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# Abstract nonconforming error estimates and application to boundary penalty methods for diffusion equations and time-harmonic Maxwell's equations

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## Abstract

We devise a novel framework for the error analysis of nonconforming finite element techniques with low-regularity solutions. The key is to use some recently-derived mollification operators that commute with the usual derivative operators. We show how to apply the approach in the context of Nitsche's boundary penalty method for a diffusion equation and for the time-harmonic Maxwell's equations. In both cases, we address the case of heterogeneous material properties. We also compare the derived error estimates to those obtained in the framework of the error analysis framework proposed by Gudi where a trimming operator is introduced to map discrete test functions into conforming test functions.

**Keywords.** Finite elements, Nonconforming methods, Error estimates, Minimum regularity, Nitsche method, Boundary penalty, Elliptic equations, Time-harmonic Maxwell equations.

## 1 Introduction

The error analysis of the finite element approximation of Partial Differential Equations (PDEs) is well understood; see, e.g., the textbooks [4, 5, 10]. The most basic result is Céa's Lemma [8] which is valid when the approximation setting is conforming (the discrete trial and test spaces are subspaces of their exact counterparts) and exactly consistent (the discrete forms are restrictions of the exact ones to the discrete spaces). Departures from this setting are usually handled in the literature by invoking Strang's Lemmas [24]. Strang's First Lemma still assumes that the approximation setting is conforming but handles the case where the discrete forms differ from their exact counterpart. Strang's Second Lemma deals with nonconforming approximation settings and is frequently invoked in the literature for the error analysis of nonconforming techniques. For instance, many authors have adopted this approach to analyze discontinuous Galerkin (dG) methods (see, e.g., [9, 11] and references therein).

One important shortcoming of Strang's Second Lemma is that one needs to insert the exact solution in the first argument of the discrete sesquilinear (or bilinear) form. Unfortunately, this is in general only possible if one assumes some additional regularity on the exact solution that goes beyond the regularity provided by the weak formulation of the model problem at hand. For instance, when approximating a diffusion equation of the form  $-\nabla \cdot (\kappa \nabla u) = f$  in some Lipschitz domain  $D$  in  $\mathbb{R}^d$ , one is essentially led to assume that  $u \in H^{1+r}(D)$  with  $r > \frac{1}{2}$  so as to make sense of the normal component  $\mathbf{n} \cdot (\kappa \nabla u)$  at the mesh interfaces. Although this assumption is not really restrictive for the Laplace equation (where  $\kappa \equiv 1$ ) since elliptic regularity guarantees the existence of an index  $r > \frac{1}{2}$  in every Lipschitz domain, it becomes unrealistic in problems with discontinuous coefficients. Similarly, for the time-harmonic Maxwell's equations of the form  $\tilde{\mu} \mathbf{A} + \nabla \times (\kappa \nabla \times \mathbf{A}) = \mathbf{f}$  in some Lipschitz domain  $D$  in  $\mathbb{R}^3$ , one is led to assume  $\kappa \nabla \times \mathbf{A} \in \mathbf{H}^r(D)$  with  $r > \frac{1}{2}$  so as to make sense of the tangential component  $\mathbf{n} \times (\kappa \nabla \times \mathbf{A})$  at the mesh interfaces, but this assumption becomes unrealistic in problems with discontinuous coefficients. Let us mention in passing that we use boldface notation for  $\mathbb{R}^d$ -valued fields in  $D$ .

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One possible way forward to overcome the limitations of Strang’s Second Lemma has been proposed by Gudi [15]. The main idea is to introduce an operator that transforms the discrete test functions into elements of the exact test space. We call this operator a trimming operator, and we call the resulting error estimate a trimmed error estimate. The reason for our terminology is that one can view the elements in the kernel of the trimming operator as discrete (test) functions that are only needed to “stabilize” the bilinear form  $a_h$  but do not contribute to the interpolatory properties of the approximation setting. We also observe that a trimming operator is one of the fundamental ingredients in the abstract setting recently devised by Veerer and Zanotti [25] to obtain quasi-optimal energy-norm error estimates for nonconforming finite element methods applied to symmetric elliptic PDEs. The trimmed error estimate in [15] has been applied to the Interior Penalty dG (IPDG) approximation of the Laplace equation with a source term  $f \in L^2(D)$  (and to a fourth-order problem also in [15], to the Stokes equations in [2], and to the linear elasticity equations in [7]). In the present work, we show how to apply the trimmed error estimate to the diffusion equation with heterogeneous material property  $\kappa$  and source term  $f \in L^p(D)$  with  $p \in (2_*, 2]$ ,  $2_* = \frac{2d}{d+2}$ , and also to the time-harmonic Maxwell’s equations with heterogeneous material properties  $\tilde{\mu}, \kappa$  and source term  $\mathbf{f} \in \mathbf{L}^2(D)$ . For simplicity, we focus, for both model problems, on the use of  $H^1$ -conforming finite elements combined with the boundary penalty method of Nitsche [20] to enforce weakly Dirichlet boundary conditions. We do not anticipate any significant difficulty in extending the present analysis to IPDG approximations for both model problems, since Nitsche’s boundary penalty method is actually the workhorse in building the IPDG approximation. The main benefit of the trimmed error analysis is that it allows one to derive error estimates as soon as the exact solution is in  $\{v \in H^1(D) \mid \nabla \cdot (\kappa \nabla v) \in L^p(D)\}$ ,  $p \in (2_*, 2]$ , for the diffusion equation, and as soon as the exact solution is in  $\{\mathbf{A} \in \mathbf{H}(\text{curl}; D) \mid \nabla \times (\kappa \nabla \times \mathbf{A}) \in \mathbf{L}^2(D)\}$  for the time-harmonic Maxwell’s equations.

One difficulty still remains with the trimmed error estimate in the case of strong contrasts in the material property  $\kappa$  since the error estimates feature a constant that is typically proportional to the square-root of a contrast factor associated with  $\kappa$  (and, in the case of Maxwell’s equations, there is also a dependency on the square-root of a local magnetic Reynolds number). These dependencies originate from the usage of the trimming operator to perform some averaging to achieve the desired conformity property, but this averaging, in turn, precludes the derivation of stability and approximation properties for the trimming operator that are local to a mesh cell. To remedy this difficulty, we devise in this work a novel approach which is based on the use of a mollification operator, and which avoids invoking any trimming on the test functions. We call the resulting error estimate a mollified error estimate. In this work, we present an abstract setting for the error analysis based on this idea and then we show how to apply it to Nitsche’s boundary penalty method to approximate the diffusion equation and the time-harmonic Maxwell’s equations. One interesting observation is that the mollification operator does not need to produce objects that are globally smooth. We show how this flexibility can be exploited to handle the case of contrasted material properties. The mollified error analysis allows one to derive error estimates as soon as the exact solution is in  $\{v \in H^1(D) \mid \kappa \nabla v \in H^r(D), \nabla \cdot (\kappa \nabla v) \in L^p(D)\}$ ,  $r > 0$ ,  $p \in (2_*, 2]$ , for the diffusion equation, and as soon as the exact solution is in  $\{\mathbf{A} \in \mathbf{H}(\text{curl}; D) \mid \kappa \nabla \times \mathbf{A} \in \mathbf{H}^r(D), \nabla \times (\kappa \nabla \times \mathbf{A}) \in \mathbf{L}^2(D)\}$ ,  $r > 0$ , for the time-harmonic Maxwell’s equations. Note that the assumptions concerning the fractional-order Sobolev spaces are minimal requirements to achieve some decay rate with respect to the mesh-size in the error estimate, and that these requirements are in general compatible with the regularity pickup estimates available in the literature for the model problems at hand (see, e.g., Jochmann [17], Bonito et al. [3] for the Maxwell’s equations).

This paper is organized as follows. In Section 2, we present the two model problems on which we will illustrate the present developments: the diffusion equation and the time-harmonic Maxwell’s equations. In Section 3, we introduce the finite element setting, the abstract discrete problem, and illustrate it by presenting Nitsche’s boundary penalty method for our two model problems. Section 4 is concerned with abstract error estimates. We first recall Strang’s Lemmas and then we present Gudi’s trimmed error estimate followed by our novel mollified error estimate. Section 5 contains some useful analysis tools. We first recall some recent results from [12] on shrinking-based mollification operators that commute with the usual derivative operators ( $\nabla$ ,  $\nabla \times$ , and  $\nabla \cdot$ ). Then, we present some inverse and functional inequalities to bound norms and integrals on mesh faces. Finally, in Section 6 and in Section 7, we show how to apply the trimmed error estimate and the mollified error estimate to our two model problems.

## 2 Model problem

We introduce in this section an abstract model problem and illustrate the setting on the diffusion equation and the time-harmonic Maxwell's equations.

### 2.1 Abstract setting

Let  $V$  and  $W$  be two Banach spaces; to stay general, we consider linear spaces over the field of complex numbers. Let  $a(\cdot, \cdot)$  be a bounded sesquilinear form on  $V \times W$ , and let  $\ell(\cdot)$  be a bounded antilinear form on  $W$ , i.e.,  $\ell \in W'$ . We consider the following abstract model problem:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u, w) = \ell(w), \quad \forall w \in W, \end{cases} \quad (2.1)$$

which we assume to be well-posed in the sense of Hadamard; that is to say, there is a unique solution and this solution depends continuously on the data. The well-posedness of the model problem (2.1) can be characterized by invoking Banach's Closed Range and Open Mapping Theorems; see Nečas [19] and Babuška and Aziz [1, p. 112].

**Theorem 2.1** (Banach–Nečas–Babuška (BNB)). *Assume that  $W$  is a reflexive Banach space. The problem (2.1) is well-posed if and only if:*

$$\inf_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} =: \alpha > 0, \quad (2.2a)$$

$$\forall w \in W, \quad [\forall v \in V, a(v, w) = 0] \implies [w = 0]. \quad (2.2b)$$

In particular, the *a priori* estimate  $\|u\|_V \leq \frac{1}{\alpha} \|\ell\|_{W'}$  holds true.

It is implicitly understood here and in what follows that the above infimum and supremum are taken over nonzero arguments.

### 2.2 Diffusion equation

To illustrate the abstract setting introduced above, let us consider a bounded Lipschitz polyhedron  $D$  in  $\mathbb{R}^d$  with  $d \geq 2$ . Let  $f \in L^p(D)$  be a source term with  $p \in (2_*, 2]$ ,  $2_* := \frac{2d}{d+2}$  (so that  $p \in (1, 2]$  if  $d = 2$ , and  $p \in (\frac{6}{5}, 2]$  if  $d = 3$ ). We consider the following model problem: find  $u : D \rightarrow \mathbb{R}$  s.t.

$$-\nabla \cdot (\kappa \nabla u) = f \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D, \quad (2.3)$$

where  $\kappa \in L^\infty(D)$  takes values a.e. in  $D$  in the interval  $[\kappa_{b,D}, \kappa_{\sharp,D}]$  with  $0 < \kappa_{b,D} \leq \kappa_{\sharp,D} < \infty$ . Since we will need later to give a pointwise meaning to  $\kappa$  on  $\partial D$ , we assume that  $\kappa \in BV(D) \cap L^\infty(D)$  (this means that  $\kappa$  is bounded and has bounded variation in  $D$ ), which then implies that  $\kappa|_{\partial D} \in L^\infty(\partial D)$ .

Let us introduce the Hilbert space  $H^1(D) = \{v \in L^2(D) \mid \nabla v \in \mathbf{L}^2(D)\}$  and its zero-trace subspace  $H_0^1(D) = \{v \in H^1(D) \mid \gamma^g(v) = 0\}$  where  $\gamma^g : H^1(D) \rightarrow H^{\frac{1}{2}}(\partial D)$  is the well-known trace operator. To be dimensionally coherent, we equip the space  $H^1(D)$  with the norm  $\|v\|_{H^1(D)} = (\|v\|_{L^2(D)}^2 + \ell_D^2 \|\nabla v\|_{\mathbf{L}^2(D)}^2)^{\frac{1}{2}}$ , where  $\ell_D$  is some length scale characteristic of  $D$ , e.g., the diameter of  $D$ . The model problem (2.3) fits the abstract setting of (2.1) with  $V = W = H_0^1(D)$  and

$$a(v, w) := \int_D \kappa \nabla v \cdot \nabla w \, dx, \quad \ell(w) := \int_D f w \, dx, \quad (2.4)$$

and its well-posedness follows from the Lax–Milgram Lemma. In particular, we have

$$|a(v, w)| \leq \kappa_{\sharp,D} \|\nabla v\|_{\mathbf{L}^2(D)} \|\nabla w\|_{\mathbf{L}^2(D)}, \quad (2.5a)$$

$$a(v, v) \geq \kappa_{b,D} \|\nabla v\|_{\mathbf{L}^2(D)}^2, \quad (2.5b)$$

for all  $v, w \in H_0^1(D)$ . Note that  $\|v\|_{H^1(D)} \leq (1 + C_{\text{PS},D}^{-2})^{\frac{1}{2}} \ell_D \|\nabla v\|_{\mathbf{L}^2(D)}$  owing to the Poincaré–Steklov inequality  $C_{\text{PS},D} \|v\|_{L^2(D)} \leq \ell_D \|\nabla v\|_{\mathbf{L}^2(D)}$  for all  $v \in H_0^1(D)$ . Note also that a Sobolev embedding implies that  $w \in L^{p'}(D)$  for all  $w \in H^1(D)$ , where  $p'$  is the conjugate number of  $p$ , i.e.,  $\frac{1}{p} + \frac{1}{p'} = 1$ , so that the linear form  $\ell(\cdot)$  is well-defined owing to Hölder's inequality.

**Remark 2.2** (Extensions). Most of what is said in the paper generalizes when lower-order terms are added to the PDE in (2.3),  $\kappa$  is tensor-valued, and non-homogeneous Dirichlet conditions are imposed.  $\square$

### 2.3 Time-harmonic Maxwell's equations

As a second example to illustrate the abstract setting introduced above, we consider the time-harmonic Maxwell's equations in a bounded Lipschitz polyhedron  $D$  in  $\mathbb{R}^3$ . Let  $\mathbf{f} \in \mathbf{L}^2(D)$  be a source term. We consider the following model problem: find  $\mathbf{A} : D \rightarrow \mathbb{R}^3$  s.t.

$$\tilde{\mu}\mathbf{A} + \nabla \times (\kappa \nabla \times \mathbf{A}) = \mathbf{f}, \quad \mathbf{A}|_{\partial D} \times \mathbf{n} = \mathbf{0}. \quad (2.6)$$

We assume that  $\tilde{\mu} \in L^\infty(D; \mathbb{C})$ ,  $\kappa \in BV(D; \mathbb{C}) \cap L^\infty(D; \mathbb{C})$ , and we set  $\mu_{\sharp, D} = \text{ess sup}_{\mathbf{x} \in D} |\tilde{\mu}(\mathbf{x})|$  and  $\kappa_{\sharp, D} = \text{ess sup}_{\mathbf{x} \in D} |\kappa(\mathbf{x})|$ . We also assume the following positivity condition: There are real numbers  $\theta$ ,  $\mu_{\flat, D} > 0$ , and  $\kappa_{\flat, D} > 0$  so that

$$\text{ess inf}_{\mathbf{x} \in D} \Re(e^{i\theta} \tilde{\mu}(\mathbf{x})) \geq \mu_{\flat, D} \quad \text{and} \quad \text{ess inf}_{\mathbf{x} \in D} \Re(e^{i\theta} \kappa(\mathbf{x})) \geq \kappa_{\flat, D}. \quad (2.7)$$

The positivity condition (2.7) fails when the two complex numbers  $\tilde{\mu}$  and  $\kappa$  are collinear and point in opposite directions. If it is the case, the model problem (2.6) is an eigenvalue problem, otherwise it is a boundary-value problem. The model problem (2.6) can be derived from the Maxwell's equations in the time-harmonic regime, i.e., under the assumption that the time variation is of the form  $e^{i\omega t}$  where  $\omega$  is the angular frequency and  $i^2 = -1$ . One example is the Helmholtz problem where  $\mathbf{A}$  stands for the electric field,  $\tilde{\mu} = -\omega^2 \epsilon + i\omega \sigma$  with  $\epsilon$  the electric permittivity and  $\sigma$  the electric conductivity,  $\kappa = \mu^{-1}$  with  $\mu$  the magnetic permeability, and  $\mathbf{f} = -i\omega \mathbf{j}_s$  with  $\mathbf{j}_s$  an imposed current. Another example is the eddy-current problem where  $\mathbf{A}$  stands for the magnetic field,  $\tilde{\mu} = i\omega \mu$ ,  $\kappa = \sigma^{-1}$ , and  $\mathbf{f} = \nabla \times (\sigma^{-1} \mathbf{j}_s)$ .

