

# Optimal stopping for change-point detection of Piecewise Deterministic Markov Processes

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# Outline

Motivation

Change-point detection problem

Numerical approximation

Numerical results

Conclusion and perspectives

# Piecewise deterministic Markov processes

## Davis (80's)

General class of **non-diffusion** dynamic stochastic **hybrid** models: **deterministic** motion punctuated by **random** jumps.

## Applications of PDMPs

Engineering systems, operations research, management science, economics, internet traffic, **dependability and safety**, **neurosciences**, **biology**, ...

- ▶ mode: **nominal**, **failures**, **breakdown**, **environment**, **number of individuals**, **response to a treatment**, ...
- ▶ Euclidean variable: **pressure**, **temperature**, **time**, **size**, **potential**, **protein level**, ...

# Impulse control problem

## Impulse control

Select

- ▶ **intervention dates**
- ▶ new **starting point** for the process at interventions

to **minimize** a cost function

- ▶ **repair** a component before breakdown
- ▶ **change** treatment before relapse
- ▶ ...

[CD 89], [Davis 93], [dSDZ 14], ...

# If the jump times are not observed?

- ▶ [BdSD 12] Optimal stopping

- ▶ jump times **observed**
- ▶ post-jump locations observed through noise

Numerical approximation of the value function and  $\epsilon$ -optimal stopping time

- ▶ [BL 17] Continuous control

- ▶ jump times and post-jump locations **observed through noise**

Optimality equation, existence of optimal policies

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Optimality equation, existence of optimal policies

No information on the jump times  $\Rightarrow$  very difficult problem

# Change-point detection

Simplest special case

- ▶ only **one** jump of the **mode** variable
- ▶ discrete noisy observations of the **continuous** variable on a regular time grid

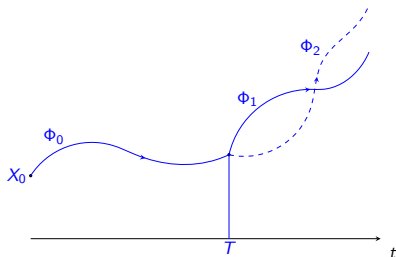
Optimal stopping = Change-point detection

Aim: numerical approximation to

- ▶ **detect** the change-point at best (not too early/late)
- ▶ **estimate** the new **mode** after the jump

## Simple PDMP model

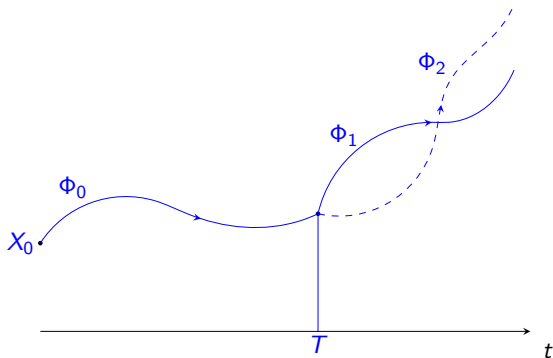
- ▶ State space  $E \times \mathbb{R} = \{0, 1, \dots, d\} \times \mathbb{R} \times \mathbb{R}$ : mode, position, time
- ▶ Starting point  $X_0 = (0, x, 0)$ , flow  $\Phi_0$
- ▶ time-dependent Jump intensity  $\lambda_0(x, u) = \lambda(u)$
- ▶ Jump kernel: position and time continuous, switch to mode  $i$  with probability  $p_i$





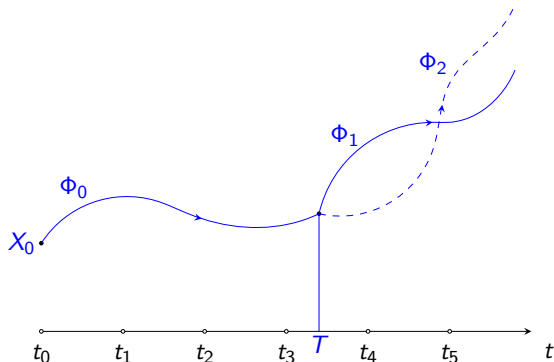
# Observations

- ▶ Observation times  $t_n = \delta n$
- ▶ Noisy observations of the positions  $Y_n = F(x_{t_n}) + \epsilon_n$



