Taking into account the rigidity of space-time in general relativity - Understanding by a simple analogy based on the concepts of the strength of materials

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Summary

The discoveries of gravitational waves linked to binary black holes coalescence have multiplied since 11 February2016. The formation of a resulting black hole from this cataclysmic event, concentrating in the final stage in a very small radius a colossal mass (50 solar masses and more), is sufficiently energetic to deform the space-time which then propagates without any attenuation during Billions of years, the associated gravitational waves. These were predicted by Einstein in his theory of general relativity a century ago. Their measurements, which are characterized by an infinitesimal $\Delta L / L$ deformation (10^{-21} , or 10^{-18} at the kilometer size of the interferometers), corresponds to the billionth of the size of an atom, today. The smallness of its deformations is the character trait of an extraordinarily great rigidity of space-time. We will show in this paper that the constant of proportionality κ between the curvature tensor $G_{\mu\nu}$ and the stress energy tensor $T_{\mu\nu}$ is a characterization of this very high rigidity. To do this we will make an analogy with the strength of materials via the theory of the beams to show it.

1rst part) – Relation between curvature, energy and rigidity of a beam – case of the strength of material

1.1) General case of a beam on 2 supports

Considering a beam of width b (m) of height h (m), on 2 simple supports, of span L (m), carried out with a material of young modulus E (MPa),with an inertia (m^4) and with an area S (m^2), that have a mass by unit of length (m = ρ S) in kg/m (cf. figure 1).

The beam has a deflection during the time $y_{(x,t)}$:

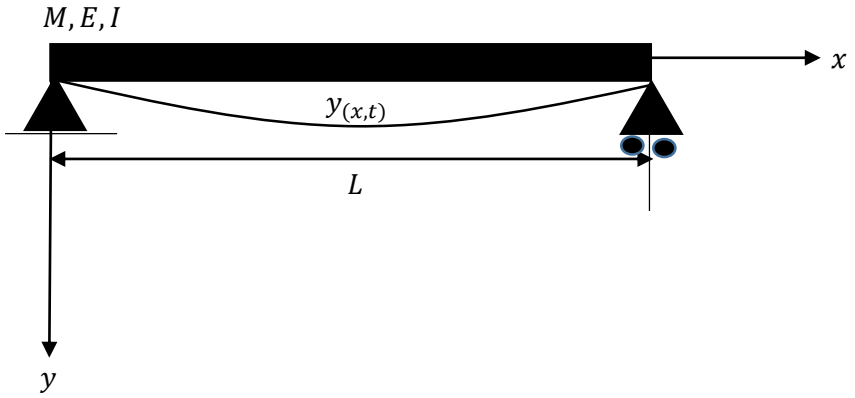

Figure 1 – Beam on 2 supports under its self-weight –

In static (deflection $y_{(x)}$ independent of the time), the fundamental relation connecting the curvature (1/R) at the bending moment $M_{(x)}$and at the second derivative of the deflection $y_{(x)}$ can be written as following:

$$\frac{d^2y_{(x)}}{dx^2} = -\frac{M_{(x)}}{EI} = \frac{1}{R} \quad (1)$$

In this expression, $M_{(x)}$ is the bending moment (N.m), and R the curvature radius (m).

The exact expression of the curvature is given in the expression (2). The term in cube root can be neglected here.

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)^3} \quad (2)$$

In addition, the elastic bending energy can be written as following:

$$U = \frac{1}{2}\int_0^L \frac{M_{(x)}^2}{EI}dx \quad (3)$$

1.2) Case of a beam on two simple supports under two equal bending moments C

To simplify, we are considering now a beam solicited by a bending moment at each extremity (cf. figure 2).

By convention, the bending moment is positive when it turns in the in the clockwise direction.