Let us introduce the Hilbert space  $\mathbf{H}(\text{curl}; D) = \{\mathbf{b} \in \mathbf{L}^2(D) \mid \nabla \times \mathbf{b} \in \mathbf{L}^2(D)\}$  and its zero-trace subspace  $\mathbf{H}_0(\text{curl}; D) = \{\mathbf{b} \in \mathbf{H}(\text{curl}; D) \mid \gamma^c(\mathbf{b}) = \mathbf{0}\}$  where  $\gamma^c : \mathbf{H}(\text{curl}; D) \rightarrow \mathbf{H}^{-\frac{1}{2}}(\partial D) := (\mathbf{H}^{\frac{1}{2}}(\partial D))'$  is the tangential trace operator s.t.

$$\langle \gamma^c(\mathbf{b}), \mathbf{l} \rangle_{\partial D} := \int_D \mathbf{b} \cdot \nabla \times \mathbf{w}(\mathbf{l}) \, dx - \int_D (\nabla \times \mathbf{b}) \cdot \mathbf{w}(\mathbf{l}) \, dx, \quad (2.8)$$

for all  $\mathbf{b} \in \mathbf{H}(\text{curl}; D)$  and all  $\mathbf{l} \in \mathbf{H}^{\frac{1}{2}}(\partial D)$  where  $\mathbf{w}(\mathbf{l}) \in \mathbf{H}^1(D)$  is a lifting of  $\mathbf{l}$  s.t.  $\gamma^s(\mathbf{w}(\mathbf{l})) = \mathbf{l}$  (component-wise) and  $\langle \cdot, \cdot \rangle_{\partial D}$  denotes the duality pairing between  $\mathbf{H}^{-\frac{1}{2}}(\partial D)$  and  $\mathbf{H}^{\frac{1}{2}}(\partial D)$ . Note that  $\gamma^c(\mathbf{b}) = \mathbf{b}|_{\partial D} \times \mathbf{n}$  whenever the field  $\mathbf{b}$  is smooth. To be dimensionally coherent, we equip the space  $\mathbf{H}(\text{curl}; D)$  with the norm  $\|\mathbf{b}\|_{\mathbf{H}(\text{curl}; D)} = (\|\mathbf{b}\|_{\mathbf{L}^2(D)}^2 + \ell_D^2 \|\nabla \times \mathbf{b}\|_{\mathbf{L}^2(D)}^2)^{\frac{1}{2}}$ . The model problem (2.6) fits the abstract setting of (2.1) with  $V = W = \mathbf{H}_0(\text{curl}; D)$  and

$$a(\mathbf{v}, \mathbf{b}) := \int_D (\tilde{\mu} \mathbf{v} \cdot \bar{\mathbf{b}} + \kappa \nabla \times \mathbf{v} \cdot \nabla \times \bar{\mathbf{b}}) \, dx, \quad \ell(\mathbf{b}) := \int_D \mathbf{f} \cdot \bar{\mathbf{b}} \, dx, \quad (2.9)$$

and its well-posedness follows from the Lax–Milgram Lemma. In particular, we have

$$|a(\mathbf{v}, \mathbf{b})| \leq \max(\mu_{\sharp, D}, \ell_D^{-2} \kappa_{\sharp, D}) \|\mathbf{v}\|_{\mathbf{H}(\text{curl}; D)} \|\mathbf{b}\|_{\mathbf{H}(\text{curl}; D)}, \quad (2.10a)$$

$$\Re(e^{i\theta} a(\mathbf{b}, \mathbf{b})) \geq \min(\mu_{\flat, D}, \ell_D^{-2} \kappa_{\flat, D}) \|\mathbf{b}\|_{\mathbf{H}(\text{curl}; D)}^2, \quad (2.10b)$$

for all  $\mathbf{v}, \mathbf{b} \in \mathbf{H}_0(\text{curl}; D)$ .

**Remark 2.3** (Extensions). Most of what is said in the paper generalizes when the non-homogeneous Dirichlet condition  $\gamma^c(\mathbf{A}) = \mathbf{g}$  is enforced in (2.6) with  $\mathbf{g}$  in the range of the trace map  $\gamma^c$ .  $\square$

## 3 Discrete problem

We now formulate a discrete version of the problem (2.1) by using the Galerkin method. The central idea in the Galerkin method consists of replacing the infinite-dimensional spaces  $V$  and  $W$  by finite-dimensional

spaces  $V_h$  and  $W_h$  that are members of sequences of spaces  $(V_h)_{h \rightarrow 0}$ ,  $(W_h)_{h \rightarrow 0}$  endowed with some approximation properties as  $h \rightarrow 0$ . The norms in  $V_h$  and  $W_h$  are denoted by  $\|\cdot\|_{V_h}$  and  $\|\cdot\|_{W_h}$ , respectively. The discrete problem is formulated as follows:

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a_h(u_h, w_h) = \ell_h(w_h), \quad \forall w_h \in W_h, \end{cases} \quad (3.1)$$

where  $a_h(\cdot, \cdot)$  is a bounded sesquilinear form on  $V_h \times W_h$  and  $\ell_h(\cdot)$  is a bounded antilinear form on  $W_h$ ; note that  $a_h(\cdot, \cdot)$  and  $\ell_h(\cdot)$  possibly differ from  $a(\cdot, \cdot)$  and  $\ell(\cdot)$ , respectively. We henceforth assume that  $\dim(V_h) = \dim(W_h)$  and that

$$\inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|a_h(v_h, w_h)|}{\|v_h\|_{V_h} \|w_h\|_{W_h}} =: \alpha_h > 0, \quad \forall h > 0, \quad (3.2)$$

so that the discrete problem (3.1) is well-posed.

### 3.1 Finite element setting

Let  $(\mathcal{T}_h)_{h>0}$  be a shape-regular sequence of meshes; we assume that each mesh covers  $D$  exactly. To avoid technical questions regarding hanging nodes, we also suppose that each mesh is matching, i.e., for all cells  $K, K' \in \mathcal{T}_h$  such that  $K \neq K'$  and  $K \cap K' \neq \emptyset$ , the set  $K \cap K'$  is a common vertex, edge, or face of both  $K$  and  $K'$  (with obvious extensions in higher space dimensions). Given a mesh  $\mathcal{T}_h$ , the elements in  $K \in \mathcal{T}_h$  are closed sets in  $\mathbb{R}^d$  by convention, and they are all assumed to be constructed from a single reference cell  $\widehat{K}$  through affine, bijective, geometric transformations  $\mathbf{T}_K: \widehat{K} \rightarrow K$ . For a mesh cell  $K \in \mathcal{T}_h$ , we define  $\check{\mathcal{T}}_K$  to be the collection of the mesh cells in  $\mathcal{T}_h$  that touch  $K$ , i.e., the mesh cells that share a vertex, an edge or a face (in dimension 3) with  $K$ . We define  $D_K := \text{int}(\cup_{K' \in \check{\mathcal{T}}_K} K')$ ; note that  $\check{\mathcal{T}}_K$  is composed of a uniformly bounded number of mesh cells owing to the shape-regularity of the mesh sequence

The set of the mesh faces is denoted  $\mathcal{F}_h$ . This set is partitioned into the subset of the interfaces denoted  $\mathcal{F}_h^\circ$  and the subset of the boundary faces denoted  $\mathcal{F}_h^\partial$ . Each interface  $F$  is oriented by choosing one unit normal vector  $\mathbf{n}_F$ . The boundary faces are oriented by using the outward normal vector that we denote  $\mathbf{n}$ . Given an interface  $F \in \mathcal{F}_h^\circ$ , we denote by  $K_l$  (left cell) and  $K_r$  (right cell) the two cells such that  $F = K_l \cap K_r$  and  $\mathbf{n}_F$  points from  $K_l$  to  $K_r$ . This convention allows us to define the notion of jump across  $F$  for a smooth enough function  $v$  as follows:

$$[[v]]_F(\mathbf{x}) = v|_{K_l}(\mathbf{x}) - v|_{K_r}(\mathbf{x}) \quad \text{a.e. } \mathbf{x} \text{ in } F. \quad (3.3)$$

We consider a reference finite element in the sense of Ciarlet  $(\widehat{K}, \widehat{P}^g, \widehat{\Sigma}^g)$ . (The superscript  $g$  is intended to remind us that this finite element will be used to build a finite-dimensional subspace composed of functions whose *gradient* in  $D$  is integrable.) We think of  $(\widehat{K}, \widehat{P}^g, \widehat{\Sigma}^g)$  as a scalar-valued finite element with some degrees of freedom that require point evaluations, for instance  $(\widehat{K}, \widehat{P}^g, \widehat{\Sigma}^g)$  could be a Lagrange finite element. The local shape functions are denoted  $(\widehat{\theta}_i)_{i \in \mathcal{N}}$ ; recall that  $\sigma_i(\widehat{\theta}_j) = \delta_{ij}$  for all  $\sigma_i \in \widehat{\Sigma}^g$ , and all  $i, j \in \mathcal{N}$ . At this point, we do not need to know the exact structure of the reference element, but we are going to assume that there exists  $k \in \mathbb{N}$  such that  $\mathbb{P}_{k,d} \subset \widehat{P}^g$ , where  $\mathbb{P}_{k,d}$  is the vector space composed of the  $d$ -variate polynomials of degree at most  $k$ .

In order to construct  $H^1$ -conforming approximation spaces based on  $(\mathcal{T}_h)_{h>0}$  using the above reference finite element, we introduce the pullback by the geometric map  $\mathbf{T}_K$  which we denote by  $\psi_K^g$ , i.e.,  $\psi_K^g(v) = v \circ \mathbf{T}_K$ . Then we set

$$P^b(\mathcal{T}_h) := \{v_h \in L^1(D) \mid \psi_K^g(v_h|_K) \in \widehat{P}^g, \forall K \in \mathcal{T}_h, \}, \quad (3.4a)$$

$$P^g(\mathcal{T}_h) := \{v_h \in P^b(\mathcal{T}_h) \mid [[v_h]]_F = 0, \forall F \in \mathcal{F}_h^\circ\}, \quad (3.4b)$$

$$P_0^g(\mathcal{T}_h) := P^g(\mathcal{T}_h) \cap H_0^1(D). \quad (3.4c)$$

(The superscript  $b$  is meant to remind us that  $P^b(\mathcal{T}_h)$  is sometimes called a broken finite element space because functions in  $P^b(\mathcal{T}_h)$  do not satisfy any matching condition across the mesh interfaces.)

### 3.2 Boundary penalty for the diffusion equation

We are going to illustrate our results on the so-called boundary penalty method of Nitsche [20]. Let us first consider the diffusion equation from Section 2.2. To avoid technicalities, we assume that there is a partition of  $D$  into  $M$  disjoint Lipschitz polyhedra  $D_1, \dots, D_M$  so that  $\kappa|_{D_i}$  is constant for all  $1 \leq i \leq M$ , and we assume that the meshes in  $(\mathcal{T}_h)_{h>0}$  are fitted to this partition, so that, for all  $h > 0$  and all  $K \in \mathcal{T}_h$ ,  $\kappa|_K$  is constant; we use the notation  $\kappa_K := \kappa|_K$ .

Let  $V_h := P^{\mathfrak{g}}(\mathcal{T}_h)$  be the  $H^1$ -conforming finite element space based on  $\mathcal{T}_h$  introduced in (3.4). For the diffusion equation, the discrete forms  $a_h(\cdot, \cdot)$  and  $\ell_h(\cdot)$  are defined by

$$a_h(v_h, w_h) := \int_D \kappa \nabla v_h \cdot \nabla w_h \, dx - \int_{\partial D} (\mathbf{n} \cdot \kappa \nabla v_h) w_h \, ds + \int_{\partial D} \eta_h v_h w_h \, ds, \quad (3.5a)$$

$$\ell_h(w_h) := \int_D f w_h \, dx, \quad (3.5b)$$

for all  $v_h, w_h \in V_h$ . It is useful to decompose the discrete bilinear form as  $a_h(\cdot, \cdot) = \tilde{a}_h(\cdot, \cdot) + s_h(\cdot, \cdot)$  where

$$\tilde{a}_h(v_h, w_h) := \int_D \kappa \nabla v_h \cdot \nabla w_h \, dx - \int_{\partial D} (\mathbf{n} \cdot \kappa \nabla v_h) w_h \, ds, \quad (3.6a)$$

$$s_h(v_h, w_h) := \int_{\partial D} \eta_h v_h w_h \, ds. \quad (3.6b)$$

The discrete bilinear form  $\tilde{a}_h(\cdot, \cdot)$  is meant to ensure a consistency property, and the discrete bilinear form  $s_h(\cdot, \cdot)$  is added for stabilization purposes. The penalty parameter  $\eta_h$  is piecewise constant on  $\partial D$  and is such that

$$\eta_{h|F} = \eta_0 \frac{\kappa_{K_F}}{h_F}, \quad \forall F \in \mathcal{F}_h^\partial, \quad (3.7)$$

where  $\eta_0 > 0$  is a user-dependent parameter to be chosen large enough (see Lemma 3.1 below), and  $K_F$  is the unique mesh cell having  $F$  as a face.

We equip the space  $V_h$  with the following norm:

$$\|v_h\|_{V_h} := \left( \|\kappa^{\frac{1}{2}} \nabla v_h\|_{L^2(D)}^2 + \|\eta_h^{\frac{1}{2}} v_h\|_{L^2(\partial D)}^2 \right)^{\frac{1}{2}}, \quad \forall v_h \in V_h. \quad (3.8)$$

Since  $\|v_h\|_{V_h} = 0$  implies that  $v_h$  is constant on  $D$  and vanishes on  $\partial D$ , and hence vanishes everywhere in  $D$ , we infer that  $\|\cdot\|_{V_h}$  is indeed a norm on  $V_h$ . Furthermore, owing to the assumed shape-regularity of the mesh sequence, there is  $c_I$ , uniform with respect to  $h$ , such that

$$\|v_h\|_{L^2(F)} \leq c_I h_F^{-\frac{1}{2}} \|v_h\|_{L^2(K_F)}, \quad (3.9)$$

for all  $v_h \in V_h$  and all  $F \in \mathcal{F}_h^\partial$ . The following stability result is classical; we simply state it without proof.

**Lemma 3.1** (Coercivity and well-posedness). *Suppose that  $\eta_h$  is defined by (3.7) with  $\eta_0 > \frac{1}{4} n_\partial c_I^2$  where  $n_\partial$  is the maximum number of boundary faces that a mesh cell can have ( $n_\partial \leq d$  for simplicial meshes). Then, the following coercivity property holds true:*

$$a_h(v_h, v_h) \geq \alpha \|v_h\|_{V_h}^2, \quad \forall v_h \in V_h, \quad (3.10)$$

with  $\alpha := \frac{\eta_0 - \frac{1}{4} n_\partial c_I^2}{1 + \eta_0}$ . Consequently, the discrete problem (3.1) is well-posed for the diffusion equation.

### 3.3 Boundary penalty for Maxwell's equations

Nitsche's boundary penalty method can also be applied to the time-harmonic Maxwell's equations from Section 2.3. For simplicity, we assume in what follows that the angle  $\theta$  in the positivity condition (2.7) is s.t.  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . This assumption allows us to weigh the stabilization term using a real parameter (i.e., without an imaginary part). We also set  $\tilde{\mu}_r := \Re(e^{i\theta} \tilde{\mu})$  and  $\kappa_r := \Re(e^{i\theta} \kappa)$ , and we assume that the ratios  $\tilde{\mu}_r/|\tilde{\mu}|$  and  $\kappa_r/|\kappa|$  are uniformly bounded from below away from zero a.e. in  $D$ . Finally, to avoid technicalities, we assume that there is a partition of  $D$  into  $M$  disjoint Lipschitz polyhedra  $D_1, \dots, D_M$  so

that  $\tilde{\mu}|_{D_i}$  and  $\kappa|_{D_i}$  are constant for all  $1 \leq i \leq M$ , and we assume that the meshes in  $(\mathcal{T}_h)_{h>0}$  are fitted to this partition, so that, for all  $h > 0$  and all  $K \in \mathcal{T}_h$ ,  $\tilde{\mu}|_K$  and  $\kappa|_K$  are constant; we use the notation  $\tilde{\mu}_{r,K} := \tilde{\mu}|_K$  and  $\kappa_{r,K} := \kappa|_K$ .

Let  $\mathbf{V}_h = \mathbf{P}^g(\mathcal{T}_h)$  be the  $\mathbf{H}^1$ -conforming finite element space based on  $\mathcal{T}_h$ , where  $\mathbf{P}^g(\mathcal{T}_h)$  is the vector-valued version of the finite element space  $P^g(\mathcal{T}_h)$  considered above for the diffusion equation. For the time-harmonic Maxwell's equations, the discrete forms  $a_h(\cdot, \cdot) = \tilde{a}_h(\cdot, \cdot) + s_h(\cdot, \cdot)$  and  $\ell_h(\cdot)$  are defined by

$$\tilde{a}_h(\mathbf{v}_h, \mathbf{b}_h) := \int_D (\tilde{\mu} \mathbf{v}_h \cdot \bar{\mathbf{b}}_h + \kappa \nabla \times \mathbf{v}_h \cdot \nabla \times \bar{\mathbf{b}}_h) \, dx + \int_{\partial D} (\mathbf{n} \times (\kappa \nabla \times \mathbf{v}_h)) \cdot \bar{\mathbf{b}}_h \, ds, \quad (3.11a)$$

$$s_h(\mathbf{v}_h, \mathbf{b}_h) := \int_{\partial D} \eta_h (\mathbf{v}_h \times \mathbf{n}) \cdot (\bar{\mathbf{b}}_h \times \mathbf{n}) \, ds, \quad (3.11b)$$

$$\ell_h(\mathbf{b}_h) := \int_D \mathbf{f} \cdot \bar{\mathbf{b}}_h \, dx, \quad (3.11c)$$

for all  $\mathbf{v}_h, \mathbf{b}_h \in \mathbf{V}_h$ . The discrete sesquilinear form  $\tilde{a}_h(\cdot, \cdot)$  is meant to ensure a consistency property, and the discrete sesquilinear form  $s_h(\cdot, \cdot)$  is added for stabilization purposes. The penalty parameter  $\eta_h$  is piecewise constant on  $\partial D$  and is such that

$$\eta_h|_F = \eta_0 \frac{\kappa_{r,K_F}}{h_F}, \quad \forall F \in \mathcal{F}_h^\partial, \quad (3.12)$$

where  $\eta_0 > 0$  is a user-dependent parameter to be chosen large enough (see Lemma 3.2 below), and  $K_F$  is the unique mesh cell having  $F$  as a face.