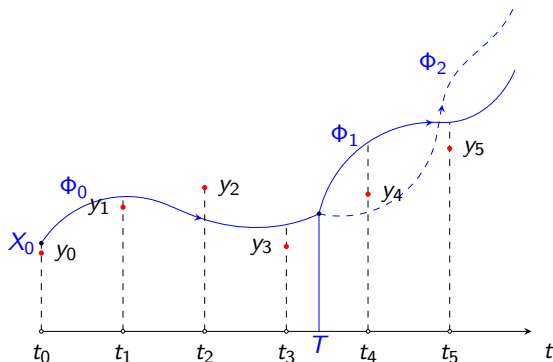
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- ▶ Cost per stage before stopping
  - ▶  $c(0, x, y) = 0$  rightfully not stopped
  - ▶  $c(m \neq 0, x, y) = \beta\delta$  lateness penalty
- ▶ Terminal cost at stopping
  - ▶  $C(m, x, y, 0) = c(m, x, y)$  no stopping before the horizon
  - ▶  $C(0, x, y, a \neq 0) = \alpha$  early stopping penalty
  - ▶  $C(m \neq 0, x, y, a = m) = 0$  good mode selection
  - ▶  $C(m \neq 0, x, y, a \neq 0, m) = \gamma$  wrong mode penalty

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Cost of admissible strategy  $(\tau, A)$

$$J(\tau, A, (m, x, y)) = \mathbb{E}_{(m, x, y)} \left[ \sum_{n=0}^{(\tau-1) \wedge N} c(X_n, Y_n) + C(X_{\tau \wedge N}, Y_{\tau \wedge N}, A) \right]$$

## Fully observed optimal stopping problem

- ▶ Filter process  $\Theta_n(A \times B) = \mathbb{P}_{(0,x,y)}(X_{\delta n} \in A \times B | \mathcal{F}_n^Y)$
- ▶  $(\Theta_n, Y_n)$  time inhomogeneous Markov chain with explicit transition kernels  $R'_n$  on  $\mathcal{P}(E) \times \mathbb{R}$
- ▶ cost functions  $c'(\theta, y) = \int c(m, x, y) d\theta(m, x)$ ,  
 $C'(\theta, y, a) = \int C(m, x, y, a) d\theta(m, x)$

## Fully observed optimal stopping problem

Minimize over all admissible strategies  $(\tau, a)$

$$J'(\tau, A, (\theta, y)) = \mathbb{E}_{(\theta, y)} \left[ \sum_{n=0}^{(\tau-1) \wedge N} c'(\Theta_n, Y_n) + C'(\Theta_{\tau \wedge N}, Y_{\tau \wedge N}, A) \right]$$



# Aim of the talk

- ▶ numerical approximation of the value function
- ▶ computable strategy

## Difficulties

- ▶ measure-valued filter process
- ▶ curse of dimensionality

# Dynamic programming

## Value function

$$\begin{aligned}
 V'(\theta, y) &= \inf_{(\tau, A)} J'(\tau, A, (\theta, y)) \\
 &= \inf_{(\tau, A)} \mathbb{E}_{(\theta, y)} \left[ \sum_{n=0}^{(\tau-1) \wedge N} c'(\Theta_n, Y_n) + C'(\Theta_{\tau \wedge N}, Y_{\tau \wedge N}, A) \right]
 \end{aligned}$$

## Dynamic programming

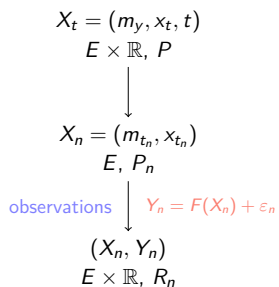
$$\begin{aligned}
 v'_N(\theta, y) &= \min_{0 \leq a \leq d} C'(\theta, y, a) \\
 v'_k(\theta, y) &= \min \left\{ \min_{1 \leq a \leq d} C'(\theta, y, a); c'(\theta, y) + R'_k v'_{k+1}(\theta, y) \right\}
 \end{aligned}$$