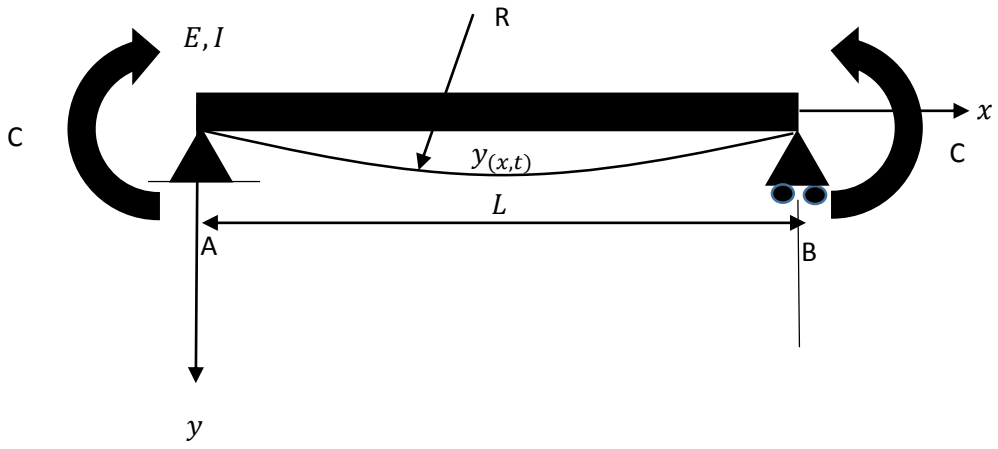

Figure 2 – Beam solicited by a moment of intensity C at each extremity –

As the reaction on the support are null, the bending equation is written as following:

$$M_{(x)} = C \quad (4)$$

Based on the equation (1) we obtain:

$$\frac{d^2y}{dx^2} = -\frac{C}{EI} = \frac{1}{R} \quad (5)$$

$$M_{(x)} = C = -\frac{EI}{R} \quad (6)$$

Introducing the expression (6) in the bending energy (3):

$$U = \frac{1}{2}\int_0^L \frac{(EI)^2}{R^2 EI} dx \quad (7)$$

As the curvature is constant, we obtain so:

$$U = \frac{1}{2}\frac{EIL}{R^2} \quad (8)$$

Or in another way in the case of the pure bending, we obtain:

$$\frac{1}{R^2} = \frac{2}{EI}\left(\frac{U}{L}\right) \quad (9)$$

We obtain so, a relation connecting the curvature at the strain energy density U/L of the beam.

Note:
By integral two times from x of the expression (5) and considering the deflection must be null on each support, we obtain the following expression of the deflection:

$$y_{(x)} = -\frac{C}{2EI}x^2 + \frac{CL}{2EI}x \quad (10)$$

And thus we find for the maximum deflection a result well known of the strength of materials:

$$y_{(L/2)} = -\frac{CL^2}{8EI} + \frac{CL^2}{4EI} = \frac{CL^2}{8EI} \quad (11)$$

If we derivate two times the equation (10) we obtain again that the curvature is constant:

$$\frac{d^2y_{(x)}}{dx^2} = -\frac{M}{EI} = -\frac{C}{EI} = \frac{1}{R}$$

In addition the rotation can be written as following:

$$\frac{dy_{(x)}}{dx} = \theta_{(x)} \quad (12)$$

By deriving from x the expression (10) we obtain:

$$\frac{dy}{dx} = -\frac{C}{EI}x + \frac{CL}{2EI} \quad (13)$$

We obtain the rotation at each extremity of the beam:

$$y'_{(0)} = \frac{CL}{2EI} = \theta_A \qquad (14)$$

And :

$$y'_{(L)} = -\frac{CL}{2EI} = \theta_B \qquad (15)$$

In addition, it is possible to write a relation between the bending moment C applied, the stiffness of the beam and the rotation θ_A and $\theta_B$at each extremity (cf. matrix calculation of a frame):

$$M_{AB} = \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B \quad (16)$$

$$M_{BA} = \frac{4EI}{L}\theta_B + \frac{2EI}{L}\theta_A \quad (17)$$

With:
M_{AB} The bending moment applied at the node A,
M_{BA} The bending moment applied at the node B.

