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Qualitative parameter estimation for a class of relaxation oscillators^{*}

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Abstract: Motivated by neuroscience applications, we introduce the concept of *qualitative estimation* as an adaptation of classical parameter estimation to nonlinear systems characterized by i) the presence of possibly many redundant parameters, ii) a small number of possible qualitatively different behaviors, iii) the presence of sharply different characteristic timescales and, consequently, iv) the generic impossibility of quantitatively modeling and fitting experimental data. As a first application, we illustrate these ideas on a class of nonlinear systems with a single unknown sigmoidal nonlinearity and two sharply separated timescales. This class of systems is shown to exhibit either global asymptotic stability or relaxation oscillations depending on a single ruling parameter and independently of the exact shape of the nonlinearity. We design and analyze a *qualitative estimator* that estimates the distance of the ruling parameter from the unknown critical value at which the transition between the two behaviors happens without using any quantitative fitting of the measured data.

Keywords: parameter estimation, relaxation oscillator, singular perturbation, neuroscience, nonlinear system, Lyapunov method

1. INTRODUCTION

Online parameter estimation of dynamical models of neuronal activity might lead to new perspectives in neurosciences. In epilepsy for instance, having access in real-time to the gains governing the excitation/inhibition balance within populations of neurons might provide important information about the on-going electrophysiological activity, and thus help developing new strategies to detect or even predict seizures. This task is challenging because of the redundancy and the large variability of biophysical parameters across populations of neurons and neuronal circuits exhibiting similar activity patterns. Disparate combinations of biophysical parameters are indeed known to lead to the same activity pattern at the cellular level Goldman et al. (2001). The same degenerated parametrization property propagates at the neuronal circuit level Marder (2011). In addition, biophysical parameters slowly vary, inducing sharp transitions between qualitatively different activity modes (spiking or bursting, healthy or epileptic, etc.) at the crossing of critical parameter sets. In this context, quantitative modeling and fitting of experimental data generically constitute an ill-posed problem and a new estimation approach is therefore needed.

Online parameter estimation for dynamical systems has been widely studied, in particular for linear systems, see

e.g., Ioannou and Sun (1996); Ljung (1999). Online estimation technique for nonlinear systems is still a developing field and the available results are more scarce, see e.g., Bastin and Gevers (1988); Farza et al. (2009); Adetola and Guay (2008); Besançon (2007); Mauroy and Goncalves (2016); Moles et al. (2003). Recently, results have appeared in the context of neurosciences. For instance, in Schiff and Sauer (2008), the authors have applied unscented nonlinear Kalman filters to estimate the states and the parameters of a spatiotemporal model of the cortex. Other stochastic filtering methods such as Monte Carlo schemes and Bayesian approach have also been considered for computational neuroscience applications Brockwell et al. (2004); David et al. (2006). While these methods can provide very interesting results, these also have a high computational cost in general, may be difficult to tune, and there is usually no convergence proof. On the other hand, deterministic schemes motivated by neuroscience applications have recently been provided in Tyukin et al. (2013); Chong et al. (2015), in which the estimation errors are ensured to converge to the origin under persistency of excitation conditions. All the estimation approaches reviewed in this introduction rely on some type of quantitative fitting of the measured data, which questions their applicability in the neuronal context.

In the present work, we cast the problem in a deterministic setting and we propose a different approach. Indeed, our objective is no longer to asymptotically estimate the unknown parameters, but first to identify analytically those which rule the type of behavior of the model, and second

