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Two Types of Rotations in Associated Stirling Numbers

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Abstract : In a previous article [1], we have found that associated Stirling numbers of first and second kind can be compacted, at any order and using a linear transformation, in a structure of arithmetical triangle. We show then a strong link between the congruence of such numbers and this common geometrical layout. It leads to non-trivial combinatorial and modular properties which would be much more difficult to find and to prove without such a structure. Amongst other things, the binomial coefficient reveals a new surprising ability to build *modular areas, angles and rotations*, geared to the order r and independent of their homonyms in terms of classical space transformations. Probably for the first time in arithmetic science, two rotations of deeply different natures, one geometrical and the other modular, are highlighted for the same Stirling numbers and their associated.

Keywords : Number Theory, Associated Stirling Numbers, Modular Rotations, Arithmetical Triangle.

AMS Classification : 11B73, 05A18.

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1 Introduction

We will prove new modular results concerning the associated Stirling numbers of both kinds through the prism of what we call the arithmetical triangle's structure. Usually, in arithmetical triangles like the so-called Pascal's triangle, the main modular properties are located on the *horizontal lines* [2][3][4][5]. Yet, surprisingly, *this is no longer the case in associated Stirling numbers arithmetical triangles at order r* : **modular properties** (excepted for $r = 1$ which corresponds to classical Stirling numbers) are **concentrated on the oblique lines of the triangles**, with slopes depending on r ...

In the following, one will provide the plan with an orthonormal reference mark and trigonometrical direction. Let's remember [1] that an integer sequence $t(i, j)$ is said to have a structure of arithmetical, or more simply is an *arithmetical triangle*, if and only if all its terms $t(i, j)$ are calculable starting from initial data and of a relation of recurrence of the type :

$$t(i, j) = g(i, j)t(i - 1, j - 1) + h(i, j)t(i - 1, j)$$

where g et h are integer functions of discrete variables i, j .

2 Geometrical rotations of the associated Stirling numbers

One is interested initially in the associated Stirling numbers of first kind at order $r \geq 1$, noted $d_r(n, k)$ in the literature : as a reminder, $d_r(n, k)$ is defined as the number of permutations of a set of n elements having exactly k cycles, all length $\geq r$.

These numbers check the relation of following recurrence (see for example [2]) :

$$d_r(n, k) = (n - 1)d_r(n - 1, k) + (n - 1)_{r-1}d_r(n - r, k - 1)$$

where $(n)_k$ indicates the downward factorial, as well as the initial conditions :

$$\begin{aligned} d_r(n, k) &= 0 & \forall n \leq kr - 1 \\ d_r(n, 1) &= (n - 1)! \end{aligned}$$

The following theorem in our previous article [1] gathers in only one structure, at any order, the various representations of the $d_r(n, k)$:

For all $r \geq 1$, the $\{d_r(n, k)\}_{n \geq 1, k \geq 1}$ has a structure of arithmetical triangle : the latter will be noted σ_r and its terms $\{D_r(m, k)\}_{m \geq 1, k \geq 1}$.

To prove it, we used R , a linear application in the space of numerical sequences s with two integer variables n et k defined by :

$$R : s(n, k) \mapsto R(s(n, k)) = s(n + k - 1, k)$$

With $s \equiv d_r$ and after setting down : $m = n - (r - 1)(k - 1)$, the application of R^{r-1} to the recurrence relation of d_r gave :

$$R^{r-1}(d_r)(m, k) = (m + (r - 1)(k - 1) - 1)R^{r-1}(d_r)(m - 1, k) + \\ (m + (r - 1)(k - 1) - 1)_{r-1}R^{r-1}(d_r)(m - 1, k - 1)$$

thus :

$$D_r(m, k) = (m + (r - 1)(k - 1) - 1)D_r(m - 1, k) + \\ (m + (r - 1)(k - 1) - 1)_{r-1}D_r(m - 1, k - 1)$$

with

$$D_r = R^{r-1}(d_r)$$

The new sequence $D_r(m, k)$ has an arithmetical triangle's structure. The latter, which puts our terms to the "immediate vicinity" from each other, will make it possible to show their modular properties step by step and in various directions.