And with our signs convention $M_{AB} = -M_{BA}$

Indeed, by using the expressions (14) and (15) in the equation (16) we obtain:

$$M_{AB} = \frac{4EI}{L}\left(\frac{CL}{2EI}\right) + \frac{2EI}{L}\left(-\frac{CL}{2EI}\right) = 2C - C = C$$

By taking into account the expressions (14) and (15) of the rotations at each extremity of the beam:

$$\theta_A = -\theta_B \quad (18)$$

The expression (16) can be written:

$$C = \frac{2EI}{L}\theta_A = k\theta_A \quad (19)$$

Where k is the rigidity (bending stiffness here) of the beam solicited by two equal moments C:

$$k = \frac{2EI}{L} \qquad (20)$$

We can now define the strain energy U of the beam in function of the bending moments applied and the rotations at the extremity.
The energy, the work of the bending moment is so:

$$\Delta U = \frac{1}{2} M \Delta\theta \quad (21)$$

In addition, under the bending moment, the beam bent and takes a curvature radius as indicated at the figure 3:

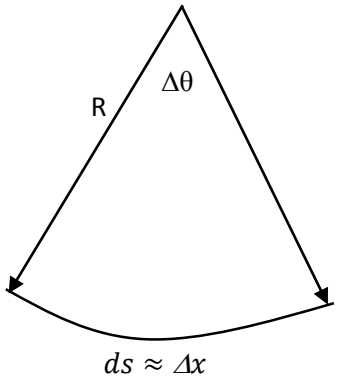

Figure 3 – Relation between the radius of curvature R of the beam and its mean fiber –

We have following the figure 3: $R\Delta\theta = ds \approx \Delta x \quad (22)$

And we have a relation between the curvature 1/R and the deflection w of the beam:

$$\frac{1}{R} = \frac{d^2 y_{(x)}}{dx^2} \quad (23)$$

By introducing the expression (23) into the expression (22) we obtain:

$$\Delta\theta = \frac{\Delta x}{R} = \frac{d^2 y_{(x)}}{dx^2} \Delta x \quad (24)$$

By introducing the expression (24) in the expression (21) the elastic strain energy takes the following value:

$$\Delta U = \frac{1}{2} M \frac{\Delta x}{R} \quad (25)$$

M is constant (M = C) and the curvature is constant also, the expression (25) becomes:

$$\Delta U = \frac{1}{2} C \frac{\Delta x}{R} \quad (26)$$

So by integral along the length L of the beam each side of the equation (26), we obtain:

$$U = \frac{1}{2} \frac{CL}{R} \quad (26)$$

By taking into account of the expression (5):

$$U = \frac{1}{2} \frac{EIL}{R^2} \quad (27)$$

So, we find again the expression (9).
Thus, we can define the total energy of the beam under the two bending moment C as defined in figure 2 in function of k, M and θ_A :

$$U = \frac{1}{2} C\theta_A + \frac{1}{2}(-C)\theta_B \quad (28)$$

And by taking into account of the rotation expression given in (14) and (15):

$$U = \frac{1}{2} C\theta_A + \frac{1}{2}(-C)\theta_B$$

With:

$$\theta_A = -\theta_B$$

$$U = C\theta_A \ (29)$$

And by taking into account of the expression (19):

$$C = \frac{2EI}{L}\theta_A = k\theta_A \quad (19)$$

We obtain:

$$U = k\theta_A^2 \ (30)$$

We can find again this expression from the equation (3):

$$U = \frac{1}{2}\int_0^L \frac{M_{(x)}^2}{EI} dx \quad (3)$$

With:

$$M = C = k\theta_A$$

$$U = \frac{1}{2}\int_0^L \frac{k^2\theta_A^2}{EI} dx \quad (31)$$

As in our case, all these terms are constant:

$$U = \frac{1}{2}\frac{k^2\theta_A^2}{EI} L \quad (32)$$

By equalizing the expression (32) with the expression of the energy obtain (27):