We equip the space  $\mathbf{V}_h$  with the following norm:

$$\|\mathbf{b}_h\|_{\mathbf{V}_h} := \left( \|\tilde{\mu}^{\frac{1}{2}} \mathbf{b}_h\|_{\mathbf{L}^2(D)}^2 + \|\kappa_r^{\frac{1}{2}} \nabla \times \mathbf{b}_h\|_{\mathbf{L}^2(D)}^2 + \|\eta_{\theta h}^{\frac{1}{2}} (\mathbf{b}_h \times \mathbf{n})\|_{\mathbf{L}^2(\partial D)}^2 \right)^{\frac{1}{2}}, \quad \forall \mathbf{b}_h \in \mathbf{V}_h, \quad (3.13)$$

where  $\eta_{\theta h} = \cos(\theta)\eta_h$ . The following stability result is proved using the same arguments as in the proof of Lemma 3.1.

**Lemma 3.2** (Coercivity and well-posedness). *Suppose that  $\eta_h$  is defined by (3.12) with  $\eta_0 > \frac{1}{4} n_{\partial} c_I^2$ . Then, the following coercivity property holds true:*

$$a_h(\mathbf{b}_h, \mathbf{b}_h) \geq \alpha \|\mathbf{b}_h\|_{\mathbf{V}_h}^2, \quad \forall \mathbf{b}_h \in \mathbf{V}_h, \quad (3.14)$$

with  $\alpha := \frac{\eta_0 - \frac{1}{4} n_{\partial} c_I^2}{1 + \eta_0}$ . Consequently, the discrete problem (3.1) is well-posed for the Maxwell's equations.

## 4 Abstract error estimates

There are many ways to investigate the approximation properties of the above discrete problem (3.1). Since  $u_h$  may not be a member of  $V$ ,  $u$  and  $u_h$  may be objects of different nature. This poses the question of defining a common ground for the discrete solution  $u_h$  and the exact solution  $u$  to measure the error. For this purpose, we assume that it is meaningful to define the linear space

$$V_b := V + V_h. \quad (4.1)$$

Obviously,  $V_b = V$  if  $V_h \subset V$  (as it happens with conforming approximations). We equip the space  $V_b$  with a norm denoted  $\|\cdot\|_{V_b}$  which we assume extends the discrete norm  $\|\cdot\|_{V_h}$  to  $V_b$ , i.e.,

$$\|v_h\|_{V_b} = \|v_h\|_{V_h}, \quad \forall v_h \in V_h. \quad (4.2)$$

The goal of this section is to bound the error  $u - u_h$  using the  $\|\cdot\|_{V_b}$ -norm. Note that even in the conforming case where the linear spaces  $V_b$  and  $V$  are identical, choosing  $\|\cdot\|_{V_b}$  to be different from  $\|\cdot\|_V$  can be useful for the error analysis.



## 4.1 A basic error identity

Our starting point is the following (relatively straightforward) error identity. Recall that the norm of any antilinear form  $\phi_h \in W'_h := \mathcal{L}(W_h; \mathbb{C})$ , is defined by  $\|\phi_h\|_{W'_h} := \sup_{w_h \in W_h} \frac{|\phi_h(w_h)|}{\|w_h\|_{W_h}}$ .

**Lemma 4.1** (Error identity). *Assume that the discrete inf-sup condition (3.2) holds. Then, the following identity holds true:*

$$\|u - u_h\|_{V_b} = \inf_{v_h \in V_h} \left[ \|u - v_h\|_{V_b} + \frac{1}{\alpha_h} \|\ell_h - a_h(v_h, \cdot)\|_{W'_h} \right]. \quad (4.3)$$

*Proof.* Let  $v_h \in V_h$ . The triangle inequality, (4.2), stability, and the fact that  $a_h(u_h, w_h) = \ell_h(w_h)$  for all  $w_h \in W_h$  imply that

$$\begin{aligned} \|u - u_h\|_{V_b} &\leq \|u - v_h\|_{V_b} + \|u_h - v_h\|_{V_b} = \|u - v_h\|_{V_b} + \|u_h - v_h\|_{V_h} \\ &\leq \|u - v_h\|_{V_b} + \frac{1}{\alpha_h} \sup_{w_h \in W_h} \frac{|a_h(u_h - v_h, w_h)|}{\|w_h\|_{W_h}} \\ &= \|u - v_h\|_{V_b} + \frac{1}{\alpha_h} \sup_{w_h \in W_h} \frac{|\ell_h(w_h) - a_h(v_h, w_h)|}{\|w_h\|_{W_h}}. \end{aligned}$$

Since  $v_h$  is arbitrary in  $V_h$  and recalling the definition of the dual norm of the discrete antilinear form  $\ell_h - a_h(v_h, \cdot)$ , we conclude that  $\|u - u_h\|_{V_b} \leq r_h$ , where  $r_h$  denotes the right-hand side of (4.3). Finally, taking  $v_h = u_h$  in the infimum and observing that  $\ell_h - a_h(u_h, \cdot)$  vanishes identically on  $W_h$ , we infer that  $\|u - u_h\|_{V_b} = r_h$ .  $\square$

**Remark 4.2** (Norms  $\|\cdot\|_{V_b}$  and  $\|\cdot\|_{V_h}$ ). The proof of Lemma 4.1 shows that the assumption (4.2) on the norms  $\|\cdot\|_{V_b}$  and  $\|\cdot\|_{V_h}$  can be relaxed by assuming that there is  $c_b$  so that  $\|v_h\|_{V_b} \leq c_b \|v_h\|_{V_h}$ , for all  $v_h \in V_h$ . Then the error identity (4.3) still holds true provided one replaces the factor  $\frac{1}{\alpha_h}$  by the factor  $\frac{c_b}{\alpha_h}$ .  $\square$

## 4.2 Strang's Lemmas

The traditional form of Strang's First Lemma consists of assuming that the approximation setting is conforming; that is to say,  $V_h \subset V$  and  $W_h \subset W$ . This implies that the linear spaces  $V$  and  $V_b$  coincide; however, these spaces may be equipped with different norms.

**Lemma 4.3** (Strang 1). *Assume the following: (i)  $V_h \subset V$  and  $W_h \subset W$ ; (ii) The sesquilinear form  $a(\cdot, \cdot)$  is bounded on  $V_b \times W_h$  with norm*

$$\|a\|_{V_b, W_h} = \sup_{v \in V_b} \sup_{w_h \in W_h} \frac{|a(v, w_h)|}{\|v\|_{V_b} \|w_h\|_{W_h}}. \quad (4.4)$$

*Then, the following error estimate holds true:*

$$\begin{aligned} \|u - u_h\|_{V_b} &\leq \inf_{v_h \in V_h} \left[ \left( 1 + \frac{\|a\|_{V_b, W_h}}{\alpha_h} \right) \|u - v_h\|_{V_b} \right. \\ &\quad \left. + \frac{1}{\alpha_h} \|\ell_h - \ell + a(v_h, \cdot) - a_h(v_h, \cdot)\|_{W'_h} \right]. \end{aligned} \quad (4.5)$$

*Proof.* This is an easy consequence of the error identity (4.3) after one has observed that

$$\ell_h(w_h) - a_h(v_h, w_h) = \ell_h(w_h) - \ell(w_h) + a(u, w_h) \pm a(v_h, w_h) - a_h(v_h, w_h),$$

since  $a(u, w_h) = \ell(w_h)$  for all  $w_h \in W_h$  since  $W_h \subset W$ . One concludes by invoking the boundedness of  $a$  on  $V_b \times W_h$ .  $\square$

The main inconvenient of the above estimate is that it assumes that the discrete setting is conforming. This shortcoming is traditionally addressed in the literature by invoking Strang's Second Lemma where one supposes that the discrete sesquilinear form  $a_h(\cdot, \cdot)$  can be extended to  $V_b \times W_h$ .

**Lemma 4.4** (Strang 2). *Assume that the discrete sesquilinear form  $a_h(\cdot, \cdot)$  is bounded on  $V_b \times W_h$  with norm*

$$\|a_h\|_{V_b, W_h} = \sup_{v \in V_b} \sup_{w_h \in W_h} \frac{|a_h(v, w_h)|}{\|v\|_{V_b} \|w_h\|_{W_h}}. \quad (4.6)$$

Then, the following error estimate holds true:

$$\|u - u_h\|_{V_b} \leq \left(1 + \frac{\|a_h\|_{V_b, W_h}}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_{V_b} + \frac{1}{\alpha_h} \|\ell_h - a_h(u, \cdot)\|_{W_h'}. \quad (4.7)$$

*Proof.* This is also an easy consequence of the error identity (4.3) after one writes

$$\ell_h(w_h) - a_h(v_h, w_h) = \ell_h(w_h) \pm a_h(u, w_h) - a_h(v_h, w_h),$$

and uses the boundedness of  $a_h$  on  $V_b \times W_h$ .  $\square$

The key problem with the above estimate is that it assumes that  $a_h(\cdot, \cdot)$  is bounded on  $V_b \times W_h$ ; in particular, it assumes that it is possible to make sense of  $a_h(u, w_h)$  and this, in turn, requires some regularity assumption on the exact solution. In the rest of this section, we present two alternative error estimates that avoid extending the discrete sesquilinear form  $a_h(\cdot, \cdot)$  to  $V_b \times W_h$ . We still need a regularity assumption on the exact solution, but this assumption is milder than that required to extend  $a_h(\cdot, \cdot)$ . To stay general, we formalize this regularity assumption by assuming that  $u \in Y$  where  $Y$  is a dense subspace of  $V$ . We set

$$V_{\sharp} := Y + V_h, \quad (4.8)$$

and we note that  $V_{\sharp}$  is a subspace of  $V_b$ . We equip the space  $V_{\sharp}$  with a norm  $\|\cdot\|_{V_{\sharp}}$  that we suppose to be (slightly) stronger than the norm  $\|\cdot\|_{V_b}$  restricted to  $V_{\sharp}$ ; specifically, we assume that

$$\|v\|_{V_b} \leq \|v\|_{V_{\sharp}}, \quad \forall v \in V_{\sharp}. \quad (4.9)$$

We also assume that the converse bound  $\|v_h\|_{V_b} \leq c_{\sharp} \|v_h\|_{V_{\sharp}}$  also holds true on the finite-dimensional subspace  $V_h$  with a constant  $c_{\sharp}$  that is uniform w.r.t.  $h$ . However, the constant  $c_{\sharp}$  may depend on the model parameters; this is the reason why we distinguish the two norms.

### 4.3 Trimmed error estimate

One possible way forward to overcome the limitations of Strang's Second Lemma has been proposed by Gudi [15]. The key idea is to introduce a so-called trimming operator  $T : W_h \rightarrow W$  that transforms the discrete test functions into objects that are conforming in  $W$ .

**Lemma 4.5** (Trimmed error estimate). *Assume that the exact solution  $u$  is in  $Y$ . Consider any map  $T : W_h \rightarrow W$  such that the following boundedness/consistency properties hold true: (i) There exists a real number  $\omega_{V_{\sharp}, W_h}^{\text{tri}}$  so that*

$$\|a(u, T(\cdot)) - a_h(v_h, T(\cdot))\|_{W_h'} \leq \omega_{V_{\sharp}, W_h}^{\text{tri}} \|u - v_h\|_{V_{\sharp}}, \quad \forall v_h \in V_h; \quad (4.10)$$

(ii) *There exists a real number  $\varpi_{V_{\sharp}, W_h}^{\text{tri}}$  so that*

$$\|\ell_h - \ell \circ T - a_h(v_h, (I - T)(\cdot))\|_{W_h'} \leq \varpi_{V_{\sharp}, W_h}^{\text{tri}} \|u - v_h\|_{V_{\sharp}}, \quad \forall v_h \in V_h, \quad (4.11)$$

where  $I$  is the identity operator in  $W_h$ . Then, the following error estimate holds true:

$$\|u - u_h\|_{V_b} \leq \left(1 + \frac{\omega_{V_{\sharp}, W_h}^{\text{tri}} + \varpi_{V_{\sharp}, W_h}^{\text{tri}}}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_{V_{\sharp}}. \quad (4.12)$$

*Proof.* This follows again from the error identity (4.3) after one writes

$$\ell_h(w_h) - a_h(v_h, w_h) = \ell_h(w_h) - \ell(T(w_h)) + a(u, T(w_h)) \pm a_h(v_h, T(w_h)) - a_h(v_h, w_h),$$

since  $a(u, T(w_h)) = \ell(T(w_h))$  for all  $w_h \in W_h$ . To conclude, one uses the boundedness/consistency properties (4.10) and (4.11) together with the assumption (4.9) on the norms.  $\square$

**Remark 4.6** (Conforming case). Whenever  $W_h \subset W$ , one can take  $T$  to be the canonical injection  $W_h \hookrightarrow W$ . In this case, the abstract error estimate (4.12) differs from that derived in Strang's First Lemma. The reason for this is that we have exploited additional boundedness properties to derive (4.12).  $\square$

## 4.4 Mollified error estimate

Although the trimmed error estimate presented in the previous section can overcome some shortcomings encountered with the use of Strang's Lemmas, as illustrated by the examples in Section 6 and in Section 7, we will also see that some difficulties remain. In particular, it is not always easy to construct a trimming operator in the context of Maxwell's equations when one does not use edge elements and the faces of the domain  $D$  are not orthogonal to one of the coordinate axes. Moreover, it is not simple to construct a trimming operator that exhibits suitable stability properties that are robust in the case of highly-contrasted coefficients. The goal of this section is to present a new approach for the error analysis that attempts to remedy these difficulties. To stay general, we decompose the discrete sesquilinear form  $a_h(\cdot, \cdot)$  as follows:

$$a_h = \tilde{a}_h + s_h, \quad (4.13)$$

where the discrete sesquilinear form  $\tilde{a}_h(\cdot, \cdot)$  is meant to ensure a consistency property, and the discrete sesquilinear form  $s_h(\cdot, \cdot)$  is added for stabilization purposes.

**Lemma 4.7** (Mollified error estimate). *Assume that the exact solution  $u$  is in  $Y$ . Suppose that there is a family  $(a_{\delta h})_{\delta \in [0, \delta_0]}$ ,  $\delta_0 > 0$ , of discrete sesquilinear forms defined on  $V_b \times W_h$  (where  $\delta \in [0, \delta_0]$  stands for some mollification parameter) such that the following holds true: (i) Asymptotic consistency (w.r.t.  $\delta$ ):*

$$\lim_{\delta \rightarrow 0} \|\ell_h - a_{\delta h}(u, \cdot)\|_{W'_h} = 0, \quad (4.14a)$$

$$\lim_{\delta \rightarrow 0} \|a_{\delta h}(v_h, \cdot) - \tilde{a}_h(v_h, \cdot)\|_{W'_h} = 0, \quad \forall v_h \in V_h. \quad (4.14b)$$

(ii) *Uniform boundedness (w.r.t.  $\delta$ ): There exists a real number  $\omega_{V_h, W_h}^{\text{mol}}$  so that*

$$\|a_{\delta h}(v, \cdot)\|_{W'_h} \leq \omega_{V_h, W_h}^{\text{mol}} \|v\|_{V_h}, \quad \forall v \in V_h, \quad \forall \delta \in [0, \delta_0]. \quad (4.15)$$

(iii) *Bounded/consistent stabilization: There exists a real number  $\sigma_{V_h, W_h}$  so that*

$$\|s_h(v_h, \cdot)\|_{W'_h} \leq \sigma_{V_h, W_h} \|u - v_h\|_{V_b}, \quad \forall v_h \in V_h. \quad (4.16)$$

Then, the following error estimate holds true:

$$\|u - u_h\|_{V_b} \leq \left(1 + \frac{\omega_{V_h, W_h}^{\text{mol}} + \sigma_{V_h, W_h}}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_{V_h}. \quad (4.17)$$

*Proof.* For all  $v_h \in V_h$ , we observe that  $\ell_h - a_h(v_h, \cdot) = \ell_{1, \delta h} + \ell_{2, \delta h}$  with the antilinear forms

$$\begin{aligned} \ell_{1, \delta h} &= \ell_h - a_{\delta h}(u, \cdot) + a_{\delta h}(v_h, \cdot) - \tilde{a}_h(v_h, \cdot), \\ \ell_{2, \delta h} &= a_{\delta h}(u - v_h, \cdot) - s_h(v_h, \cdot). \end{aligned}$$

Owing to (4.14a)-(4.14b) and letting  $\delta \rightarrow 0$ , we infer that

$$\|\ell_h - a_h(v_h, \cdot)\|_{V'_h} \leq \|\ell_{2, \delta h}\|_{W'_h} \leq (\omega_{V_h, W_h}^{\text{mol}} + \sigma_{V_h, W_h}) \|u - v_h\|_{V_b},$$

where the last bound follows from (4.15)-(4.16). We invoke the error identity (4.3) to conclude.  $\square$

## 5 Analysis tools

We introduce in this section some analysis tools that are useful to realize the above program.