$$v'_0 = V'$$

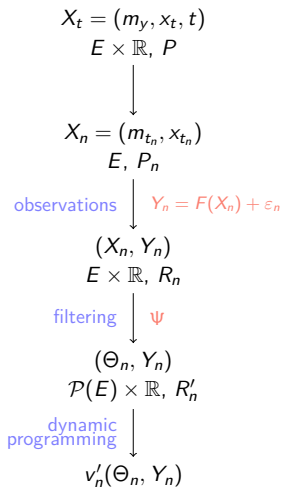
# Discretization

$$X_t = (m_y, x_t, t) \\ E \times \mathbb{R}, P$$

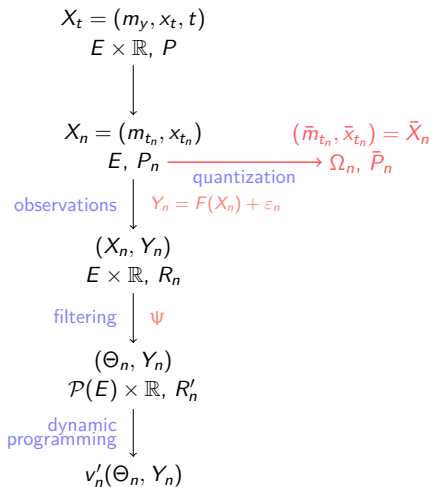
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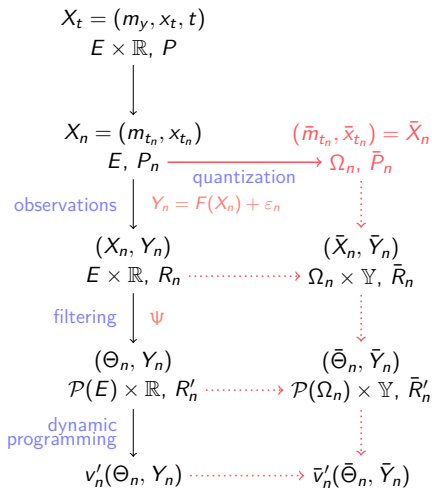
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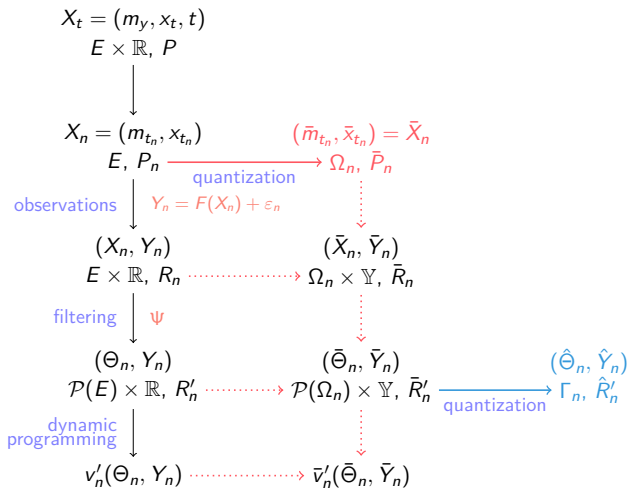
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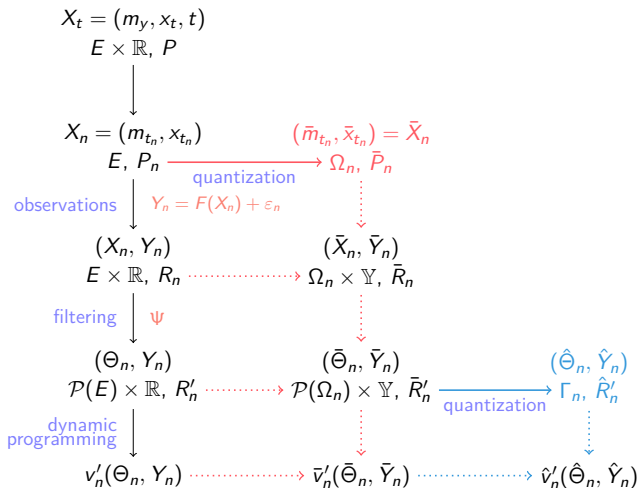


## Discretization





## Discretization



# Quantization

[P 98], [PPP 04], [PRS05], ...

