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MULTIFRACTAL ANALYSIS OF THE BIRKHOFF SUMS OF SAINT-PETERSBURG POTENTIAL

DONG HAN KIM, LINGMIN LIAO, MICHAL RAMS, AND BAO-WEI WANG

ABSTRACT. Let $((0, 1], T)$ be the doubling map in the unit interval and φ be the Saint-Petersburg potential, defined by $\varphi(x) = 2^n$ if $x \in (2^{-n-1}, 2^{-n}]$ for all $n \geq 0$. We consider asymptotic properties of the Birkhoff sum $S_n(x) = \varphi(x) + \dots + \varphi(T^{n-1}(x))$. With respect to the Lebesgue measure, the Saint-Petersburg potential is not integrable and it is known that $\frac{1}{n \log n} S_n(x)$ converges to $\frac{1}{\log 2}$ in probability. We determine the Hausdorff dimension of the level set $\{x : \lim_{n \rightarrow \infty} S_n(x)/n = \alpha\}$ ($\alpha > 0$), as well as that of the set $\{x : \lim_{n \rightarrow \infty} S_n(x)/\Psi(n) = \alpha\}$ ($\alpha > 0$), when $\Psi(n) = n \log n$, n^a or 2^{n^γ} for $a > 1$, $\gamma > 0$. The fast increasing Birkhoff sum of the potential function $x \mapsto 1/x$ is also studied.

1. INTRODUCTION

Let T be the doubling map on the unit interval $(0, 1]$ defined by

$$Tx = 2x - [2x] + 1,$$

where $[x]$ is the smallest integer larger than or equal to x . Let ϵ_1 be the function defined by $\epsilon_1(x) = [2x] - 1$ and $\epsilon_n(x) := \epsilon_1(T^{n-1}x)$ for $n \geq 2$. Then each real number $x \in (0, 1]$ can be expanded into an infinite series as

$$x = \frac{\epsilon_1(x)}{2} + \dots + \frac{\epsilon_n(x)}{2^n} + \dots \quad (1.1)$$

We call (1.1) the binary expansion of x and also write it as

$$x = [\epsilon_1(x)\epsilon_2(x)\dots].$$

The Saint-Petersburg potential is a function $\varphi : (0, 1] \rightarrow \mathbb{R}$ defined as

$$\varphi(x) = 2^n \text{ if } x \in (2^{-n-1}, 2^{-n}], \forall n \geq 0.$$

We remark that the definition of φ is equivalent to

$$\varphi(x) = 2^n \text{ where } n \geq 0 \text{ is the smallest integer such that } \epsilon_{n+1}(x) = 1.$$

and is also equivalent to

$$\varphi(x) = 2^n \text{ if the binary expansion of } x \text{ begins with } 0^n 1,$$

where 0^n ($n \geq 0$) means a block with n consecutive zeros.

The name of Saint-Petersburg potential is motivated by the famous Saint-Petersburg game in probability theory. The Saint-Petersburg potential is of infinite expectation with respect to the Lebesgue measure. Furthermore, it increases exponentially fast near to the point 0.

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In this paper, we are concerned with the following Birkhoff sums of the Saint-Petersburg potential:

$$\forall n \geq 1, \quad S_n(x) := \varphi(x) + \varphi(T(x)) + \cdots + \varphi(T^{n-1}(x)), \quad x \in (0, 1].$$

Let

$$I = \{x \in (0, 1] : \epsilon_1(x) = 1\}.$$

Define the hitting time of $x \in (0, 1]$ to I as

$$n(x) := \inf\{n \geq 0 : T^n x \in I\}.$$

Then

$$n(x) = n \quad \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \quad \text{for all } n \geq 0.$$

Using $n(x)$, we define a new dynamical system $\widehat{T} : (0, 1] \rightarrow (0, 1]$ by

$$\widehat{T}(x) = T^{n(x)+1}(x) = 2^{n+1}\left(x - \frac{1}{2^{n+1}}\right) \quad \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \quad \text{for all } n \geq 0,$$

called the acceleration of T , in order that φ and $\varphi \circ \widehat{T}$ are independent. Let

$$\widehat{S}_n(x) := \varphi(x) + \varphi(\widehat{T}(x)) + \cdots + \varphi(\widehat{T}^{n-1}(x)), \quad x \in (0, 1].$$

The convergence in probability of $\widehat{S}_n(x)$ is well known (e.g. [6, p.253]) which states that for any $\epsilon > 0$, the Lebesgue measure λ of

$$\left\{x \in (0, 1] : \left|\frac{\widehat{S}_n(x)}{n \log n} - \frac{1}{\log 2}\right| \geq \epsilon\right\}$$

tends to 0 as $n \rightarrow \infty$.

Let $\{\Psi_n\}_{n \geq 1}$ be an increasing sequence such that $\Psi_n \rightarrow \infty$ as $n \rightarrow \infty$. Then it was shown in [5] that Lebesgue almost surely either

$$\lim_{n \rightarrow \infty} \frac{\widehat{S}_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n \rightarrow \infty} \frac{\widehat{S}_n(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n \geq 1} \lambda(\{x \in (0, 1] : \varphi(x) \geq \Psi_n\}) < \infty \quad \text{or} \quad = \infty.$$

Let $n_1 = n_1(x) = n(x) + 1$ and $n_k = n_k(x) = n_1(\widehat{T}^{k-1}x) = n(\widehat{T}^{k-1}x)$ for $k \geq 2$. It is direct to see that

$$\forall \ell \geq 1, \quad S_{n_1 + \cdots + n_\ell}(x) = 2\widehat{S}_\ell(x) - \ell.$$

Moreover, the ergodicity of T (of \widehat{T}) implies Lebesgue almost surely

$$\lim_{\ell \rightarrow \infty} \frac{n_1(x) + \cdots + n_\ell(x)}{\ell} = \int_0^1 (n(x) + 1) d\lambda(x) = 2.$$

Combining these two facts together, we obtain the same convergence results as above if we replace \widehat{S}_n by S_n . In particular, the average $S_n(x)/(n \log n)$ converges to $1/\log 2$ in probability, and almost surely (with respect to the Lebesgue measure) either

$$\lim_{n \rightarrow \infty} \frac{S_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n \rightarrow \infty} \frac{S_n(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n \geq 1} \lambda(\{x \in (0, 1] : \varphi(x) \geq \Psi_n\}) < \infty \quad \text{or} \quad = \infty,$$

where $\{\Psi_n\}_{n \geq 1}$ is an increasing sequence such that $\Psi_n \rightarrow \infty$ as $n \rightarrow \infty$. Recall that φ has infinite expectation with respect to the Lebesgue measure. We thus have $S_n(x)/n$ converges to infinity for Lebesgue almost all points.

In this article, we want to further study the asymptotic behavior of the Birkhoff sum $S_n(x)$ of the Saint-Petersburg potential. We give a complete multifractal analysis of $S_n(x)$.

First, for any $\alpha \geq 1$, we consider the level set

$$E(\alpha) = \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} S_n(x) = \alpha \right\}.$$

For $t \in \mathbb{R}$ and $q > 0$, define

$$P(t, q) := \log \sum_{j=1}^{\infty} 2^{-tj - q(2^j - 1)}.$$

Then P is a real-analytic function. Furthermore, for each $q > 0$, there is a unique $t(q) > 0$ such that $P(t(q), q) = 0$. This function $q \mapsto t(q)$ is real-analytic, strictly decreasing and convex.