Abusing the language, for any fixed n_0 and $p_0 \geq 1$, the linear transformation R is a plane geometrical rotation, of centre $s(n_0, 1)$ and angle $(\mathcal{D}_{n_0, p_0}, \mathcal{D}_{n_0, p_0-1})$, of \mathcal{D}_{n_0, p_0} , where $\mathcal{D}_{n', p'}$ is a general line of elements $s(n, k)$ starting from $s(n', 1)$ (invariant of R) and passing through $s(p', 2)$. Geometrically speaking, the subscripts of its elements (n, k) check the following discrete equation [1] :

$$(k - 1)(n' - p') + (n - n') = 0 \quad \forall n, k \geq 1$$

We can represent R in the following way :

	$k \rightarrow$			
	s(1,1)
	s(2,1)
	s(3,1)	s(3,2)	...	s(n,k)
$n \downarrow$	s(4,1)	s(4,2)
	...	s(5,2)	s(5,3)	...
	s(6,3)	...
	s(7,4)

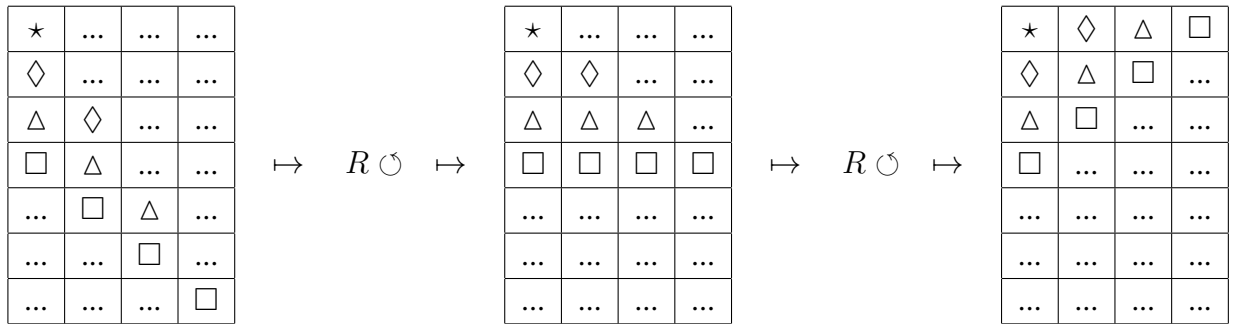
$$\mapsto R \circlearrowleft \mapsto$$

s(1,1)
s(2,1)	s(3,2)
s(3,1)	s(4,2)	s(5,3)	...
s(4,1)	s(5,2)	s(6,3)	s(7,4)
...
...
...

$$\mapsto R \circlearrowleft \mapsto$$

s(1,1)	s(3,2)	s(5,3)	s(7,4)
s(2,1)	s(4,2)	s(6,3)	...
s(3,1)	s(5,2)
s(4,1)
...
...
...

Or, more schematically :



As for the associated Stirling numbers of second kind at any order, they check the law of following recurrence (see for example [2]) :

$$s_r(n, k) = k s_r(n - 1, k) + \binom{n - 1}{r - 1} s_r(n - r, k - 1)$$

The terms in s_r are of the same form as the d_r in their recurrence relation. It was then possible to define $r - 1$ rotations of the same type as R and to extend the theorem above to the $s_r(n, k)$:

For all $r \geq 1$, the $\{s_r(n, k)\}_{\substack{n \geq 1 \\ k \geq 1}}$ has a structure of arithmetical triangle, noted π_r with $\{S_r(m, k)\}_{\substack{m \geq 1 \\ k \geq 1}}$ its terms.

The lines associated with a natural integer n in π_r or σ_r are the whole of the terms corresponding to $s_r(n, k)$ or $d_r(n, k)$ for all k . It was said that n divides its associated line if it divides of it all the terms except the first. For all positive integer r and all integer h , Σ_r^h and Π_r^h were defined as the arithmetical structures obtained after h iterations of R applied respectively to the arithmetical

triangles σ_r and π_r .

Those preliminary results will help us to introduce a new kind of plane transformations, directly related to modular properties of the associated Stirling numbers.

