



HAL
open science

Modeling stochastic dynamics of agents with multi-leveled opinions and binary actions

Vineeth Varma, Irinel-Constantin Morarescu

► **To cite this version:**

Vineeth Varma, Irinel-Constantin Morarescu. Modeling stochastic dynamics of agents with multi-leveled opinions and binary actions. IEEE Conference on Decision and Control 2017, IEEE, Dec 2017, Melbourne, Australia. hal-01561708

HAL Id: hal-01561708

<https://hal.science/hal-01561708>

Submitted on 20 Jul 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Modeling Stochastic Dynamics of Agents with Multi-leveled Opinions and Binary Actions

Vineeth S Varma and Irinel-Constantin Morărescu

Abstract—A stochastic multi-agent opinion dynamics model is proposed and analyzed, in which, the multi-leveled opinion of each agent is influenced by a random neighbor’s binary action, determined by its opinion level. This model is shown to asymptotically produce consensus with a finite number of connected agents. On the other hand, when the number of agents is large, the time to achieve consensus can become exponentially high, potentially resulting in other equilibrium points based on the structure of the connectivity graph. Numerical simulations validate the proposed analysis, and demonstrate these other equilibrium points, that exist when a large number of agents are present.

Index Terms—Opinion dynamics, Markov chains, agent based models

I. INTRODUCTION

Opinion dynamics in social networks is a challenging problem that has received a lot of interest in the last decades. A major issue is related to the development of realistic models that can capture the behavior of real large scale social network [1]. There are mainly two classes of models: those considering that opinions can evolve in a discrete set and those considering a continuous set of values that can be taken by each agent. The models in the first class come from statistical physics, and the most employed are the Ising [2], voter [3] and Sznajd [4] models. When the opinions are not constrained to a discrete set, we can find in the literature two popular models: the Deffuant [5] and the Hegselmann-Krause [6] models. The last ones are often identified in the literature as bounded confidence models and the main difference between them is that the former suppose that each agent updates its opinion by using only the opinion of a random neighbor while the later assume that the update is done by using the opinion of all the neighbors. It is worth noting here that Hegselmann-Krause model was adapted in [7] to a model of opinion dynamics with decaying confidence.

In order to more accurately describe the opinion dynamics and to model more realistic behaviors, a mix of continuous opinion with discrete actions (CODA) was proposed in [8]. This model reflects the fact that even if we often face binary choices or actions, the opinion behind evolves in a continuous space of values. A multi-agent system that approach CODA model was proposed and analyzed in [9]. It was shown that this deterministic model leads to a variety of asymptotic behaviors including consensus.

Starting from the modeling in [9], we propose in this paper a stochastic model with discrete opinion and binary actions. In other words we assume that opinions can be approximated by a given number of levels between within $(0, 1)$ while the actions can be only 0 or 1 (see Section II for more details). As in [9] we consider that extremal opinions (close to 0 or 1) present more inertia. This corresponds to the fact that people that strongly believe in one action 0 or 1 are more difficult to influence. In our opinion, considering stochastic behavior, instead of deterministic, results in a more realistic model.

Many studies in the literature focus on the emergence of consensus in social networks [10], [11], [5], [12]. Our goal here is to analyze the asymptotic behavior of the opinions in the network under the proposed stochastic dynamics. Nevertheless, one main result of the paper states that as long as the network is connected, the opinions will asymptotically reach consensus. Moreover, the consensus value is close to one of the extreme opinions 0 or 1.

In this paper we basically reformulate the model from [9] as an interactive Markov chain. Similar approaches for the Deffuant and Hegselmann-Krause models have been considered in literature (see for instance [13], [14]). One advantage of this approach is that, it also allows analysis of the behavior of infinite populations partitioned into a certain number of opinion classes. Even in the Markov chains approach, the novelty of our paper comes from the fact that instead of assuming that agents access the opinion of the neighbors, we only suppose that the quantized and binary version of the opinion (called action) is accessible.