### 5.1 Mollification operators

Smoothing by mollification (i.e., by convolution with a smooth kernel) is an important tool for the analysis and approximation of PDEs that has been introduced by Leray [18, p. 206], Sobolev [23, p. 487], and Friedrichs [14, p. 136–139]. The goal of this section is to define mollification operators that commute with the usual differential operators, and that converge optimally when the function to be smoothed is defined

over a Lipschitz domain  $D$  in  $\mathbb{R}^d$ . We use the shrinking technique of  $D$  in [12] to avoid the need to extend the function to be smoothed outside  $D$ .

The starting point is to observe that Proposition 2.3 in Hofmann et al. [16] implies the existence of a vector field  $\mathbf{j} \in C^\infty(\mathbb{R}^d)$  that is globally transversal on  $\partial D$  (i.e., there is a real number  $\gamma > 0$  such that  $\mathbf{n}(\mathbf{x}) \cdot \mathbf{j}(\mathbf{x}) \geq \gamma$  at a.e. point  $\mathbf{x}$  on  $\partial D$  where  $\mathbf{n}$  is the unit normal vector pointing outward  $D$ ) and  $\|\mathbf{j}(\mathbf{x})\|_{\ell^2} = 1$  for all  $\mathbf{x} \in \partial D$ . Then, one can show the following: (i) The map

$$\varphi_\delta : \mathbb{R}^d \ni \mathbf{x} \mapsto \mathbf{x} - \delta \mathbf{j}(\mathbf{x}) \in \mathbb{R}^d. \quad (5.1)$$

is in  $C^\infty(\mathbb{R}^d)$  for all  $\delta \in [0, 1]$ ; (ii) For all  $k \in \mathbb{N}$ , there is  $c$  such that  $\max_{\mathbf{x} \in D} \|D^k \varphi_\delta(\mathbf{x}) - D^k \mathbf{x}\|_{\ell^2} \leq c \ell_D^{-k} \delta$ , for all  $\delta \in [0, 1]$ ; (iii) There is  $r > 0$  so that

$$\varphi_\delta(D) + B(\mathbf{0}, \delta r) \subset D, \quad \forall \delta \in [0, 1]. \quad (5.2)$$

Let us consider the following kernel:

$$\rho(\mathbf{y}) = \begin{cases} \eta \exp\left(-\frac{1}{1-\|\mathbf{y}\|_{\ell^2}^2}\right), & \text{if } \|\mathbf{y}\|_{\ell^2} < 1, \\ 0, & \text{if } \|\mathbf{y}\|_{\ell^2} \geq 1, \end{cases} \quad (5.3)$$

where  $\eta$  is chosen so that  $\int_{\mathbb{R}^d} \rho(\mathbf{y}) \, d\mathbf{y} = \int_{B(\mathbf{0}, 1)} \rho(\mathbf{y}) \, d\mathbf{y} = 1$ . Let  $\delta \in [0, 1]$  and let  $f \in L^1(D; \mathbb{R}^q)$  with  $q = 1$  if we consider scalar-valued functions and  $q = d$  if we consider vector-valued functions.

We define a mollification operator as follows:

$$\mathcal{K}_\delta(f)(\mathbf{x}) := \int_{B(\mathbf{0}, 1)} \rho(\mathbf{y}) \mathbb{K}_\delta(\mathbf{x}) f(\varphi_\delta(\mathbf{x}) + (\delta r)\mathbf{y}) \, d\mathbf{y}, \quad \forall \mathbf{x} \in D, \quad (5.4)$$

where  $\mathbb{K}_\delta : D \rightarrow \mathbb{R}^{q \times q}$  is a smooth field. Note that the definition (5.4) makes sense owing to (5.2). The examples we have in mind for the field  $\mathbb{K}_\delta$  (inspired by Schöberl [21, 22]) are  $\mathbb{K}_\delta^g(\mathbf{x}) = 1$  ( $q = 1$ ),  $\mathbb{K}_\delta^c(\mathbf{x}) = \mathbb{J}_\delta^T(\mathbf{x})$  ( $q = d = 3$ ),  $\mathbb{K}_\delta^d(\mathbf{x}) = \det(\mathbb{J}_\delta(\mathbf{x})) \mathbb{J}_\delta^{-1}(\mathbf{x})$  ( $q = d$ ), and  $\mathbb{K}_\delta^b(\mathbf{x}) = \det(\mathbb{J}_\delta(\mathbf{x}))$  ( $q = 1$ ), where  $\mathbb{J}_\delta$  is the Jacobian matrix of  $\varphi_\delta$  at  $\mathbf{x} \in D$ . The mollification operator built using the field  $\mathbb{K}_\delta^x$  is denoted  $\mathcal{K}_\delta^x$  with  $x \in \{g, c, d, b\}$ . In what follows, we just state the main properties of the mollification operator  $\mathcal{K}_\delta$ , and we refer the reader to [12] for proofs.

**Lemma 5.1** (Smoothness). *For all  $f \in L^1(D; \mathbb{R}^q)$  and all  $\delta \in (0, 1]$ ,  $\mathcal{K}_\delta(f) \in C^\infty(\overline{D}; \mathbb{R}^q)$ , i.e.,  $\mathcal{K}_\delta(f) \in C^\infty(D; \mathbb{R}^q)$  and  $\mathcal{K}_\delta(f)$  as well as all its derivatives admit a continuous extension to  $\overline{D}$ .*

Let  $p \in [1, \infty]$ . Let us set  $Z^{g,p}(D) = W^{1,p}(D) = \{f \in L^p(D) \mid \nabla f \in L^p(D)\}$ ,  $Z^{c,p}(D) = \{\mathbf{g} \in L^p(D) \mid \nabla \times \mathbf{g} \in L^p(D)\}$  (for  $d = 3$ ), and  $Z^{d,p}(D) = \{\mathbf{g} \in L^p(D) \mid \nabla \cdot \mathbf{g} \in L^p(D)\}$ .

**Lemma 5.2** (Commuting). *The following holds true: (i)  $\nabla \mathcal{K}_\delta^g(f) = \mathcal{K}_\delta^c(\nabla f)$ , for all  $f \in Z^{g,p}(D)$ ; (ii)  $\nabla \times \mathcal{K}_\delta^c(\mathbf{g}) = \mathcal{K}_\delta^d(\nabla \times \mathbf{g})$ , for all  $\mathbf{g} \in Z^{c,p}(D)$  (for  $d = 3$ ); (iii)  $\nabla \cdot \mathcal{K}_\delta^d(\mathbf{g}) = \mathcal{K}_\delta^b(\nabla \cdot \mathbf{g})$ , for all  $\mathbf{g} \in Z^{d,p}(D)$ ; that is to say, the following diagrams commute:*

$$\begin{array}{ccccccc} Z^{g,p}(D) & \xrightarrow{\nabla} & Z^{c,p}(D) & \xrightarrow{\nabla \times} & Z^{d,p}(D) & \xrightarrow{\nabla \cdot} & L^p(D) \\ \downarrow \mathcal{K}_\delta^g & & \downarrow \mathcal{K}_\delta^c & & \downarrow \mathcal{K}_\delta^d & & \downarrow \mathcal{K}_\delta^b \\ C^\infty(\overline{D}) & \xrightarrow{\nabla} & C^\infty(\overline{D}) & \xrightarrow{\nabla \times} & C^\infty(\overline{D}) & \xrightarrow{\nabla \cdot} & C^\infty(\overline{D}) \end{array}$$

**Theorem 5.3** (Convergence). (i) *There are  $c, \delta_0 > 0$ , uniform, such that  $\|\mathcal{K}_\delta(f)\|_{L^p(D; \mathbb{R}^q)} \leq c \|f\|_{L^p(D; \mathbb{R}^q)}$  for all  $f \in L^p(D; \mathbb{R}^q)$ , all  $\delta \in [0, \delta_0]$ , and all  $p \in [1, \infty]$ . Moreover,*

$$\lim_{\delta \rightarrow 0} \|\mathcal{K}_\delta(f) - f\|_{L^p(D; \mathbb{R}^q)} = 0, \quad \forall f \in L^p(D; \mathbb{R}^q), \quad \forall p \in [1, \infty), \quad (5.5)$$

(ii) *There is  $c$ , uniform, such that for all  $f \in W^{s,p}(D; \mathbb{R}^q)$ , all  $\delta \in [0, \delta_0]$ , all  $s \in (0, 1]$ , and all  $p \in [1, \infty)$  ( $p \in [1, \infty]$  if  $s = 1$ ),*

$$\|\mathcal{K}_\delta(f) - f\|_{L^p(D; \mathbb{R}^q)} \leq c \ell_D^{-s} \delta^s \|f\|_{W^{s,p}(D; \mathbb{R}^q)}. \quad (5.6)$$

**Corollary 5.4** (Convergence of derivatives). *The following holds true: (i)  $\lim_{\delta \rightarrow 0} \|\nabla(\mathcal{K}_\delta^g(f) - f)\|_{L^p(D)} = 0$ ,  $\forall f \in Z^{g,p}(D)$ , and if  $\nabla f \in \mathbf{W}^{s,p}(D)$ ,  $\|\nabla(\mathcal{K}_\delta^g(f) - f)\|_{L^p(D)} \leq c \ell_D^{-s} \delta^s \|\nabla f\|_{\mathbf{W}^{s,p}(D)}$ ; (ii)  $\lim_{\delta \rightarrow 0} \|\nabla \times (\mathcal{K}_\delta^g(\mathbf{g}) - \mathbf{g})\|_{L^p(D)} = 0$ ,  $\forall \mathbf{g} \in \mathbf{Z}^{c,p}(D)$ , and if  $\nabla \times \mathbf{g} \in \mathbf{W}^{s,p}(D)$ ,  $\|\nabla \times (\mathcal{K}_\delta^g(\mathbf{g}) - \mathbf{g})\|_{L^p(D)} \leq c \ell_D^{-s} \delta^s \|\nabla \times \mathbf{g}\|_{\mathbf{W}^{s,p}(D)}$ ; (iii)  $\lim_{\delta \rightarrow 0} \|\nabla \cdot (\mathcal{K}_\delta^d(\mathbf{g}) - \mathbf{g})\|_{L^p(D)} = 0$ ,  $\forall \mathbf{g} \in \mathbf{Z}^{d,p}(D)$ , and if  $\nabla \cdot \mathbf{g} \in W^{s,p}(D)$ ,  $\|\nabla \cdot (\mathcal{K}_\delta^d(\mathbf{g}) - \mathbf{g})\|_{L^p(D)} \leq c \ell_D^{-s} \delta^s \|\nabla \cdot \mathbf{g}\|_{W^{s,p}(D)}$ . In the above statements, convergence holds for all  $p \in [1, \infty)$ , and convergence rates hold with  $c$  uniform for all  $\delta \in [0, \delta_0]$ , all  $s \in (0, 1]$ , and all  $p \in [1, \infty)$  ( $p \in [1, \infty]$  if  $s = 1$ ).*

**Remark 5.5** (Convergence in  $D$ ). Corollary 5.4(i) strengthens the original result by Friedrichs where strong convergence of the gradient only occurs in compact subsets of  $D$  (see, e.g., [6, Thm. 9.2]). Note though that Corollary 5.4(i) is valid for Lipschitz domains, whereas the original result by Friedrichs is valid for any open set.  $\square$

**Remark 5.6** (Density). Lemma 5.1, together with Lemma 5.2 and (5.5), implies that  $C^\infty(\overline{D}; \mathbb{R}^q)$  is dense in  $Z^{x,p}(D)$  for all  $x \in \{g, c, d\}$ .  $\square$

## 5.2 Inverse inequalities on faces

Let  $F \in \mathcal{F}_h^\circ$  be an interface. Let  $l \geq 0$ . Let  $P_l(F)$  be composed of the restriction of  $d$ -variate polynomials of degree at most  $l$  to  $F$ ; more precisely, we define  $P_l(F) := \mathbb{P}_{l,d-1} \circ \mathbf{T}_F^{-1}$  where  $\mathbf{T}_F : \widehat{F}^{d-1} \rightarrow F$  is an affine bijective map and  $\widehat{F}^{d-1}$  is the unit simplex in  $\mathbb{R}^{d-1}$  if  $\widehat{K}$  is the unit simplex of  $\mathbb{R}^d$  or  $\widehat{F}^{d-1}$  is the unit cuboid of  $\mathbb{R}^{d-1}$  if  $\widehat{K}$  is the unit cuboid of  $\mathbb{R}^d$ .

**Lemma 5.7** (Verfürth's inverse inequality). *Let  $F \in \mathcal{F}_h^\circ$ , let  $l \geq 0$ , and let  $P_l(F)$  be defined as above. Let  $\Phi_F : L^2(F) \rightarrow H^{-1}(D_F)$  be the map s.t.  $\Phi_F(r)(\varphi) := \int_F r \varphi \, ds$  for all  $\varphi \in H_0^1(D_F)$  and all  $r \in L^2(F)$ , where  $D_F$  is the interior of the set of the points of the two cells sharing  $F$ . Let  $\tilde{h}_F := \frac{|D_F|}{|F|}$  and let  $p \in (1, \infty)$ . Then there is  $c$ , uniform w.r.t.  $h$  (but depending on  $l$ ), such that*

$$\|g\|_{L^2(F)} \leq c \tilde{h}_F^{-\frac{1}{2} + d(\frac{1}{2} - \frac{1}{p})} \|\Phi_F(g)\|_{W_0^{1,p'}(D_F)'}, \quad \forall g \in P_l(F), \quad (5.7)$$

where  $p'$  is the conjugate number of  $p$  (i.e.,  $\frac{1}{p} + \frac{1}{p'} = 1$ ) and the dual space  $W_0^{1,p'}(D_F)'$  is equipped with the norm  $\|\Phi_F(g)\|_{W_0^{1,p'}(D_F)'} = \sup_{\varphi \in W_0^{1,p'}(D_F); \|\nabla \varphi\|_{L^{p'}(D_F)} = 1} |\Phi_F(g)(\varphi)|$ .

*Proof.* The proof hinges on the use of suitable bubble functions introduced by Verfürth; see [26, §3.6] for the proof with  $p = 2$ . The adaptation for  $p \neq 2$  is straightforward and is omitted for brevity.  $\square$

We now consider a boundary face  $F \in \mathcal{F}_h^\partial$  and show that the tangential component of a discrete function can be estimated in a mesh-dependent way by the local distance of this discrete function to the functions in  $\mathbf{H}_0(\text{curl}; D)$ .