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to estimate the distance between these parameters and critical values at which a change of activity occurs. We illustrate these ideas on a class of nonlinear systems with a single sigmoidal nonlinearity and two sharply separated timescales. In order to capture salient properties of neuronal dynamics, we suppose that the exact functional form of the nonlinearity is unknown, as well as the timescale separation. We firstly show that, independently of the particular expression of the sigmoid nonlinearity, a single parameter rules the transition between global asymptotic stability and almost global convergence to a relaxation limit cycle. This part of the paper is inspired by the work in Drion et al. (2015). The parameter redundancy has algebraically been tackled by extracting the few ruling parameters governing neuronal dynamics and their critical values. Those parameters define *qualitative* models that provide a constructive geometric framework to analyze modulation and robustness of neuronal activity in a principled way Franci et al. (2014). Based on the assumption that the input and the states of the system are known, we subsequently design an online *qualitative estimator* that estimates the distance of the ruling parameter to the unknown critical value at which the rest/oscillation transition happens. The designed estimator therefore provides online information about the system activity type and whether a change of activity is prone to appear without any quantitative fitting of the measured data. The estimator design is guided by singularity theory Golubitsky and Schaeffer (1985) and its convergence and robustness properties are ensured by the joint use of Lyapunov analysis and geometric singular perturbation theory Fenichel (1979). The idea of qualitative estimation is also related to the problem of steering a system toward an *a priori* unknown bifurcation point Moreau and Sontag (2003).

The proofs of the results are omitted for space reasons, and these can be found in Tang et al. (2016).

Notation. The usual Euclidean distance is denoted by $|\cdot|$. For a function $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, the associated infinity norm is denoted by $\|h\|_{[0,\infty)} = \sup_{s \in [0,\infty)} |h(s)|$, when it is well-defined. We use $\text{sgn}(x)$ to denote the sign function from \mathbb{R} to $\{-1, 0, 1\}$ with $\text{sgn}(0) = 0$. For any function $f : \mathbb{R} \rightarrow \mathbb{R}$, we denote the range of f as $f(\mathbb{R}) = \{z : z = f(x) \text{ for some } x \in \mathbb{R}\}$. Let A, B be two non-empty subsets in \mathbb{R}^n , their Hausdorff distance is noted by $d_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b|\}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{K}_∞ if it is zero at zero, strictly increasing and unbounded.

2. PROBLEM STATEMENT

2.1 A class of two-time-scale nonlinear systems

We consider the nonlinear system

$$\begin{cases} \dot{x}_f = -x_f + S(\beta x_f + u - x_s), & (1a) \\ \dot{x}_s = \varepsilon(x_f - x_s), & (1b) \end{cases}$$

where $x_f, x_s \in \mathbb{R}$ are the state variables, $\beta \in \mathbb{R}$ is an *unknown* parameter, $u \in \mathbb{R}$ is the input, which is assumed to be a known constant, and $0 < \varepsilon \ll 1$ is a small parameter, which is *unknown*. The mapping $S : \mathbb{R} \rightarrow \mathbb{R}$ is an *unknown* sigmoid function, which is assumed to satisfy the following properties.

Assumption 1. The following properties hold.

- a) S is smooth.
- b) $S(0) = 0$.
- c) $S'(a) > 0$ for all $a \in \mathbb{R}$ (monotonicity), and $\text{argmax}_{a \in \mathbb{R}} S'(a) = 0$ (sector-valued).
- d) $\text{sgn}(S''(a)) = -\text{sgn}(a)$ for all $a \in \mathbb{R}$. \square

Standard sigmoid functions such as $a \mapsto c_1 \tanh(c_2 a)$, $a \mapsto c_1 \arctan(c_2 a)$, $a \mapsto \frac{c_1}{1+e^{-c_2 a}} - \frac{c_1}{2}$ with suitable $c_1, c_2 \in \mathbb{R}$, verify Assumption 1.

Due to the small parameter ε , system (1) evolves according to two time scales. We follow the standard approach of singular perturbation theory (Kokotović et al. (1986); Fenichel (1979)) to decompose system (1) into two subsystems, which represent the fast dynamics and the slow dynamics, respectively, called the *layer* and the *reduced* subsystems. By setting $\varepsilon = 0$ in (1b), we obtain the layer dynamics behavior

$$\begin{cases} \dot{x}_f = -x_f + S(\beta x_f + u - x_s), & (2a) \\ \dot{x}_s = 0. & (2b) \end{cases}$$

In (2b), the slow variable x_s is treated as a constant parameter.