## Quantization of a random variable $X \in L^2(\mathbb{R}^q)$

Approximate  $X$  by  $\hat{X}$  taking **finitely** many values such that  $\|X - \hat{X}\|_2$  is **minimum**

- ▶ Find a finite weighted grid  $\Gamma$  with  $|\Gamma| = N_\Gamma$
- ▶ Set  $\hat{X} = p_\Gamma(X)$  closest neighbor projection

## Asymptotic properties

If  $E[|X|^{2+\eta}] < +\infty$  for some  $\eta > 0$  then

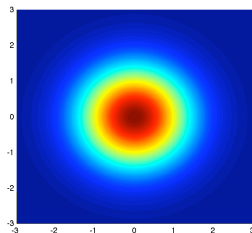
$$\lim_{N_\Gamma \rightarrow \infty} N_\Gamma^{1/q} \min_{|\Gamma| \leq N_\Gamma} \|X - \hat{X}^\Gamma\|_2 = C$$

# Algorithms

There exist algorithms providing

- ▶  $\Gamma$
- ▶ law of  $\hat{X}$
- ▶ transition probabilities for quantization of Markov chains

Example:  $\mathcal{N}(0, I_2)$ :

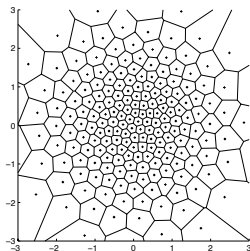


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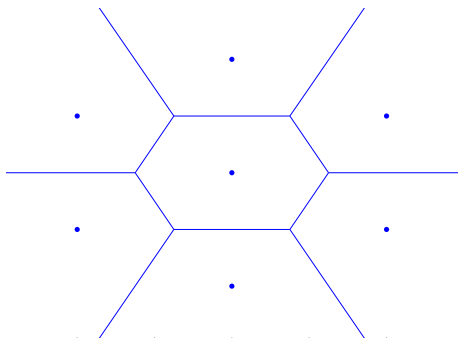
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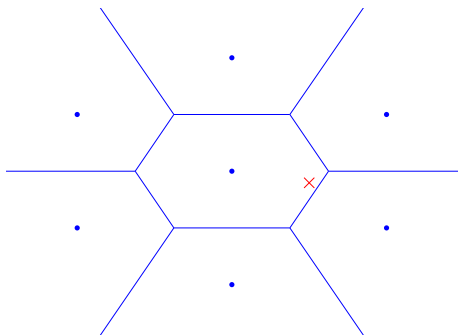
# Grids construction

Model  $\longrightarrow$  simulator of trajectories  $\longrightarrow$  grids



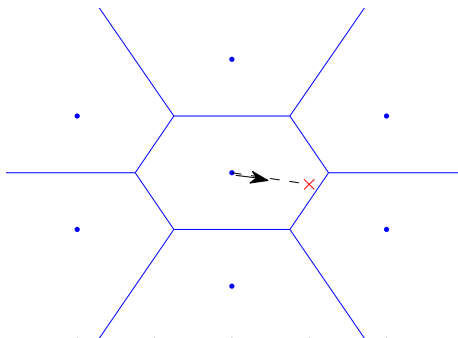
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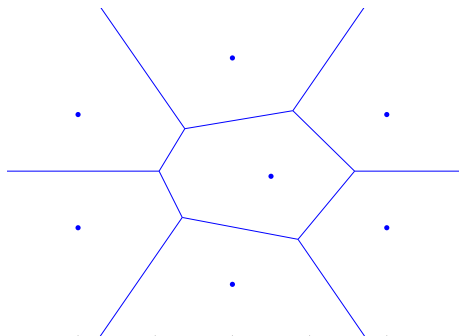
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# Assets and drawbacks of quantization

## Assets

- ▶ a **simulator** of the target law is enough to build the grids
- ▶ automatic construction of grids
- ▶ convergence rate for  $\mathbb{E}[|f(X) - f(\hat{X})|]$  if  $f$  **lipschitz**
- ▶ **empirical** error measure by Monte Carlo