**Lemma 5.8** (Bound on tangential component). *Let  $F \in \mathcal{F}_h^\partial$  and let  $K_F$  be the unique mesh cell of which  $F$  is a boundary face. Let  $l \geq 0$  and let  $P_l(K_F)$  be composed of the  $d$ -variate polynomials of degree at most  $l$  with values in  $\mathbb{R}^d$  ( $d = 3$ ). Let  $\tilde{h}_F := \frac{|K_F|}{|F|}$ . Then there is  $c$ , uniform w.r.t.  $h$  (but depending on  $l$ ), such that*

$$\|\mathbf{b} \times \mathbf{n}\|_{L^2(F)} \leq c \inf_{\mathbf{B} \in \mathbf{H}_0(\text{curl}; D)} \left( \tilde{h}_F^{\frac{1}{2}} \|\nabla \times (\mathbf{B} - \mathbf{b})\|_{L^2(K_F)} + \tilde{h}_F^{-\frac{1}{2}} \|\mathbf{B} - \mathbf{b}\|_{L^2(K_F)} \right), \quad \forall \mathbf{b} \in P_l(K_F). \quad (5.8)$$

*Proof.* Let us set  $\mathbf{H}_{0,F}^1(K_F) = \{\varphi \in \mathbf{H}^1(K_F) \mid \varphi|_{\partial K \setminus F} = 0\}$ . Owing to the regularity of the mesh sequence, the following Poincaré-Steklov inequality holds true: There is  $C_{\text{PS},\delta} > 0$  s.t.  $C_{\text{PS},\delta} \|\varphi\|_{L^2(K_F)} \leq \tilde{h}_F \|\nabla \varphi\|_{L^3(K_F)}$ , for all  $\varphi \in \mathbf{H}_{0,F}^1(K_F)$ . We equip the dual space  $\mathbf{H}_{0,F}^1(K_F)'$  with the norm  $\|\Phi\|_{\mathbf{H}_{0,F}^1(K_F)'} = \sup_{\varphi \in \mathbf{H}_{0,F}^1(K_F); \|\nabla \varphi\|_{L^3(K_F)} = 1} |\Phi(\varphi)|$ . Using as above Verfürth's argument based on suitable bubble functions, one can show that

$$\|\mathbf{b} \times \mathbf{n}\|_{L^2(F)} \leq c \tilde{h}_F^{-\frac{1}{2}} \sup_{\varphi \in \mathbf{H}_{0,F}^1(K_F); \|\nabla \varphi\|_{L^3(K_F)} = 1} \left| \int_F (\mathbf{b} \times \mathbf{n}) \cdot \varphi \, ds \right|.$$

Integrating by parts on  $K_F$ , we infer that

$$\int_F (\mathbf{b} \times \mathbf{n}) \cdot \boldsymbol{\varphi} \, ds = \int_{K_F} (\mathbf{b} \cdot \nabla \times \boldsymbol{\varphi} - \boldsymbol{\varphi} \cdot \nabla \times \mathbf{b}) \, dx.$$

Moreover, for all  $\mathbf{B} \in \mathbf{H}_0(\text{curl}; D)$ , we have  $\int_{K_F} (\mathbf{B} \cdot \nabla \times \boldsymbol{\varphi} - \boldsymbol{\varphi} \cdot \nabla \times \mathbf{B}) \, dx = \int_D (\mathbf{B} \cdot \nabla \times \tilde{\boldsymbol{\varphi}} - \tilde{\boldsymbol{\varphi}} \cdot \nabla \times \mathbf{B}) \, dx = 0$ , where  $\tilde{\boldsymbol{\varphi}} \in \mathbf{H}^1(D)$  is the zero-extension of  $\boldsymbol{\varphi}$  to  $D$ . We infer that

$$\int_F (\mathbf{b} \times \mathbf{n}) \cdot \boldsymbol{\varphi} \, ds = - \int_{K_F} ((\mathbf{B} - \mathbf{b}) \cdot \nabla \times \boldsymbol{\varphi} - \boldsymbol{\varphi} \cdot \nabla \times (\mathbf{B} - \mathbf{b})) \, dx.$$

We conclude the proof by applying the Cauchy–Schwarz inequality, applying the Poincaré–Steklov inequality to bound  $\|\boldsymbol{\varphi}\|_{L^2(K_F)}$ , and passing to the infimum over  $\mathbf{B} \in \mathbf{H}_0(\text{curl}; D)$ .  $\square$

**Remark 5.9** (Bound on trace of scalar-valued functions). A similar result to that of Lemma 5.8 for scalar-valued functions is available. Let  $l \geq 0$  and let  $P_l(K_F)$  and  $\tilde{h}_F$  be defined as above. Then there is  $c$ , uniform w.r.t.  $h$  (but depending on  $l$ ), such that

$$\|v\|_{L^2(F)} \leq c \inf_{u \in H_0^1(D)} \left( \tilde{h}_F^{\frac{1}{2}} \|\nabla(u - v)\|_{L^2(K_F)} + \tilde{h}_F^{-\frac{1}{2}} \|u - v\|_{L^2(K_F)} \right), \quad \forall v \in P_l(K_F). \quad (5.9)$$

One way of proving this bound is to observe that  $\|v\|_{L^2(F)} = \|u - v\|_{L^2(F)}$ , for all  $u \in H_0^1(D)$ , and to invoke the well-known multiplicative trace inequality in  $H^1(K_F)$ . One can also prove this bound using Lemma 5.7, but the advantage of using the multiplicative trace inequality is that the constant  $c$  in (5.9) is independent of the polynomial degree  $l$  and that (5.9) is actually valid for all  $v \in H^1(K_F)$ .  $\square$

### 5.3 Boundedness of trace operators on faces

We now mention two lemmas that are useful when Verfürth’s inverse inequality cannot be employed. This is the case when one needs to estimate a generalized integral involving the trace of a function living in some infinite-dimensional functional space.

**Lemma 5.10** (Normal component). *Let  $r > 0$  and let  $p \in (2_*, 2]$ ,  $2_* = \frac{2d}{d+2}$ . Then, for all  $F \in \mathcal{F}_h^\partial$ , the linear operator  $\gamma_F^\dagger : \mathbf{C}^\infty(\overline{D}) \rightarrow (P^\mathfrak{g}(\mathcal{T}_h))'$  s.t.  $\langle \gamma_F^\dagger(\boldsymbol{\phi}), w_h \rangle_{(P^\mathfrak{g}(\mathcal{T}_h))', P^\mathfrak{g}(\mathcal{T}_h)} = \int_F (\mathbf{n} \cdot \boldsymbol{\phi}) w_h \, ds$  for all  $w_h \in P^\mathfrak{g}(\mathcal{T}_h)$ , can be extended by continuity to the space  $\mathbf{Y} = \{\boldsymbol{\phi} \in \mathbf{L}^2(K) \mid \boldsymbol{\phi} \in \mathbf{H}^r(K), \nabla \cdot \boldsymbol{\phi} \in L^p(K)\}$ . In particular, there is  $c$ , uniform w.r.t.  $h$ , such that the following holds true:*

$$\left| \int_F (\mathbf{n} \cdot \boldsymbol{\phi}) w_h \, ds \right| \leq c h_F^{-\frac{1}{2}} \|w_h\|_{L^2(F)} \times \left( h_K^r |\boldsymbol{\phi}|_{\mathbf{H}^r(K)} + h_K^{d(\frac{1}{2_*} - \frac{1}{p})} \|\nabla \cdot \boldsymbol{\phi}\|_{L^p(K)} + \|\boldsymbol{\phi}\|_{L^2(K)} \right), \quad (5.10)$$

for all  $w_h \in P^\mathfrak{g}(\mathcal{T}_h)$  and all  $\boldsymbol{\phi} \in \mathbf{Y}$ , where  $K$  is the unique mesh cell of which  $F$  is a face. Note that in (5.10) we abuse the notation by identifying the action of the linear form to an integral over  $F$ .

*Proof.* We only sketch the proof. Classical arguments (employing the contravariant Piola transformation and the pullback by the geometric map  $\mathbf{T}_K$  to work on the reference cell  $\widehat{K}$  where one invokes norm equivalence in finite dimension) allow one to prove that

$$\left| \int_F (\mathbf{n} \cdot \boldsymbol{\phi}) w_h \, ds \right| \leq c h_F^{\frac{d-1}{2}} \|w_h\|_{L^2(F)} \left( h_K^{-\frac{d}{s}} \|\boldsymbol{\phi}\|_{L^s(K)} + h_K^{1-\frac{d}{q}} \|\nabla \cdot \boldsymbol{\phi}\|_{L^q(K)} \right), \quad (5.11)$$

provided  $s > 2$  and  $q \geq \frac{ds}{d+s}$ . Let us set  $d_r = \max(d-2r, d-2)$ ; since  $r > 0$ , we have  $d_r < d$ , and since  $d \geq 2$ , we have  $d_r \geq 0$ ; hence,  $2 < \frac{2d}{d_r}$ . Let us take  $s = \min(\frac{2d}{d_r}, \frac{dp}{d-p})$  and note that  $s > 2$  (since  $\frac{dp}{d-p} > 2$  iff  $p > 2_*$ ); moreover,  $s \leq \frac{dp}{d-p}$  implies that  $q = p$  is a legitimate choice (apply the increasing function  $t \mapsto \frac{dt}{d+t}$  to the previous inequality). From the embedding  $\mathbf{H}^r(\widehat{K}) \hookrightarrow \mathbf{L}^s(\widehat{K})$  valid because  $s \leq \frac{2d}{d_r}$  (indeed, if  $d-2r > 0$ ,  $s \leq \frac{2d}{d_r} \leq \frac{2d}{d-2r}$  which is the critical exponent, and if  $d-2r \leq 0$ ,  $\mathbf{H}^r(\widehat{K}) \hookrightarrow \mathbf{L}^\infty(\widehat{K})$ ), one readily shows that

$$h_K^{-\frac{d}{s}} \|\boldsymbol{\phi}\|_{L^s(K)} \leq c h_K^{-\frac{d}{2}} \left( \|\boldsymbol{\phi}\|_{L^2(K)} + h_K^r |\boldsymbol{\phi}|_{\mathbf{H}^r(K)} \right).$$

Putting all these bounds together, we obtain (5.10).  $\square$

**Lemma 5.11** (Tangential component). *Let  $r > 0$  and let  $p \in (2_*, 2]$ ,  $2_* = \frac{2d}{d+2}$ . Then, for all  $F \in \mathcal{F}_h^\partial$ , the linear operator  $\gamma_F^c : \mathbf{C}^\infty(\overline{D}) \rightarrow (\mathbf{P}^g(\mathcal{T}_h))'$  s.t.  $\langle \gamma_F^c(\phi), \mathbf{b}_h \rangle_{(\mathbf{P}^g(\mathcal{T}_h))', \mathbf{P}^g(\mathcal{T}_h)} = \int_F (\mathbf{n} \times \phi) \bar{\mathbf{b}}_h \, ds$  for all  $\mathbf{b}_h \in \mathbf{P}^g(\mathcal{T}_h)$ , can be extended by continuity to the space  $\mathbf{Y} = \{\phi \in \mathbf{L}^2(K) \mid \phi \in \mathbf{H}^r(K), \nabla \times \phi \in L^p(K)\}$ . In particular, there is  $c$ , uniform w.r.t.  $h$ , such that the following holds true:*

$$\left| \int_F (\mathbf{n} \times \phi) \bar{\mathbf{b}}_h \, ds \right| \leq c h_F^{-\frac{1}{2}} \|\mathbf{b}_h\|_{L^2(F)} \times \left( h_K^r |\phi|_{\mathbf{H}^r(K)} + h_K^{d(\frac{1}{2_*} - \frac{1}{p})} \|\nabla \times \phi\|_{L^p(K)} + \|\phi\|_{L^2(K)} \right), \quad (5.12)$$

for all  $\mathbf{b}_h \in \mathbf{P}^g(\mathcal{T}_h)$  and all  $\phi \in \mathbf{Y}$ , where  $K$  is the unique mesh cell of which  $F$  is a face. Note that in (5.12) we abuse the notation by identifying the action of the linear form to an integral over  $F$ .

*Proof.* Similar to that of Lemma 5.10. □

## 6 Application to the diffusion equation

In this section, we show how the trimmed error estimate from Lemma 4.5 and the mollified error estimate from Lemma 4.7 can be applied to the approximation of the diffusion equation using the boundary penalty method described in Section 3.2. The discrete spaces are  $W_h = V_h = \mathbf{P}^g(\mathcal{T}_h)$ , and the space  $V_b = H_0^1(D) + V_h$  is equipped with the norm  $\|\cdot\|_{V_b}$  that extends to  $V_b$  the norm  $\|\cdot\|_{V_h}$  originally defined by (3.8) on  $V_h$ . The discrete forms  $a_h(\cdot, \cdot)$  and  $\ell_h(\cdot)$  are defined by (3.5).

### 6.1 Trimmed error estimate

We define the trimming operator  $T : \mathbf{P}^g(\mathcal{T}_h) \rightarrow P_0^g(\mathcal{T}_h)$  such that, for all  $w_h \in \mathbf{P}^g(\mathcal{T}_h)$ ,  $T(w_h)|_K$  is defined, for all  $K \in \mathcal{T}_h$ , by zeroing out all the degrees of freedom of  $w_h$  that are attached to vertices, edges, and faces located at the boundary  $\partial D$ . This type of construction has been recently analyzed in [13] in the more general context of quasi-interpolation operators in canonical finite element spaces with prescribed boundary conditions. Let  $\mathcal{T}_h^\partial$  be the collection of the mesh cells touching the boundary; note that  $w_h - T(w_h)$  vanishes on all the mesh cells in  $\mathcal{T}_h \setminus \mathcal{T}_h^\partial$  but does not on the mesh cells in  $\mathcal{T}_h^\partial$ . For all  $K \in \mathcal{T}_h^\partial$ , one can prove that the following bounds hold true for all  $w_h \in \mathbf{P}^g(\mathcal{T}_h)$  with  $c$  uniform w.r.t.  $h$ : If  $\partial K \cap \partial D$  is composed of one or more boundary faces, then

$$h_K \|\nabla(w_h - T(w_h))\|_{L^2(K)} + \|w_h - T(w_h)\|_{L^2(K)} \leq c h_K^{\frac{1}{2}} \|w_h\|_{L^2(\partial K \cap \partial D)}, \quad (6.1)$$

whereas if  $\partial K \cap \partial D$  is a manifold of dimension  $d' < d - 1$ , then

$$h_K \|\nabla(w_h - T(w_h))\|_{L^2(K)} + \|w_h - T(w_h)\|_{L^2(K)} \leq c h_K^{\frac{1}{2}} \|w_h\|_{L^2(F)}, \quad \forall F \in \mathcal{F}_K^\partial, \quad (6.2)$$

where  $\mathcal{F}_K^\partial := \{F \in \mathcal{F}_h^\partial \mid \partial K \cap \partial D \subseteq F\}$  is the collection of the boundary faces containing the manifold  $\partial K \cap \partial D$ . We introduce the contrast factor

$$\xi_\kappa = \max_{K \in \mathcal{T}_h^\partial} \frac{\kappa_K}{\max_{F \in \mathcal{F}_K^\partial} \kappa_{K_F}}, \quad (6.3)$$

where we recall that, for all  $F \in \mathcal{F}_K^\partial \subset \mathcal{F}_h^\partial$ ,  $K_F$  is the unique mesh cell having  $F$  as boundary face. Finally, let us set

$$Y = \{v \in H_0^1(D) \mid \nabla \cdot (\kappa \nabla v) \in L^p(D)\}, \quad (6.4)$$

with  $p \in (2_*, 2]$ ,  $2_* = \frac{2d}{d+2}$ , and let us equip the space  $V_\sharp = Y + V_h$  with the norm

$$\|v\|_{V_\sharp} = \left( \|v\|_{V_b}^2 + \sum_{K \in \mathcal{T}_h^\partial} \kappa_K^{-1} h_K^{2d(\frac{1}{2_*} - \frac{1}{p})} \|\nabla \cdot (\kappa \nabla v)\|_{L^p(K)}^2 \right)^{\frac{1}{2}}. \quad (6.5)$$

**Lemma 6.1** (Trimmed error estimate). *The assumptions of Lemma 4.5 hold true with the trimming operator  $T : \mathbf{P}^g(\mathcal{T}_h) \rightarrow P_0^g(\mathcal{T}_h)$  defined above, where the constants  $\omega_{V_\sharp, V_h}^{\text{tri}}$  and  $\varpi_{V_\sharp, V_h}^{\text{tri}}$  are proportional to  $\xi_\kappa^{\frac{1}{2}}$  with the contrast factor  $\xi_\kappa$  defined by (6.3).*

*Proof.* (1) Let us verify that (4.10) holds. Let  $(v_h, w_h) \in V_h \times V_h$ . Since  $T(w_h) \in H_0^1(D)$ , we infer that

$$a(u, T(w_h)) - a_h(v_h, T(w_h)) = \int_D \kappa \nabla(u - v_h) \cdot \nabla T(w_h) \, dx \leq \|u - v_h\|_{V_h} \|\kappa^{\frac{1}{2}} \nabla T(w_h)\|_{L^2(D)}.$$

We have  $T(w_h) = w_h$  on all  $K \in \mathcal{T}_h \setminus \mathcal{T}_h^\partial$  so that we only need to bound  $\|\kappa^{\frac{1}{2}} \nabla T(w_h)\|_{L^2(K)}$  for all  $K \in \mathcal{T}_h^\partial$ . In this case, the triangle inequality implies that

$$\|\kappa^{\frac{1}{2}} \nabla T(w_h)\|_{L^2(K)} \leq \|\kappa^{\frac{1}{2}} \nabla w_h\|_{L^2(K)} + \kappa_K^{\frac{1}{2}} \|\nabla \hat{w}_h\|_{L^2(K)},$$

where we have set  $\hat{w}_h := w_h - T(w_h)$ . If  $\partial K \cap \partial D$  is composed of one or more boundary faces, we use the approximation property (6.1) together with the regularity of the mesh sequence to infer that

$$\kappa_K^{\frac{1}{2}} \|\nabla \hat{w}_h\|_{L^2(K)} \leq c \kappa_K^{\frac{1}{2}} h_K^{\frac{1}{2}} \|w_h\|_{L^2(\partial K \cap \partial D)} \leq c' \left( \sum_{\substack{F \in \mathcal{F}_h^\partial \\ F \subseteq \partial K \cap \partial D}} \|\eta_h^{\frac{1}{2}} w_h\|_{L^2(F)}^2 \right)^{\frac{1}{2}}.$$

Instead, if  $\partial K \cap \partial D$  is a manifold of dimension  $d' < d-1$ , we use the approximation property (6.2) together with the regularity of the mesh sequence to infer that, for all  $F \in \mathcal{F}_K^\partial$ ,

$$\kappa_K^{\frac{1}{2}} \|\nabla \hat{w}_h\|_{L^2(K)} \leq c \left( \frac{\kappa_K}{\kappa_{K_F}} \right)^{\frac{1}{2}} \kappa_{K_F}^{\frac{1}{2}} h_F^{-\frac{1}{2}} \|w_h\|_{L^2(F)} \leq c' \left( \frac{\kappa_K}{\kappa_{K_F}} \right)^{\frac{1}{2}} \|\eta_h^{\frac{1}{2}} w_h\|_{L^2(F)}.$$

We now take a boundary face in  $\mathcal{F}_K^\partial$ , say  $F_*$ , s.t.  $\kappa_{K_F}$  is maximal so as to make the above upper bound as small as possible. We obtain

$$\kappa_K^{\frac{1}{2}} \|\nabla \hat{w}_h\|_{L^2(K)} \leq c' \left( \frac{\kappa_K}{\max_{F \in \mathcal{F}_K^\partial} \kappa_{K_F}} \right)^{\frac{1}{2}} \|\eta_h^{\frac{1}{2}} w_h\|_{L^2(F_*)} \leq c' \left( \max_{K \in \mathcal{T}_h^\partial} \frac{\kappa_K}{\max_{F \in \mathcal{F}_K^\partial} \kappa_{K_F}} \right)^{\frac{1}{2}} \|\eta_h^{\frac{1}{2}} w_h\|_{L^2(F_*)}.$$

Recalling the definition (6.3) of the contrast factor  $\xi_\kappa$ , we infer that  $\kappa_K^{\frac{1}{2}} \|\nabla \hat{w}_h\|_{L^2(K)} \leq c' \xi_\kappa^{\frac{1}{2}} \|\eta_h^{\frac{1}{2}} w_h\|_{L^2(F_*)}$  with  $F_* \in \mathcal{F}_K^\partial \subset \mathcal{F}_h^\partial$ . It is now straightforward to complete the proof of (4.10).