To account for slow variations of x_s , we rescale time as $\tau = \varepsilon t$, hence $\frac{d}{dt} = \varepsilon \frac{d}{d\tau}$. Then system (1) becomes

$$\begin{cases} \varepsilon x'_f := -x_f + S(\beta x_f + u - x_s), & (3a) \\ x'_s = x_f - x_s, & (3b) \end{cases}$$

where \prime stands for $\frac{d}{d\tau}$. Setting $\varepsilon = 0$ in (3a), we obtain the reduced dynamics

$$\begin{cases} 0 = -x_f + S(\beta x_f + u - x_s), & (4a) \\ x'_s = x_f - x_s. & (4b) \end{cases}$$

The reduced dynamics evolves in the *slow time* τ and is an algebro-differential equation. In particular it defines a one-dimensional vector field over the *critical manifold*

$$S^0_{u,\beta} := \{(x_s, x_f) \in \mathbb{R}^2 : -x_f + S(\beta x_f + u - x_s) = 0\}. \quad (5)$$

The critical manifold $S^0_{u,\beta}$ depends on u and β . However, since we assume u and β constant, for simplicity we omit the index u and β in the rest of the paper. The critical manifold will therefore simply be denoted by S^0 . Let

$$\beta_c := \frac{1}{S'(0)}, \quad (6)$$

which is well-defined according to item c) of Assumption 1. This parameter will play a key role in the sequel.

2.2 Stability analysis

We first state stability properties for system (1) in the next proposition.

Proposition 1. Consider system (1), the following holds.

- (1) For any $\beta - \beta_c < 0$ and any constant input u , there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}]$, system (1) has a uniformly globally exponentially stable fixed point. Moreover, all trajectories converge in an $\mathcal{O}(\varepsilon)$ -time to an $\mathcal{O}(\varepsilon)$ -neighborhood of the critical manifold S^0 defined in (5).
- (2) a) For all $0 < \beta - \beta_c < 1$, there exists $\bar{u} > 0$, such that, for any constant input u in $(-\bar{u}, \bar{u})$,

there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}]$, system (1) possesses an almost globally attractive and locally exponentially stable periodic orbit P^ε . In particular, for any initial condition, except that in the unique unstable fixed point of (1), the solution converges to P^ε . Moreover, P^ε is strongly attractive, i.e. any solution in a neighborhood of the periodic orbit P^ε converges to it exponentially fast with decay rate $e^{-K/\varepsilon}$, for some constant $K > 0$.

- b) Let T^ε be the period of P^ε and $p: \mathbb{R} \rightarrow P^\varepsilon$, with $p(t) = p(t + T^\varepsilon)$ for all $t \in \mathbb{R}$, be the associated periodic solution of (1). Let P^0 be the singular limit of P^ε . The Hausdorff distance between P^ε and P^0 verifies $d_H(P^\varepsilon, P^0) = \mathcal{O}(\varepsilon^{2/3})$. Moreover, for all $t_0 \in \mathbb{R}$, there exists $\delta T^\varepsilon \subset [t_0, t_0 + T^\varepsilon)$ and $|\delta T^\varepsilon| = \mathcal{O}(\varepsilon)T^\varepsilon$, such that $|p(t) - S^0| < \mathcal{O}(\varepsilon)$ for all $t \in [t_0, t_0 + T^\varepsilon) \setminus \delta T^\varepsilon$. \square

Proposition 1 indicates that the solution to system (1) converges either to a stable fixed point or to a stable limit cycle. The transition between the two behaviors depends only on the single parameter $\beta - \beta_c$ and not on the exact expression of the sigmoid S .

Remark 1. The proof of Proposition 1 relies on differential geometry arguments ((Fenichel, 1979, Theorem 1,2,3), (Krupa and Szmolyan, 2001b,a, Theorem 2.1)). We mention that the approach in Kokotović et al. (1986); Khalil (2002) for singularly perturbed nonlinear systems is not applicable in the context of this work because, when $\beta > \beta_c$, the critical manifold of system (1) possesses singularities, which implies that (4a) no longer has isolated roots as often assumed. \square

2.3 Objective

Our objective is to detect online in which type of activity system (1) is, that is, whether solutions converge to a fixed point or a limit cycle (oscillation) and whether we are close to a change of activity. According to Proposition 1, the two types of behaviors depend only on the value of β . In the simple, academic setting of this paper, we might approximate the sigmoid function S by Taylor expansion up to some order and quantitatively estimate the expansion coefficients together with ε and β . However, due to the generic impossibility of quantitatively modeling and fitting experimental neuronal data, this approach would not overcome, in a real experimental setting, the limitation of quantitative estimation reviewed in the introduction. Instead, we aim at *qualitatively* estimating the distance of the ruling parameter β from critical value without using any type of quantitative fitting. Indeed, this distance is all we need to assess online whether the system exhibits oscillation or has a globally exponentially stable fixed point and whether it is near to a change of activity.