On obtient :

$$\frac{1}{2}\frac{k^2\theta_A^2}{EI} L = \frac{1}{2}\frac{EIL}{R^2} \ (33)$$

After simplification:

$$\frac{k^2\theta_A^2}{(EI)^2} = \frac{1}{R^2} \ (34)$$

By equalizing this expression with the expression (9) we obtain:

$$\frac{1}{R^2} = \frac{2}{EI}\left(\frac{U}{L}\right) = \frac{k^2\theta_A^2}{(EI)^2} \quad (35)$$

So :

$$\frac{k^2\theta_A^2}{(EI)^2} = \frac{2}{EI}\left(\frac{U}{L}\right)$$

$$k^2\theta_A^2 = \frac{2(EI)^2}{EI}\left(\frac{U}{L}\right)$$

$$k^2\theta_A^2 = \frac{2EI}{L}U \quad (36)$$

With the expression (20):

$$k = \frac{2EI}{L} \quad (20)$$

The expression (36) becomes:

$$k^2\theta_A^2 = kU$$

So:

$$k\theta_A^2 = U$$

That is the expression (30) already demonstrated.

By resuming the expression (35) and taking into account of the expression (30) we obtain so:

$$\frac{1}{R^2} = \frac{2}{EI}\left(\frac{U}{L}\right) = \frac{k^2\theta_A^2}{(EI)^2} \quad (35)$$

$$\frac{1}{R^2} = \frac{2}{EI}\left(\frac{U}{L}\right) = \frac{k}{(EI)^2}\,U \quad (36)$$

With the stiffness k:

$$k = \frac{2EI}{L} \quad (20)$$

We demonstrate so that the factor of proportionality between the energy and the curvature is linked at the rigidity k of the beam:

$$\boldsymbol{\frac{1}{R^2} = \frac{2}{EIL}U = \frac{k}{(EI)^2}\,U}\ (\mathbf{37})$$

The dimensional equation (if we notice U the energy) is the following:

$$\frac{U}{\frac{N}{m^2}\times m^4 \times m} = \frac{U}{N.m^3} = \frac{U}{\frac{kgm}{s^2}\times m^3} = \boxed{\frac{s^2}{kg\times m}\times\frac{U}{m^3}} = N.m \times \frac{U}{(N.m^2)^2} = \frac{U}{Nm^3} = \frac{U}{\frac{kgm}{s^2}\times m^3} \quad (38)$$

2nd part) Rigidity of the space time - Case of the general relativity

2.1) Expression of the Einstein equation

The Einstein equation is the following:

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad (39)$$

The developed form of the Einstein equation is the following:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (40)$$

We remind that the components of the stress energy tensor $T_{\mu\nu}$ have the dimension of an energy density:

G = 6.6726x10^{-11} m^{3}kg^{-1}s^{-2} 1J = 1W.s = 1 N.m = 1kg.m^2.s^{-2} kg

$$Curvature = \frac{8\pi G}{c^4} \times \frac{Energy}{Volume} = \frac{8\pi G}{c^2} \frac{Mass}{Volume} \quad (41)$$

1/m^2 (m/s)^4 (m^3)

In this equation, the curvature, and the space time energy are connected (cf. figure 3).