The contribution of this work can be briefly summarized as follows. Firstly we propose and analyze the asymptotic behavior of a novel stochastic model for opinion dynamics based on binary actions. Secondly we analyze some particular cases of networks when the number of agents is assumed to be infinite.

The rest of the paper is organized as follows. Section II introduces the main notation and concepts necessary for the model description. The analysis of the asymptotic behavior of opinions described by this stochastic model is provided in Section III. The results are valid for generic connected networks with finite number of agents. Next, in Section IV, we analyze particular networks with infinite number of agents. The results of our numerical studies and simulations are illustrated in Section V. The paper ends with some concluding remarks and perspectives.

V. S. Varma and I.-C. Morărescu are with the Université de Lorraine, CRAN, UMR 7039 and the CNRS, CRAN, UMR 7039, France, vineeth.satheeskumar-varma@univ-lorraine.fr, constantin.morarescu@univ-lorraine.fr

II. MODEL

Denote the opinion of agent i as $x_i(t) \in \Theta$ for all $i \in \mathcal{K}$, where $\mathcal{K} = \{1, 2, \dots, K\}$ and $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$ is the discrete set of possible opinions, such that $\theta_n \in (0, 1) \setminus \{0.5\}$ and $\theta_n < \theta_{n+1}$ for all $n \in \{1, 2, \dots, N\}$, where N is positive and *even* and Θ is constructed such that $\theta_{N/2} < 0.5$ and $\theta_{N/2+1} > 0.5$. Let us introduce some graph notions allowing us to define the interaction structure in the social network under consideration.

Definition 1 (Directed graph): A directed graph G is a couple $(\mathcal{K}, \mathcal{E})$ with \mathcal{K} being a finite set denoting the vertices and $\mathcal{E} \subseteq \mathcal{K} \times \mathcal{K}$ the set of directed edges. To each edge $(i, j) \in \mathcal{E}$ we assign a value $\rho_{i,j} \geq 0$ representing the weight/trust that agent j gives to agent i . We say that agent j is a neighbor of agent i if $\rho_{i,j} > 0$.

Agent i is said to be *connected* with agent j if G contains a directed path from i to j , i.e. if there exists at least one sequence $(i = i_1, i_2, \dots, i_{p+1} = j)$ such that $\rho_{k,k+1} > 0$, $\forall k \in \{1, 2, \dots, p\}$.

Definition 2 (Strongly connected): The graph G is strongly connected if any two distinct agents $i, j \in \mathcal{K}$ are connected.

In the sequel we suppose the following holds true.

Assumption 1: The graph $(\mathcal{K}, \mathcal{E})$ modeling the interaction in the network is strongly connected.

The action taken by agent $x_i(t)$ is given by

$$q_i(t) = \lfloor x_i(t) \rfloor,$$

where $\lfloor x_i(t) \rfloor$ is the nearest integer function. This means that if an agent has an opinion more than 0.5, it will take the action 1 and 0 otherwise. This kind of opinion quantization is suitable for many practical applications, for example, an agent may support the left or right political party, with various opinion levels (opinions close to 0 or 1 represents stronger opinion), however in the election, the agent can vote with exactly two choices (left or right).

A. Population shares

Let the total population share of users with opinion θ_n be denoted by $p_n(t)$ such that $\sum_n p_n(t) = 1$, calculated as

$$p_n(t) = \frac{\sum_{i=1}^K \delta_{x_i(t), \theta_n}}{K} \quad (1)$$

where δ is the Kronecker delta function. Define $p^-(t)$ as the population share corresponding to the *action* 0, and $p^+(t)$ as the population share corresponding to the *action* 1, calculated as

$$p^-(t) = \frac{\sum_{i=1}^K \delta_{q_i(t), 0}}{K} = \sum_{n=1}^{N/2} p_n(t) \quad (2)$$

and

$$p^+(t) = \frac{\sum_{i=1}^K \delta_{q_i(t), 1}}{K} = \sum_{n=N/2+1}^N p_n(t) \quad (3)$$

