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A new result about prime numbers:

$$\lim_{n \rightarrow +\infty} \frac{n}{p_n - n(\ln n + \ln \ln n - 1)} = +\infty$$

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Abstract

In this short paper we propose a new result about prime numbers:

$$\lim_{n \rightarrow +\infty} \frac{n}{p_n - n(\ln n + \ln \ln n - 1)} = +\infty$$

Keywords: Prime numbers, Dusart

Preliminaries.

We write $\ln_2 n$ instead of $\ln \ln n$.

Let p_n denote the n^{th} prime number.

In 1999 [1] Pierre Dusart showed that :

$$n(\ln n + \ln_2 n - 1) < p_n < n(\ln n + \ln_2 n - 0.9484) \quad \text{for } n \geq 39017$$

In [2] and [3] it is also proved that:

$$p_n \leq n(\ln n + \ln_2 n - 1 + \frac{\ln_2 n - 2}{\ln n}) \quad \text{for } n \geq 688383$$

Theorem.

$$\lim_{n \rightarrow +\infty} \frac{n}{p_n - n(\ln n + \ln_2 n - 1)} = +\infty$$

Proof. Remember that (in [2]):

$$p_n \leq n(\ln n + \ln_2 n - 1 + \frac{\ln_2 n - 2}{\ln n}) \quad \text{for } n \geq 688383$$

We deduce:

$$n(\ln n + \ln_2 n - 1 + \frac{\ln_2 n - 2}{\ln n}) - n(\ln n + \ln_2 n - 1) \geq p_n - n(\ln n + \ln_2 n - 1) \quad \text{for } n \geq 688383$$

Hence:

$$\frac{n}{n(\ln n + \ln_2 n - 1 + \frac{\ln_2 n - 2}{\ln n}) - n(\ln n + \ln_2 n - 1)} \leq \frac{n}{p_n - n(\ln n + \ln_2 n - 1)} \quad \text{for } n \geq 688383$$

We have:

$$\frac{n}{n(\ln n + \ln_2 n - 1 + \frac{\ln_2 n - 2}{\ln n}) - n(\ln n + \ln_2 n - 1)} = \frac{n}{\frac{n \ln_2 n - 2}{\ln n}}$$

$$\frac{n}{\frac{n \ln_2 n - 2}{\ln n}} = \frac{\ln n}{\ln_2 n - 2}$$

And:

$$\lim_{n \rightarrow +\infty} \frac{\ln n}{\ln_2 n - 2} = +\infty$$

Because:

$$\frac{n}{n(\ln n + \ln_2 n - 1 + \frac{\ln_2 n - 2}{\ln n}) - n(\ln n + \ln_2 n - 1)} \leq \frac{n}{p_n - n(\ln n + \ln_2 n - 1)} \quad \text{for } n \geq 688383$$

We deduce:

$$\lim_{n \rightarrow +\infty} \frac{n}{p_n - n(\ln n + \ln_2 n - 1)} = +\infty$$

Consequence.

Let ϵ be a function defined by $\epsilon(n)$, n is a natural number and $n \geq 688383$

If:

$$\frac{n}{p_n - (n(\ln n + \ln_2 n - 1))} \geq \epsilon(n)$$

We have:

$$\frac{p_n - (n(\ln n + \ln_2 n - 1))}{n} \leq \frac{1}{\epsilon(n)}$$

And:

$$p_n - (n(\ln n + \ln_2 n - 1)) \leq \frac{n}{\epsilon(n)}$$

$$p_n - (n(\ln n + \ln_2 n - 1)) \leq \frac{n}{\epsilon(n)} = p_n \leq (n(\ln n + \ln_2 n - 1)) + \frac{n}{\epsilon(n)}$$

$$p_n \leq (n(\ln n + \ln_2 n - 1)) + \frac{n}{\epsilon(n)} = p_n \leq (n(\ln n + \ln_2 n - 1 + \frac{1}{\epsilon(n)}))$$

Because:

$$\lim_{n \rightarrow +\infty} \frac{n}{p_n - n(\ln n + \ln_2 n - 1)} = +\infty$$

The n^{th} prime number is smaller than $n(\ln n + \ln_2 n - 1 + \frac{1}{\epsilon(n)})$ $\lim_{n \rightarrow +\infty} \epsilon(n) = +\infty$

Example. The function in [2] and previously described. We have $\epsilon(n) = \frac{\ln n}{\ln_2 n - 2}$ because

$$\frac{n}{p_n - (n(\ln n + \ln_2 n - 1))} \geq \frac{\ln n}{\ln_2 n - 2}$$

Let ϕ be a function defined by $\phi(n)$, n is a natural number and $n \geq 688383$

If:

$$p_n \leq \phi(n) \quad \text{for } n \geq 688383$$

We have:

$$\frac{n}{p_n - (n(\ln n + \ln_2 n - 1))} \geq \frac{n}{\phi(n) - (n(\ln n + \ln_2 n - 1))}$$

And:

$$\epsilon(n) = \frac{n}{\phi(n) - (n(\ln n + \ln_2 n - 1))}$$

Consequently:

$$p_n \leq n(\ln n + \ln_2 n - 1 + \frac{\phi(n) - (n(\ln n + \ln_2 n - 1))}{n}) \quad \lim_{n \rightarrow +\infty} \frac{\phi(n) - (n(\ln n + \ln_2 n - 1))}{n} = 0$$

References

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