Denote by \dim_H the Hausdorff dimension. The function $\alpha \mapsto \dim_H E(\alpha)$, called the Birkhoff spectrum of the Saint-Petersburg potential φ , is proved to be the Legendre transformation of the function $q \mapsto t(q)$.

Theorem 1.1. *For any $\alpha \geq 1$ we have*

$$\dim_H E(\alpha) = \inf_{q > 0} \{t(q) + q\alpha\}.$$

Consequently, $\dim_H E(1) = 0$ and the function $\alpha \mapsto \dim_H E(\alpha)$ is real-analytic, strictly increasing, concave, and has limit 1 as $\alpha \rightarrow \infty$.

The Birkhoff spectrum of a continuous potential was obtained for full shifts [14], for topologically mixing subshifts of finite type [4], and for repellers of a topologically mixing $C^{1+\epsilon}$ expanding map [2]. A continuous potential in a compact space is bounded, hence these classical results are all for bounded potentials. Our Theorem 1.1 gives a Birkhoff spectrum for an unbounded function with a singular point. To prove Theorem 1.1, we will transfer our question to a Birkhoff spectrum problem of an interval map with infinitely many branches and we will apply the techniques developed in [9] for continued fraction dynamical system and in [8] for general expanding interval maps with infinitely many branches.

We also study the Birkhoff sums $S_n(x)$ of fast increasing rates. Let $\Psi : \mathbb{N} \rightarrow \mathbb{N}$ be an increasing function. For $\beta \in [0, \infty]$, consider the level set

$$E_{\Psi}(\beta) := \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{\Psi(n)} S_n(x) = \beta \right\}.$$

Theorem 1.2. *If $\Psi(n)$ is one of the following*

$$\Psi(n) = n \log n, \quad \Psi(n) = n^a \quad (a > 1), \quad \Psi(n) = 2^{n^\gamma} \quad (0 < \gamma < 1/2),$$

then for any $\beta \in [0, \infty]$, $\dim_H E_{\Psi}(\beta) = 1$.

If $\Psi(n) = 2^{n^\gamma}$ with $1/2 \leq \gamma < 1$, then for any $\beta \in (0, \infty)$, the set $E_{\Psi}(\beta)$ is empty, and $\dim_H E_{\Psi}(\beta) = 1$ for $\beta = 0, +\infty$.

If $\Psi(n) = 2^{n^\gamma}$ with $\gamma \geq 1$, then for any $\beta \in (0, \infty]$, the set $E_{\Psi}(\beta)$ is empty, and $\dim_H E_{\Psi}(\beta) = 1$ for $\beta = 0$.

We remark that by the above discussion on the convergence of $S_n(x)$, for all cases in Theorem 1.2, the sets $E_{\Psi}(0)$ has full measure, and thus obviously has full Hausdorff dimension.

From the definition of $S_n(x)$, we see that for the integer n such that $\epsilon_n(x) = 1$, one has $S_n(x) = S_{n-1}(x) + 1$, which implies for all x , $\liminf_{n \rightarrow \infty} \frac{S_n(x)}{S_{n-1}(x)} = 1$. Thus if $\liminf_{n \rightarrow \infty} \frac{\Psi(n)}{\Psi(n-1)} > 1$, then for any $\beta \in (0, \infty)$, the set $E_\Psi(\beta)$ is empty. By the definition of $S_n(x)$, we can also check that for all x , for the integer n such that $\epsilon_n(x) = 1$, we have $S_n(x) \leq 2^n - 1$. This implies $\liminf_{n \rightarrow \infty} S_n(x)/2^n \leq 1$ (See also the formula (3.15) in Section 3). Hence, for 'regular' growth functions Ψ we only need to consider exponential and subexponential growth rates.

However, if we pick a point x with dyadic expansion consisting mostly of 0's, with infinitely many 1's but in large distances from each other, then the Birkhoff sum $S_{n_i}(x)$ may grow arbitrarily fast on some subsequence n_i . Thus for any increasing $\Psi(n)$ there exists a point x such that $\limsup_{n \rightarrow \infty} S_n(x)/\Psi(n) = \infty$.

Our study on the Saint-Petersburg potential is an attempt of multifractal analysis of unbounded potential functions on the doubling map dynamical system. However, the Saint-Petersburg potential is locally constant and not continuous. One might think of an another unbounded potential function $g : x \mapsto 1/x$ which is close to the Saint-Petersburg potential but is continuous. In fact, our method for studying the fast increasing Birkhoff sum of Saint-Petersburg potential also works for the fast increasing Birkhoff sum of the potential $g : x \mapsto 1/x$.

Denote by $S_n g(x)$ the Birkhoff sum

$$S_n g(x) := g(x) + g(T(x)) + \cdots + g(T^{n-1}(x)), \quad x \in (0, 1].$$

For $\beta \in [0, \infty]$, let

$$F_\Psi(\beta) := \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{\Psi(n)} S_n g(x) = \beta \right\}.$$

Theorem 1.3. *If $\Psi(n)$ is one of the following*

$$\Psi(n) = n \log n, \quad \Psi(n) = n^a \quad (a > 1), \quad \Psi(n) = 2^{n^\gamma} \quad (0 < \gamma < 1/2),$$

then for any $\beta \in [0, \infty]$, $\dim_H F_\Psi(\beta) = 1$.

If $\Psi(n) = 2^{n^\gamma}$ with $1/2 \leq \gamma < 1$, then for any $\beta \in (0, \infty)$, the set $F_\Psi(\beta)$ is empty, and $\dim_H F_\Psi(\beta) = 1$ for $\beta = 0, +\infty$.

If $\Psi(n) = 2^{n^\gamma}$ with $\gamma \geq 1$, then for any $\beta \in (0, \infty]$, the set $F_\Psi(\beta)$ is empty, and $\dim_H F_\Psi(\beta) = 1$ for $\beta = 0$.

We remark that these multifractal analysis on the Birkhoff sums of fast increasing rates have been done for some special potentials in continued fraction dynamical system ([9, 11, 12]).

2. BIRKHOFF SPECTRUM OF THE SAINT-PETERSBURG POTENTIAL

In this section, we will obtain the Birkhoff spectrum of the Saint-Petersburg potential, i.e. the Hausdorff dimension of the following level set:

$$E(\alpha) := \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad (\alpha \geq 1).$$

We will transfer our question to a Birkhoff spectrum problem for an interval map with infinitely many branches.