Proposition 1 proves that almost all trajectories of system (1) converge to an $\mathcal{O}(\varepsilon)$ -neighborhood of the critical manifold S^0 in both the fixed point and the limit cycle cases, at least for most time in the latter case. Based on singularity theory Golubitsky and Schaeffer (1985), Franci and Sepulchre (2014) and in view of Assumption 1, the qualitative shape of the critical manifold S^0 is the same as that of the set

$$\{(x_f, x_s) : -x_f^3 + (\beta - \beta_c)x_f + u - x_s = 0\}, \quad (7)$$

whose definition is independent of S . Qualitatively, the shape of this set, as well as of S^0 , is fully determined by the sign of $\beta - \beta_c$ (see Fig.1). When $\beta < \beta_c$, it is the graph of a monotone decreasing function. For $\beta = \beta_c$, it has a point of infinite slope, corresponding to the hysteresis singularity, and, when $\beta > \beta_c$, it is S-shaped. We exploit this information in the next section to design and analyze the estimator.

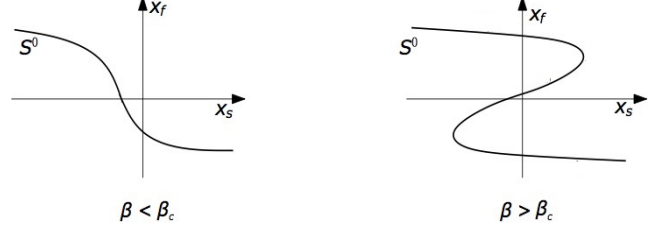


Fig. 1: Geometrical form of the critical manifold

3. QUALITATIVE ESTIMATOR

3.1 Estimator design

In the following, we construct the estimate of $\beta - \beta_c$ based on the sole information provided by (7) about the qualitative shape of the critical manifold. We first make the following assumption.

Assumption 2. Both the fast x_f and the slow x_s variables are measured in system (1) and the control input u is known and constant. \square

We propose the following nonlinear parameter estimator

$$\dot{\hat{\beta}} = -kx_f(-x_f^3 + \hat{\beta}x_f + u - x_s) := \hat{f}(\hat{\beta}, x_f, u - x_s), \quad (8)$$

where $k > 0$ is a design parameter. We note that \hat{f} is independent of S and ε , its expression is only related to the set (7). The influence of the design parameter k and the robustness of the estimator will be briefly discussed in Section 4.

3.2 Steady-state properties of the qualitative estimator

Steady-state of $\hat{\beta}$ satisfies $-x_f^3 + \hat{\beta}x_f + u - x_s = 0$. Hence, we implicitly define the function $\hat{\beta}^*(x_f, x_s, u)$ such that

$$-x_f^3 + \hat{\beta}^*(x_f, x_s, u)x_f + u - x_s = 0, \quad (9)$$

for $u, \beta \in \mathbb{R}$ and x_f, x_s on S^0 , i.e. x_f, x_s are related by the critical manifold equation

$$-x_f + S(\beta x_f + u - x_s) = 0. \quad (10)$$

Recalling that the sigmoid function S is invertible on $S(\mathbb{R})$ according to item c) of Assumption 1, we can explicitly invert (10) as follows

$$x_s = -S^{-1}(x_f) + \beta x_f + u. \quad (11)$$

Replacing this expression for x_s in (9), we obtain the explicit expression of $\hat{\beta}^*(x_f, x_s, u)$

$$\hat{\beta}^*(x_f, x_s, u) = \frac{x_f^3 - S^{-1}(x_f) + \beta x_f}{x_f}. \quad (12)$$

We see that $\hat{\beta}^*$ depends only on x_f and β . We therefore write $\hat{\beta}^*(x_f, \beta)$ in the following.