(2) Let us verify (4.11). Let  $(v_h, w_h) \in V_h \times V_h$  and let us set  $e_h := u - v_h$  and (as above)  $\hat{w}_h := w_h - T(w_h)$ . A direct calculation shows that

$$\begin{aligned} & \ell_h(w_h) - \ell(T(w_h)) - a_h(v_h, (I - T)(w_h)) \\ &= \int_D f \hat{w}_h \, dx - \int_D \kappa \nabla v_h \cdot \nabla \hat{w}_h \, dx + \int_{\partial D} (\mathbf{n} \cdot \kappa \nabla v_h) \hat{w}_h \, ds - \int_{\partial D} \eta_h v_h \hat{w}_h \, ds \\ &= \sum_{K \in \mathcal{T}_h^\partial} \int_K -\nabla \cdot (\kappa \nabla e_h) \hat{w}_h \, dx - \sum_{F \in \mathcal{F}_h^{\circ\partial}} \int_F [[\kappa \nabla v_h]] \cdot \mathbf{n}_F \hat{w}_h \, ds - \int_{\partial D} \eta_h v_h \hat{w}_h \, ds, \end{aligned}$$

where  $\mathcal{F}_h^{\circ\partial}$  is the collection of the mesh interfaces that touch the boundary (note that  $\hat{w}_h$  vanishes on all the remaining interfaces in  $\mathcal{F}_h^\circ$ ). The Cauchy–Schwarz inequality leads to  $\|\ell_h - \ell \circ T - a_h(v_h, (I - T)(\cdot))\|_{V_h'} \leq \mathfrak{T}_1 \mathfrak{T}_2$  with  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  defined by

$$\begin{aligned} \mathfrak{T}_1 &= \left( \sum_{K \in \mathcal{T}_h^\partial} \kappa_K^{-1} h_K^{2d(\frac{1}{2^*} - \frac{1}{p})} \|\nabla \cdot (\kappa \nabla e_h)\|_{L^p(K)}^2 + \sum_{F \in \mathcal{F}_h^{\circ\partial}} \kappa_{K_F}^{-1} \tilde{h}_F \|[ [\kappa \nabla v_h] ] \cdot \mathbf{n}_F\|_{L^2(F)}^2 + \|\eta_h^{\frac{1}{2}} e_h\|_{L^2(\partial D)}^2 \right)^{\frac{1}{2}}, \\ \mathfrak{T}_2 &= \left( \sum_{K \in \mathcal{T}_h^\partial} \kappa_K h_K^{2d(\frac{1}{p} - \frac{1}{2^*})} \|\hat{w}_h\|_{L^{p'}(K)}^2 + \sum_{F \in \mathcal{F}_h^{\circ\partial}} \kappa_{K_F} \tilde{h}_F^{-1} \|\hat{w}_h\|_{L^2(F)}^2 + \|\eta_h^{\frac{1}{2}} w_h\|_{L^2(\partial D)}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where, for all  $F \in \mathcal{F}_h^{\circ\partial}$ ,  $K_F$  is the mesh cell sharing  $F$  and having the larger value of  $\kappa_K$  (the choice of  $K_F$  is irrelevant if both cells give the same value),  $\tilde{h}_F$  is defined in Lemma 5.7, and where  $p'$  is the conjugate number of  $p$ . Moreover, in the last term defining  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$ , we have exploited the fact that  $u$  and  $T(w_h)$  have zero trace at the boundary  $\partial D$  so that  $\|\eta_h^{\frac{1}{2}} v_h\|_{L^2(\partial D)} = \|\eta_h^{\frac{1}{2}} e_h\|_{L^2(\partial D)}$  and  $\|\eta_h^{\frac{1}{2}} \hat{w}_h\|_{L^2(\partial D)} = \|\eta_h^{\frac{1}{2}} w_h\|_{L^2(\partial D)}$ .



(2a) Bound on  $\mathfrak{T}_1$ . We need to bound  $\|[\kappa \nabla v_h] \cdot \mathbf{n}_F\|_{L^2(F)}$  and to this purpose we use Lemma 5.7. This is possible since, by assumption,  $\kappa$  is piecewise constant on the mesh  $\mathcal{T}_h$  and, therefore,  $[\kappa \nabla v_h] \cdot \mathbf{n}_F \in P_l(F)$  for some polynomial degree  $l \geq 0$ . We infer that

$$\tilde{h}_F^{\frac{1}{2}} \|[\kappa \nabla v_h] \cdot \mathbf{n}_F\|_{L^2(F)} \leq c \tilde{h}_F^{d(\frac{1}{2} - \frac{1}{p})} \sup_{\substack{\varphi \in W_0^{1,p'}(D_F) \\ \|\nabla \varphi\|_{L^{p'}(D_F)} = 1}} \int_F [[\kappa \nabla v_h] \cdot \mathbf{n}_F] \varphi \, ds.$$

Let  $\varphi \in W_0^{1,p'}(D_F)$  be s.t.  $\|\nabla \varphi\|_{L^{p'}(D_F)} = 1$ . By definition of the jump and using the divergence formula, we have

$$\begin{aligned} \int_F [[\kappa \nabla v_h] \cdot \mathbf{n}_F] \varphi \, ds &= \int_F (\kappa \nabla v_h)|_{K_l} \cdot \mathbf{n}_{K_l} \varphi \, ds + \int_F (\kappa \nabla v_h)|_{K_r} \cdot \mathbf{n}_{K_r} \varphi \, ds \\ &= \sum_{K \in \mathcal{T}_F} \int_K \nabla \cdot (\varphi \kappa \nabla v_h) \, dx = \sum_{K \in \mathcal{T}_F} \int_K (\varphi \nabla \cdot (\kappa \nabla v_h) + \kappa \nabla v_h \cdot \nabla \varphi) \, dx, \end{aligned}$$

where  $K_l, K_r$  are the two mesh cells sharing the interface  $F$  and where we have set  $\mathcal{T}_F = \{K_l, K_r\}$ . Moreover, since  $p' \geq 2$ , the zero-extension of  $\varphi$  to  $D$  is in  $H_0^1(D)$ , and this implies that  $\sum_{K \in \mathcal{T}_F} \int_K (\varphi \nabla \cdot (\kappa \nabla v_h) + \kappa \nabla v_h \cdot \nabla \varphi) \, dx = 0$ . Combining these two relations, we infer that

$$\sum_{K \in \mathcal{T}_F} \int_K (\varphi \nabla \cdot (\kappa \nabla e_h) + \kappa \nabla e_h \cdot \nabla \varphi) \, dx = 0.$$

Invoking Hölder's inequality and the Poincaré–Steklov inequality in  $W_0^{1,p}(K)$ , for all  $K \in \mathcal{T}_F$ , we infer that

$$\left| \int_K \varphi \nabla \cdot (\kappa \nabla e_h) \, dx \right| \leq \|\varphi\|_{L^{p'}(K)} \|\nabla \cdot (\kappa \nabla e_h)\|_{L^p(K)} \leq c h_K \|\nabla \varphi\|_{L^{p'}(K)} \|\nabla \cdot (\kappa \nabla e_h)\|_{L^p(K)}.$$

Moreover, invoking Hölder's inequality, that  $p \leq 2$ , and mesh regularity, we infer that

$$\left| \int_K \kappa \nabla e_h \cdot \nabla \varphi \, dx \right| \leq \|\nabla \varphi\|_{L^{p'}(K)} \|\kappa \nabla e_h\|_{L^p(K)} \leq c h_K^{d(\frac{1}{p} - \frac{1}{2})} \|\nabla \varphi\|_{L^{p'}(K)} \|\kappa \nabla e_h\|_{L^2(K)}.$$

Putting the above bounds together and since  $K_F$  has been chosen so that  $\kappa_{K_F} = \max_{K \in \mathcal{T}_F} \kappa_K$ , we conclude that  $\mathfrak{T}_1 \leq c \|e_h\|_{V_\sharp}$ .

(2b) Bound on  $\mathfrak{T}_2$ . Applying an inverse inequality from  $L^{p'}(K)$  to  $L^2(K)$ , for all  $K \in \mathcal{T}_h^\partial$ , we infer that  $h_K^{d(\frac{1}{p} - \frac{1}{2^*})} \|\hat{w}_h\|_{L^{p'}(K)} \leq c h_K^{-1} \|\hat{w}_h\|_{L^2(K)}$ . Moreover, applying an inverse trace inequality on  $K_F$ , for all  $F \in \mathcal{F}_h^{\partial\partial}$ , and invoking mesh regularity, we infer that  $\tilde{h}_F^{-\frac{1}{2}} \|\hat{w}_h\|_{L^2(F)} \leq c h_{K_F}^{-1} \|\hat{w}_h\|_{L^2(K_F)}$ . Finally, using the approximation property (6.2) on all  $K \in \mathcal{T}_h^\partial$  and recalling the definition of the contrast factor  $\xi_\kappa$ , we conclude that  $\mathfrak{T}_2 \leq c \xi_\kappa^{\frac{1}{2}} \|w_h\|_{V_h}$ . This completes the proof of (4.11).  $\square$

## 6.2 Mollified error estimate

Recall the decomposition  $a_h(\cdot, \cdot) = \tilde{a}_h(\cdot, \cdot) + s_h(\cdot, \cdot)$  with the bilinear forms  $\tilde{a}_h(\cdot, \cdot)$  and  $s_h(\cdot, \cdot)$  defined by (3.6). Let us set

$$Y = \{v \in H_0^1(D) \mid \kappa \nabla v \in \mathbf{H}^r(D^\partial), \nabla \cdot (\kappa \nabla v) \in L^p(D)\}, \quad (6.6)$$

with  $r > 0$ ,  $p \in (2_*, 2]$ ,  $2_* = \frac{2d}{d+2}$ , and where  $D^\partial = \cup_{F \in \mathcal{F}_h^\partial} D_{K_F}$  is the collection of all the points in  $D$  belonging to a mesh cell touching a mesh cell having (at least) a boundary face. Let us equip the space  $V_\sharp = Y + V_h$  with the norm

$$\begin{aligned} \|v\|_{V_\sharp} &= \left( \|\eta_h^{\frac{1}{2}} v\|_{L^2(\partial D)}^2 + \sum_{K \in \mathcal{T}_h} \kappa_K^{-1} \|\kappa \nabla v\|_{L^2(D_K)}^2 \right. \\ &\quad \left. + \sum_{F \in \mathcal{F}_h^\partial} \kappa_{K_F}^{-1} \left( h_{K_F}^r |\kappa \nabla v|_{\mathbf{H}^r(D_{K_F})} + h_{K_F}^{d(\frac{1}{2^*} - \frac{1}{p})} \|\nabla \cdot (\kappa \nabla v)\|_{L^p(D_{K_F})} \right)^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (6.7)$$

Since  $\kappa_K^{-1} \|\kappa \nabla v\|_{L^2(D_K)}^2 \geq \|\kappa^{\frac{1}{2}} \nabla v\|_{L^2(K)}^2$ , the bound  $\|v\|_{V_\sharp} \leq \|v\|_{V_h}$  holds true for all  $v \in V_\sharp$ .

**Lemma 6.2** (Mollified error estimate). *The assumptions of Lemma 4.7 hold true with the bilinear form*

$$a_{\delta h}(v, w_h) = \int_D \mathcal{K}_\delta^d(\kappa \nabla v) \cdot \nabla w_h \, dx - \int_{\partial D} (\mathbf{n} \cdot \mathcal{K}_\delta^d(\kappa \nabla v)) w_h \, ds, \quad (6.8)$$

for all  $(v, w_h) \in V_b \times V_h$  and all  $\delta \in [0, \delta_0]$  with  $\delta_0 > 0$  small enough.

*Proof.* (1) Let us verify the property (4.14a). Let  $w_h \in V_h$ . We observe that integration by parts together with the commuting property (iii) from Lemma 5.2 implies that

$$a_{\delta h}(u, w_h) = \int_D -\nabla \cdot (\mathcal{K}_\delta^d(\kappa \nabla u)) w_h \, dx = \int_D -\mathcal{K}_\delta^b(\nabla \cdot (\kappa \nabla u)) w_h \, dx.$$

Since  $\ell_h(w_h) = \int_D f w_h \, dx$  and the exact solution is such that  $-\nabla \cdot (\kappa \nabla u) = f$ , we infer that

$$\ell_h(w_h) - a_{\delta h}(u, w_h) = \int_D (f - \mathcal{K}_\delta^b(f)) w_h \, dx.$$

This implies that  $|\ell_h(w_h) - a_{\delta h}(u, w_h)| \leq \|f - \mathcal{K}_\delta^b(f)\|_{L^p(D)} \|w_h\|_{L^{p'}(D)}$ . Since  $\|w_h\|_{L^{p'}(D)} \leq c \|\nabla w_h\|_{L^2(D)} \leq c \kappa_{b,D}^{-\frac{1}{2}} \|w_h\|_{V_h}$  owing to a Sobolev embedding, we can conclude that  $\|\ell_h - a_{\delta h}(u, \cdot)\|_{V_h'} \rightarrow 0$  as  $\delta \rightarrow 0$  by invoking Theorem 5.3. This proves (4.14a). Let us verify the property (4.14b). For all  $v_h \in V_h$ , we have

$$a_{\delta h}(v_h, w_h) - \tilde{a}_h(v_h, w_h) = \int_D (\mathcal{K}_\delta^d(\mathbf{g}_h) - \mathbf{g}_h) \cdot \nabla w_h \, dx - \int_{\partial D} (\mathbf{n} \cdot (\mathcal{K}_\delta^d(\mathbf{g}_h) - \mathbf{g}_h)) w_h \, ds,$$

with  $\mathbf{g}_h = \kappa \nabla v_h$ . The Cauchy–Schwarz inequality implies that

$$\|a_{\delta h}(v_h, \cdot) - \tilde{a}_h(v_h, \cdot)\|_{V_h'} \leq \kappa_{b,D}^{-\frac{1}{2}} \|\mathcal{K}_\delta^d(\mathbf{g}_h) - \mathbf{g}_h\|_{L^2(D)} + c \kappa_{b,D}^{-\frac{1}{2}} h^{\frac{1}{2}} \|\mathcal{K}_\delta^d(\mathbf{g}_h) - \mathbf{g}_h\|_{L^2(\partial D)}.$$

The first term tends to zero as  $\delta \rightarrow 0$  owing to Theorem 5.3. The passage to the limit on the second term is a little more subtle: one uses that  $\mathbf{g}_h$  is bounded in  $D$  and has bounded variation in  $D$ . This proves (4.14b).