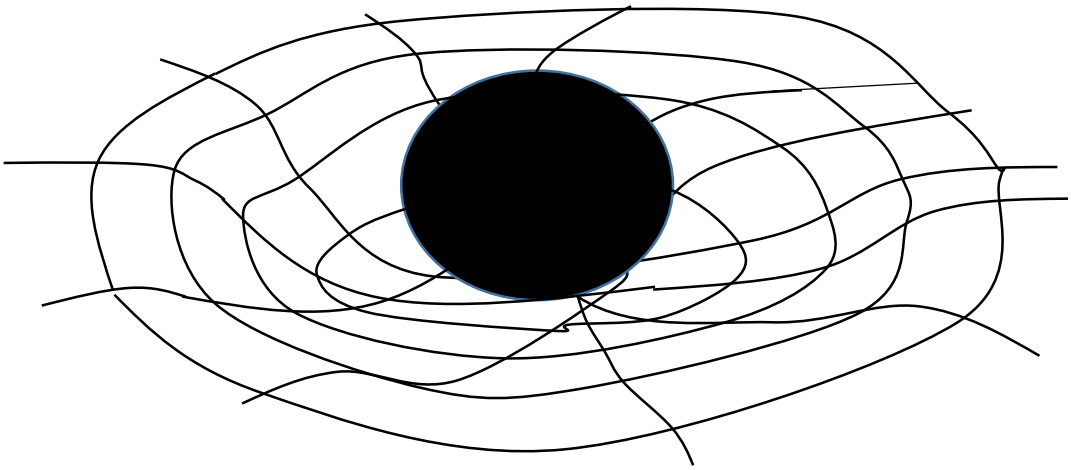

Figure 3 – Symbolic view of the curvature of the space time projected in a space plane –

$G_{\mu\nu}$ is the Einstein tensor.

$T_{\mu\nu}$ is the stress energy tensor.

2.2) Factor of proportionnality between curvature and energy – stiffness of the space time.

The factor of proportionality between the curvature and the energy is so:

$$\kappa = \frac{8\pi G}{c^4} \quad (42)$$

In addition, the dimensional equation of κ is the following:

$$\kappa = \frac{L^3T^4}{MT^2L^4} = \frac{T^2}{ML} = \frac{s^2}{kgm} \quad (43)$$

And the energy is in fact the energy density in the Einstein equation, so:

$$G_{\mu v} = \kappa T_{\mu v}$$

We notice: $T_{\mu v} = \frac{U_{\mu v}}{V}$

We obtain so via the dimensional equation:

$$\kappa \times \frac{U_{\mu v}}{V} => \frac{L^3T^4}{MT^2L^4} = \frac{T^2}{ML} = \frac{s^2}{kgm} \times \frac{U_{\mu v}}{m^3} \quad (44)$$

We find so again the dimensional equation (38) associated at the equation (37) in the case of a beam in bending solicited by two equal moments (one at each extremity).

Conclusions

We have so, a perfect analogy between the expressions (37) and (44), one issued of the beam theory in strength of materials and the other issued of the general relativity in 4 dimensions of the space time [24]. The curvature of the space time is so well proportional at the energy that is present and at the stiffness of the fabric of the space time.

$$\frac{\mathbf{1}}{\boldsymbol{R^2}} = \frac{\mathbf{2}}{\boldsymbol{EIL}}\boldsymbol{U} = \frac{\boldsymbol{k}}{\boldsymbol{(EI)^2}}\ \boldsymbol{U}$$

$$\boldsymbol{R_{\mu v}} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{g_{\mu v}R} + \boldsymbol{\Lambda g_{\mu v}} = \frac{\boldsymbol{8\pi G}}{\boldsymbol{c^4}}\boldsymbol{T_{\mu v}}$$

The term $k = \frac{2EI}{L}$ is the rigidity of a beam submitted at two bending moments (one at each extremity).

The term $\kappa = \frac{\boldsymbol{8\pi G}}{\boldsymbol{c^4}}$ is linked at the rigidity of the space time by analogy with the formula obtained for the beam.

Numerical application:

G = 6.6740831x10^{-11} m^3 kg^{-1} m^2

c = 299792458 m/s

$$\kappa = 2.0765799E^{-43}\ s^2kg^{-1}\text{-}m^{-1}$$

This extremely small number translate the extreme rigidity (1/flexibility) of the space time.

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