B. Opinion dynamics

In this work, we look at the evolution of opinions of the agents based on their mutual influence. We also account for the *inertia* of opinion, i.e., when the opinion of the agent is closer to 0.5, he is more likely to shift as he is less decisive, whereas someone with a strong opinion (close to 1 or 0) is less likely to shift his opinion as he is more convinced by his opinion. The opinion of agent j may shift towards the action of agent i based on the *inertia function* $f(x_j)$ where $f: [0, 1] \rightarrow [0, 1]$, for example $f(x) = 4x(1-x)$. That is, an agent with an opinion close to 0 or 1 is more likely to disregard the opinion of the neighbors and keep its opinion unchanged, while an agent with an opinion close to 0.5 is highly likely to change its opinion, this behavior is modeled by the function f .¹

At each time instant t , agent j is influenced by the action of one random neighboring agent. We introduce

$$\pi_{i,j} = \frac{\rho_{i,j}}{\sum_{k=1}^K \rho_{k,j}}, \forall i, j \in \mathcal{K}.$$

such that, the probability of agent j being influenced by the action of agent i is given by $f(x_j(t))\pi_{i,j}$. We denote by $X_{-j}(t) = (x_1(t), \dots, x_{j-1}(t), x_{j+1}(t), \dots, x_K(t))$ the vector of all opinions except that of j . Additionally, by $\lfloor X_{-j}(t) \rfloor$ we denote the vector of actions of all agents except that of j . Notice that $\lfloor X_{-j}(t) \rfloor$ takes a value $\bar{a}_{-j} = (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_K) \in \{0, 1\}^{K-1}$.

This results in a stochastic opinion dynamics for agent j given by:

$$\Pr(x_j(t+1) = \theta_m | x_j(t) = \theta_n, \lfloor X_{-j}(t) \rfloor = \bar{a}_{-j}) = \begin{cases} 0 & \text{if } |m-n| > 1 \\ f(\theta_n) \sum_{k \in \mathcal{K} \setminus \{j\}} \pi_{k,j} (1 - a_k) & \text{if } m = n-1 \\ f(\theta_n) \sum_{k \in \mathcal{K} \setminus \{j\}} \pi_{k,j} a_k & \text{if } m = n+1 \\ 1 - f(\theta_n) & \text{if } m = n \\ & \& 1 < m, n < N \\ 1 - f(\theta_n) \left[\sum_{k \in \mathcal{K} \setminus \{j\}} \pi_{k,j} a_k \right] & \text{if } m = n = 1 \\ 1 - f(\theta_n) \left[\sum_{k \in \mathcal{K} \setminus \{j\}} \pi_{k,j} (1 - a_k) \right] & \text{if } m = n = N \end{cases} \quad (4)$$

for all $m, n \in \{1, 2, \dots, N\}$ and for all $j \in \mathcal{K}$. The above equation can be explained by following the assumptions we have made on our opinion dynamics. The first line indicates that opinion levels can not jump by more than one level in a single time step. The second and third lines indicate the jump probabilities (to left and right), when the initial opinion of the agent is a non-extreme opinion. When the initial opinions are at the extreme right (θ_N) or left (θ_1), then these probabilities will be modified as the opinions can't jump any further to one side, therefore the modified jump probabilities are given in the final three lines.

¹Note that $\frac{1}{f(x_j)}$ would be the actual 'inertia' as a higher f implies a higher willingness to change its opinion.

III. ASYMPTOTIC BEHAVIOR OF OPINIONS IN FINITE NETWORKS

We denote the combined states of the overall network by α_m where $\alpha_m \in \Theta^K$ for all $m \in \{1, 2, \dots, N^K\}$. We order α_m such that $\alpha_1 = (\theta_1, \theta_1, \dots, \theta_1)$ and $\alpha_2 = (\theta_N, \theta_N, \dots, \theta_N)$. Now define a sequence of random variables $X(t) = (x_1(t), x_2(t), \dots, x_K(t))$, where the realizations of $X(t)$ belong to Θ^K , such that:

$$\Pr(X(t+1) = \alpha_m | X(t) = \alpha_n, X(t-1) = \alpha_{n'}, \dots) = \prod_{k=1}^K \Pr(x_k(t+1) = \theta_{m_k} | X(t) = \alpha_n) \quad (5)$$