2.1. Transference lemma. Recall that the Saint-Petersburg potential φ is given by

$$\varphi(x) = 2^n, \text{ if } x = [0^n 1, \dots]$$

where $x = [\epsilon_1 \epsilon_2, \dots]$ denotes the digit sequence in the binary expansion of x . Recall also the definition of hitting time $n(x)$ and the acceleration \widehat{T} of the doubling map T in Section 1. Define a new potential function

$$\phi(x) := 2^{n(x)+1} - 1, \quad x \in (0, 1].$$

In fact, ϕ is nothing but the function satisfying

$$\phi(x) = \sum_{j=0}^{n(x)} \varphi(T^j x).$$

With the notation $n_1 = n(x) + 1 \geq 1$, and $n_k = n(\widehat{T}^{k-1}x) + 1$ for $k \geq 2$ given in Section 1, we have

$$\phi(\widehat{T}x) = \sum_{j=0}^{n(\widehat{T}x)} \varphi(T^j(\widehat{T}x)) = \sum_{j=n_1}^{n_2-1} \varphi(T^j x).$$

Hence,

$$\sum_{j=0}^{n_1+\dots+n_\ell-1} \varphi(T^j x) = \sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x) = 2^{n_1} + \dots + 2^{n_\ell} - \ell. \quad (2.1)$$

Note that the derivative of \widehat{T} satisfies

$$|\widehat{T}'|(x) = 2^{n(x)+1} = 2^{n_1} = \phi(x) + 1. \quad (2.2)$$

We have

$$n_1 + \dots + n_\ell = \sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x).$$

Recall the set in question:

$$E(\alpha) = \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad (\alpha \geq 1).$$

Define

$$\widetilde{E}(\alpha) := \left\{ x \in (0, 1] : \lim_{\ell \rightarrow \infty} \frac{\sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x)} = \alpha \right\} \quad (\alpha \geq 1).$$

The following lemma shows the two level sets are the same.

Lemma 2.1. *For all $\alpha \geq 1$, we have $E(\alpha) = \widetilde{E}(\alpha)$.*

Proof. It is evident that $E(\alpha) \subset \widetilde{E}(\alpha)$, because, as discussed above,

$$\frac{\sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x)} = \frac{1}{n_1 + \dots + n_\ell} \sum_{j=0}^{n_1+\dots+n_\ell-1} \varphi(T^j x). \quad (2.3)$$

Now, we show the other direction. Take an $x \in \widetilde{E}(\alpha)$, express x in its binary expansion

$$x = [0^{n_1-1} 1 0^{n_2-1} 1 \dots 0^{n_\ell-1} 1 \dots].$$

In fact, $n_\ell - 1$ is the recurrence time for $n(\widehat{T}^{\ell-1}x)$, for each $\ell \geq 1$.

By (2.1), we have, at present,

$$\lim_{\ell \rightarrow \infty} \frac{1}{n_1 + \cdots + n_\ell} \sum_{j=0}^{n_1 + \cdots + n_\ell - 1} \varphi(T^j x) = \alpha.$$

So, we are required to check it holds for all n .

For any $\epsilon > 0$, there exists $\ell_0 \in \mathbb{N}$ such that, for any $\ell \geq \ell_0$,

$$\alpha - \epsilon \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell} \leq \alpha + \epsilon. \quad (2.4)$$

For any $n_1 + \cdots + n_\ell < n < n_1 + \cdots + n_\ell + n_{\ell+1}$ with $\ell \geq \ell_0$, it is trivial that

$$\frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell + n_{\ell+1}} \leq \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1}{n_1 + \cdots + n_\ell}.$$

Thus, it suffices to show that

$$2^{n_{\ell+1}} = o(n_1 + \cdots + n_\ell), \quad (2.5)$$

which also implies

$$n_{\ell+1} = o(n_1 + \cdots + n_\ell).$$

Let M_0 be a large integer such that, for all $M \geq M_0$, $2^M \geq 4\alpha M$. So, when $n_{\ell+1} \leq M_0$, there is nothing to prove. So, we always assume $2^{n_{\ell+1}} \geq 4\alpha n_{\ell+1}$.

By (2.4), we have

$$\begin{aligned} 2^{n_1} + \cdots + 2^{n_\ell} - \ell &\geq (\alpha - \epsilon)(n_1 + \cdots + n_\ell), \\ 2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1 &\leq (\alpha + \epsilon)(n_1 + \cdots + n_\ell + n_{\ell+1}). \end{aligned}$$

These give

$$2^{n_{\ell+1}} \leq 2\epsilon(n_1 + \cdots + n_\ell) + (\alpha + \epsilon)n_{\ell+1} + 1.$$

So, we have

$$2^{n_{\ell+1}} \leq 4\epsilon(n_1 + \cdots + n_\ell). \quad \square$$

2.2. Dimension of $\tilde{E}(\alpha)$. Now we calculate the Hausdorff dimension of the set $\tilde{E}(\alpha)$. At first, we give a notation.

- For each finite word $w \in \bigcup_{n \geq 1} \{0, 1\}^n$ of length n , a T -dyadic cylinder of order n is defined as

$$I_n(w) = \{x \in (0, 1] : (\epsilon_1(x), \dots, \epsilon_n(x)) = w\}.$$

- For $(n_1, \dots, n_\ell) \in (\mathbb{N} \setminus \{0\})^\ell$, a \hat{T} -dyadic cylinder of order ℓ is defined as

$$D_\ell(n_1, \dots, n_\ell) = \{x \in (0, 1] : n_k(x) = n_k, 1 \leq k \leq \ell\}.$$

Proof of Theorem 1.1. To calculate the Hausdorff dimension of $\tilde{E}(\alpha)$, we construct a suitable measure supported on $\tilde{E}(\alpha)$. The Gibbs measures derived from the Ruelle-Perron-Frobenius transfer operator are good candidates for such a measure.

In fact, by considering the inverse branches $U_i : x \mapsto \frac{x+1}{2^{i+1}}$ ($i \geq 0$) of \hat{T} , we can code the dynamical system $([0, 1], \hat{T})$ by the conformal infinite iterated function system $(U_i)_{i \geq 0}$ which satisfies the open set condition ([10, Section 1]).

Consider the potential function with two parameters

$$\psi_{t,q} := -t \log |\hat{T}'| - (\log 2) \cdot q\phi \quad (t \in \mathbb{R}, q > 0).$$

Then $(\psi_{t,q} \circ U_i)_{i \geq 0}$ is a family of strong Hölder family ([10, Page 30]). Hence, we can define a Ruelle operator

$$\mathcal{L}_{t,q} f(x) := \sum_{y \in \hat{T}^{-1}x} e^{\psi_{t,q}(y)} f(y),$$

on the Banach space of continuous functions on the corresponding infinite symbolic space ([10, Page 31]).

By the Ruelle-Perron-Frobenius transfer operator theory [10, Theorems 2.9 and 2.10], for any $q > 0$ (to satisfy the condition 2.2 of [10]), we can find an eigenvalue $\lambda_{t,q}$ and an eigenfunction $h_{t,q}$ for $\mathcal{L}_{t,q}$ and an eigenfunction $\nu_{t,q}$ for the conjugate operator $\mathcal{L}_{t,q}^*$. Then the pressure function $P(t, q) = \log \lambda_{t,q}$ and the ergodic Gibbs measure $\mu_{t,q}$ is given by $h_{t,q} \cdot \nu_{t,q}$.