The next lemma provides important properties of $\hat{\beta}^*(x_f, \beta)$.

In particular, it ensures that $\hat{\beta}^*(x_f, \beta)$ is well-defined when $x_f = 0$. Furthermore it formally characterises in which sense $\hat{\beta}^*$ provides qualitative information about $\beta - \beta_c$.

Lemma 1. The following holds.

- 1) $\hat{\beta}^*$ is smooth on $S(\mathbb{R}) \times \mathbb{R}$.
- 2) For all $\beta \in \mathbb{R}$, $\frac{\partial \hat{\beta}^*(x_f, \beta)}{\partial \beta} > 0$ when $\beta < \beta_c$ and $x_f \in S(\mathbb{R})$, while $\frac{\partial \hat{\beta}^*(x_f, \beta)}{\partial \beta} > 0$ when $0 < \beta - \beta_c < 1$ and x_f is in a neighborhood of the origin.
- 3) $\hat{\beta}^*(x_f, \beta) = \beta - \beta_c + \mathcal{O}(x_f^2)$ for any $x_f \in S(\mathbb{R})$ and $\beta \in \mathbb{R}$. \square

Lemma 1 is informative about the qualitative dependence of $\hat{\beta}^*$ on $\beta - \beta_c$. According to item 2) of Lemma 1, $\hat{\beta}^*$ is strictly increasing in β , and hence in $\beta - \beta_c$, in the resting activity (i.e. $\beta < \beta_c$). If a change of activity is prone to appear (i.e. β is close to β_c), then necessarily x_f will lie in a neighborhood of the origin (for u small), which relates to the point of infinite slope, corresponding to the hysteresis singularity of the critical manifold. In this case, $\hat{\beta}^*$ will change its sign from negative to positive according to item 3) of Lemma 1, and then it will grow according to the second part of item 2) of Lemma 1. The same holds in the other direction, meaning that if $\hat{\beta}^*$ goes from positive to negative values. Hence, $\hat{\beta}^*$ allows to detect a change of activity in system (1). If $\hat{\beta}$ converges to $\hat{\beta}^*$ as time tends to infinity, then we can distinguish the different behaviors of the system based on $\hat{\beta}$. This is the purpose of the next subsection.

3.3 Stability analysis

Firstly, we discuss stability property of (8) with (x_f, x_s, u) as input in Lemma 2. Then we study the different behaviors of the three-dimensional overall system, i.e. system (1) and estimator (8) in Proposition 2. Finally, we show the link between $\hat{\beta}$ and $\hat{\beta}^*$ in Theorem 1.

The next lemma states an incremental input-to-state stability property (in the semiglobal sense) of system (8) with (x_f, x_s, u) as input. In the sequel we use the triplet (x_f, x_s, u) to denote a solution to system (1) with arbitrary initial condition and input u .

Lemma 2. Let $\Delta > 0$ and \mathcal{S}_Δ be the set of solutions (x_f, x_s, u) to system (1) such that: (i) $\max(\|x_f\|_{[0, +\infty)}, \|x_s\|_{[0, +\infty)}, \|u\|_{[0, +\infty)}) \leq \Delta$; (ii) there exist $t^*(\Delta) \geq 0$ and $a(\Delta) > 0$, such that for any $t \geq t^*(\Delta)$,

$$\int_{t^*(\Delta)}^t x_f^2(\tau) d\tau \geq a(\Delta) (t - t^*(\Delta)). \quad (13)$$