## Drawbacks

- ▶ computation time
- ▶ curse of dimension
- ▶ open questions of convergence of the algorithms

# Convergence

## Technical assumptions

$$\begin{aligned}
 |v'_0(\delta_{(0,x_0)}, y_0) - \bar{v}'_0(\delta_{(0,x_0)}, y_0)| &\leq \sum_{n=0}^{N-1} a_n \mathbb{E}[|\bar{X}_n - X_n|] \\
 &= O(N_\Omega^{-1})
 \end{aligned}$$

$$\begin{aligned}
 &|\hat{v}'_0(\delta_{(0,x_0)}, y_0) - \bar{v}'_0(\delta_{(0,x_0)}, y_0)| \\
 &\leq \sum_{n=0}^N c_n \left( \mathbb{E} \left[ |\hat{Y}_n - \bar{Y}_n| \right] + \mathbb{E} \left[ \|\hat{\Theta}_n - \bar{\Theta}_n\|_{n,1} \right] \right) \\
 &= O(N_\Gamma^{-1}/N_\Omega)
 \end{aligned}$$

# Candidate computable strategy

## Dynamic programming

- ▶  $\hat{v}'_N(\hat{\theta}, \hat{y}) = \min_{0 \leq a \leq d} C'(\hat{\theta}, \hat{y}, a)$
- ▶  $\hat{v}'_k(\hat{\theta}, \hat{y}) = \min \left\{ \min_{1 \leq a \leq d} C'(\hat{\theta}, \hat{y}, a); c'(\hat{\theta}, \hat{y}) + \hat{R}'_k \hat{v}'_{k+1}(\hat{\theta}, \hat{y}) \right\}$

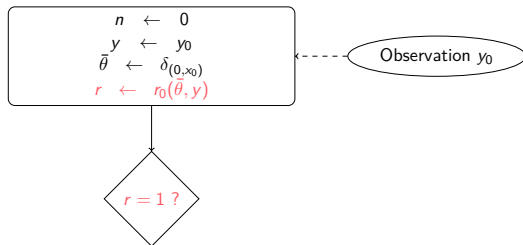
## Set

- ▶  $r_N(\cdot) = 0, a_N(\cdot) = 0$  if  $\hat{v}'_N(\text{proj}_{\Gamma_N}(\cdot)) = C'(\text{proj}_{\Gamma_N}(\cdot), 0)$
- ▶  $r_N(\cdot) = 1, a_N(\cdot) = i$  if  $\hat{v}'_N(\text{proj}_{\Gamma_N}(\cdot)) = C'(\text{proj}_{\Gamma_N}(\cdot), i)$
- ▶  $r_n(\cdot) = 0$  if  $\hat{v}'_n(\text{proj}_{\Gamma_n}(\cdot)) = \hat{R}'_n \hat{v}'_{n+1}(\text{proj}_{\Gamma_n}(\cdot))$
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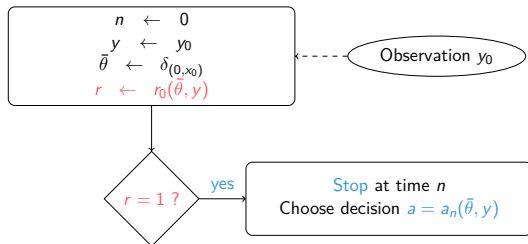
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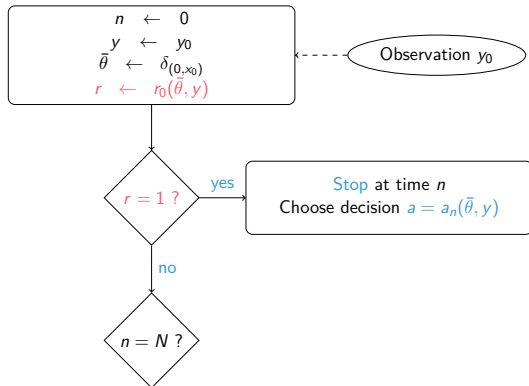
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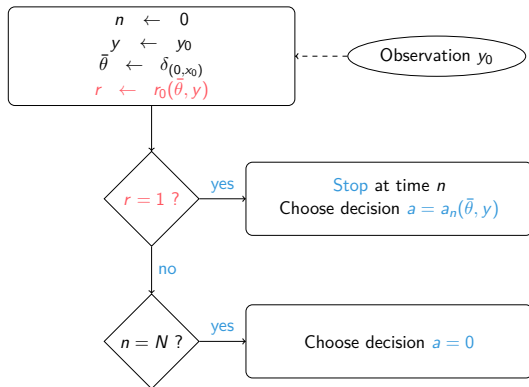
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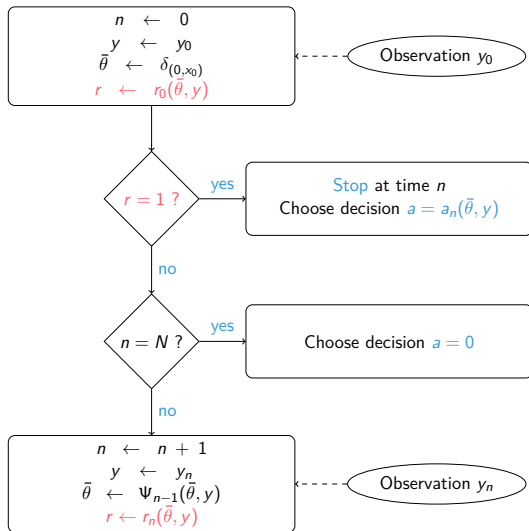