The pressure function can be computed by (see [10, Pages 31 and 48])

$$P(t, q) = \lim_{\ell \rightarrow \infty} \frac{1}{\ell} \log \left(\sum_{(n_1, \dots, n_\ell) \in (\mathbb{N} \setminus \{0\})^\ell} \exp \sup_{x \in D_\ell(n_1, \dots, n_\ell)} (S_\ell \psi_{t,q}(x)) \right).$$

Note that for all $n \geq 1$,

$$|\widehat{T}'|(x) = 2^n, \text{ and } \phi(x) = 2^n - 1 \text{ if } x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}} \right] = D_1(n).$$

Then for $x \in D_\ell(n_1, \dots, n_\ell)$,

$$S_\ell \psi_{t,q}(x) = -(\log 2) \cdot t \cdot \sum_{j=1}^{\ell} n_j + (\log 2) \cdot q \cdot \sum_{j=1}^{\ell} (2^{n_j} - 1).$$

Thus

$$\begin{aligned} & \sum_{(n_1, \dots, n_\ell) \in (\mathbb{N} \setminus \{0\})^\ell} \exp \sup_{x \in D_\ell(n_1, \dots, n_\ell)} (S_\ell \psi_{t,q}(x)) \\ &= \sum_{n_1=1}^{\infty} \dots \sum_{n_\ell=1}^{\infty} \left(\prod_{j=1}^{\ell} 2^{-tn_j} \cdot \prod_{j=1}^{\ell} 2^{q(2^{n_j}-1)} \right) = \left(\sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)} \right)^\ell. \end{aligned}$$

Hence

$$P(t, q) = \log \sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)}.$$

Now we calculate the local dimension of the Gibbs measure $\mu_{t,q}$. Let $D_\ell(x)$ be the \widehat{T} -dyadic cylinder containing x of order ℓ . By the Gibbs property of $\mu_{t,q}$,

$$\begin{aligned} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} &= \frac{S_\ell \psi_{t,q}(x) - \ell P(t, q)}{-S_\ell \log |\widehat{T}'|(x)} \\ &= \frac{-t S_\ell \log |\widehat{T}'|(x) - (\log 2) \cdot q S_\ell \phi(x) - \ell P(t, q)}{-S_\ell \log |\widehat{T}'|(x)} \quad (2.6) \\ &= t + q \frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}'|(x)} + \frac{\ell P(t, q)}{S_\ell \log |\widehat{T}'|(x)}. \end{aligned}$$

• UPPER BOUND. For each $q > 0$, let $t(q)$ be the the number such that $P(t(q), q) = 0$. (The existence of $t(q)$ comes from the facts that $P(t, q)$ is real-analytic and that for fixed $q > 0$, $P(t, q) > 0$ when $t \rightarrow -\infty$ and $P(t, q) < 0$ when $t \rightarrow +\infty$.) Then for all $x \in \widetilde{E}(\alpha)$, we have

$$\begin{aligned} \liminf_{r \rightarrow 0} \frac{\log \mu_{t,q}(B(x, r))}{\log r} &\leq \liminf_{\ell \rightarrow \infty} \frac{\log \mu_{t,q}(B(x, |D_\ell(x)|))}{\log |D_\ell(x)|} \\ &\leq \liminf_{\ell \rightarrow \infty} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} = t(q) + q\alpha, \end{aligned}$$

where for the second inequality the trivial inclusion $D_\ell(x) \subset B(x, |D_\ell(x)|)$ is used. By Billingsley Lemma (see e.g. [3, Proposition 4.9.]), this gives an upper bound of the Hausdorff dimension of $\tilde{E}(\alpha)$. Thus we have

$$\dim_H \tilde{E}(\alpha) \leq \inf_{q>0} \{t(q) + q\alpha\}.$$

• **LOWER BOUND.** By the real-analyticity of $P(t, q)$ and the implicit function theorem, the function $q \mapsto t(q)$ is also real-analytic. Thus there exists q_0 such that the following infimum is attained

$$\inf_{q>0} \{t(q) + q\alpha\}.$$

Then we have

$$t'(q_0) + \alpha = 0. \quad (2.7)$$

To prove the lower bound, we first show two claims.

Claim (A): The measure $\mu_{t(q_0), q_0}$ is supported on E_α .

On the one hand, since $P(t(q), q) = 0$,

$$\frac{\partial P}{\partial t} t'(q) + \frac{\partial P}{\partial q} = 0. \quad (2.8)$$

On the other hand, by the ergodicity of the measure $\mu_{t, q}$, we have for $\mu_{t, q}$ almost all x ,

$$\lim_{\ell \rightarrow \infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}^\ell(x)|} = \frac{\int \phi d\mu_{t, q}}{\int \log |\widehat{T}'| d\mu_{t, q}} \cdot \log 2.$$

By Ruelle-Perron-Frobenius transfer operator theory ([10], Proposition 6.5),

$$\int (\log 2) \cdot \phi d\mu_{t, q} = -\frac{\partial P}{\partial q} \quad \text{and} \quad \int \log |\widehat{T}'| d\mu_{t, q} = -\frac{\partial P}{\partial t}.$$

Thus by (2.8) and then (2.7), for $\mu_{t(q_0), q_0}$ almost all x ,

$$\lim_{\ell \rightarrow \infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}^\ell(x)|} = \frac{\frac{\partial P}{\partial q}}{\frac{\partial P}{\partial t}} = -t'(q_0) = \alpha.$$

This shows Claim (A).

Claim (B): For $\mu_{t(q_0), q_0}$ almost all x ,

$$\lim_{n \rightarrow \infty} \frac{\log \mu_{t(q_0), q_0}(I_n(x))}{\log 2^{-n}} = t(q_0) + q_0\alpha,$$

where $I_n(x)$ is the T -dyadic cylinder of order n containing x .

On the one hand, by (2.6) and then by (2.8) and (2.7), one has for $\mu_{t(q_0), q_0}$ almost all x

$$\lim_{\ell \rightarrow \infty} \frac{\log \mu_{t(q_0), q_0}(D_\ell(x))}{\log |D_\ell(x)|} = t(q_0) + q_0 \frac{\frac{\partial P}{\partial q}}{\frac{\partial P}{\partial t}} = t(q_0) + q_0\alpha. \quad (2.9)$$

On the other hand, note that for any $x \in E(\alpha)$, if the binary expansion of x is $x = [0^{n_1-1}10^{n_2-1}1\dots]$, then for any $\delta > 0$, for ℓ large enough,

$$(\alpha - \delta)\ell \leq 2^{n_1} + \dots + 2^{n_\ell} - \ell = S_\ell \phi(x) \leq (\alpha + \delta)\ell.$$

Hence

$$n_\ell = O(\log \ell),$$

which implies

$$\lim_{\ell \rightarrow \infty} \frac{\log |D_\ell(x)|}{\log |D_{\ell+1}(x)|} = \lim_{\ell \rightarrow \infty} \frac{n_1 + \dots + n_\ell}{n_1 + \dots + n_\ell + n_{\ell+1}} = 1. \quad (2.10)$$

Thus

$$\lim_{n \rightarrow \infty} \frac{\log \mu_{t(q_0), q_0}(I_n(x))}{\log 2^{-n}} = \lim_{\ell \rightarrow \infty} \frac{\log \mu_{t(q_0), q_0}(D_\ell(x))}{\log |D_\ell(x)|}.$$

This shows Claim (B).

