

$$x = \frac{\epsilon_1(x)}{2} + \dots + \frac{\epsilon_n(x)}{2^n} + \dots \quad (1.1)$$

We call (1.1) the binary expansion of  $x$  and also write it as

$$x = [\epsilon_1(x)\epsilon_2(x)\dots].$$

The Saint-Petersburg potential is a function  $\varphi : (0, 1] \rightarrow \mathbb{R}$  defined as

$$\varphi(x) = 2^n \text{ if } x \in (2^{-n-1}, 2^{-n}], \forall n \geq 0.$$

We remark that the definition of  $\varphi$  is equivalent to

$$\varphi(x) = 2^n \text{ where } n \geq 0 \text{ is the smallest integer such that } \epsilon_{n+1}(x) = 1.$$

and is also equivalent to

$$\varphi(x) = 2^n \text{ if the binary expansion of } x \text{ begins with } 0^n 1,$$

where  $0^n$  ( $n \geq 0$ ) means a block with  $n$  consecutive zeros.

The name of Saint-Petersburg potential is motivated by the famous Saint-Petersburg game in probability theory. The Saint-Petersburg potential is of infinite expectation with respect to the Lebesgue measure. Furthermore, it increases exponentially fast near to the point 0.

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In this paper, we are concerned with the following Birkhoff sums of the Saint-Petersburg potential:

$$\forall n \geq 1, \quad S_n(x) := \varphi(x) + \varphi(T(x)) + \cdots + \varphi(T^{n-1}(x)), \quad x \in (0, 1].$$

Let

$$I = \{x \in (0, 1] : \epsilon_1(x) = 1\}.$$

Define the hitting time of  $x \in (0, 1]$  to  $I$  as

$$n(x) := \inf\{n \geq 0 : T^n x \in I\}.$$

Then

$$n(x) = n \quad \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \quad \text{for all } n \geq 0.$$

Using  $n(x)$ , we define a new dynamical system  $\widehat{T} : (0, 1] \rightarrow (0, 1]$  by

$$\widehat{T}(x) = T^{n(x)+1}(x) = 2^{n+1}\left(x - \frac{1}{2^{n+1}}\right) \quad \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \quad \text{for all } n \geq 0,$$

called the acceleration of  $T$ , in order that  $\varphi$  and  $\varphi \circ \widehat{T}$  are independent. Let

$$\widehat{S}_n(x) := \varphi(x) + \varphi(\widehat{T}(x)) + \cdots + \varphi(\widehat{T}^{n-1}(x)), \quad x \in (0, 1].$$

The convergence in probability of  $\widehat{S}_n(x)$  is well known (e.g. [6, p.253]) which states that for any  $\epsilon > 0$ , the Lebesgue measure  $\lambda$  of

$$\left\{x \in (0, 1] : \left|\frac{\widehat{S}_n(x)}{n \log n} - \frac{1}{\log 2}\right| \geq \epsilon\right\}$$

tends to 0 as  $n \rightarrow \infty$ .

Let  $\{\Psi_n\}_{n \geq 1}$  be an increasing sequence such that  $\Psi_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then it was shown in [5] that Lebesgue almost surely either

$$\lim_{n \rightarrow \infty} \frac{\widehat{S}_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n \rightarrow \infty} \frac{\widehat{S}_n(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n \geq 1} \lambda(\{x \in (0, 1] : \varphi(x) \geq \Psi_n\}) < \infty \quad \text{or} \quad = \infty.$$

Let  $n_1 = n_1(x) = n(x) + 1$  and  $n_k = n_k(x) = n_1(\widehat{T}^{k-1}x) = n(\widehat{T}^{k-1}x)$  for  $k \geq 2$ . It is direct to see that

$$\forall \ell \geq 1, \quad S_{n_1 + \cdots + n_\ell}(x) = 2\widehat{S}_\ell(x) - \ell.$$

Moreover, the ergodicity of  $T$  (of  $\widehat{T}$ ) implies Lebesgue almost surely

$$\lim_{\ell \rightarrow \infty} \frac{n_1(x) + \cdots + n_\ell(x)}{\ell} = \int_0^1 (n(x) + 1) d\lambda(x) = 2.$$

Combining these two facts together, we obtain the same convergence results as above if we replace  $\widehat{S}_n$  by  $S_n$ . In particular, the average  $S_n(x)/(n \log n)$  converges to  $1/\log 2$  in probability, and almost surely (with respect to the Lebesgue measure) either

$$\lim_{n \rightarrow \infty} \frac{S_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n \rightarrow \infty} \frac{S_n(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n \geq 1} \lambda(\{x \in (0, 1] : \varphi(x) \geq \Psi_n\}) < \infty \quad \text{or} \quad = \infty,$$

where  $\{\Psi_n\}_{n \geq 1}$  is an increasing sequence such that  $\Psi_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Recall that  $\varphi$  has infinite expectation with respect to the Lebesgue measure. We thus have  $S_n(x)/n$  converges to infinity for Lebesgue almost all points.

In this article, we want to further study the asymptotic behavior of the Birkhoff sum  $S_n(x)$  of the Saint-Petersburg potential. We give a complete multifractal analysis of  $S_n(x)$ .

First, for any  $\alpha \geq 1$ , we consider the level set

$$E(\alpha) = \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} S_n(x) = \alpha \right\}.$$

For  $t \in \mathbb{R}$  and  $q > 0$ , define

$$P(t, q) := \log \sum_{j=1}^{\infty} 2^{-tj - q(2^j - 1)}.$$

Then  $P$  is a real-analytic function. Furthermore, for each  $q > 0$ , there is a unique  $t(q) > 0$  such that  $P(t(q), q) = 0$ . This function  $q \mapsto t(q)$  is real-analytic, strictly decreasing and convex.

Denote by  $\dim_H$  the Hausdorff dimension. The function  $\alpha \mapsto \dim_H E(\alpha)$ , called the Birkhoff spectrum of the Saint-Petersburg potential  $\varphi$ , is proved to be the Legendre transformation of the function  $q \mapsto t(q)$ .

**Theorem 1.1.** *For any  $\alpha \geq 1$  we have*

$$\dim_H E(\alpha) = \inf_{q > 0} \{t(q) + q\alpha\}.$$

*Consequently,  $\dim_H E(1) = 0$  and the function  $\alpha \mapsto \dim_H E(\alpha)$  is real-analytic, strictly increasing, concave, and has limit 1 as  $\alpha \rightarrow \infty$ .*

The Birkhoff spectrum of a continuous potential was obtained for full shifts [14], for topologically mixing subshifts of finite type [4], and for repellers of a topologically mixing  $C^{1+\epsilon}$  expanding map [2]. A continuous potential in a compact space is bounded, hence these classical results are all for bounded potentials. Our Theorem 1.1 gives a Birkhoff spectrum for an unbounded function with a singular point. To prove Theorem 1.1, we will transfer our question to a Birkhoff spectrum problem of an interval map with infinitely many branches and we will apply the techniques developed in [9] for continued fraction dynamical system and in [8] for general expanding interval maps with infinitely many branches.