3 Modular rotations of the associated Stirling numbers

Our study will concentrate on π_r but, as we will see it, it can easily spread with the other triangle σ_r . There is the following theorem :

Theorem

Let the arithmetical triangle π_r be, with $\rho_r^0(n)$ the line in π_r associated with a natural integer n , $\eta_r^0(n)$ the horizontal line of π_r sharing its non-zero first term with $\rho_r^0(n)$ and $\delta_r^0(n)$ the line which forms an angle of $-\frac{\pi}{4}$ with $\eta_r^0(n)$. Let finally $\mathcal{A}_r^0(n)$ be the whole of the elements of π_r ranging in the area between $\delta_r^0(n)$ and $\rho_r^0(n)$, except $\delta_r^0(n)$ (see figure 1).

There is then for all prime numbers $n \geq 2$ and for all $r \geq 1$:

$$\mathcal{A}_r^0(n) \equiv 0 \pmod{n}$$

Proof :

Let n be a prime number ≥ 2 and $r \geq 1$. We consider $\rho_r^0(n)$ and its intersection C with the orthogonal line to $\eta_r^0(n)$ in B . C will enable us to leave the triangle π_r and to avoid thus certain "effects edge" in our proof (cf. following figure 2).

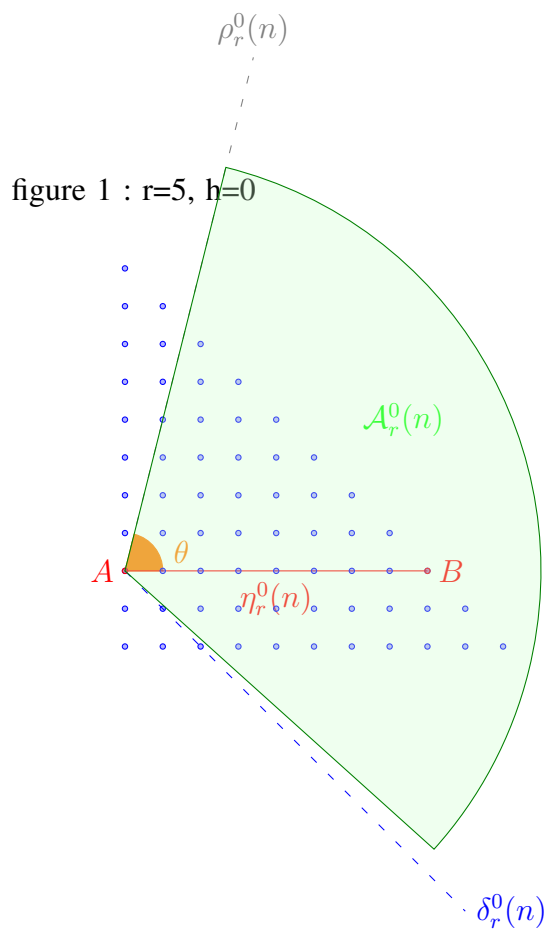


figure 2

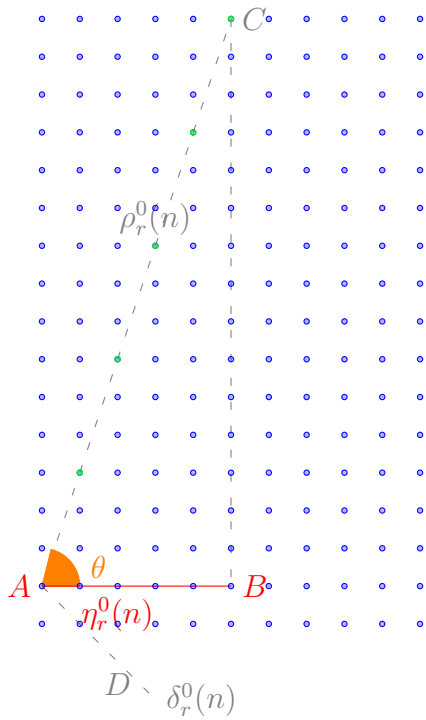
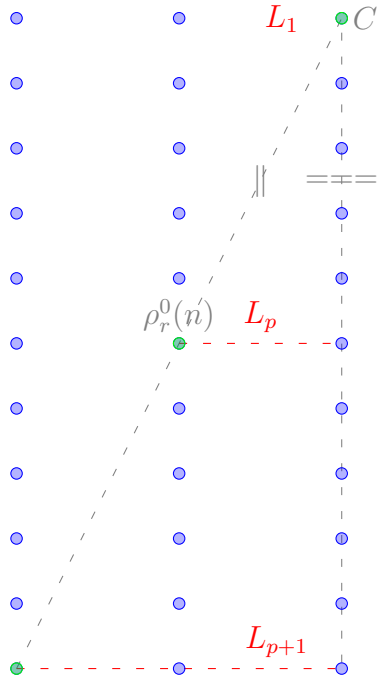