To conclude the desired lower bound, we apply the classical mass distribution principle (see [3, Proposition 4.2]). Since the Hausdorff dimension will not be changed if we replace the δ -coverings by T -dyadic cylinder coverings (see [3, Section 2.4]), the lower bound of the Hausdorff dimension can be given by the mass transference principle on T -dyadic cylinders. By the above two claims and Egorov's theorem, for any $\eta > 0$, there exists an integer N_0 such that the set

$$\left\{ x \in E_\alpha : \mu(I_n(x)) \leq |I_n(x)|^{t(q_0) + q_0 \alpha - \eta}, \quad n \geq N \right\}$$

is of $\mu_{t(q_0), q_0}$ positive measure. So, it implies that

$$\dim_H E_\alpha \geq t(q_0) + q_0 \alpha - \eta.$$

Note that ([10, Lemma 7.5]) the function $q \mapsto t(q)$ is a decreasing convex function such that

$$t(0) = 1, \quad \lim_{q \rightarrow \infty} (t(q) + q) = 0,$$

and

$$\lim_{q \rightarrow 0^+} t'(q) = -\infty, \quad \lim_{q \rightarrow +\infty} t'(q) = -1.$$

Therefore, we have proved for any $\alpha \in (1, +\infty)$

$$\dim_H(\tilde{E}(\alpha)) = \inf_{q > 0} \{t(q) + q\alpha\},$$

which is Legendre transformation. All the properties stated in Theorem 1.1 are satisfied by the function $\alpha \mapsto \dim_H(\tilde{E}(\alpha))$ which is the same function as $\alpha \mapsto \dim_H(E(\alpha))$ by Lemma 2.1.

For the end point $\alpha = 1$, it suffices to note that the level set $E(1)$ is nothing but the set of numbers with frequency of the digit 1 in its binary expansion being 1. Thus the Hausdorff dimension of $E(1)$ is 0. Hence, the Legendre transformation formula for the Hausdorff dimension of $E(\alpha)$ ($\alpha > 1$) also holds for $\alpha = 1$. \square

3. FAST INCREASING BIRKHOFF SUM

At first, we give two simple observations.

Lemma 3.1. *Let W be an integer such that $2^t \leq W < 2^{t+1}$ for some positive integer t . For any $0 \leq n \leq t$, among the integers between W and $W(1+2^{-n})$, there is one $V = V(W, n)$ whose binary expansion of V has at most $n+2$ digits 1 and ends with at least $t-n$ zeros.*

Proof. By the assumption, we have $2^{-n}W \geq 2^{t-n}$. Thus among the $2^{-n}W$ consecutive integers from W to $W(1+2^{-n})$ there is at least one integer which is divisible by 2^{t-n} which means there is an integer $\ell \geq 1$ such that

$$W \leq \ell 2^{k-n} \leq W(1+2^{-n}).$$

Let $V = \ell 2^{t-n}$ and note that V is an integer whose binary expansion ends with at least $t-n$ zeros. Since $\ell 2^{t-n} \leq W(1+2^{-n}) < 2^{t+2}$, we conclude that $\ell 2^{t-n}$ has at most $(t+2) - (t-n) = n+2$ digits 1 in its binary expansion. \square

In the follows, the base of the logarithm is taken to be 2.

Lemma 3.2. *For each integer W , and any integer $n \leq \log W$, we can find a word w with length*

$$|w| \leq (n+2)(2 + \log W)$$

and for any $x \in I_{|w|}(w)$

$$W \leq \sum_{j=0}^{|w|-1} \varphi(T^j x) \leq W(1 + 2^{-n}).$$

Proof. Let V be an integer given in Lemma 3.1. Then $W \leq V \leq W(1 + 2^{-n})$. Moreover if we write this number V in binary expansion:

$$V = 2^{t_1} + \cdots + 2^{t_p},$$

one has that $\lfloor \log W \rfloor + 1 \geq t_1 > \cdots > t_p \geq \lfloor \log W \rfloor - n$ and $p \leq n + 2$. Consider the word

$$w = (10^{t_1-1}1, 10^{t_2-1}1, \dots, 10^{t_p-1}1)$$

here the word $10^{t_p-1}1$ is 1 when $t_p = 0$. Then we can check that the length of w satisfies

$$|w| = (t_1 + 1) + \cdots + (t_p + 1) \leq p(t_1 + 1) \leq (n+2)(2 + \log W),$$

and for any $x \in I_{|w|}(w)$,

$$\sum_{j=0}^{|w|-1} \varphi(T^j x) = V.$$

Hence, the proof is completed. \square

We also need the following lemmas whose proofs are left for the reader.

Lemma 3.3. *For any $m \geq 1$, define*

$$F_m = \{x \in (0, 1] : \epsilon_{km}(x) = 1, \text{ for all } k \geq 1\}.$$

Then $\dim_H F_m = \frac{m-1}{m}$.

Lemma 3.4. [13, Lemma 4] *Given a subset \mathbb{J} of positive integers and an infinite sequence $\{a_k\}_{k=1}^\infty$ of 0's and 1's, let*

$$E(\mathbb{J}, \{a_k\}_{k=1}^\infty) = \left\{ x \in (0, 1] : \epsilon_k(x) = a_k, \text{ for all } k \in \mathbb{J} \right\}.$$

If the density of \mathbb{J} is zero, that is,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Card}\{k \leq n : k \in \mathbb{J}\} = 0$$

then $\dim_H E(\mathbb{J}, \{a_k\}_{k=1}^\infty) = 1$.

Before the proof Theorem 1.2, we show the following lemma.

Lemma 3.5. *Let $\Psi : \mathbb{N} \rightarrow \mathbb{N}$ be an increasing function such that $\Psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$. Assume that there exists a subsequence N_k satisfying the following conditions*

$$N_k - N_{k-1} \rightarrow \infty, \quad \Psi(N_k) - \Psi(N_{k-1}) \rightarrow \infty, \quad (3.1)$$

and

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} \rightarrow 1, \quad \frac{\log(\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \rightarrow 0, \quad (3.2)$$

as $k \rightarrow \infty$. Then the set

$$E_\Psi(1) = \left\{ x \in (0, 1] : \lim_{n \rightarrow \infty} \frac{1}{\Psi(n)} S_n(x) = 1 \right\}$$

has Hausdorff dimension 1.

Proof. Fix a large integer m and write

$$\mathcal{U} = \left\{ u = (\epsilon_1, \dots, \epsilon_m) : \epsilon_m = 1, \epsilon_i \in \{0, 1\}, i \neq m \right\}.$$

To avoid the abuse of notation, by the first assumption of (3.1), we assume $N_k - N_{k-1} \gg m$ for all $k \geq 1$ by setting $N_0 = 0$ and $\Psi(N_0) = 0$.

For each $k \geq 1$, we write

$$W_k := \Psi(N_k) - \Psi(N_{k-1})$$

and let $\{n_k\}$ be a sequence of integers tending to ∞ such that

$$n_k \leq \log W_k, \quad n_k \cdot \frac{\log(\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \rightarrow 0.$$

By the second assumptions of (3.1) and (3.2), this sequence of $n_k \geq 0$ do exist.

Now for W_k and n_k , let w_k be the word given in Lemma 3.2. Then the length a_k of w_k satisfies

$$\begin{aligned} a_k &\leq (n_k + 2)(2 + \log W_k) \\ &= (n_k + 2)(2 + \log(\Psi(N_k) - \Psi(N_{k-1}))) = o(N_k - N_{k-1}) \end{aligned} \quad (3.3)$$

and for any $x \in I_{a_k}(w_k)$,

$$W_k \leq \sum_{j=0}^{a_k-1} \varphi(T^j x) \leq W_k(1 + 2^{-n_k}). \quad (3.4)$$

Define t_k, ℓ_k to be the integers satisfying

$$N_k - N_{k-1} - a_k = t_k m + \ell_k,$$

for some $0 \leq \ell_k < m$.