Then, for $(x_{f1}, x_{s1}, u_1), (x_{f2}, x_{s2}, u_2) \in \mathcal{S}_\Delta$, for any $M > 0$ and $|\hat{\beta}_1(0)|, |\hat{\beta}_2(0)| \leq M$, there exist strictly positive constants $\ell_{\Delta, M}, \omega_{\Delta, M}$ and \mathcal{K}_∞ function $\gamma_{\Delta, M}$, such that

the corresponding solutions $\hat{\beta}_1, \hat{\beta}_2$ to system (8) satisfy for any $t \geq 0$

$$\begin{aligned} |\hat{\beta}_1(t) - \hat{\beta}_2(t)| &\leq \ell_{\Delta, M} e^{-\omega_{\Delta, M} t} |\hat{\beta}_1(0) - \hat{\beta}_2(0)| \\ &+ \gamma_{\Delta, M} (\|x_{f1} - x_{f2}\|_{[0, t)} + \|x_{s1} - x_{s2}\|_{[0, t)} \\ &+ \|u_1 - u_2\|_{[0, t)}). \end{aligned} \quad (14)$$

\square

Similar to semiglobal ISS with respect to inputs as defined in Angeli et al. (2000), property (14) indicates that the estimator (8) is semiglobal incremental ISS with respect to initial conditions and inputs. Condition (13) is similar to a persistency of excitation condition. In the present paper, we assume it and we verify it online in simulations. Conditions on the initial states of system (1) and its input to ensure it a priori will be investigated in future work.

To finalize the analysis, we consider the overall system, that is (1) and (8), which is three-dimensional and given by

$$\begin{cases} \dot{\hat{\beta}} = -kx_f(-x_f^3 + \hat{\beta}x_f + u - x_s), & (15a) \\ \dot{x}_f = -x_f + S(\beta x_f + u - x_s), & (15b) \\ \dot{x}_s = \varepsilon(x_f - x_s). & (15c) \end{cases}$$

Note that system (15) is a cascade of system (15b)-(15c) with (15a). We state the convergence property of system (15) in the next proposition.

Proposition 2. Consider system (15).

- (1) For any $\beta - \beta_c < 0$, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, $x_f(0), x_s(0) \in \mathbb{R}$ and constant input u such that the corresponding solution (x_f, x_s, u) to (15b)-(15c) is in the set \mathcal{S}_Δ for some $\Delta > 0$, and for any $\hat{\beta}(0) \in \mathbb{R}$, the corresponding solution to system (15) asymptotically converges to a fixed point.
- (2) For any $0 < \beta - \beta_c < 1$, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, $x_f(0), x_s(0) \in \mathbb{R}$ and constant input u such that the corresponding solution (x_f, x_s, u) to (15b)-(15c) is in the set \mathcal{S}_Δ for some $\Delta > 0$, and for any $\hat{\beta}(0) \in \mathbb{R}$, the corresponding solution to system (15) asymptotically converge to a periodic limit cycle. \square

Proposition 2 ensures that the estimate $\hat{\beta}$ converges either to a fixed point or to a limit cycle depending on the sign of $\beta - \beta_c$. Now we are ready to state the link between the estimate $\hat{\beta}$ and $\hat{\beta}^*$ defined by (12) in the following theorem.

Theorem 1. Consider system (15).

- (1) For any $\beta - \beta_c < 0$, let u and ε be defined as in item (1) of Proposition 2, and let x_f^* be the x_f -component of the fixed point of (15). Then $\hat{\beta}(t)$ converges to $\hat{\beta}^*(x_f^*, \beta)$ as time goes to infinity.
- (2) For any $0 < \beta - \beta_c < 1$, let u and ε be defined as in item (2) of Proposition 2, and let $x_f^{lc}(t + \theta)$ be the T^ε -periodic asymptotic orbit of $x_f(t)$ along the limit cycle to which converges x_f . Then, for all $t_0 \geq 0$, there exists $\delta T^\varepsilon \subset [t_0, t_0 + T^\varepsilon)$ with $|\delta T^\varepsilon| = \mathcal{O}(\varepsilon)T^\varepsilon$, such that $\hat{\beta}(t) - \hat{\beta}^*(x_f^{lc}(t + \theta), \beta) = \mathcal{O}(\varepsilon)$ for all $t \in [t_0, t_0 + T^\varepsilon) \setminus \delta T^\varepsilon$. \square