We also study the Birkhoff sums  $S_n(x)$  of fast increasing rates. Let  $\Psi : \mathbb{N} \rightarrow \mathbb{N}$  be an increasing function. For  $\beta \in [0, \infty]$ , consider the level set

$$E_{\Psi}(\beta) := \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{\Psi(n)} S_n(x) = \beta \right\}.$$

**Theorem 1.2.** *If  $\Psi(n)$  is one of the following*

$$\Psi(n) = n \log n, \quad \Psi(n) = n^a \quad (a > 1), \quad \Psi(n) = 2^{n^\gamma} \quad (0 < \gamma < 1/2),$$

*then for any  $\beta \in [0, \infty]$ ,  $\dim_H E_{\Psi}(\beta) = 1$ .*

*If  $\Psi(n) = 2^{n^\gamma}$  with  $1/2 \leq \gamma < 1$ , then for any  $\beta \in (0, \infty)$ , the set  $E_{\Psi}(\beta)$  is empty, and  $\dim_H E_{\Psi}(\beta) = 1$  for  $\beta = 0, +\infty$ .*

*If  $\Psi(n) = 2^{n^\gamma}$  with  $\gamma \geq 1$ , then for any  $\beta \in (0, \infty]$ , the set  $E_{\Psi}(\beta)$  is empty, and  $\dim_H E_{\Psi}(\beta) = 1$  for  $\beta = 0$ .*

We remark that by the above discussion on the convergence of  $S_n(x)$ , for all cases in Theorem 1.2, the sets  $E_{\Psi}(0)$  has full measure, and thus obviously has full Hausdorff dimension.

From the definition of  $S_n(x)$ , we see that for the integer  $n$  such that  $\epsilon_n(x) = 1$ , one has  $S_n(x) = S_{n-1}(x) + 1$ , which implies for all  $x$ ,  $\liminf_{n \rightarrow \infty} \frac{S_n(x)}{S_{n-1}(x)} = 1$ . Thus if  $\liminf_{n \rightarrow \infty} \frac{\Psi(n)}{\Psi(n-1)} > 1$ , then for any  $\beta \in (0, \infty)$ , the set  $E_\Psi(\beta)$  is empty. By the definition of  $S_n(x)$ , we can also check that for all  $x$ , for the integer  $n$  such that  $\epsilon_n(x) = 1$ , we have  $S_n(x) \leq 2^n - 1$ . This implies  $\liminf_{n \rightarrow \infty} S_n(x)/2^n \leq 1$  (See also the formula (3.15) in Section 3). Hence, for 'regular' growth functions  $\Psi$  we only need to consider exponential and subexponential growth rates.

However, if we pick a point  $x$  with dyadic expansion consisting mostly of 0's, with infinitely many 1's but in large distances from each other, then the Birkhoff sum  $S_{n_i}(x)$  may grow arbitrarily fast on some subsequence  $n_i$ . Thus for any increasing  $\Psi(n)$  there exists a point  $x$  such that  $\limsup_{n \rightarrow \infty} S_n(x)/\Psi(n) = \infty$ .

Our study on the Saint-Petersburg potential is an attempt of multifractal analysis of unbounded potential functions on the doubling map dynamical system. However, the Saint-Petersburg potential is locally constant and not continuous. One might think of an another unbounded potential function  $g : x \mapsto 1/x$  which is close to the Saint-Petersburg potential but is continuous. In fact, our method for studying the fast increasing Birkhoff sum of Saint-Petersburg potential also works for the fast increasing Birkhoff sum of the potential  $g : x \mapsto 1/x$ .

Denote by  $S_n g(x)$  the Birkhoff sum

$$S_n g(x) := g(x) + g(T(x)) + \cdots + g(T^{n-1}(x)), \quad x \in (0, 1].$$

For  $\beta \in [0, \infty]$ , let

$$F_\Psi(\beta) := \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{\Psi(n)} S_n g(x) = \beta \right\}.$$

**Theorem 1.3.** *If  $\Psi(n)$  is one of the following*

$$\Psi(n) = n \log n, \quad \Psi(n) = n^a \quad (a > 1), \quad \Psi(n) = 2^{n^\gamma} \quad (0 < \gamma < 1/2),$$

then for any  $\beta \in [0, \infty]$ ,  $\dim_H F_\Psi(\beta) = 1$ .

If  $\Psi(n) = 2^{n^\gamma}$  with  $1/2 \leq \gamma < 1$ , then for any  $\beta \in (0, \infty)$ , the set  $F_\Psi(\beta)$  is empty, and  $\dim_H F_\Psi(\beta) = 1$  for  $\beta = 0, +\infty$ .

If  $\Psi(n) = 2^{n^\gamma}$  with  $\gamma \geq 1$ , then for any  $\beta \in (0, \infty]$ , the set  $F_\Psi(\beta)$  is empty, and  $\dim_H F_\Psi(\beta) = 1$  for  $\beta = 0$ .

We remark that these multifractal analysis on the Birkhoff sums of fast increasing rates have been done for some special potentials in continued fraction dynamical system ([9, 11, 12]).

## 2. BIRKHOFF SPECTRUM OF THE SAINT-PETERSBURG POTENTIAL

In this section, we will obtain the Birkhoff spectrum of the Saint-Petersburg potential, i.e. the Hausdorff dimension of the following level set:

$$E(\alpha) := \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad (\alpha \geq 1).$$

We will transfer our question to a Birkhoff spectrum problem for an interval map with infinitely many branches.

**2.1. Transference lemma.** Recall that the Saint-Petersburg potential  $\varphi$  is given by

$$\varphi(x) = 2^n, \text{ if } x = [0^n 1, \dots]$$

where  $x = [\epsilon_1 \epsilon_2, \dots]$  denotes the digit sequence in the binary expansion of  $x$ . Recall also the definition of hitting time  $n(x)$  and the acceleration  $\widehat{T}$  of the doubling map  $T$  in Section 1. Define a new potential function

$$\phi(x) := 2^{n(x)+1} - 1, \quad x \in (0, 1].$$

In fact,  $\phi$  is nothing but the function satisfying

$$\phi(x) = \sum_{j=0}^{n(x)} \varphi(T^j x).$$

With the notation  $n_1 = n(x) + 1 \geq 1$ , and  $n_k = n(\widehat{T}^{k-1}x) + 1$  for  $k \geq 2$  given in Section 1, we have

$$\phi(\widehat{T}x) = \sum_{j=0}^{n(\widehat{T}x)} \varphi(T^j(\widehat{T}x)) = \sum_{j=n_1}^{n_2-1} \varphi(T^j x).$$

Hence,

$$\sum_{j=0}^{n_1+\dots+n_\ell-1} \varphi(T^j x) = \sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x) = 2^{n_1} + \dots + 2^{n_\ell} - \ell. \quad (2.1)$$

Note that the derivative of  $\widehat{T}$  satisfies

$$|\widehat{T}'|(x) = 2^{n(x)+1} = 2^{n_1} = \phi(x) + 1. \quad (2.2)$$

We have

$$n_1 + \dots + n_\ell = \sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x).$$

Recall the set in question:

$$E(\alpha) = \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad (\alpha \geq 1).$$

Define

$$\widetilde{E}(\alpha) := \left\{ x \in (0, 1] : \lim_{\ell \rightarrow \infty} \frac{\sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x)} = \alpha \right\} \quad (\alpha \geq 1).$$

The following lemma shows the two level sets are the same.