which by definition, is a Markov process and represents the opinion dynamics of all the K agents. Here, the term $\Pr(x_k(t+1) = \theta_{m_k} | X(t) = \alpha_n)$ can be calculated based on (4) as

$$\Pr(x_k(t+1) = \theta_{m_k} | X(t) = \alpha_n) = \begin{cases} 0 & \text{if } |m_k - n_k| > 1 \\ f(\theta_{n_k}) Q_k^-(\alpha_n) & \text{if } m_k = n_k - 1 \\ f(\theta_{n_k}) Q_k^+(\alpha_n) & \text{if } m_k = n_k + 1 \\ 1 - f(\theta_{n_k}) & \text{if } m = n \in (1, N) \\ 1 - f(\theta_{n_k}) Q_k^+(\alpha_n) & \text{if } m_k = n_k = 1 \\ 1 - f(\theta_{n_k}) Q_k^-(\alpha_n) & \text{if } m_k = n_k = N \end{cases} \quad (6)$$

where $Q_k^-(\alpha_n) = \sum_{j=1}^K \pi_{j,k} (1 - [\theta_{n_j}])$ and $Q_k^+(\alpha_n) = \sum_{j=1}^K \pi_{j,k} [\theta_{n_j}]$.

Let $\underline{\xi}(t)$ be the probability distribution of the system over the states $(\alpha_1, \alpha_2, \dots)$ at time t , i.e., $\xi_m(t) = \Pr(X(t) = \alpha_m)$. The Markov process in matrix form can be written as

$$\underline{\xi}(t+1) = M \underline{\xi}(t) \quad (7)$$

where the elements of the *transition matrix* M are

$$M_{m,n} = \Pr(X(t+1) = \alpha_m | X(t) = \alpha_n)$$

and are obtained from (5) and (6).

Definition 3 (Absorbing state): A state α of a Markov process is called an absorbing state, if and only if

$$\Pr(X(t+1) = \alpha | X(t) = \alpha) = 1.$$

Proposition 1: Under Assumption 1, the Markov process $X(t)$ defined in (5) has exactly two absorbing states $\alpha_1 = (\theta_1, \theta_1, \dots, \theta_1)$ and $\alpha_2 = (\theta_N, \theta_N, \dots, \theta_N)$ if the graph G is strongly connected.

Proof in Appendix A.

Based on the results in [15], we could also compute the probability of reaching absorbing states α_1 or α_2 by expanding M and rewriting (7) as

$$\underline{\xi}(t+1) = \begin{bmatrix} I & 0 \\ R_1 & R_2 \end{bmatrix} \underline{\xi}(t) \quad (8)$$

where I is the 2×2 identity matrix, and R_1 and R_2 are the remaining elements of M (M takes this form as α_1 and α_2 are absorbing states).

Corollary 1: Under Assumption 1, the Markov process $X(t)$ defined in (5) will asymptotically almost surely reach one of the two defined absorbing states irrespective of the starting state. The probability to reach a specific absorbing state can be calculated as $B = (I' - R_1)^{-1} R_2$, where I' is the $(N^K - 2) \times (N^K - 2)$ identity matrix, $B_{m,n}$, $m \in \{1, 2\}$, $n \in \{1, 2, \dots, N^K - 2\}$ is the probability to reach the absorbing state α_m from a non-absorbing state α_n . This result directly follows from [15].

IV. BEHAVIOR OF OPINIONS IN LARGE-SCALE NETWORKS

Many real social networks contain large number of individuals/agents. In this case we can approximate the finite horizon behavior of the agents by the asymptotic behavior of a network with an infinite number of individuals. Indeed the convergence in such networks may be very slow and the transient behavior can be studied by analyzing the infinite population networks (see [13], [14]). Therefore, the goal of this section is to provide an approximation of the transient behavior of large-scale networks. Unlike the previous section, we can analytically study only some particular network structures, such as all-to-all communications and clustered communications. Beside the two absorbing state emphasized before, in large networks, we can also reach other *population equilibrium points* as defined below, even if the individual opinion of agents vary in time.