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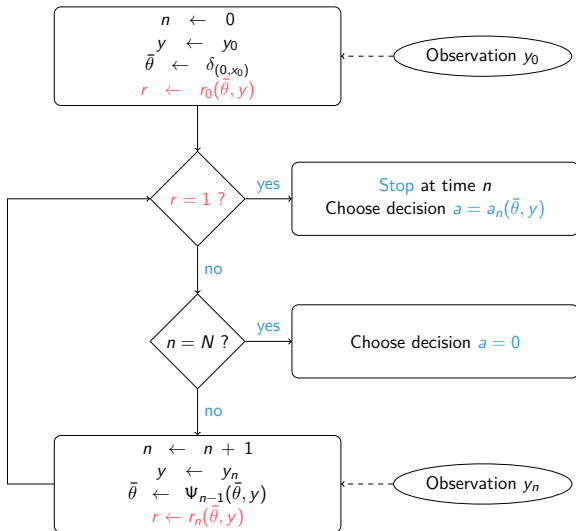




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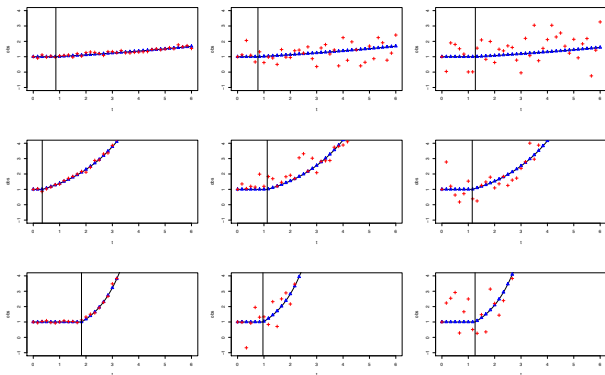


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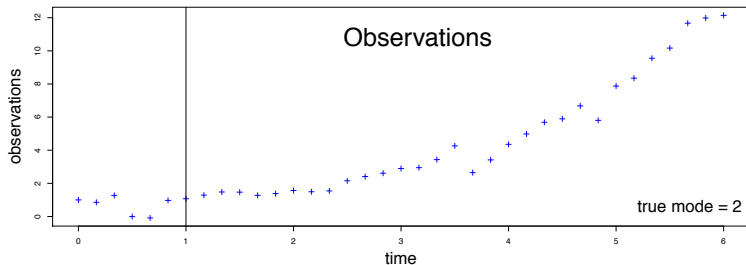
# Example 1

- ▶  $d = 3$ ,  $p_i = 1/3$ ,  $x_0 = 1$
- ▶  $\Phi_0(x, t) = x$ ,  $\Phi_1(x, t) = xe^{0.1t}$ ,  $\Phi_2(x, t) = xe^{0.5t}$ ,  
 $\Phi_3(x, t) = xe^{1t}$
- ▶  $\beta = 1$  (late detection),  $\gamma = 1.5$  (wrong mode),  $\delta = 1/6$