**Lemma 2.1.** *For all  $\alpha \geq 1$ , we have  $E(\alpha) = \widetilde{E}(\alpha)$ .*

*Proof.* It is evident that  $E(\alpha) \subset \widetilde{E}(\alpha)$ , because, as discussed above,

$$\frac{\sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x)} = \frac{1}{n_1 + \dots + n_\ell} \sum_{j=0}^{n_1+\dots+n_\ell-1} \varphi(T^j x). \quad (2.3)$$

Now, we show the other direction. Take an  $x \in \widetilde{E}(\alpha)$ , express  $x$  in its binary expansion

$$x = [0^{n_1-1} 1 0^{n_2-1} 1 \dots 0^{n_\ell-1} 1 \dots].$$

In fact,  $n_\ell - 1$  is the recurrence time for  $n(\widehat{T}^{\ell-1}x)$ , for each  $\ell \geq 1$ .

By (2.1), we have, at present,

$$\lim_{\ell \rightarrow \infty} \frac{1}{n_1 + \cdots + n_\ell} \sum_{j=0}^{n_1 + \cdots + n_\ell - 1} \varphi(T^j x) = \alpha.$$

So, we are required to check it holds for all  $n$ .

For any  $\epsilon > 0$ , there exists  $\ell_0 \in \mathbb{N}$  such that, for any  $\ell \geq \ell_0$ ,

$$\alpha - \epsilon \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell} \leq \alpha + \epsilon. \quad (2.4)$$

For any  $n_1 + \cdots + n_\ell < n < n_1 + \cdots + n_\ell + n_{\ell+1}$  with  $\ell \geq \ell_0$ , it is trivial that

$$\frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell + n_{\ell+1}} \leq \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1}{n_1 + \cdots + n_\ell}.$$

Thus, it suffices to show that

$$2^{n_{\ell+1}} = o(n_1 + \cdots + n_\ell), \quad (2.5)$$

which also implies

$$n_{\ell+1} = o(n_1 + \cdots + n_\ell).$$

Let  $M_0$  be a large integer such that, for all  $M \geq M_0$ ,  $2^M \geq 4\alpha M$ . So, when  $n_{\ell+1} \leq M_0$ , there is nothing to prove. So, we always assume  $2^{n_{\ell+1}} \geq 4\alpha n_{\ell+1}$ .

By (2.4), we have

$$\begin{aligned} 2^{n_1} + \cdots + 2^{n_\ell} - \ell &\geq (\alpha - \epsilon)(n_1 + \cdots + n_\ell), \\ 2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1 &\leq (\alpha + \epsilon)(n_1 + \cdots + n_\ell + n_{\ell+1}). \end{aligned}$$

These give

$$2^{n_{\ell+1}} \leq 2\epsilon(n_1 + \cdots + n_\ell) + (\alpha + \epsilon)n_{\ell+1} + 1.$$

So, we have

$$2^{n_{\ell+1}} \leq 4\epsilon(n_1 + \cdots + n_\ell). \quad \square$$

**2.2. Dimension of  $\tilde{E}(\alpha)$ .** Now we calculate the Hausdorff dimension of the set  $\tilde{E}(\alpha)$ . At first, we give a notation.

- For each finite word  $w \in \bigcup_{n \geq 1} \{0, 1\}^n$  of length  $n$ , a  $T$ -dyadic cylinder of order  $n$  is defined as

$$I_n(w) = \{x \in (0, 1] : (\epsilon_1(x), \dots, \epsilon_n(x)) = w\}.$$

- For  $(n_1, \dots, n_\ell) \in (\mathbb{N} \setminus \{0\})^\ell$ , a  $\hat{T}$ -dyadic cylinder of order  $\ell$  is defined as

$$D_\ell(n_1, \dots, n_\ell) = \{x \in (0, 1] : n_k(x) = n_k, 1 \leq k \leq \ell\}.$$

*Proof of Theorem 1.1.* To calculate the Hausdorff dimension of  $\tilde{E}(\alpha)$ , we construct a suitable measure supported on  $\tilde{E}(\alpha)$ . The Gibbs measures derived from the Ruelle-Perron-Frobenius transfer operator are good candidates for such a measure.

In fact, by considering the inverse branches  $U_i : x \mapsto \frac{x+1}{2^{i+1}}$  ( $i \geq 0$ ) of  $\hat{T}$ , we can code the dynamical system  $([0, 1], \hat{T})$  by the conformal infinite iterated function system  $(U_i)_{i \geq 0}$  which satisfies the open set condition ([10, Section 1]).

Consider the potential function with two parameters

$$\psi_{t,q} := -t \log |\hat{T}'| - (\log 2) \cdot q\phi \quad (t \in \mathbb{R}, q > 0).$$

Then  $(\psi_{t,q} \circ U_i)_{i \geq 0}$  is a family of strong Hölder family ([10, Page 30]). Hence, we can define a Ruelle operator

$$\mathcal{L}_{t,q} f(x) := \sum_{y \in \hat{T}^{-1}x} e^{\psi_{t,q}(y)} f(y),$$

on the Banach space of continuous functions on the corresponding infinite symbolic space ([10, Page 31]).

By the Ruelle-Perron-Frobenius transfer operator theory [10, Theorems 2.9 and 2.10], for any  $q > 0$  (to satisfy the condition 2.2 of [10]), we can find an eigenvalue  $\lambda_{t,q}$  and an eigenfunction  $h_{t,q}$  for  $\mathcal{L}_{t,q}$  and an eigenfunction  $\nu_{t,q}$  for the conjugate operator  $\mathcal{L}_{t,q}^*$ . Then the pressure function  $P(t, q) = \log \lambda_{t,q}$  and the ergodic Gibbs measure  $\mu_{t,q}$  is given by  $h_{t,q} \cdot \nu_{t,q}$ .

The pressure function can be computed by (see [10, Pages 31 and 48])

$$P(t, q) = \lim_{\ell \rightarrow \infty} \frac{1}{\ell} \log \left( \sum_{(n_1, \dots, n_\ell) \in (\mathbb{N} \setminus \{0\})^\ell} \exp \sup_{x \in D_\ell(n_1, \dots, n_\ell)} (S_\ell \psi_{t,q}(x)) \right).$$

Note that for all  $n \geq 1$ ,

$$|\widehat{T}'|(x) = 2^n, \text{ and } \phi(x) = 2^n - 1 \text{ if } x \in \left( \frac{1}{2^n}, \frac{1}{2^{n-1}} \right] = D_1(n).$$

Then for  $x \in D_\ell(n_1, \dots, n_\ell)$ ,

$$S_\ell \psi_{t,q}(x) = -(\log 2) \cdot t \cdot \sum_{j=1}^{\ell} n_j + (\log 2) \cdot q \cdot \sum_{j=1}^{\ell} (2^{n_j} - 1).$$

Thus

$$\begin{aligned} & \sum_{(n_1, \dots, n_\ell) \in (\mathbb{N} \setminus \{0\})^\ell} \exp \sup_{x \in D_\ell(n_1, \dots, n_\ell)} (S_\ell \psi_{t,q}(x)) \\ &= \sum_{n_1=1}^{\infty} \dots \sum_{n_\ell=1}^{\infty} \left( \prod_{j=1}^{\ell} 2^{-tn_j} \cdot \prod_{j=1}^{\ell} 2^{q(2^{n_j}-1)} \right) = \left( \sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)} \right)^\ell. \end{aligned}$$