Definition 4 (Equilibrium in population): When $p_n(t') = p_n(t)$ for all $t' > t$ and all $n \in \{1, 2, \dots, N\}$, then the system is at equilibrium in population.

We start by analyzing an all-to-all connected network. Although many real networks are not actually all-to-all connected, real networks often have communities or clusters which are locally all-to-all connected, and these clusters can be studied with the machinery we develop in the following section.

A. All-to-all identically connected network

Consider an all-to-all connected graph, i.e. $\rho_{i,j} = 1$ for all i, j . As every agent has identical interactions with the rest, we can do a large number or mean field approximation (using the central limit theorem) to study the dynamics of the population shares. For finite K , the number of agents that migrate from opinion θ_m at time t to opinion θ_n at time $t+1$ is a stochastic variable and can be expressed as the sum of $K p_m(t)$ independent and identically distributed² random variables (recall that $p_m(t)$ is the population share of agents with opinion θ_m). However, as $K \rightarrow \infty$, the total number of agents migrating converges to a deterministic value (the expectation) if we apply the central limit theorem. Thus, we transform the agent dynamics given in (6), into a

²Each agent has the same distribution due to the symmetric structure of the graph.

deterministic population dynamics given by

$$\begin{aligned}
p_1(t+1) &= p_1(t) - f(\theta_1)p_1(t) \left(\sum_{n=N/2+1}^N p_n(t) \right) + \\
&\quad p_2(t)f(\theta_2) \left(\sum_{n=1}^{N/2} p_n(t) \right) \\
p_m(t+1) &= (1 - f(\theta_m))p_m(t) + \\
&\quad f(\theta_{m-1})p_{m-1}(t) \left(\sum_{n=N/2+1}^N p_n(t) \right) + \\
&\quad f(\theta_{m+1})p_{m+1}(t) \left(\sum_{n=1}^{N/2} p_n(t) \right) \\
p_N(t+1) &= p_N(t) - f(\theta_N)p_N(t) \left(\sum_{n=1}^{N/2} p_n(t) \right) + \\
&\quad p_{N-1}(t)f(\theta_{N-1}) \left(\sum_{n=N/2+1}^N p_n(t) \right)
\end{aligned} \tag{9}$$

where the second equation holds for all $m \in \{2, 3, \dots, N-1\}$. With this, we have transformed a stochastic dynamic system with an infinitely large number of agents into a deterministic quadratic dynamical system. Define the population share vector $\underline{p}(t) = (p_1(t), \dots, p_N(t))^T$ (where T is the transpose) and define the matrix $\mathbf{P}(t)$ as

$$\begin{bmatrix}
1 - f(\theta_1)p^+(t) & f(\theta_2)p^-(t) & 0 & \dots & 0 \\
f(\theta_1)p^+(t) & 1 - f(\theta_2) & \dots & \dots & 0 \\
0 & f(\theta_2)p^+(t) & \dots & \dots & 0 \\
\dots & & & & \\
\dots & \dots & \dots & \dots & f(\theta_N)p^-(t) \\
0 & \dots & \dots & \dots & 1 - f(\theta_N)p^-(t)
\end{bmatrix}$$

Then, we can rewrite (9) in matrix form as

$$\underline{p}(t+1) = \mathbf{P}(t)\underline{p}(t) \tag{10}$$

Since $\mathbf{P}(t)$ is time dependent only through $\underline{p}(t)$, at steady state, when $\underline{p}(t+1) = \underline{p}(t)$, we have $\mathbf{P}(t+1) = \mathbf{P}(t)$ and so $\underline{p}(t)$ will be an eigen-vector of $\mathbf{P}(t)$ at the steady state.