Let w_k ($k \geq 1$) be given as the above. We define a Cantor subset of $E_\Psi(1)$ as follows.

Level 1 of the Cantor subset. Define

$$E_1 = \left\{ I_{N_1}(u_1, \dots, u_{t_1}, 1^{\ell_1}, w_1) : u_i \in \mathcal{U}, 1 \leq i \leq t_1 \right\}.$$

For simplicity, we use $I_{N_1}(U_1)$ to denote a general cylinder in E_1 .

Level 2 of the Cantor subset. This level is composed by sublevels for each cylinder $I_{N_1}(U_1) \in E_1$. Fix an element $I_{N_1} = I_{N_1}(U_1) \in E_1$. Define

$$E_2(I_{N_1}(U_1)) = \left\{ I_{N_2}(U_1, u_1, \dots, u_{t_2}, 1^{\ell_2}, w_2) : u_i \in \mathcal{U}, 1 \leq i \leq t_2 \right\}.$$

Then

$$E_2 = \bigcup_{I_{N_1} \in E_1} E_2(I_{N_1}).$$

For simplicity, we use $I_{N_2}(U_2)$ to denote a general cylinder in E_2 .

From Level k to $k+1$. Fix $I_{N_k}(U_k) \in E_k$. Define

$$E_{k+1}(I_{N_k}(U_k)) = \left\{ I_{N_{k+1}}(U_k, u_1, \dots, u_{t_{k+1}}, 1^{\ell_{k+1}}, w_{k+1}) : u_i \in \mathcal{U}, 1 \leq i \leq t_{k+1} \right\}.$$

Then

$$E_{k+1} = \bigcup_{I_{N_k} \in E_k} E_{k+1}(I_{N_k}).$$

Up to now we have constructed a sequence of nested sets $\{E_k\}_{k \geq 1}$. Set

$$F = \bigcap_{k \geq 1} E_k.$$

We claim that

$$F \subset E(\Psi).$$

In fact, for all $x \in F$, by construction, for each $k \geq 1$,

$$\begin{aligned} & \sum_{n=N_{k-1}}^{N_k-1} \varphi(T^n x) \\ = & \sum_{n=N_{k-1}}^{N_{k-1}+t_k m-1} \varphi(T^n x) + \sum_{n=N_{k-1}+t_k m}^{N_{k-1}+t_k m+\ell_k-1} \varphi(T^n x) + \sum_{n=N_{k-1}+t_k m+\ell_k}^{N_k-1} \varphi(T^n x) \\ = & t_k O(2^m) + \ell_k + W_k(1 + O(2^{-n_k})) \\ = & O\left(\frac{(N_k - N_{k-1})2^m}{m}\right) + (\Psi(N_k) - \Psi(N_{k-1}))(1 + O(2^{-n_k})). \end{aligned}$$

Since $n_k \rightarrow \infty$ which implies $2^{-n_k} \rightarrow 0$ as $k \rightarrow \infty$, we have

$$\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k)(1 + o(1)) + O\left(\frac{N_k 2^m}{m}\right).$$

By the assumption $\Psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$, we then deduce

$$\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k) + o(\Psi(N_k)),$$

Thus

$$\lim_{k \rightarrow \infty} \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k)} = 1. \quad (3.5)$$

While, for each $N_{k-1} < N \leq N_k$

$$\frac{\sum_{n=0}^{N_{k-1}-1} \varphi(T^n x)}{\Psi(N_k)} \leq \frac{\sum_{n=0}^{N-1} \varphi(T^n x)}{\Psi(N)} \leq \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_{k-1})}.$$

So by the first assumption of (3.2), we deduce from (3.5) that

$$\lim_{n \rightarrow \infty} \frac{1}{\Psi(n)} S_n(x) = 1.$$

This proves $x \in E_{\Psi}(1)$ and hence $F \subset E_{\Psi}(1)$.

In the following, we will construct a Hölder continuous function from F to F_m . Recall that

$$F_m = \{x \in (0, 1] : \epsilon_{km}(x) = 1, \text{ for all } k \geq 1\}.$$

Define

$$\begin{aligned} f : F & \rightarrow F_m \\ x & \mapsto y \end{aligned}$$

where y is obtained by eliminating the digits $\{(\epsilon_{N_k - \ell_k - a_k + 1}, \dots, \epsilon_{N_k})\}_{k \geq 1}$ in the binary expansion of x . Now we calculate the Hölder exponent of f .

Take two points $x_1, x_2 \in F$ closed enough. Let n be the smallest integer such that $\epsilon_n(x_1) \neq \epsilon_n(x_2)$ and k be the integer such that $N_k < n \leq N_{k+1}$. Note that by the construction of F , the digits sequence

$$\{(\epsilon_{N_k - \ell_k - a_k + 1}, \dots, \epsilon_{N_k})\}_{k \geq 1} \text{ and } \{\epsilon_{N_k + tm}\}_{1 \leq t \leq t_{k+1}}$$

are the same for all $x \in F$. So we must have

$$N_k < n < N_{k+1} - \ell_{k+1} - a_{k+1}. \quad (3.6)$$

Since n is strictly less than $N_{k+1} - \ell_{k+1} - a_{k+1}$ and $\epsilon_{N_k+tm}(x_1) = \epsilon_{N_k+tm}(x_2) = 1$ for all $1 \leq t \leq t_{k+1}$, thus, at most m steps after the position n , saying n' , $\epsilon_{n'}(x_1) = \epsilon_{n'}(x_2) = 1$. So it follows that

$$|x_1 - x_2| \geq \frac{1}{2^{n+m}}.$$

Again by the construction and the definition of the map f , we have $y_1 = f(x_1)$ and $y_2 = f(x_2)$ have common digits up to the position $n-1-(\ell_1+a_1)-\dots-(\ell_k+a_k)$. Thus, it follows

$$|f(x_1) - f(x_2)| \leq \frac{1}{2^{n-1-(\ell_1+a_1)-\dots-(\ell_k+a_k)}}.$$

Recall that $\ell_k < m$ and $a_1 + \dots + a_k = o(N_k)$ (see (3.3)) and also that $N_k/k \rightarrow \infty$ as $k \rightarrow \infty$ (by (3.1)). We have

$$1 \geq \frac{n-1-(\ell_1+a_1)-\dots-(\ell_k+a_k)}{n+m} \geq \frac{n-1-km-o(N_k)}{n+m} = 1+o(1),$$

which implies that f is $(1-\eta)$ -Hölder for any $\eta > 0$. Thus

$$\dim_H F \geq (1-\eta) \dim_H F_m.$$

By Lemma 3.3, we then have

$$\dim_H F \geq (1-\eta) \frac{m-1}{m}.$$

By the arbitrariness of $\eta > 0$ and letting $m \rightarrow \infty$, we conclude that $\dim_H E(\Psi) = 1$. This finishes the proof. \square

Proof of Theorem 1.2. In all the three parts of Theorem 1.2, the case of $\beta = 0$ is a direct consequence of Theorem 1.1.

(I). Assume that Ψ is one of the functions $\Psi(n) = n \log n$, $\Psi(n) = n^a$ ($a > 1$), $\Psi(n) = 2^{n^\gamma}$ with $0 < \gamma < 1/2$.