(2) Let us now verify (4.15). Let  $v \in V_b$ . The Cauchy–Schwarz inequality together with the bound on the normal component from Lemma 5.10 implies that

$$\begin{aligned} \|a_{\delta h}(v, \cdot)\|_{V_h'} &\leq \left( \sum_{K \in \mathcal{T}_h} \kappa_K^{-1} \|\phi\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \\ &\quad + c \left( \sum_{F \in \mathcal{F}_h^\partial} \kappa_{K_F}^{-1} (h_{K_F}^r |\phi|_{\mathbf{H}^r(K_F)} + h_{K_F}^{d(\frac{1}{2^*} - \frac{1}{p})}) \|\nabla \cdot \phi\|_{L^p(K_F)} + \|\phi\|_{L^2(K_F)}^2 \right)^{\frac{1}{2}}, \end{aligned} \quad (6.9)$$

where  $\phi = \mathcal{K}_\delta^d(\kappa \nabla v)$  and  $K_F$  is the unique mesh cell of which  $F \in \mathcal{F}_h^\partial$  is a boundary face. We choose  $\delta_0$  to be small enough so that  $\varphi_\delta(K) \subset D_K$  for all  $K \in \mathcal{T}_h$  and all  $\delta \in [0, \delta_0]$  (recall that  $D_K$  denotes the subset of  $D$  defined as the collection of the points located in the mesh cells touching  $K$  and that  $D_K$  is composed of a uniformly bounded number of mesh cells owing to shape-regularity of the mesh sequence). Using the local stability properties  $\|\phi\|_{L^2(K)} \leq c \|\kappa \nabla v\|_{L^2(D_K)}$  and  $|\phi|_{\mathbf{H}^r(K)} \leq c |\kappa \nabla v|_{\mathbf{H}^r(D_K)}$  and the fact that  $\nabla \cdot \phi = \mathcal{K}_\delta^b(\nabla \cdot (\kappa \nabla v))$  together with a similar local  $L^2$ -stability property for  $\mathcal{K}_\delta^b$ , we infer that

$$\begin{aligned} \|a_{\delta h}(v, \cdot)\|_{V_h'} &\leq c \left( \sum_{K \in \mathcal{T}_h} \kappa_K^{-1} \|\kappa \nabla v\|_{L^2(D_K)}^2 \right)^{\frac{1}{2}} \\ &\quad + c \left( \sum_{F \in \mathcal{F}_h^\partial} \kappa_{K_F}^{-1} (h_{K_F}^r |\kappa \nabla v|_{\mathbf{H}^r(D_{K_F})} + h_{K_F}^{d(\frac{1}{2^*} - \frac{1}{p})}) \|\nabla \cdot (\kappa \nabla v)\|_{L^p(D_{K_F})} \right)^{\frac{1}{2}}, \end{aligned}$$

where the last term in (6.9) has been regrouped with the summation over the mesh cells since  $\kappa_{K_F}^{-1} \|\phi\|_{L^2(K_F)}^2 \leq c \kappa_{K_F}^{-1} \|\kappa \nabla v\|_{L^2(D_{K_F})}^2$ . This proves (4.15) owing to the definition (6.7) for the norm  $\|\cdot\|_{V_b}$ .

(3) Let us finally verify (4.16). Since the stabilization bilinear form is symmetric and positive semidefinite, we infer that  $\|s_h(v_h, \cdot)\|_{V_h'} \leq \|\eta_h^{\frac{1}{2}} v_h\|_{L^2(\partial D)}$  for all  $v_h \in V_h$ , and the expected bound (4.16) readily follows since the exact solution  $u$  has a zero trace on  $\partial D$ .  $\square$

The norm  $\|\cdot\|_{V_{\sharp}}$  defined by (6.7) is not fully satisfactory since it depends on the contrast factor  $\xi_{\kappa}$ . We now show that this shortcoming can be addressed by modifying the mollified bilinear form  $a_{\delta h}$ . Recall the partition of  $D$  into  $D_1, \dots, D_M$ . For all  $1 \leq i \leq M$ , we define the mollification operators  $\mathcal{K}_{\delta, i}$  as in Section 5.1 using a shrinking-based technique in the subdomain  $D_i$ . Then we define the broken mollification operator  $\hat{\mathcal{K}}_{\delta}$  such that, for all  $f \in L^1(D; \mathbb{R}^q)$ , we have  $\hat{\mathcal{K}}_{\delta}(f)|_{D_i} = \mathcal{K}_{\delta, i}(f|_{D_i})$ . Note that  $\hat{\mathcal{K}}_{\delta}(f)$  is no longer smooth globally in  $D$ , but we have  $\hat{\mathcal{K}}_{\delta}(f)|_{D_i} \in C^{\infty}(\bar{D}_i)$ . The space  $Y$  is still defined by (6.6), but we now equip the space  $V_{\sharp} = Y + V_h$  with the norm

$$\|v\|_{V_{\sharp}} = \left( \|v\|_{V_b}^2 + \sum_{1 \leq i \leq M} \sum_{F \in \mathcal{F}_h^{\partial} \cap \partial D_i} \kappa_i^{-1} \left( h_{K_F}^r |\kappa_i \nabla v|_{\mathbf{H}^r(D_i, \kappa_F)} + h_{K_F}^{d(\frac{1}{2^*} - \frac{1}{p})} \|\nabla \cdot (\kappa_i \nabla v)\|_{L^p(D_i, \kappa_F)} \right)^2 \right)^{\frac{1}{2}}, \quad (6.10)$$

where we have set  $\kappa_i = \kappa|_{D_i}$  and  $D_{i, K_F} = D_{K_F} \cap D_i$ .

**Lemma 6.3** (Mollified error estimate). *The assumptions of Lemma 4.7 hold true with the bilinear form*

$$\hat{a}_{\delta h}(v, w_h) = \int_D \hat{\mathcal{K}}_{\delta}^{\mathbf{d}}(\kappa \nabla v) \cdot \nabla w_h \, dx - \int_{\partial D} (\mathbf{n} \cdot \hat{\mathcal{K}}_{\delta}^{\mathbf{d}}(\kappa \nabla v)) w_h \, ds, \quad (6.11)$$

for all  $(v, w_h) \in V_b \times V_h$  and all  $\delta \in [0, \delta_0]$  with  $\delta_0 > 0$  small enough. Moreover, the stability constants invoked in Lemma 4.7 are uniform with respect to the contrast factor  $\xi_{\kappa}$ .

*Proof.* The main change w.r.t. the proof of Lemma 6.2 concerns the verification of (4.14a). Let  $\mathcal{K}_{\delta}^{\mathbf{d}}$  be the global mollification operator considered above. Since  $\mathcal{K}_{\delta}^{\mathbf{d}}(\kappa \nabla u)$  is globally smooth over  $D$  and the test function  $w_h$  is continuous, we infer that

$$\ell_h(w_h) - \hat{a}_{\delta h}(u, w_h) = \int_D (f - \hat{\mathcal{K}}_{\delta}^{\mathbf{b}}(f)) w_h \, dx + \sum_{1 \leq i \leq M} \int_{\partial D_i^{\circ}} \mathbf{n}_i \cdot (\hat{\mathcal{K}}_{\delta}^{\mathbf{d}}(\kappa \nabla u) - \mathcal{K}_{\delta}^{\mathbf{d}}(\kappa \nabla u)) w_h \, ds,$$

where  $\partial D_i^{\circ} = \partial D_i \cap D$  is the part of the boundary  $\partial D_i$  located in the interior of  $D$  and  $\mathbf{n}_i$  is the unit outward-pointing normal vector to  $\partial D_i$ . Let us denote  $\Phi_{1, \delta}$  and  $\Phi_{2, \delta}$  the two linear forms (acting on  $w_h$ ) on the above right-hand side. It is clear that  $\|\Phi_{1, \delta}\|_{V_h'} \rightarrow 0$  as  $\delta \rightarrow 0$ . Considering  $\Phi_{2, \delta}$ , we add/subtract  $\mathbf{n}_i \cdot (\kappa \nabla u)|_{D_i}$  (this is legitimate since  $u \in Y$  provided integrals are understood as the action of linear forms on  $w_h$  as in Lemma 5.10) and we obtain

$$\langle \Phi_{2, \delta}, w_h \rangle_{V_h', V_h} = \sum_{1 \leq i \leq M} \sum_{F \subset \partial D_i^{\circ}} \int_F \mathbf{n}_i \cdot (\mathbf{e}_{\delta, i, F, 1} - \mathbf{e}_{\delta, i, F, 2}) w_h \, ds,$$

with  $\mathbf{e}_{\delta, i, F, 1} = \mathcal{K}_{\delta, i}^{\mathbf{d}}(\phi_i) - \phi_i$ ,  $\mathbf{e}_{\delta, i, F, 2} = \mathcal{K}_{\delta}^{\mathbf{d}}(\phi)|_{D_i} - \phi_i$ ,  $\phi = \kappa \nabla u$ , and  $\phi_i = \phi|_{D_i}$ . The bound (5.11) from the proof of Lemma 5.10 together with the triangle inequality then implies that

$$\left| \int_F \mathbf{n}_i \cdot (\mathbf{e}_{\delta, i, F, 1} - \mathbf{e}_{\delta, i, F, 2}) w_h \, ds \right| \leq c h_F^{-\frac{1}{2}} \|w_h\|_{L^2(F)} \\ \times \left( h_K^{-\frac{d}{s}} (\|\mathbf{e}_{\delta, i, F, 1}\|_{L^s(K)} + \|\mathbf{e}_{\delta, i, F, 2}\|_{L^s(K)}) + h_K^{1-\frac{d}{q}} (\|\nabla \cdot \mathbf{e}_{\delta, i, F, 1}\|_{L^q(K)} + \|\nabla \cdot \mathbf{e}_{\delta, i, F, 2}\|_{L^q(K)}) \right),$$

with  $s > 2$  and  $q = \frac{ds}{d+s}$ . Since  $u \in Y$ , one readily shows using a Sobolev embedding that indeed  $\phi \in L^s(D)$  for some  $s > 2$  and since  $p > 2^*$ ,  $s$  can be chosen so that  $\nabla \cdot \phi \in L^q(D)$ . Using the commuting properties of  $\mathcal{K}_{\delta, i}^{\mathbf{d}}$  and of  $\mathcal{K}_{\delta}^{\mathbf{d}}$ , we infer that the above bound tends to zero as  $\delta \rightarrow 0$ , and we conclude that  $\|\Phi_{2, \delta}\|_{V_h'} \rightarrow 0$  as  $\delta \rightarrow 0$  (recall that  $h$  is fixed in this reasoning). Furthermore, the verification of (4.14b) and of (4.15) is similar to that in the proof of Lemma 6.2 upon reasoning on each subdomain. In particular, when proving (4.15), we exploit the fact that the broken mollification operator is local to each subdomain so as to consider the norm  $\|\cdot\|_{V_{\sharp}}$  defined by (6.10). Finally, the verification of (4.16) is unmodified.  $\square$

**Remark 6.4** (Partition of  $D$ ). In the above proof, it is possible to consider that the partition of  $D$  into  $D_1, \dots, D_M$  coincides with the current mesh, so that the broken mollification operator is local to each mesh cell.  $\square$

## 7 Application to the time-harmonic Maxwell's equations

In this section, we show how the trimmed error estimate from Lemma 4.5 and the mollified error estimate from Lemma 4.7 can be applied to the approximation of the time-harmonic Maxwell's equations using the boundary penalty method described in Section 3.3. The discrete space is  $\mathbf{V}_h = \mathbf{P}^g(\mathcal{T}_h)$ , and the space  $\mathbf{V}_b = \mathbf{H}_0(\text{curl}; D) + \mathbf{V}_h$  can be equipped with the norm  $\|\cdot\|_{\mathbf{V}_b}$  that extends to  $\mathbf{V}_b$  the norm  $\|\cdot\|_{\mathbf{V}_h}$  originally defined by (3.13) on  $\mathbf{V}_h$ ; note in particular that functions in  $\mathbf{V}_b$  have a well-defined tangential trace on  $\partial D$ . Indeed, any function  $\mathbf{b} \in \mathbf{V}_b$  can be written as  $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$  with  $\mathbf{b}_1 \in \mathbf{H}_0(\text{curl}; D)$  and  $\mathbf{b}_2 \in \mathbf{V}_h$ , and we have  $\gamma^c(\mathbf{b}) = \mathbf{b}_2|_{\partial D} \times \mathbf{n}$ . Finally, the discrete forms  $a_h(\cdot, \cdot)$  and  $\ell_h(\cdot)$  are defined by (3.11).

### 7.1 Trimmed error estimate

We define the trimming operator  $T : \mathbf{P}^g(\mathcal{T}_h) \rightarrow \mathbf{P}^g(\mathcal{T}_h) \cap \mathbf{H}_0(\text{curl}; D) = \{\mathbf{b}_h \in \mathbf{P}^g(\mathcal{T}_h) \mid \mathbf{b}_h|_{\partial D} \times \mathbf{n} = \mathbf{0}\}$  such that, for all  $\mathbf{b}_h \in \mathbf{P}^g(\mathcal{T}_h)$ ,  $T(\mathbf{b}_h)|_K$ , for all  $K \in \mathcal{T}_h$ , is defined by zeroing out all the degrees of freedom of the tangential component of  $\mathbf{b}_h$  at the boundary. Note that the trimming operator couples the Cartesian components of  $\mathbf{b}_h$  if the faces composing the boundary  $\partial D$  are not orthogonal to the coordinate axes. We have  $T(\mathbf{b}_h) = \mathbf{b}_h$  on all  $K \in \mathcal{T}_h \setminus \mathcal{T}_h^\partial$ , whereas for all  $K \in \mathcal{T}_h^\partial$ , one can prove the following bounds for all  $\mathbf{b}_h \in \mathbf{P}^g(\mathcal{T}_h)$  with  $c$  uniform w.r.t.  $h$ : If  $\partial K \cap \partial D$  is composed of one or more boundary faces, then

$$h_K \|\nabla(\mathbf{b}_h - T(\mathbf{b}_h))\|_{\mathbf{L}^2(K)} + \|\mathbf{b}_h - T(\mathbf{b}_h)\|_{\mathbf{L}^2(K)} \leq c h_K^{\frac{1}{2}} \|\mathbf{b}_h \times \mathbf{n}\|_{\mathbf{L}^2(\partial K \cap \partial D)}, \quad (7.1)$$

whereas if  $\partial K \cap \partial D$  is a manifold of dimension  $d' < d - 1$ , then

$$h_K \|\nabla(\mathbf{b}_h - T(\mathbf{b}_h))\|_{\mathbf{L}^2(K)} + \|\mathbf{b}_h - T(\mathbf{b}_h)\|_{\mathbf{L}^2(K)} \leq c h_K^{\frac{1}{2}} \|\mathbf{b}_h \times \mathbf{n}\|_{\mathbf{L}^2(F)}, \quad \forall F \in \mathcal{F}_K^\partial, \quad (7.2)$$

where we recall that  $\mathcal{F}_K^\partial$  is the collection of the boundary faces containing the manifold  $\partial K \cap \partial D$ . We introduce the contrast factor  $\xi_{\kappa_r}$  for the parameter  $\kappa_r$  which is defined similarly to (6.3) by replacing  $\kappa$  by  $\kappa_r$ . We also define the local magnetic Reynolds numbers  $\zeta_{\mu\kappa, F} = \mu_{r, K_F} h_{K_F}^2 / \kappa_{r, K_F}$ , for all  $F \in \mathcal{F}_h^\partial$ , and we define the global magnetic Reynolds number as  $\zeta_{\mu\kappa} = \max_{F \in \mathcal{F}_h^\partial} \zeta_{\mu\kappa, F}$ . Finally, let us set

$$\mathbf{Y} = \{v \in \mathbf{H}_0(\text{curl}; D) \mid \nabla \times (\kappa \nabla \times v) \in \mathbf{L}^2(D)\}, \quad (7.3)$$

and let us equip the space  $\mathbf{Y}$  with the norm

$$\|\mathbf{b}\|_{\mathbf{V}_\sharp} = \left( \|\mathbf{b}\|_{\mathbf{V}_b}^2 + \sum_{F \in \mathcal{F}_h^\partial} \kappa_{r, K_F} h_{K_F}^{-2} \|\mathbf{b}\|_{\mathbf{L}^2(K_F)}^2 + \sum_{K \in \mathcal{T}_h^\partial} \mu_{r, K}^{-1} \|\mathbf{b}\|_{\mathbf{L}^2(K)}^2 + \kappa_{r, K}^{-1} h_K^2 \|\nabla \times (\kappa \nabla \times \mathbf{b})\|_{\mathbf{L}^2(K)}^2 \right)^{\frac{1}{2}}. \quad (7.4)$$

**Lemma 7.1** (Trimmed error estimate). *The assumptions of Lemma 4.5 hold true with the trimming operator  $T : \mathbf{P}^g(\mathcal{T}_h) \rightarrow \mathbf{P}^g(\mathcal{T}_h) \cap \mathbf{H}_0(\text{curl}; D)$  defined above, where the constants  $\omega_{\mathbf{V}_\sharp, \mathbf{V}_h}^{\text{tri}}$  and  $\varpi_{\mathbf{V}_\sharp, \mathbf{V}_h}^{\text{tri}}$  are proportional to  $\xi_{\kappa_r}^{\frac{1}{2}}$  and to  $\zeta_{\mu\kappa}^{\frac{1}{2}}$ , where  $\xi_{\kappa_r}$  is the contrast factor for  $\kappa_r$  and  $\zeta_{\mu\kappa}$  is the global magnetic Reynolds number.*

*Proof.* We only highlight the differences with respect to the proof of Lemma 6.1.