Hence

$$P(t, q) = \log \sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)}.$$

Now we calculate the local dimension of the Gibbs measure  $\mu_{t,q}$ . Let  $D_\ell(x)$  be the  $\widehat{T}$ -dyadic cylinder containing  $x$  of order  $\ell$ . By the Gibbs property of  $\mu_{t,q}$ ,

$$\begin{aligned} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} &= \frac{S_\ell \psi_{t,q}(x) - \ell P(t, q)}{-S_\ell \log |\widehat{T}'|(x)} \\ &= \frac{-t S_\ell \log |\widehat{T}'|(x) - (\log 2) \cdot q S_\ell \phi(x) - \ell P(t, q)}{-S_\ell \log |\widehat{T}'|(x)} \quad (2.6) \\ &= t + q \frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}'|(x)} + \frac{\ell P(t, q)}{S_\ell \log |\widehat{T}'|(x)}. \end{aligned}$$

• UPPER BOUND. For each  $q > 0$ , let  $t(q)$  be the the number such that  $P(t(q), q) = 0$ . (The existence of  $t(q)$  comes from the facts that  $P(t, q)$  is real-analytic and that for fixed  $q > 0$ ,  $P(t, q) > 0$  when  $t \rightarrow -\infty$  and  $P(t, q) < 0$  when  $t \rightarrow +\infty$ .) Then for all  $x \in \widetilde{E}(\alpha)$ , we have

$$\begin{aligned} \liminf_{r \rightarrow 0} \frac{\log \mu_{t,q}(B(x, r))}{\log r} &\leq \liminf_{\ell \rightarrow \infty} \frac{\log \mu_{t,q}(B(x, |D_\ell(x)|))}{\log |D_\ell(x)|} \\ &\leq \liminf_{\ell \rightarrow \infty} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} = t(q) + q\alpha, \end{aligned}$$



where for the second inequality the trivial inclusion  $D_\ell(x) \subset B(x, |D_\ell(x)|)$  is used. By Billingsley Lemma (see e.g. [3, Proposition 4.9.]), this gives an upper bound of the Hausdorff dimension of  $\tilde{E}(\alpha)$ . Thus we have

$$\dim_H \tilde{E}(\alpha) \leq \inf_{q>0} \{t(q) + q\alpha\}.$$

• **LOWER BOUND.** By the real-analyticity of  $P(t, q)$  and the implicit function theorem, the function  $q \mapsto t(q)$  is also real-analytic. Thus there exists  $q_0$  such that the following infimum is attained

$$\inf_{q>0} \{t(q) + q\alpha\}.$$

Then we have

$$t'(q_0) + \alpha = 0. \quad (2.7)$$

To prove the lower bound, we first show two claims.

Claim (A): The measure  $\mu_{t(q_0), q_0}$  is supported on  $E_\alpha$ .

On the one hand, since  $P(t(q), q) = 0$ ,

$$\frac{\partial P}{\partial t} t'(q) + \frac{\partial P}{\partial q} = 0. \quad (2.8)$$

On the other hand, by the ergodicity of the measure  $\mu_{t, q}$ , we have for  $\mu_{t, q}$  almost all  $x$ ,

$$\lim_{\ell \rightarrow \infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}^\ell(x)|} = \frac{\int \phi d\mu_{t, q}}{\int \log |\widehat{T}'| d\mu_{t, q}} \cdot \log 2.$$

By Ruelle-Perron-Frobenius transfer operator theory ([10], Proposition 6.5),

$$\int (\log 2) \cdot \phi d\mu_{t, q} = -\frac{\partial P}{\partial q} \quad \text{and} \quad \int \log |\widehat{T}'| d\mu_{t, q} = -\frac{\partial P}{\partial t}.$$

Thus by (2.8) and then (2.7), for  $\mu_{t(q_0), q_0}$  almost all  $x$ ,

$$\lim_{\ell \rightarrow \infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}^\ell(x)|} = \frac{\frac{\partial P}{\partial q}}{\frac{\partial P}{\partial t}} = -t'(q_0) = \alpha.$$

This shows Claim (A).

Claim (B): For  $\mu_{t(q_0), q_0}$  almost all  $x$ ,

$$\lim_{n \rightarrow \infty} \frac{\log \mu_{t(q_0), q_0}(I_n(x))}{\log 2^{-n}} = t(q_0) + q_0\alpha,$$

where  $I_n(x)$  is the  $T$ -dyadic cylinder of order  $n$  containing  $x$ .

On the one hand, by (2.6) and then by (2.8) and (2.7), one has for  $\mu_{t(q_0), q_0}$  almost all  $x$

$$\lim_{\ell \rightarrow \infty} \frac{\log \mu_{t(q_0), q_0}(D_\ell(x))}{\log |D_\ell(x)|} = t(q_0) + q_0 \frac{\frac{\partial P}{\partial q}}{\frac{\partial P}{\partial t}} = t(q_0) + q_0\alpha. \quad (2.9)$$

On the other hand, note that for any  $x \in E(\alpha)$ , if the binary expansion of  $x$  is  $x = [0^{n_1-1} 1 0^{n_2-1} 1 \dots]$ , then for any  $\delta > 0$ , for  $\ell$  large enough,

$$(\alpha - \delta)\ell \leq 2^{n_1} + \dots + 2^{n_\ell} - \ell = S_\ell \phi(x) \leq (\alpha + \delta)\ell.$$

Hence

$$n_\ell = O(\log \ell),$$

which implies

$$\lim_{\ell \rightarrow \infty} \frac{\log |D_\ell(x)|}{\log |D_{\ell+1}(x)|} = \lim_{\ell \rightarrow \infty} \frac{n_1 + \dots + n_\ell}{n_1 + \dots + n_\ell + n_{\ell+1}} = 1. \quad (2.10)$$

Thus

$$\lim_{n \rightarrow \infty} \frac{\log \mu_{t(q_0), q_0}(I_n(x))}{\log 2^{-n}} = \lim_{\ell \rightarrow \infty} \frac{\log \mu_{t(q_0), q_0}(D_\ell(x))}{\log |D_\ell(x)|}.$$

This shows Claim (B).

To conclude the desired lower bound, we apply the classical mass distribution principle (see [3, Proposition 4.2]). Since the Hausdorff dimension will not be changed if we replace the  $\delta$ -coverings by  $T$ -dyadic cylinder coverings (see [3, Section 2.4]), the lower bound of the Hausdorff dimension can be given by the mass transference principle on  $T$ -dyadic cylinders. By the above two claims and Egorov's theorem, for any  $\eta > 0$ , there exists an integer  $N_0$  such that the set

$$\left\{ x \in E_\alpha : \mu(I_n(x)) \leq |I_n(x)|^{t(q_0) + q_0 \alpha - \eta}, \quad n \geq N \right\}$$

is of  $\mu_{t(q_0), q_0}$  positive measure. So, it implies that

$$\dim_H E_\alpha \geq t(q_0) + q_0 \alpha - \eta.$$

Note that ([10, Lemma 7.5]) the function  $q \mapsto t(q)$  is a decreasing convex function such that

$$t(0) = 1, \quad \lim_{q \rightarrow \infty} (t(q) + q) = 0,$$

and

$$\lim_{q \rightarrow 0^+} t'(q) = -\infty, \quad \lim_{q \rightarrow +\infty} t'(q) = -1.$$

Therefore, we have proved for any  $\alpha \in (1, +\infty)$

$$\dim_H(\tilde{E}(\alpha)) = \inf_{q > 0} \{t(q) + q\alpha\},$$

which is Legendre transformation. All the properties stated in Theorem 1.1 are satisfied by the function  $\alpha \mapsto \dim_H(\tilde{E}(\alpha))$  which is the same function as  $\alpha \mapsto \dim_H(E(\alpha))$  by Lemma 2.1.