figure 3



Let n be a fixed prime number. One will show by recurrence that all the elements $S_r(m, k)$ of π_r contained in the triangle ABC are such as :

$$S_r(m, k) \equiv 0 \pmod{n}$$

Let us note L_p the lines of π_r located on the p -nth line starting from C and on the right of $\rho_r^0(n)$ (cf. figure 3). The L_p are defined until AB excluded. We can show by recurrence on p that the L_p check the relation above.

- for $p = 1$: the line L_1 contains only one null term in C , thus divisible by n .
- let us suppose our relation of congruence checked for L_p .

Let the following line L_{p+1} be. In the triangle of Pascal, one knows that the congruence of a line is transmitted to that from under by simple linearity - except the first term of the line. In the same way here, since all the column BC is zero, one can show using L_p that all the elements of L_{p+1} are divisible by n , except its first term noted $L_{p+1}(1)$. We will then have two possibilities :

- if $L_{p+1}(1)$ belongs to $\rho_r^0(n)$, then $L_{p+1}(1) = S_r(m, k)$ with by definition :

$$m = n - (k - 1)(r - 1)$$

Yet, one can show in a combinatorial way that [2] :

$$S_r(m, k) = s_r(n, k) = \frac{1}{k!} \sum_{\substack{\alpha_1 \geq r \dots \alpha_k \geq r \\ \alpha_1 + \alpha_2 + \dots + \alpha_k = n}} \frac{n!}{\alpha_1! \alpha_2! \dots \alpha_k!}$$

Given some simple modular properties, this leads to :

$$S_r(m, k) \equiv 0 \pmod{n}$$

To generalize our theorem to σ_r , it would be sufficient to have a similar modular property of the kind :

$$d_r(n, k) \equiv 0 \pmod{n}$$

that we will adapt to σ_r , like for π_r , using $D_r(m, k) \equiv R^{r-1} [d_r(n, k)]$ with always :

$$m = n - (k - 1)(r - 1)$$

- if $L_{p+1}(1)$ does not belong to $\rho_r^0(n)$, then we can consider it as equal to one of the terms of the sequence (for a fixed k) : $S_r(m + 1, k), S_r(m + 2, k), \dots, S_r(m + r - 1, k)$.

Those terms check the following recurrence relations :

$$\begin{aligned} S_r(m + 1, k) &= k.S_r(m, k) + \binom{n}{r-1} S_r(m, k - 1) \\ S_r(m + 2, k) &= k.S_r(m + 1, k) + \binom{n+1}{r-1} S_r(m + 1, k - 1) \\ &\vdots \\ S_r(m + r - 1, k) &= k.S_r(m + r - 2, k) + \binom{n+r-2}{r-1} S_r(m + r - 2, k - 1) \end{aligned}$$

Since n is prime, then :

$$\begin{aligned} \binom{n}{r-1} &\equiv 0 \pmod{n} \\ \binom{n+1}{r-1} &\equiv 0 \pmod{n} \\ &\vdots \\ \binom{n+r-2}{r-1} &\equiv 0 \pmod{n} \end{aligned}$$

Thus, according to the first equations of our two systems :

$$S_r(m, k) \equiv 0 \pmod{n} \Rightarrow S_r(m + 1, k) \equiv 0 \pmod{n}$$

In the same way, according to the second equations :

$$S_r(m + 1, k) \equiv 0 \pmod{n} \Rightarrow S_r(m + 2, k) \equiv 0 \pmod{n}$$

$$S_r(m + r - 2, k) \equiv 0 \pmod{n} \Rightarrow S_r(m + r - 1, k) \equiv 0 \pmod{n}$$

Thus, all the first possible terms of L_{p+1} are divisible by n , which involves, according to the assumption on L_p that *all* the terms of L_{p+1} are divisible by n . It is then possible for us to extend this property to all the set \mathcal{A}_r^0 starting from the same property of "linear transmission of congruence" mentioned above.