(I₁). $0 < \beta < \infty$. It suffices to consider the dimension of $E_\Psi(1)$ i.e. $\beta = 1$, since for other $\beta \in (0, \infty)$, we need only replace $\Psi(n)$ by $\beta\Psi(n)$.

To show $\dim_H E_\Psi(1) = 1$, we can apply Lemma 3.5 directly. If $\Psi(n) = n \log n$, we can choose $N_k = k^2$. For $\Psi(n) = n^a$ ($a > 1$), we can also choose $N_k = k^2$. Suppose now $\Psi(n) = 2^{n^\gamma}$ with $0 < \gamma < 1/2$. Let $\delta > 0$ be small such that

$$\frac{\gamma}{1-\gamma} + \delta\gamma < 1 \quad (3.7)$$

which is possible since $\gamma < 1/2$. Take

$$N_k = \lfloor k^{\frac{1}{1-\gamma} + \delta} \rfloor. \quad (3.8)$$

Then we have

$$N_{k+1} - N_k \approx k^{\frac{\gamma}{1-\gamma} + \delta}, \quad (3.9)$$

and

$$\begin{aligned} \log(\Psi(N_{k+1}) - \Psi(N_k)) &\approx \log(\Psi'(N_k)(N_{k+1} - N_k)) \\ &\approx N_k^\gamma + \log(N_{k+1} - N_k) \approx N_k^\gamma. \end{aligned}$$

Here we write $A \approx B$ when $A/B \rightarrow 1$. This shows the validity of (3.1). Moreover,

$$\frac{\log(\Psi(N_{k+1}) - \Psi(N_k))}{N_{k+1} - N_k} \approx \frac{k^{\frac{\gamma}{1-\gamma} + \gamma\delta}}{k^{\frac{\gamma}{1-\gamma} + \delta}} = k^{-\delta(1-\gamma)} \rightarrow 0 \quad (k \rightarrow \infty).$$

Thus the second assumption of (3.2) is satisfied. At last, for the first assumption in (3.2), by (3.7)

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} = 2^{(k-1)^{\frac{\gamma}{1-\gamma} + \delta\gamma} - k^{\frac{\gamma}{1-\gamma} + \delta\gamma}} \rightarrow 1.$$

Hence Lemma 3.5 applies.

(I₂). If $\beta = \infty$, we may choose $\tilde{\Psi}(n) = 2^{n^\eta}$ for some $0 < \eta < \frac{1}{2}$ such that $E_{\tilde{\Psi}}(1) \subset E_{\Psi}(\infty)$. Then $\dim_H E_{\Psi}(\infty) = 1$ follows from (I₁).

(II). Now suppose that $\Psi(n) = 2^{n^\gamma}$ with $1/2 \leq \gamma < 1$.

(II₁). Let $\beta \in (0, \infty)$. We will prove that $E_{\Psi}(\beta)$ is empty. On the contrary, suppose there is $x \in E_{\Psi}(\beta)$, which has binary expansion

$$x = [0^{n_1-1}10^{n_2-1}1 \dots 0^{n_\ell-1}1 \dots]. \quad (3.10)$$

Then, by (2.1) we have

$$\begin{aligned} \frac{S_{n_1+n_2+\dots+n_\ell}(x)}{\Psi(n_1+n_2+\dots+n_\ell)} &= \frac{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell}{2^{(n_1+n_2+\dots+n_\ell)\gamma}} \rightarrow \beta, \\ \frac{S_{n_1+n_2+\dots+n_{\ell+1}}(x)}{\Psi(n_1+n_2+\dots+n_{\ell+1})} &= \frac{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell + 2^{n_{\ell+1}-1}}{2^{(n_1+n_2+\dots+n_{\ell+1})\gamma}} \rightarrow \beta. \end{aligned} \quad (3.11)$$

Since

$$\frac{2^{(n_1+n_2+\dots+n_\ell)\gamma}}{2^{(n_1+n_2+\dots+n_{\ell+1})\gamma}} \rightarrow 1,$$

by dividing the two limits of (3.11), we deduce that

$$\frac{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell + 2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell} = 1 + \frac{2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell} \rightarrow 1,$$

which implies that

$$\frac{S_{n_1+n_2+\dots+n_{\ell+1}}(x)}{S_{n_1+n_2+\dots+n_\ell}(x)} = 1 + \frac{2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell} \rightarrow 1.$$

Combining with (3.11), we get

$$1 \leftarrow \frac{\Psi(n_1 + \dots + n_{\ell+1})}{\Psi(n_1 + \dots + n_\ell)} = \frac{2^{(n_1+n_2+\dots+n_\ell+n_{\ell+1})\gamma}}{2^{(n_1+n_2+\dots+n_\ell)\gamma}}.$$

Thus

$$\begin{aligned} &(n_1 + n_2 + \dots + n_\ell + n_{\ell+1})^\gamma - (n_1 + n_2 + \dots + n_\ell)^\gamma \\ &= (n_1 + n_2 + \dots + n_\ell)^\gamma \left(\left(1 + \frac{n_{\ell+1}}{n_1 + n_2 + \dots + n_\ell} \right)^\gamma - 1 \right) \\ &\approx \frac{\gamma n_{\ell+1}}{(n_1 + n_2 + \dots + n_\ell)^{1-\gamma}} \rightarrow 0. \end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $k_0 \geq 1$ such that for all $j > k_0$,

$$n_j < \varepsilon(n_1 + n_2 + \dots + n_{j-1})^{1-\gamma}. \quad (3.12)$$

Then for any $k_0 < j \leq \ell$

$$n_j < \varepsilon(n_1 + n_2 + \dots + n_\ell)^{1-\gamma}.$$

This implies

$$\begin{aligned} S_{n_1+n_2+\dots+n_\ell}(x) &= 2^{n_1} + 2^{n_2} + \dots + 2^{n_\ell} - \ell \\ &\leq M + \ell 2^{\varepsilon(n_1+n_2+\dots+n_\ell)^{1-\gamma}} - \ell, \end{aligned}$$

with $M := 2^{n_1} + \dots + 2^{n_{k_0}}$. Thus we have

$$\frac{S_{n_1+n_2+\dots+n_\ell}(x)}{\Psi(n_1+n_2+\dots+n_\ell)} < \frac{M + \ell 2^{\varepsilon(n_1+n_2+\dots+n_\ell)^{1-\gamma}} - \ell}{2^{(n_1+n_2+\dots+n_\ell)\gamma}}. \quad (3.13)$$

By observing $n_j \geq 1$, we deduce that the upper bound of (3.13) converges to 0 for $1/2 \leq \gamma < 1$, a contradiction to (3.11). Hence $E_{\Psi}(\beta)$ is an empty set.

(II₂). $\beta = \infty$. Fix $\delta \in (\gamma, 1)$ and take a large integer K such that $2^{K\delta} > 1$. Consider the set of points such that at every position $2^k, k > K$ in their binary expansions, they have a string of zeros of length $2^{\delta k}$, i.e.

$$E := \left\{ x \in (0, 1] : \epsilon_{2^k+1} = \cdots = \epsilon_{2^k+2^{\delta k}} = 0, \text{ for all } k \geq K \right\}.$$

On the one hand, $E \subset E_\psi(\infty)$, since for any $n \in (2^k, 2^{k+1}]$ for some $k \geq K$,

$$S_n(x) > 2^{2^{\delta k}} \geq 2^{(n/2)^\delta} \gg 2^{n^\gamma}.$$

On the other hand, the set E has dimension 1 guaranteed by Lemma 3.4.