For the end point  $\alpha = 1$ , it suffices to note that the level set  $E(1)$  is nothing but the set of numbers with frequency of the digit 1 in its binary expansion being 1. Thus the Hausdorff dimension of  $E(1)$  is 0. Hence, the Legendre transformation formula for the Hausdorff dimension of  $E(\alpha)$  ( $\alpha > 1$ ) also holds for  $\alpha = 1$ .  $\square$

### 3. FAST INCREASING BIRKHOFF SUM

At first, we give two simple observations.

**Lemma 3.1.** *Let  $W$  be an integer such that  $2^t \leq W < 2^{t+1}$  for some positive integer  $t$ . For any  $0 \leq n \leq t$ , among the integers between  $W$  and  $W(1+2^{-n})$ , there is one  $V = V(W, n)$  whose binary expansion of  $V$  has at most  $n+2$  digits 1 and ends with at least  $t-n$  zeros.*

*Proof.* By the assumption, we have  $2^{-n}W \geq 2^{t-n}$ . Thus among the  $2^{-n}W$  consecutive integers from  $W$  to  $W(1+2^{-n})$  there is at least one integer which is divisible by  $2^{t-n}$  which means there is an integer  $\ell \geq 1$  such that

$$W \leq \ell 2^{k-n} \leq W(1+2^{-n}).$$

Let  $V = \ell 2^{t-n}$  and note that  $V$  is an integer whose binary expansion ends with at least  $t-n$  zeros. Since  $\ell 2^{t-n} \leq W(1+2^{-n}) < 2^{t+2}$ , we conclude that  $\ell 2^{t-n}$  has at most  $(t+2) - (t-n) = n+2$  digits 1 in its binary expansion.  $\square$

In the follows, the base of the logarithm is taken to be 2.

**Lemma 3.2.** *For each integer  $W$ , and any integer  $n \leq \log W$ , we can find a word  $w$  with length*

$$|w| \leq (n+2)(2 + \log W)$$

and for any  $x \in I_{|w|}(w)$

$$W \leq \sum_{j=0}^{|w|-1} \varphi(T^j x) \leq W(1 + 2^{-n}).$$

*Proof.* Let  $V$  be an integer given in Lemma 3.1. Then  $W \leq V \leq W(1 + 2^{-n})$ . Moreover if we write this number  $V$  in binary expansion:

$$V = 2^{t_1} + \cdots + 2^{t_p},$$

one has that  $\lfloor \log W \rfloor + 1 \geq t_1 > \cdots > t_p \geq \lfloor \log W \rfloor - n$  and  $p \leq n + 2$ . Consider the word

$$w = (10^{t_1-1}1, 10^{t_2-1}1, \dots, 10^{t_p-1}1)$$

here the word  $10^{t_p-1}1$  is 1 when  $t_p = 0$ . Then we can check that the length of  $w$  satisfies

$$|w| = (t_1 + 1) + \cdots + (t_p + 1) \leq p(t_1 + 1) \leq (n + 2)(2 + \log W),$$

and for any  $x \in I_{|w|}(w)$ ,

$$\sum_{j=0}^{|w|-1} \varphi(T^j x) = V.$$

Hence, the proof is completed.  $\square$

We also need the following lemmas whose proofs are left for the reader.

**Lemma 3.3.** *For any  $m \geq 1$ , define*

$$F_m = \{x \in (0, 1] : \epsilon_{km}(x) = 1, \text{ for all } k \geq 1\}.$$

Then  $\dim_H F_m = \frac{m-1}{m}$ .

**Lemma 3.4.** [13, Lemma 4] *Given a subset  $\mathbb{J}$  of positive integers and an infinite sequence  $\{a_k\}_{k=1}^\infty$  of 0's and 1's, let*

$$E(\mathbb{J}, \{a_k\}_{k=1}^\infty) = \left\{ x \in (0, 1] : \epsilon_k(x) = a_k, \text{ for all } k \in \mathbb{J} \right\}.$$

If the density of  $\mathbb{J}$  is zero, that is,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Card}\{k \leq n : k \in \mathbb{J}\} = 0$$

then  $\dim_H E(\mathbb{J}, \{a_k\}_{k=1}^\infty) = 1$ .

Before the proof Theorem 1.2, we show the following lemma.

**Lemma 3.5.** *Let  $\Psi : \mathbb{N} \rightarrow \mathbb{N}$  be an increasing function such that  $\Psi(n)/n \rightarrow \infty$  as  $n \rightarrow \infty$ . Assume that there exists a subsequence  $N_k$  satisfying the following conditions*

$$N_k - N_{k-1} \rightarrow \infty, \quad \Psi(N_k) - \Psi(N_{k-1}) \rightarrow \infty, \quad (3.1)$$

and

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} \rightarrow 1, \quad \frac{\log(\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \rightarrow 0, \quad (3.2)$$

as  $k \rightarrow \infty$ . Then the set

$$E_\Psi(1) = \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{\Psi(n)} S_n(x) = 1 \right\}$$

has Hausdorff dimension 1.

*Proof.* Fix a large integer  $m$  and write

$$\mathcal{U} = \left\{ u = (\epsilon_1, \dots, \epsilon_m) : \epsilon_m = 1, \epsilon_i \in \{0, 1\}, i \neq m \right\}.$$

To avoid the abuse of notation, by the first assumption of (3.1), we assume  $N_k - N_{k-1} \gg m$  for all  $k \geq 1$  by setting  $N_0 = 0$  and  $\Psi(N_0) = 0$ .

For each  $k \geq 1$ , we write

$$W_k := \Psi(N_k) - \Psi(N_{k-1})$$

and let  $\{n_k\}$  be a sequence of integers tending to  $\infty$  such that

$$n_k \leq \log W_k, \quad n_k \cdot \frac{\log(\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \rightarrow 0.$$

By the second assumptions of (3.1) and (3.2), this sequence of  $n_k \geq 0$  do exist.

Now for  $W_k$  and  $n_k$ , let  $w_k$  be the word given in Lemma 3.2. Then the length  $a_k$  of  $w_k$  satisfies

$$\begin{aligned} a_k &\leq (n_k + 2)(2 + \log W_k) \\ &= (n_k + 2)(2 + \log(\Psi(N_k) - \Psi(N_{k-1}))) = o(N_k - N_{k-1}) \end{aligned} \quad (3.3)$$

and for any  $x \in I_{a_k}(w_k)$ ,

$$W_k \leq \sum_{j=0}^{a_k-1} \varphi(T^j x) \leq W_k(1 + 2^{-n_k}). \quad (3.4)$$

Define  $t_k, \ell_k$  to be the integers satisfying

$$N_k - N_{k-1} - a_k = t_k m + \ell_k,$$

for some  $0 \leq \ell_k < m$ .

Let  $w_k$  ( $k \geq 1$ ) be given as the above. We define a Cantor subset of  $E_\Psi(1)$  as follows.

*Level 1 of the Cantor subset.* Define

$$E_1 = \left\{ I_{N_1}(u_1, \dots, u_{t_1}, 1^{\ell_1}, w_1) : u_i \in \mathcal{U}, 1 \leq i \leq t_1 \right\}.$$

For simplicity, we use  $I_{N_1}(U_1)$  to denote a general cylinder in  $E_1$ .