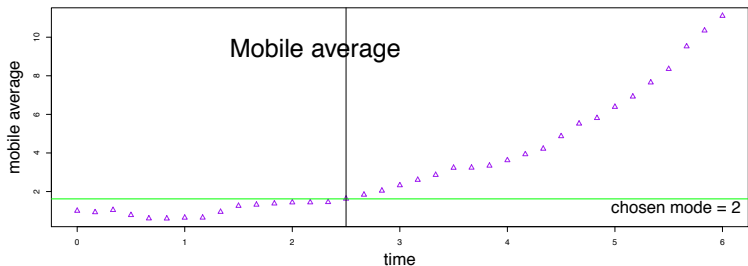
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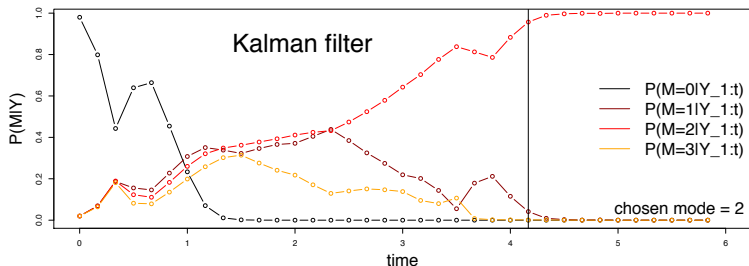
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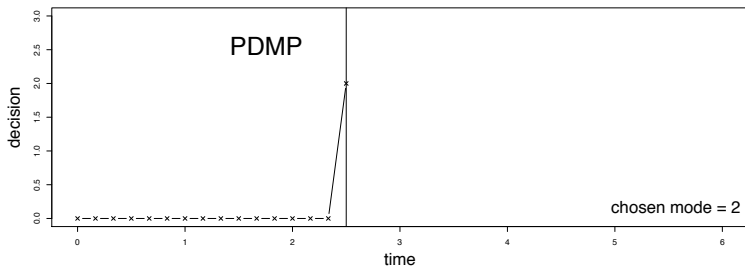
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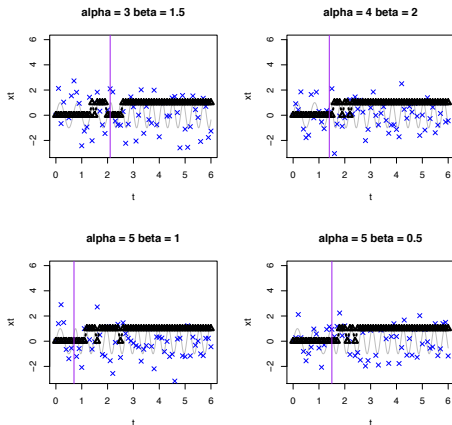
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$\alpha$	$\sigma^2$	Moving Average threshold=2				Kalman				PDMP			
		window				threshold				Nb grid points			
		2	3	4	5	0.5	0.75	0.9	cal	30	50	75	100
3	0.1	<b>0.40</b>	<b>0.40</b>	<b>0.40</b>	0.41	2.34	0.61	<b>0.42</b>	<b>0.42</b>	<b>0.70</b>	<b>0.70</b>	<b>0.70</b>	<b>0.70</b>
	0.5	0.93	0.81	0.76	<b>0.73</b>	1.44	0.54	0.51	<b>0.49</b>	0.78	0.79	0.77	<b>0.76</b>
	1	1.73	1.42	1.29	<b>1.16</b>	1.18	<b>0.58</b>	0.63	0.62	0.99	1.04	<b>0.98</b>	1.01
4	0.1	<b>0.40</b>	<b>0.40</b>	<b>0.40</b>	0.41	3.06	0.69	<b>0.42</b>	<b>0.42</b>	0.69	0.71	0.69	<b>0.68</b>
	0.5	0.95	0.81	0.76	<b>0.73</b>	1.76	0.56	0.51	<b>0.50</b>	0.73	<b>0.71</b>	0.72	0.72
	1	2.05	1.57	1.39	<b>1.22</b>	1.36	<b>0.60</b>	0.63	0.62	<b>0.92</b>	<b>0.92</b>	0.95	0.95
5	0.1	<b>0.40</b>	<b>0.40</b>	<b>0.40</b>	0.41	3.78	0.78	<b>0.42</b>	<b>0.42</b>	0.68	0.69	<b>0.67</b>	0.69
	0.5	0.97	0.81	0.76	<b>0.73</b>	2.08	0.59	0.51	<b>0.50</b>	0.72	<b>0.69</b>	0.72	0.72
	1	2.37	1.73	1.48	<b>1.28</b>	1.54	<b>0.61</b>	0.63	0.62	<b>0.92</b>	0.94	0.93	<b>0.92</b>
6	0.1	<b>0.40</b>	<b>0.40</b>	<b>0.40</b>	0.41	4.50	0.86	<b>0.42</b>	0.43	<b>0.68</b>	<b>0.68</b>	<b>0.68</b>	0.69
	0.5	0.98	0.82	0.76	<b>0.73</b>	2.40	0.62	0.51	<b>0.50</b>	0.70	0.70	0.70	<b>0.69</b>
	1	2.69	1.88	1.57	<b>1.35</b>	1.72	0.63	0.63	<b>0.62</b>	0.90	<b>0.89</b>	0.91	<b>0.89</b>