When $\beta < \beta_c$, according to item 1) of Proposition 2, the estimate converges to a fixed point as time increases, and item 1) of Theorem 1 specifies that its value is $\hat{\beta}^*$. When $0 < \beta - \beta_c < 1$, the estimate converges to a periodic limit cycle in view of item 2) of Proposition 2, and item 2) of Theorem 1 indicates that the estimate value is in an $\mathcal{O}(\varepsilon)$ -neighborhood of $\hat{\beta}^*$ for all the time, after a sufficiently long time, except during jumps and each is of length $|\delta T^\varepsilon| = \mathcal{O}(\varepsilon)T^\varepsilon$. We see that $\hat{\beta}^*$ is not a constant in this case but a state-dependent signal, which is unusual in parameter estimation. In practice, we can average the obtained value of $\hat{\beta}$ over a sliding window.

4. NUMERICAL ILLUSTRATIONS

We analyze the results provided by the proposed qualitative estimator in simulations in this section. We first consider the sigmoid function $S : x \mapsto \tanh(x)$, for which $\beta_c = 1$. We choose $\varepsilon = 0.001, u = -0.01$. Figure 2 illustrates the time evolutions of the states x_f, x_s of system (1) for different values of $\beta - \beta_c$. We find that x_f converges to a constant value as time increases, as well as x_s , when $\beta - \beta_c < 0$, which is consistent with item 1) of Proposition 1. Both x_f and x_s converge to an oscillatory behavior when $0 < \beta - \beta_c < 1$, which is in agreement with item 2) of Proposition 1.

We have designed the estimator as in (8) with $k = 5$. To check that the estimator is able to detect a change of activity of the system, we use a step signal for β . In particular, $\beta = 0.9$ on $[0, 500]$ and $\beta = 1.1$ on $[500, 1000]$, which leads to a change of sign of $\beta - \beta_c$ at $t = 500$. We observe in Fig. 3 that, when $\beta - \beta_c < 0$, the estimate converges to a constant value as time increases, corresponding to the resting activity. When $0 < \beta - \beta_c < 1$, the estimate tends to a periodic function, which is strictly positive. Hence, it indicates the oscillation activity. Moreover, the estimate value evolves from negative to positive when the change of sign of $\beta - \beta_c$ occurs. This implies the change of the system's behavior. It could be interpreted as, for example in epilepsy a change from rest to seizure. We have tested different values of k in (8). The simulations indicate that the speed of convergence of $\hat{\beta}$ increases with k . However, this also leads to bigger spikes at "jumps", which may provide negative estimate value during a very short interval.

To test the robustness of our approach, we next consider the following sigmoid functions: $S_1 : x \mapsto \tanh(x), S_2 : x \mapsto \frac{x}{\sqrt{1+x^2}}, S_3 : x \mapsto \frac{1}{1+e^{-x}} - \frac{1}{2}$. The corresponding critical values are $\beta_{c1} = \beta_{c2} = 1$ and $\beta_{c3} = 4$. We emphasize that even though the nonlinearities are different, the estimator remains the same as defined in (8). Figure 4 shows the relationship between $\hat{\beta}$ and $\beta - \beta_c$ for different input u and perturbation parameter ε . Note that, when $\beta - \beta_c < 0$, the value of $\hat{\beta}$ is chosen as the constant value, to which the estimate converges. When $0 < \beta - \beta_c < 1$, the estimate tends to a periodic behavior. Then $\hat{\beta}$ is selected as the average value of this periodic function (over a period). We observe that for $\beta - \beta_c < 0$, the estimate value $\hat{\beta}$ is negative and it increases as $\beta - \beta_c$ increases. When $0 < \beta - \beta_c < 1$, $\hat{\beta}$ is positive. Moreover, when $\beta - \beta_c = 0$, the estimate is around the origin. Hence the estimate $\hat{\beta}$ qualitatively

infers the distance of $\beta - \beta_c$ to the origin. We remark that the estimator works robustly to variations of the sigmoid shape.

To further evaluate the robustness of the scheme, we have added small additive measurement noises given by $d_{out}(t) = 0.005 \sin(50t)$ for x_f, x_s and $d_{in} = 0.008$ for input u . It is found from Fig. 5 that the estimator still provides good results in this case.