*Level 2 of the Cantor subset.* This level is composed by sublevels for each cylinder  $I_{N_1}(U_1) \in E_1$ . Fix an element  $I_{N_1} = I_{N_1}(U_1) \in E_1$ . Define

$$E_2(I_{N_1}(U_1)) = \left\{ I_{N_2}(U_1, u_1, \dots, u_{t_2}, 1^{\ell_2}, w_2) : u_i \in \mathcal{U}, 1 \leq i \leq t_2 \right\}.$$

Then

$$E_2 = \bigcup_{I_{N_1} \in E_1} E_2(I_{N_1}).$$

For simplicity, we use  $I_{N_2}(U_2)$  to denote a general cylinder in  $E_2$ .

*From Level  $k$  to  $k+1$ .* Fix  $I_{N_k}(U_k) \in E_k$ . Define

$$E_{k+1}(I_{N_k}(U_k)) = \left\{ I_{N_{k+1}}(U_k, u_1, \dots, u_{t_{k+1}}, 1^{\ell_{k+1}}, w_{k+1}) : u_i \in \mathcal{U}, 1 \leq i \leq t_{k+1} \right\}.$$

Then

$$E_{k+1} = \bigcup_{I_{N_k} \in E_k} E_{k+1}(I_{N_k}).$$

Up to now we have constructed a sequence of nested sets  $\{E_k\}_{k \geq 1}$ . Set

$$F = \bigcap_{k \geq 1} E_k.$$

We claim that

$$F \subset E(\Psi).$$

In fact, for all  $x \in F$ , by construction, for each  $k \geq 1$ ,

$$\begin{aligned} & \sum_{n=N_{k-1}}^{N_k-1} \varphi(T^n x) \\ = & \sum_{n=N_{k-1}}^{N_{k-1}+t_k m-1} \varphi(T^n x) + \sum_{n=N_{k-1}+t_k m}^{N_{k-1}+t_k m+\ell_k-1} \varphi(T^n x) + \sum_{n=N_{k-1}+t_k m+\ell_k}^{N_k-1} \varphi(T^n x) \\ = & t_k O(2^m) + \ell_k + W_k(1 + O(2^{-n_k})) \\ = & O\left(\frac{(N_k - N_{k-1})2^m}{m}\right) + (\Psi(N_k) - \Psi(N_{k-1}))(1 + O(2^{-n_k})). \end{aligned}$$

Since  $n_k \rightarrow \infty$  which implies  $2^{-n_k} \rightarrow 0$  as  $k \rightarrow \infty$ , we have

$$\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k)(1 + o(1)) + O\left(\frac{N_k 2^m}{m}\right).$$

By the assumption  $\Psi(n)/n \rightarrow \infty$  as  $n \rightarrow \infty$ , we then deduce

$$\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k) + o(\Psi(N_k)),$$

Thus

$$\lim_{k \rightarrow \infty} \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k)} = 1. \quad (3.5)$$

While, for each  $N_{k-1} < N \leq N_k$

$$\frac{\sum_{n=0}^{N_{k-1}-1} \varphi(T^n x)}{\Psi(N_k)} \leq \frac{\sum_{n=0}^{N-1} \varphi(T^n x)}{\Psi(N)} \leq \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_{k-1})}.$$

So by the first assumption of (3.2), we deduce from (3.5) that

$$\lim_{n \rightarrow \infty} \frac{1}{\Psi(n)} S_n(x) = 1.$$

This proves  $x \in E_{\Psi}(1)$  and hence  $F \subset E_{\Psi}(1)$ .

In the following, we will construct a Hölder continuous function from  $F$  to  $F_m$ . Recall that

$$F_m = \{x \in (0, 1] : \epsilon_{km}(x) = 1, \text{ for all } k \geq 1\}.$$

Define

$$\begin{aligned} f : F & \rightarrow F_m \\ x & \mapsto y \end{aligned}$$

where  $y$  is obtained by eliminating the digits  $\{(\epsilon_{N_k-\ell_k-a_k+1}, \dots, \epsilon_{N_k})\}_{k \geq 1}$  in the binary expansion of  $x$ . Now we calculate the Hölder exponent of  $f$ .

Take two points  $x_1, x_2 \in F$  closed enough. Let  $n$  be the smallest integer such that  $\epsilon_n(x_1) \neq \epsilon_n(x_2)$  and  $k$  be the integer such that  $N_k < n \leq N_{k+1}$ . Note that by the construction of  $F$ , the digits sequence

$$\{(\epsilon_{N_k-\ell_k-a_k+1}, \dots, \epsilon_{N_k})\}_{k \geq 1} \text{ and } \{\epsilon_{N_k+tm}\}_{1 \leq t \leq t_{k+1}}$$

are the same for all  $x \in F$ . So we must have

$$N_k < n < N_{k+1} - \ell_{k+1} - a_{k+1}. \quad (3.6)$$

Since  $n$  is strictly less than  $N_{k+1} - \ell_{k+1} - a_{k+1}$  and  $\epsilon_{N_k+tm}(x_1) = \epsilon_{N_k+tm}(x_2) = 1$  for all  $1 \leq t \leq t_{k+1}$ , thus, at most  $m$  steps after the position  $n$ , saying  $n'$ ,  $\epsilon_{n'}(x_1) = \epsilon_{n'}(x_2) = 1$ . So it follows that

$$|x_1 - x_2| \geq \frac{1}{2^{n+m}}.$$

Again by the construction and the definition of the map  $f$ , we have  $y_1 = f(x_1)$  and  $y_2 = f(x_2)$  have common digits up to the position  $n-1-(\ell_1+a_1)-\dots-(\ell_k+a_k)$ . Thus, it follows

$$|f(x_1) - f(x_2)| \leq \frac{1}{2^{n-1-(\ell_1+a_1)-\dots-(\ell_k+a_k)}}.$$

Recall that  $\ell_k < m$  and  $a_1 + \dots + a_k = o(N_k)$  (see (3.3)) and also that  $N_k/k \rightarrow \infty$  as  $k \rightarrow \infty$  (by (3.1)). We have

$$1 \geq \frac{n-1-(\ell_1+a_1)-\dots-(\ell_k+a_k)}{n+m} \geq \frac{n-1-km-o(N_k)}{n+m} = 1+o(1),$$

which implies that  $f$  is  $(1-\eta)$ -Hölder for any  $\eta > 0$ . Thus

$$\dim_H F \geq (1-\eta) \dim_H F_m.$$

By Lemma 3.3, we then have

$$\dim_H F \geq (1-\eta) \frac{m-1}{m}.$$

By the arbitrariness of  $\eta > 0$  and letting  $m \rightarrow \infty$ , we conclude that  $\dim_H E(\Psi) = 1$ . This finishes the proof.  $\square$

*Proof of Theorem 1.2.* In all the three parts of Theorem 1.2, the case of  $\beta = 0$  is a direct consequence of Theorem 1.1.

(I). Assume that  $\Psi$  is one of the functions  $\Psi(n) = n \log n$ ,  $\Psi(n) = n^a$  ( $a > 1$ ),  $\Psi(n) = 2^{n^\gamma}$  with  $0 < \gamma < 1/2$ .

(I<sub>1</sub>).  $0 < \beta < \infty$ . It suffices to consider the dimension of  $E_\Psi(1)$  i.e.  $\beta = 1$ , since for other  $\beta \in (0, \infty)$ , we need only replace  $\Psi(n)$  by  $\beta\Psi(n)$ .