Let us specify an important detail : when $L_p = AB$, recurrence stops and the element D of the following line (cf. figure 2) is never divisible by n . Indeed, in D the term $S_r(m+1, 2)$ is such as :

$$\begin{aligned} S_r(m+1, 2) &= 2S_r(m, 2) + \binom{n+r-1}{r-1} S_r(m, 1) \\ &= 2S_r(m, 2) + \binom{n+r-1}{r-1} \end{aligned}$$

since A (associated to n) checks $m = n - (r-1)(1-1) = n$ and $S_r(m, 1) = 1$.

However, as $S_r(m, 2)$ belongs to AB , it is divisible by n but $\binom{n+r-1}{r-1}$ are not it, which implies that $S_r(m+1, 2)$ is not divisible by n : if this property had not been checked (it was seen that held only with the coefficient in front of $S_r(m, 1) = 1$) then $\delta_r^0(n)$ would *entirely* belong to \mathcal{A}_r^0 and thus would not be any more one asymptote.

For σ_r , one would have rather used the following system :

$$\begin{aligned} D_r(m+1, k) &= n.D_r(m, k) + (r-1)! \binom{n}{r-1} D_r(m, k-1) \\ D_r(m+2, k) &= (n+1).D_r(m+1, k) + (r-1)! \binom{n+1}{r-1} D_r(m+1, k-1) \\ &\vdots \\ D_r(m+r-1, k) &= (n+r-2).D_r(m+r-2, k) + (r-1)! \binom{n+r-2}{r-1} D_r(m+r-2, k-1) \end{aligned}$$

and, for n prime :

$$\begin{aligned} (r-1)! \binom{n}{r-1} &\equiv 0 \quad [n] \\ (r-1)! \binom{n+1}{r-1} &\equiv 0 \quad [n] \\ &\vdots \\ (r-1)! \binom{n+r-2}{r-1} &\equiv 0 \quad [n] \end{aligned}$$

The rest rises exactly from the same manner as for π_r : *QED*.

Definitions : one will call *modular angle associated with a natural integer n* the surface $\mathcal{A}_r^0(n)$. Its associated geometrical angle is : $\theta + \frac{\pi}{4}$, where :

$$\theta = \arctan(r - 1)$$

One will also speak about *modular rotation* in π_r or σ_r when the parameter r varies. These definitions are generalizable to any arithmetical triangle.

From the proof of the main theorem on modular rotations above, it appears that the binomial coefficients play an essential role in the construction of the angle θ , which results from the equation of $\rho_r^0(n)$:

$$m + (k - 1)(r - 1) = n$$

with n fixed and $\forall k, m$.

This angle depends only on the order r which then can be considered as the main parameter of modular rotations. For this reason, we may underline that **a modular rotation is not the result of any classical geometrical rotation of the plane**. Indeed, as we saw above, the arithmetical triangle structure is "geometrically fixed", whereas only the modular angle varies with r .

4 Conclusion

This work proposed to study some modular properties of the associated Stirling numbers of both kinds at any order r , after having "compacted" them in the state of arithmetical triangles π_r and σ_r , according to our previous paper [1]. Then, it appeared that those properties could be described in terms of angles depending on the "rotatory" parameter r . We introduced then some "modular" equivalents to classical geometrical concepts such as rotations, areas, angles. Those different types of plane transformations may be independent from each other : the geometrical structure of arithmetical triangle is preserved by modular rotation. Such an amazing and direct link between the two worlds of arithmetic and geometry must be explored. This will be the object of our future work.

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