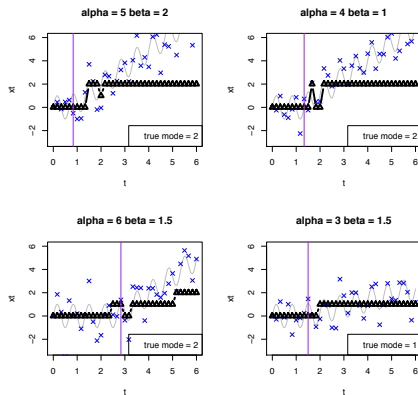
## Example 2

- ▶  $d = 1$ ,  $x_0 = (0, 0)$
- ▶  $\Phi_0((x, u), t) = (\sin(3\pi(u + t)), u + t)$ ,  
 $\Phi_1((x, u), t) = (\sin(5\pi(u + t)), u + t)$
- ▶  $\delta = 1/6$ , noise variance 1



## Example 3

- ▶  $d = 2$ ,  $x_0 = (0, 0)$
- ▶  $\Phi_0((x, u), t) = (\sin(3\pi(u + t)), u + t)$ ,
- ▶  $\Phi_1((x, u), t) = (\sin(3\pi(u + t)) + 0.5t, u + t)$ ,
- ▶  $\Phi_2((x, u), t) = (\sin(3\pi(u + t)) + 1.5t, u + t)$
- ▶  $\delta = 1/6$ , noise variance 1



# Conclusion and perspectives

- ▶ Change-point detection method for **continuous-time** jump dynamics, able to **detect** a jump and **select** the post-jump mode
- ▶ For general flows but dimension 1 (+ time)

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### To be done

- ▶ Real data applications
- ▶ Theoretical validity of the stopping rule
- ▶ Allow to stop **between** observations
- ▶ **Several** jumps
- ▶ Stop **and restart** the process from a new point

## Reference

- [BL 17] N. Bäuerle, D. Lange *Optimal control of partially observed PDMPs*
- [BdSD 12] A. Brandejsky, B. de Saporta, F. Dufour *Optimal stopping for partially observed PDMPs*
- [CD 89] O. Costa, M. Davis *Impulse control of piecewise-deterministic processes*
- [Davis 93] M. Davis, *Markov models and optimization*
- [dSDZ 14] B. de Saporta, F. Dufour, H. Zhang *Numerical methods for simulation and optimization of PDMPs: application to reliability*
- [P 98] G. Pagès *A space quantization method for numerical integration*
- [PPP 04] G. Pagès, H. Pham, J. Printems *An optimal Markovian quantization algorithm for multi-dimensional stochastic control problems*
- [PRS 05] H. Pham, W. Runggaldier, A.f Sellami *Approximation by quantization of the filter process and applications to optimal stopping problems under partial observation*