To show  $\dim_H E_\Psi(1) = 1$ , we can apply Lemma 3.5 directly. If  $\Psi(n) = n \log n$ , we can choose  $N_k = k^2$ . For  $\Psi(n) = n^a$  ( $a > 1$ ), we can also choose  $N_k = k^2$ . Suppose now  $\Psi(n) = 2^{n^\gamma}$  with  $0 < \gamma < 1/2$ . Let  $\delta > 0$  be small such that

$$\frac{\gamma}{1-\gamma} + \delta\gamma < 1 \tag{3.7}$$

which is possible since  $\gamma < 1/2$ . Take

$$N_k = \lfloor k^{\frac{1}{1-\gamma} + \delta} \rfloor. \tag{3.8}$$

Then we have

$$N_{k+1} - N_k \approx k^{\frac{\gamma}{1-\gamma} + \delta}, \tag{3.9}$$

and

$$\begin{aligned} \log(\Psi(N_{k+1}) - \Psi(N_k)) &\approx \log(\Psi'(N_k)(N_{k+1} - N_k)) \\ &\approx N_k^\gamma + \log(N_{k+1} - N_k) \approx N_k^\gamma. \end{aligned}$$

Here we write  $A \approx B$  when  $A/B \rightarrow 1$ . This shows the validity of (3.1). Moreover,

$$\frac{\log(\Psi(N_{k+1}) - \Psi(N_k))}{N_{k+1} - N_k} \approx \frac{k^{\frac{\gamma}{1-\gamma} + \gamma\delta}}{k^{\frac{\gamma}{1-\gamma} + \delta}} = k^{-\delta(1-\gamma)} \rightarrow 0 \quad (k \rightarrow \infty).$$

Thus the second assumption of (3.2) is satisfied. At last, for the first assumption in (3.2), by (3.7)

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} = 2^{(k-1)^{\frac{\gamma}{1-\gamma} + \delta\gamma} - k^{\frac{\gamma}{1-\gamma} + \delta\gamma}} \rightarrow 1.$$

Hence Lemma 3.5 applies.

(I<sub>2</sub>). If  $\beta = \infty$ , we may choose  $\tilde{\Psi}(n) = 2^{n^\eta}$  for some  $0 < \eta < \frac{1}{2}$  such that  $E_{\tilde{\Psi}}(1) \subset E_{\Psi}(\infty)$ . Then  $\dim_H E_{\Psi}(\infty) = 1$  follows from (I<sub>1</sub>).

(II). Now suppose that  $\Psi(n) = 2^{n^\gamma}$  with  $1/2 \leq \gamma < 1$ .

(II<sub>1</sub>). Let  $\beta \in (0, \infty)$ . We will prove that  $E_{\Psi}(\beta)$  is empty. On the contrary, suppose there is  $x \in E_{\Psi}(\beta)$ , which has binary expansion

$$x = [0^{n_1-1}10^{n_2-1}1 \dots 0^{n_\ell-1}1 \dots]. \quad (3.10)$$

Then, by (2.1) we have

$$\begin{aligned} \frac{S_{n_1+n_2+\dots+n_\ell}(x)}{\Psi(n_1+n_2+\dots+n_\ell)} &= \frac{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell}{2^{(n_1+n_2+\dots+n_\ell)\gamma}} \rightarrow \beta, \\ \frac{S_{n_1+n_2+\dots+n_{\ell+1}}(x)}{\Psi(n_1+n_2+\dots+n_{\ell+1})} &= \frac{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell + 2^{n_{\ell+1}-1}}{2^{(n_1+n_2+\dots+n_{\ell+1})\gamma}} \rightarrow \beta. \end{aligned} \quad (3.11)$$

Since

$$\frac{2^{(n_1+n_2+\dots+n_\ell)\gamma}}{2^{(n_1+n_2+\dots+n_{\ell+1})\gamma}} \rightarrow 1,$$

by dividing the two limits of (3.11), we deduce that

$$\frac{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell + 2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell} = 1 + \frac{2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell} \rightarrow 1,$$

which implies that

$$\frac{S_{n_1+n_2+\dots+n_{\ell+1}}(x)}{S_{n_1+n_2+\dots+n_\ell}(x)} = 1 + \frac{2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell} \rightarrow 1.$$

Combining with (3.11), we get

$$1 \leftarrow \frac{\Psi(n_1 + \dots + n_{\ell+1})}{\Psi(n_1 + \dots + n_\ell)} = \frac{2^{(n_1+n_2+\dots+n_\ell+n_{\ell+1})\gamma}}{2^{(n_1+n_2+\dots+n_\ell)\gamma}}.$$

Thus

$$\begin{aligned} &(n_1 + n_2 + \dots + n_\ell + n_{\ell+1})^\gamma - (n_1 + n_2 + \dots + n_\ell)^\gamma \\ &= (n_1 + n_2 + \dots + n_\ell)^\gamma \left( \left( 1 + \frac{n_{\ell+1}}{n_1 + n_2 + \dots + n_\ell} \right)^\gamma - 1 \right) \\ &\approx \frac{\gamma n_{\ell+1}}{(n_1 + n_2 + \dots + n_\ell)^{1-\gamma}} \rightarrow 0. \end{aligned}$$

Therefore, for any  $\varepsilon > 0$ , there exists  $k_0 \geq 1$  such that for all  $j > k_0$ ,

$$n_j < \varepsilon(n_1 + n_2 + \dots + n_{j-1})^{1-\gamma}. \quad (3.12)$$

Then for any  $k_0 < j \leq \ell$

$$n_j < \varepsilon(n_1 + n_2 + \dots + n_\ell)^{1-\gamma}.$$

This implies

$$\begin{aligned} S_{n_1+n_2+\dots+n_\ell}(x) &= 2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell \\ &\leq M + \ell 2^{\varepsilon(n_1+n_2+\dots+n_\ell)^{1-\gamma}} - \ell, \end{aligned}$$

with  $M := 2^{n_1} + \dots + 2^{n_{k_0}}$ . Thus we have

$$\frac{S_{n_1+n_2+\dots+n_\ell}(x)}{\Psi(n_1+n_2+\dots+n_\ell)} < \frac{M + \ell 2^{\varepsilon(n_1+n_2+\dots+n_\ell)^{1-\gamma}} - \ell}{2^{(n_1+n_2+\dots+n_\ell)\gamma}}. \quad (3.13)$$

By observing  $n_j \geq 1$ , we deduce that the upper bound of (3.13) converges to 0 for  $1/2 \leq \gamma < 1$ , a contradiction to (3.11). Hence  $E_{\Psi}(\beta)$  is an empty set.

(II<sub>2</sub>).  $\beta = \infty$ . Fix  $\delta \in (\gamma, 1)$  and take a large integer  $K$  such that  $2^{K\delta} > 1$ . Consider the set of points such that at every position  $2^k, k > K$  in their binary expansions, they have a string of zeros of length  $2^{\delta k}$ , i.e.

$$E := \left\{ x \in (0, 1] : \epsilon_{2^k+1} = \cdots = \epsilon_{2^k+2^{\delta k}} = 0, \text{ for all } k \geq K \right\}.$$

On the one hand,  $E \subset E_\psi(\infty)$ , since for any  $n \in (2^k, 2^{k+1}]$  for some  $k \geq K$ ,

$$S_n(x) > 2^{2^{\delta k}} \geq 2^{(n/2)^\delta} \gg 2^{n^\gamma}.$$

On the other hand, the set  $E$  has dimension 1 guaranteed by Lemma 3.4.