5. CONCLUSION

We have, to the best of our knowledge, introduced for the first time the concept of *qualitative estimation*, which allows to detect online the behavioral state of a multiple-timescale nonlinear system, independently of large uncertainties on the system nonlinearities and without using any type of quantitative fitting. This is achieved by first extracting the system ruling parameter(s) and by subsequently designing an estimator of the distance of the ruling parameter(s) to some critical value(s) at which the behavioral transition happens. Timescale separation is instrumental in the estimator design, because it allows to reduce the qualitative estimation problem to the estimation of the qualitative shape of the system critical manifold. As a first illustration, we have focused on a class of two-dimensional nonlinear systems with two time-scales and a single nonlinearity either exhibiting resting or relaxation oscillation behaviors.

Future extensions will focus on generalizing the theory to higher-dimensional systems, different unknown nonlinearities, and multiple possible qualitative behaviors. We will also investigate the case where only the fast variable is measured, and not the full state as done in the paper.

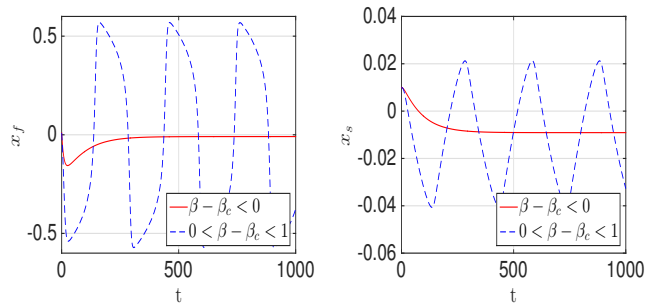


Fig. 2: States x_f and x_s of system (1) for different values of β .

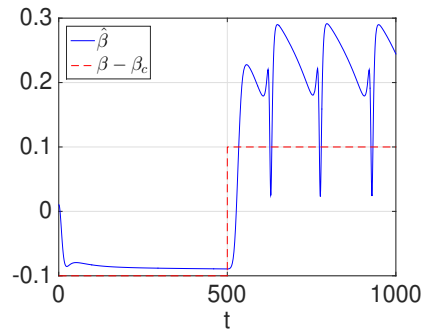


Fig. 3: Variable $\hat{\beta}$ for two different values of β .

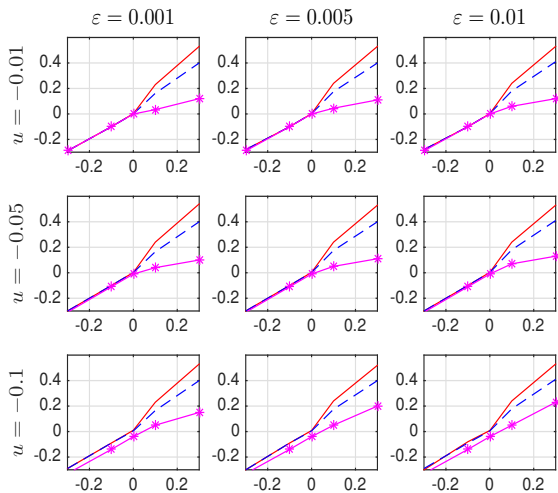


Fig. 4: Robustness with respect to different sigmoid functions. Vertical axis is $\hat{\beta}$ and horizontal axis is $\beta - \beta_c$. Different S functions: $S_1 : x \mapsto \tanh(x)$ (red solid line), $S_2 : x \mapsto \frac{x}{\sqrt{1+x^2}}$ (blue dashed line), $S_3 : x \mapsto \frac{1}{1+e^{-x}} - \frac{1}{2}$ (pink $-*$ line).

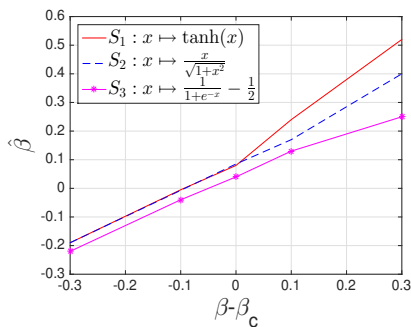


Fig. 5: Robustness with respect to additive measurement noises for different sigmoid functions.

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