(1) Verification of (4.10). Let  $(\mathbf{v}_h, \mathbf{b}_h) \in \mathbf{V}_h \times \mathbf{V}_h$ . Since  $T(\mathbf{b}_h) \in \mathbf{H}_0(\text{curl}; D)$ , we infer that

$$\begin{aligned} a(\mathbf{A}, T(\mathbf{b}_h)) - a_h(\mathbf{v}_h, T(\mathbf{b}_h)) &= \int_D (\tilde{\mu}(\mathbf{A} - \mathbf{v}_h) \cdot \overline{T(\mathbf{b}_h)} + \kappa \nabla \times (\mathbf{A} - \mathbf{v}_h) \cdot \nabla \times \overline{T(\mathbf{b}_h)}) \, dx \\ &\leq \|\mathbf{A} - \mathbf{v}_h\|_{\mathbf{V}_b} (\|\mu_r^{\frac{1}{2}} T(\mathbf{b}_h)\|_{\mathbf{L}^2(D)} + \|\kappa_r^{\frac{1}{2}} \nabla \times T(\mathbf{b}_h)\|_{\mathbf{L}^2(D)}). \end{aligned}$$

We have  $T(\mathbf{b}_h) = \mathbf{b}_h$  on all  $K \in \mathcal{T}_h \setminus \mathcal{T}_h^\partial$ , so that we only need to bound  $T(\mathbf{b}_h)$  on all  $K \in \mathcal{T}_h^\partial$ . Reasoning as in the proof of Lemma 6.1 and estimating the approximation properties of  $\nabla \times T(\mathbf{b}_h)$  by those of  $\nabla T(\mathbf{b}_h)$ , we infer that, for all  $K \in \mathcal{T}_h^\partial$ , if  $\partial K \cap \partial D$  is composed of one or more boundary faces, then

$$\|\kappa_r^{\frac{1}{2}} \nabla \times T(\mathbf{b}_h)\|_{\mathbf{L}^2(K)} \leq \|\kappa_r^{\frac{1}{2}} \nabla \times \mathbf{b}_h\|_{\mathbf{L}^2(K)} + c \|\eta_{\theta h}^{\frac{1}{2}}(\mathbf{b}_h \times \mathbf{n})\|_{\mathbf{L}^2(\partial K \cap \partial D)},$$

whereas if the manifold  $\partial K \cap \partial D$  is of dimension  $d' < d - 1$ , then

$$\|\kappa_r^{\frac{1}{2}} \nabla \times T(\mathbf{b}_h)\|_{\mathbf{L}^2(K)} \leq \|\kappa_r^{\frac{1}{2}} \nabla \times \mathbf{b}_h\|_{\mathbf{L}^2(K)} + c \xi_{\kappa_r}^{\frac{1}{2}} \|\eta_{\theta h}^{\frac{1}{2}}(\mathbf{b}_h \times \mathbf{n})\|_{\mathbf{L}^2(F)},$$

where  $F$  is a boundary face in  $\mathcal{F}_K^\partial$  s.t.  $\kappa_{r,K_F}$  is maximal. The reasoning to bound  $\|\mu_r^{\frac{1}{2}}T(\mathbf{b}_h)\|_{\mathbf{L}^2(K)}$  for all  $K \in \mathcal{T}_h^\partial$  is similar and leads to the additional dependency on the factor  $\zeta_{\mu\kappa}$ .

(2) Verification of (4.11). Let  $(\mathbf{v}_h, \mathbf{b}_h) \in \mathbf{V}_h \times \mathbf{V}_h$  and let us set  $\mathbf{e}_h := \mathbf{A} - \mathbf{v}_h$  and  $\mathbf{d}_h := \mathbf{b}_h - T(\mathbf{b}_h)$ . A direct calculation shows that

$$\begin{aligned} & \ell_h(\mathbf{b}_h) - \ell(T(\mathbf{b}_h)) - a_h(\mathbf{v}_h, (I - T)(\mathbf{b}_h)) \\ &= \int_D \mathbf{f} \cdot \bar{\mathbf{d}}_h \, dx - \int_D (\tilde{\mu} \mathbf{v}_h \cdot \bar{\mathbf{d}}_h + \kappa \nabla \times \mathbf{v}_h \cdot \nabla \times \bar{\mathbf{d}}_h) \, dx - \int_{\partial D} (\mathbf{n} \times (\kappa \nabla \times \mathbf{v}_h)) \bar{\mathbf{d}}_h \, ds - \int_{\partial D} \eta_h(\mathbf{v}_h \times \mathbf{n}) \cdot (\bar{\mathbf{d}}_h \times \mathbf{n}) \, ds \\ &= \sum_{K \in \mathcal{T}_h^\partial} \int_K (\tilde{\mu} \mathbf{e}_h \cdot \bar{\mathbf{d}}_h + \nabla \times (\kappa \nabla \times \mathbf{e}_h) \cdot \bar{\mathbf{d}}_h) \, dx + \sum_{F \in \mathcal{F}_h^\partial} \int_F \llbracket \kappa \nabla \times \mathbf{v}_h \rrbracket \times \mathbf{n}_F \cdot \bar{\mathbf{d}}_h \, ds - \int_{\partial D} \eta_h(\mathbf{v}_h \times \mathbf{n}) \cdot (\bar{\mathbf{d}}_h \times \mathbf{n}) \, ds, \end{aligned}$$

where we recall that  $\mathcal{F}_h^\partial$  is the collection of the mesh interfaces that touch the boundary. The Cauchy–Schwarz inequality leads to  $\|\ell_h - \ell \circ T - a_h(\mathbf{v}_h, (I - T)(\cdot))\|_{\mathbf{V}_h'} \leq c \mathfrak{T}_1 \mathfrak{T}_2$  with  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  defined by

$$\begin{aligned} \mathfrak{T}_1 &= \left( \sum_{K \in \mathcal{T}_h^\partial} \mu_{r,K}^{-1} \|\mathbf{e}_h\|_{\mathbf{L}^2(K)}^2 + \kappa_{r,K}^{-1} h_K^2 \|\nabla \times (\kappa \nabla \times \mathbf{e}_h)\|_{\mathbf{L}^2(K)}^2 + \sum_{F \in \mathcal{F}_h^\partial} \kappa_{r,K_F}^{-1} \tilde{h}_F \|\llbracket \kappa \nabla \times \mathbf{v}_h \rrbracket \times \mathbf{n}_F\|_{\mathbf{L}^2(F)}^2 \right. \\ &\quad \left. + \|\eta_{\theta h}^{\frac{1}{2}}(\mathbf{v}_h \times \mathbf{n})\|_{\mathbf{L}^2(\partial D)}^2 \right)^{\frac{1}{2}}, \\ \mathfrak{T}_2 &= \left( \sum_{K \in \mathcal{T}_h^\partial} (\mu_{r,K} + \kappa_{r,K} h_K^{-2}) \|\mathbf{d}_h\|_{\mathbf{L}^2(K)}^2 + \sum_{F \in \mathcal{F}_h^\partial} \kappa_{r,K_F} \tilde{h}_F^{-1} \|\mathbf{d}_h\|_{\mathbf{L}^2(F)}^2 + \|\eta_{\theta h}^{\frac{1}{2}}(\mathbf{b}_h \times \mathbf{n})\|_{\mathbf{L}^2(\partial D)}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where, for all  $F \in \mathcal{F}_h^\partial$ ,  $K_F$  is the mesh cell sharing  $F$  and having the larger value of  $\kappa_{\sharp,K}$  (the choice of  $K_F$  is irrelevant if both cells give the same value), and  $\tilde{h}_F$  is defined in Lemma 5.7.

(2a) Bound on  $\mathfrak{T}_1$ . The bound on the terms composing the summation over  $K \in \mathcal{T}_h^\partial$  is straightforward. To bound  $\|\llbracket \kappa \nabla \times \mathbf{v}_h \rrbracket \times \mathbf{n}_F\|_{\mathbf{L}^2(F)}$ , for all  $F \in \mathcal{F}_h^\partial$ , we use Lemma 5.7 (with  $p = 2$ ). This is possible since, by assumption,  $\kappa$  is piecewise constant on the mesh  $\mathcal{T}_h$  and, therefore,  $\llbracket \kappa \nabla \times \mathbf{v}_h \rrbracket \times \mathbf{n}_F \in \mathbf{P}_l(F)$  for some polynomial degree  $l \geq 0$ . Finally, to bound  $\|\eta_{\theta h}^{\frac{1}{2}}(\mathbf{v}_h \times \mathbf{n})\|_{\mathbf{L}^2(F)}$ , for all  $F \in \mathcal{F}_h^\partial$ , we invoke Lemma 5.8 where we pick  $\mathbf{B} = \mathbf{A} \in \mathbf{H}_0(\text{curl}; D)$  in the infimum. Using the regularity of the mesh sequence, we infer that

$$\|\eta_{\theta h}^{\frac{1}{2}}(\mathbf{v}_h \times \mathbf{n})\|_{\mathbf{L}^2(F)} \leq c \left( \kappa_{r,K_F}^{\frac{1}{2}} \|\nabla \times (\mathbf{A} - \mathbf{v}_h)\|_{\mathbf{L}^2(K_F)} + \kappa_{r,K_F}^{\frac{1}{2}} h_{K_F}^{-1} \|\mathbf{A} - \mathbf{v}_h\|_{\mathbf{L}^2(K_F)} \right),$$

and the conclusion is straightforward.

(2b) Bound on  $\mathfrak{T}_2$ . Reasoning as in the proof of Lemma 6.1., we infer that  $|\mathfrak{T}_2| \leq c \xi_{\kappa_r}^{\frac{1}{2}} \zeta_{\mu\kappa}^{\frac{1}{2}} \|\mathbf{b}_h\|_{\mathbf{V}_h}$ .  $\square$

## 7.2 Mollified error estimate

Let us set

$$\mathbf{Y} = \{\mathbf{b} \in \mathbf{H}_0(\text{curl}; D) \mid \kappa \nabla \times \mathbf{b} \in \mathbf{H}^r(D^\partial), \nabla \times (\kappa \nabla \times \mathbf{b}) \in \mathbf{L}^2(D)\}, \quad (7.5)$$

with  $r > 0$  and  $D^\partial$  defined in Section 6.2. Let us equip the space  $\mathbf{V}_\sharp = \mathbf{Y} + \mathbf{V}_h$  with the norm

$$\begin{aligned} \|\mathbf{b}\|_{\mathbf{V}_\sharp} &= \left( \|\mathbf{b}\|_{\mathbf{V}_b}^2 + \sum_{F \in \mathcal{F}_h^\partial} \kappa_{r,K_F} h_{K_F}^{-2} \|\mathbf{b}\|_{\mathbf{L}^2(K_F)}^2 \right. \\ &\quad \left. + \sum_{F \in \mathcal{F}_h^\partial} \kappa_{r,K_F}^{-1} (h_{K_F}^r |\kappa \nabla \times \mathbf{b}|_{\mathbf{H}^r(D_{K_F})} + h_{K_F} \|\nabla \times (\kappa \nabla \times \mathbf{b})\|_{\mathbf{L}^2(D_{K_F})})^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (7.6)$$

**Lemma 7.2** (Mollified error estimate). *The assumptions of Lemma 4.7 hold true with the bilinear form*

$$a_{\delta h}(v, w_h) = \int_D (\tilde{\mu} \mathbf{v} \cdot \bar{\mathbf{b}}_h + \mathcal{K}_\delta^c(\kappa \nabla \times \mathbf{v}) \cdot \nabla \times \bar{\mathbf{b}}_h) \, dx + \int_{\partial D} (\mathbf{n} \times \mathcal{K}_\delta^c(\kappa \nabla \times \mathbf{v})) \cdot \bar{\mathbf{b}}_h \, ds, \quad (7.7)$$

for all  $(\mathbf{v}, \mathbf{b}_h) \in \mathbf{V}_b \times \mathbf{V}_h$  and all  $\delta \in [0, \delta_0]$  with  $\delta_0 > 0$  small enough.

*Proof.* We only highlight the differences with respect to the proof of Lemma 6.2.

(1) Verification of (4.14a)-(4.14b). Concerning (4.14a), we invoke the commuting property (ii) from Lemma 5.2 to infer that

$$\ell_h(\mathbf{b}_h) - a_{\delta h}(\mathbf{A}, \mathbf{b}_h) = \int_D (\mathbf{f} - \mathcal{K}_\delta^d(\mathbf{f})) \cdot \bar{\mathbf{b}}_h \, dx,$$

and we can conclude as above. Concerning (4.14b), we obtain

$$a_{\delta h}(\mathbf{v}_h, \mathbf{b}_h) - \tilde{a}_h(\mathbf{v}_h, \mathbf{b}_h) = \int_D (\mathcal{K}_\delta^c(\mathbf{g}_h) - \mathbf{g}_h) \cdot \nabla \times \bar{\mathbf{b}}_h \, dx + \int_{\partial D} (\mathbf{n} \times (\mathcal{K}_\delta^c(\mathbf{g}_h) - \mathbf{g}_h)) \cdot \bar{\mathbf{b}}_h \, ds,$$

with  $\mathbf{g}_h = \kappa \nabla \times \mathbf{v}_h$ , and we can again conclude as above.

(2) Verification of (4.15). Let  $\mathbf{v} \in V_b$ . We now invoke the bound on the tangential component from Lemma 5.11 (with  $p = 2$ ) to infer that

$$\begin{aligned} \|a_{\delta h}(\mathbf{v}, \cdot)\|_{V'_h} &\leq \left( \sum_{K \in \mathcal{T}_h} \kappa_{r,K}^{-1} \|\phi\|_{L^2(K)}^2 \right)^{\frac{1}{2}} + c \|\tilde{\mu}_r^{\frac{1}{2}} \mathbf{v}\|_{L^2(D)} \\ &\quad + c' \left( \sum_{F \in \mathcal{F}_h^\partial} \kappa_{r,K_F}^{-1} (h_{K_F}^r |\phi|_{\mathbf{H}^r(K_F)} + h_{K_F} \|\nabla \times \phi\|_{L^2(K_F)} + \|\phi\|_{L^2(K_F)})^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where  $\phi = \mathcal{K}_\delta^c(\kappa \nabla \times \mathbf{v})$ , and we can conclude as above.

(3) Verification of (4.16). Here, we proceed slightly differently by invoking Lemma 5.8 where we pick  $\mathbf{B} = \mathbf{A} \in \mathbf{H}_0(\text{curl}; D)$  in the infimum. Using the regularity of the mesh sequence, we infer that

$$\|\eta_{\theta h}^{\frac{1}{2}}(\mathbf{v}_h \times \mathbf{n})\|_{L^2(F)} \leq c \left( \kappa_{r,K_F}^{\frac{1}{2}} \|\nabla \times (\mathbf{A} - \mathbf{v}_h)\|_{L^2(K_F)} + \kappa_{r,K_F}^{\frac{1}{2}} h_{K_F}^{-1} \|\mathbf{A} - \mathbf{v}_h\|_{L^2(K_F)} \right),$$

and the conclusion is straightforward owing to the definition of the norm  $\|\cdot\|_{V_\sharp}$  (note that the last term above is the one responsible for the appearance of the contribution  $\sum_{F \in \mathcal{F}_h^\partial} \kappa_{r,K_F} h_{K_F}^{-2} \|\mathbf{b}\|_{L^2(K_F)}^2$  to the norm  $\|\cdot\|_{V_\sharp}$  defined by (7.6)).  $\square$

Finally, to achieve a robust estimate w.r.t. the contrast in the parameter  $\kappa$ , we consider the broken mollification operators  $\hat{\mathcal{K}}_\delta$  introduced at the end of Section 6.2. The space  $\mathbf{Y}$  is still defined by (7.5), but we now equip the space  $\mathbf{V}_\sharp = \mathbf{Y} + \mathbf{V}_h$  with the norm

$$\begin{aligned} \|\mathbf{b}\|_{V_\sharp} &= \left( \|\mathbf{b}\|_{V_b}^2 + \sum_{F \in \mathcal{F}_h^\partial} \kappa_{r,K_F} h_{K_F}^{-2} \|\mathbf{b}\|_{L^2(K_F)}^2 \right. \\ &\quad \left. + \sum_{1 \leq i \leq M} \sum_{F \in \mathcal{F}_h^\partial \cap D_i} \kappa_{r,i}^{-1} (h_{K_F}^r |\kappa_i \nabla \times \mathbf{b}|_{\mathbf{H}^r(D_i, K_F)} + h_{K_F} \|\nabla \times (\kappa_i \nabla \times \mathbf{b})\|_{L^2(D_i, K_F)})^2 \right)^{\frac{1}{2}}, \end{aligned} \quad (7.8)$$

where we have set  $\kappa_{r,i} = \kappa_{r|D_i}$ ,  $\kappa_i = \kappa|D_i$ , and  $D_{i,K_F} = D_{K_F} \cap D_i$ .

**Lemma 7.3** (Mollified error estimate). *The assumptions of Lemma 4.7 hold true with the bilinear form*

$$\hat{a}_{\delta h}(v, w_h) = \int_D (\tilde{\mu} \mathbf{v} \cdot \bar{\mathbf{b}}_h + \hat{\mathcal{K}}_\delta^c(\kappa \nabla \times \mathbf{v}) \cdot \nabla \times \bar{\mathbf{b}}_h) \, dx + \int_{\partial D} (\mathbf{n} \times \hat{\mathcal{K}}_\delta^c(\kappa \nabla \times \mathbf{v})) \cdot \bar{\mathbf{b}}_h \, ds, \quad (7.9)$$

for all  $(\mathbf{v}, \mathbf{b}_h) \in \mathbf{V}_b \times \mathbf{V}_h$  and all  $\delta \in [0, \delta_0]$  with  $\delta_0 > 0$  small enough. Moreover, the stability constants invoked in Lemma 4.7 are uniform with respect to the contrast factor  $\xi_\kappa$ .

*Proof.* Use the same arguments as in the proof of Lemma 6.3.  $\square$

**Remark 7.4** (Partition of  $D$ ). In the above proof, it is possible to consider that the partition of  $D$  into  $D_1, \dots, D_M$  coincides with the current mesh, so that the broken mollification operator is local to each mesh cell.  $\square$

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