(III). Suppose that  $\Psi(n) = 2^{n^\gamma}$  with  $\gamma \geq 1$  and let  $\beta \in (0, +\infty]$ . Assume that there exists  $x \in E_\Psi(\beta)$  for some  $\beta \in (0, +\infty)$ . Write the binary expansion of  $x$  as (3.10). Then by (2.1),

$$\begin{aligned} \frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} &= \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}} \rightarrow \beta, \\ \frac{S_{n_1+n_2+\cdots+n_\ell-1}(x)}{\Psi(n_1+n_2+\cdots+n_\ell-1)} &= \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell - 1}{2^{(n_1+n_2+\cdots+n_\ell-1)^\gamma}} \rightarrow \beta. \end{aligned} \quad (3.14)$$

However,

$$\frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell}{2^{n_1 + 2^{n_2} + \cdots + 2^{n_\ell} - \ell - 1}} \rightarrow 1 \quad \text{but} \quad \frac{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}{2^{(n_1+n_2+\cdots+n_\ell-1)^\gamma}} \geq 2,$$

which is a contradiction. Hence  $E_\Psi(\beta)$  is empty when  $\beta \in (0, +\infty)$ .

When  $\beta = +\infty$ , by (2.1), we have

$$\liminf_{n \rightarrow \infty} \frac{S_n(x)}{2^n} \leq 1. \quad (3.15)$$

So,

$$\liminf_{n \rightarrow \infty} \frac{S_n(x)}{\Psi(n)} \leq 1.$$

This shows that  $E_\Psi(\infty)$  is also empty.  $\square$

#### 4. THE POTENTIAL $1/x$

In fact, the techniques in Section 3 can be applied to the continuous potential  $g : x \mapsto 1/x$  on  $(0, 1]$  which has a singularity at 0.

*Proof of Theorem 1.3.* We first show that if  $\Psi(n)$  is one of the following

$$\Psi(n) = n \log n, \quad \Psi(n) = n^a \quad (a > 1), \quad \Psi(n) = 2^{n^\gamma} \quad (0 < \gamma < 1/2),$$

then for any  $\beta \in [0, \infty]$ ,  $\dim_H F_\Psi(\beta) = 1$ .

We note that if  $x \in (0, 1]$  has binary expansion  $x = [0^n 1^s \dots]$ , then  $\varphi(x) = 2^n$  and

$$2^n \leq g(x) \leq 2^n + 2^{n-s+1} = 2^n(1 + 2^{-s+1}). \quad (4.1)$$

In Lemma 3.2, for an integer  $W$ , and for any integer  $n \leq \log W$ , we can construct instead of the words  $w = (10^{t_1-1}1, 10^{t_2-1}1, \dots, 10^{t_p-1}1)$ , the following word

$$w = (10^{t_1-1}1^{s+1}, 10^{t_2-1}1^{s+1}, \dots, 10^{t_p-1}1^{s+1}).$$

Then the length of the word satisfies

$$|w| = \sum_{i=1}^p (t_i + s + 1) \leq p(t_1 + s + 1) \leq (n + 2)(\log W + s + 2). \quad (4.2)$$



By (4.1), for any  $x \in I_{|w|}(w)$ ,

$$W + s(n+2) \leq \sum_{j=0}^{|w|-1} g(T^j x) \leq W(1+2^{-n}) \cdot (1+2^{-s}) + 2s(n+2).$$

For each  $k \geq 1$ , we still write

$$W_k := \Psi(N_k) - \Psi(N_{k-1})$$

and let  $n_k, s_k$  be a sequence of integers tending to  $\infty$  such that

$$n_k \cdot \frac{\log(\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \rightarrow 0, \quad (4.3)$$

$$\frac{n_k \cdot s_k}{N_k - N_{k-1}} \rightarrow 0, \quad (4.4)$$

and

$$\frac{n_k \cdot s_k}{\Psi(N_k) - \Psi(N_{k-1})} \rightarrow 0. \quad (4.5)$$

By (3.1) and (3.2), these two sequences of  $n_k \geq 0, s_k \geq 0$  do exist.

Now for  $W_k$  and  $n_k, s_k$ , let  $w_k$  be the word given as above. Then by (4.3) and (4.4), the length  $a_k$  of  $w_k$  satisfies

$$\begin{aligned} a_k &\leq (n_k + 2)(\log W_k + s_k + 2) \\ &= (n_k + 2)(\log(\Psi(N_k) - \Psi(N_{k-1})) + s_k + 2) \\ &= o(N_k - N_{k-1}) \end{aligned} \quad (4.6)$$

and for any  $x \in I_{a_k}(w_k)$ ,

$$\begin{aligned} W_k + s_k(n_k + 2) &\leq \sum_{j=0}^{a_k-1} g(T^j x) \\ &\leq W_k(1+2^{-n_k}) \cdot (1+2^{-s_k}) + 2s_k(n_k + 2). \end{aligned} \quad (4.7)$$

Hence by (4.5) we still have the same estimation:

$$\sum_{n=0}^{N_k-1} g(T^n x) = \Psi(N_k) + o(\Psi(N_k))$$

and the rest of the proof is the same as (I) of the proof of Theorem 1.2.

We can repeat the same arguments in Section 3 and show that for potential  $g$ , the set  $F_\Psi(\beta)$  is empty if  $\beta \in (0, \infty), \Psi(n) = 2^{n^\gamma} (1/2 \leq \gamma < 1)$  or  $\beta \in (0, \infty], \Psi(n) = 2^{n^\gamma} (\gamma \geq 1)$ .

In fact, by definition, if there exists  $x \in F_\Psi(\beta)$ , with its binary expansion

$$x = [0^{n_1-1} 1 0^{n_2-1} 1 \dots 0^{n_\ell-1} 1 0^{n_{\ell+1}-1} 1 \dots],$$

then

$$\frac{S_{n_1+n_2+\dots+n_\ell} g(x)}{\Psi(n_1+n_2+\dots+n_\ell)} \rightarrow \beta, \quad \frac{S_{n_1+n_2+\dots+n_\ell+1} g(x)}{\Psi(n_1+n_2+\dots+n_\ell+1)} \rightarrow \beta.$$

Thus

$$\frac{S_{n_1+n_2+\dots+n_\ell} g(x)}{S_{n_1+n_2+\dots+n_\ell+1} g(x)} \rightarrow 1.$$

Observing  $\varphi \leq g \leq 2\varphi$ , we have

$$\frac{2^{n_{\ell+1}}}{S_{n_1+n_2+\dots+n_\ell} g(x)} \rightarrow 0,$$

which then implies

$$\frac{S_{n_1+n_2+\dots+n_\ell}g(x)}{S_{n_1+n_2+\dots+n_{\ell+1}}g(x)} \rightarrow 1.$$

By the definition of  $x \in F_\Psi(\beta)$ , we have

$$\frac{\Psi(n_1 + n_2 + \dots + n_{\ell+1})}{\Psi(n_1 + n_2 + \dots + n_\ell)} \rightarrow 1.$$

This further implies the same inequality with (3.12) and the rest of proof is the same as (II<sub>1</sub>) and (III) by noting  $\varphi \leq g \leq 2\varphi$ .

For  $\Psi(n) = 2^{n^\gamma}$  ( $1/2 \leq \gamma < 1$ ), we can also prove  $\dim_H F_\Psi(\infty) = 1$  by the same proof as (II<sub>2</sub>).  $\square$

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