(III). Suppose that $\Psi(n) = 2^{n^\gamma}$ with $\gamma \geq 1$ and let $\beta \in (0, +\infty]$. Assume that there exists $x \in E_\Psi(\beta)$ for some $\beta \in (0, +\infty)$. Write the binary expansion of x as (3.10). Then by (2.1),

$$\begin{aligned} \frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} &= \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}} \rightarrow \beta, \\ \frac{S_{n_1+n_2+\cdots+n_\ell-1}(x)}{\Psi(n_1+n_2+\cdots+n_\ell-1)} &= \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell - 1}{2^{(n_1+n_2+\cdots+n_\ell-1)^\gamma}} \rightarrow \beta. \end{aligned} \quad (3.14)$$

However,

$$\frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell}{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell - 1} \rightarrow 1 \quad \text{but} \quad \frac{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}{2^{(n_1+n_2+\cdots+n_\ell-1)^\gamma}} \geq 2,$$

which is a contradiction. Hence $E_\Psi(\beta)$ is empty when $\beta \in (0, +\infty)$.

When $\beta = +\infty$, by (2.1), we have

$$\liminf_{n \rightarrow \infty} \frac{S_n(x)}{2^n} \leq 1. \quad (3.15)$$

So,

$$\liminf_{n \rightarrow \infty} \frac{S_n(x)}{\Psi(n)} \leq 1.$$

This shows that $E_\Psi(\infty)$ is also empty. \square

4. THE POTENTIAL $1/x$

In fact, the techniques in Section 3 can be applied to the continuous potential $g : x \mapsto 1/x$ on $(0, 1]$ which has a singularity at 0.

Proof of Theorem 1.3. We first show that if $\Psi(n)$ is one of the following

$$\Psi(n) = n \log n, \quad \Psi(n) = n^a \quad (a > 1), \quad \Psi(n) = 2^{n^\gamma} \quad (0 < \gamma < 1/2),$$

then for any $\beta \in [0, \infty]$, $\dim_H F_\Psi(\beta) = 1$.

We note that if $x \in (0, 1]$ has binary expansion $x = [0^n 1^s \dots]$, then $\varphi(x) = 2^n$ and

$$2^n \leq g(x) \leq 2^n + 2^{n-s+1} = 2^n(1 + 2^{-s+1}). \quad (4.1)$$

In Lemma 3.2, for an integer W , and for any integer $n \leq \log W$, we can construct instead of the words $w = (10^{t_1-1}1, 10^{t_2-1}1, \dots, 10^{t_p-1}1)$, the following word

$$w = (10^{t_1-1}1^{s+1}, 10^{t_2-1}1^{s+1}, \dots, 10^{t_p-1}1^{s+1}).$$

Then the length of the word satisfies

$$|w| = \sum_{i=1}^p (t_i + s + 1) \leq p(t_1 + s + 1) \leq (n + 2)(\log W + s + 2). \quad (4.2)$$

By (4.1), for any $x \in I_{|w|}(w)$,

$$W + s(n+2) \leq \sum_{j=0}^{|w|-1} g(T^j x) \leq W(1+2^{-n}) \cdot (1+2^{-s}) + 2s(n+2).$$

For each $k \geq 1$, we still write

$$W_k := \Psi(N_k) - \Psi(N_{k-1})$$

and let n_k, s_k be a sequence of integers tending to ∞ such that

$$n_k \cdot \frac{\log(\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \rightarrow 0, \quad (4.3)$$

$$\frac{n_k \cdot s_k}{N_k - N_{k-1}} \rightarrow 0, \quad (4.4)$$

and

$$\frac{n_k \cdot s_k}{\Psi(N_k) - \Psi(N_{k-1})} \rightarrow 0. \quad (4.5)$$

By (3.1) and (3.2), these two sequences of $n_k \geq 0, s_k \geq 0$ do exist.

Now for W_k and n_k, s_k , let w_k be the word given as above. Then by (4.3) and (4.4), the length a_k of w_k satisfies

$$\begin{aligned} a_k &\leq (n_k + 2)(\log W_k + s_k + 2) \\ &= (n_k + 2)(\log(\Psi(N_k) - \Psi(N_{k-1})) + s_k + 2) \\ &= o(N_k - N_{k-1}) \end{aligned} \quad (4.6)$$

and for any $x \in I_{a_k}(w_k)$,

$$\begin{aligned} W_k + s_k(n_k + 2) &\leq \sum_{j=0}^{a_k-1} g(T^j x) \\ &\leq W_k(1+2^{-n_k}) \cdot (1+2^{-s_k}) + 2s_k(n_k + 2). \end{aligned} \quad (4.7)$$

Hence by (4.5) we still have the same estimation:

$$\sum_{n=0}^{N_k-1} g(T^n x) = \Psi(N_k) + o(\Psi(N_k))$$

and the rest of the proof is the same as (I) of the proof of Theorem 1.2.

We can repeat the same arguments in Section 3 and show that for potential g , the set $F_\Psi(\beta)$ is empty if $\beta \in (0, \infty), \Psi(n) = 2^{n^\gamma} (1/2 \leq \gamma < 1)$ or $\beta \in (0, \infty], \Psi(n) = 2^{n^\gamma} (\gamma \geq 1)$.

In fact, by definition, if there exists $x \in F_\Psi(\beta)$, with its binary expansion

$$x = [0^{n_1-1} 1 0^{n_2-1} 1 \dots 0^{n_\ell-1} 1 0^{n_{\ell+1}-1} 1 \dots],$$

then

$$\frac{S_{n_1+n_2+\dots+n_\ell} g(x)}{\Psi(n_1+n_2+\dots+n_\ell)} \rightarrow \beta, \quad \frac{S_{n_1+n_2+\dots+n_\ell+1} g(x)}{\Psi(n_1+n_2+\dots+n_\ell+1)} \rightarrow \beta.$$

Thus

$$\frac{S_{n_1+n_2+\dots+n_\ell} g(x)}{S_{n_1+n_2+\dots+n_\ell+1} g(x)} \rightarrow 1.$$

Observing $\varphi \leq g \leq 2\varphi$, we have

$$\frac{2^{n_{\ell+1}}}{S_{n_1+n_2+\dots+n_\ell} g(x)} \rightarrow 0,$$

which then implies

$$\frac{S_{n_1+n_2+\dots+n_\ell}g(x)}{S_{n_1+n_2+\dots+n_{\ell+1}}g(x)} \rightarrow 1.$$

By the definition of $x \in F_\Psi(\beta)$, we have

$$\frac{\Psi(n_1 + n_2 + \dots + n_{\ell+1})}{\Psi(n_1 + n_2 + \dots + n_\ell)} \rightarrow 1.$$

This further implies the same inequality with (3.12) and the rest of proof is the same as (II₁) and (III) by noting $\varphi \leq g \leq 2\varphi$.

For $\Psi(n) = 2^{n^\gamma}$ ($1/2 \leq \gamma < 1$), we can also prove $\dim_H F_\Psi(\infty) = 1$ by the same proof as (II₂). \square

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