Theorem 1: When $\rho_{i,j} = 1 \forall i, j \in \mathcal{K}$ and $K \rightarrow \infty$, if at some t , $p^+(t) = 1$ (or 0), i.e., all the agents have the same action 1 or 0, then $p^+(t') = 1$ (or 0) for all $t' > t$ and asymptotically $p_N(t') = 1$ (or $p_1(t') = 1$) as $t' \rightarrow \infty$ which corresponds to the two stable equilibria of the system. When $N > 2$, the system also allows for one unstable equilibrium point at $p^+(t) = 0.5$ with the distribution following

$$p_n f(\theta_n) = p_m f(\theta_m), \forall m, n \in \{1, 2, \dots, N\}.$$

See Appendix B.

Special case of $N = 2$: We observe that when $N = 2$, i.e., when there are exactly two opinion levels, equation (9) can be simplified into

$$\begin{aligned}
p_1(t+1) &= p_1(t) - f(\theta_1)p_1(t)p_2(t) + \\
&\quad p_2(t)f(\theta_2)p_1(t) \\
p_2(t+1) &= p_2(t) - f(\theta_2)p_2(t)p_1(t) + \\
&\quad p_1(t)f(\theta_1)p_2(t)
\end{aligned} \tag{11}$$

As $f(\theta)$ is symmetric by definition, we get $p_1(t+1) = p(t)$ and $p_2(t+1) = p(t)$. Therefore, when there are a very large number of agents and exactly two opinion levels, then any population distribution becomes an equilibrium.

B. Networks partitioned into clusters

In this section, we introduce the concept of cluster/community that basically illustrates that in a social networks we may have groups of agents that strongly interact with each other and interact less with agents outside the group.

Definition 5 (Homogeneous cluster): A set of agents C defines a homogeneous cluster when the following properties are satisfied:

- 1) $C \subset \mathcal{K}$ and $\rho_{i,j} = 1$ for all $i, j \in C$.
- 2) All agents within the cluster are homogeneously influenced by agents outside the cluster, i.e. $\rho_{i,j} = \rho_{i,k}, \forall i \in \mathcal{K} \setminus C, j, k \in C$.
- 3) Denote by $\lambda \in [0, 1]$, the net influence ratio on an agent inside C by other agents in C , where λ is given by:

$$\lambda = \frac{\sum_{i \in C} \rho_{i,j}}{\sum_{k \in \mathcal{K}} \rho_{k,j}} \quad \forall j \in C \tag{12}$$

which is identical for any j due to (2). Then, C defines a cluster if and only if $\lambda > 0.5$.

With this definition, if $|C| \rightarrow \infty$, denote by ϕ_n the population fraction of agents inside C sharing the opinion θ_n . We define by $\phi^{\rightarrow 1}(t) \in [0, 1]$ the effective influence on any agent in C to shift their opinions to 1, i.e.,

$$\phi^{\rightarrow 1}(t) = \sum_{k \in \mathcal{K} \setminus \{j\}} \pi_{k,j} q_k(t) \quad j \in C$$

which is identical for any j due to (2) in Definition 5. Similarly $\phi^{\rightarrow 0} \in [0, 1]$ is the effective influence on agents in C to shift their opinion closer to 0, which is given by

$$\phi^{\rightarrow 0}(t) = \sum_{k \in \mathcal{K} \setminus \{j\}} \pi_{k,j} (1 - q_k(t)) \quad j \in C$$

which results in $\phi^{\rightarrow 1}(t) + \phi^{\rightarrow 0}(t) = 1$. In the sequel, we keep the time dependence implicit and we will simply use $\phi^{\rightarrow 0}$ or $\phi^{\rightarrow 1}$ instead of $\phi^{\rightarrow 0}(t)$ or $\phi^{\rightarrow 1}(t)$. This notation can be exploited to simplify the transition probabilities for opinions of agents in C to be

$$\begin{aligned}
\Pr(x_k(t+1) = \theta_{m_k} | x_k(t) = \theta_{n_k}, [X_{-j}(t)] = \bar{a}_{-j}) = \\
\begin{cases}
0 & \text{if } |m_k - n_k| > 1 \\
f(\theta_{n_k})\phi^{\rightarrow 0} & \text{if } m_k = n_k - 1 \\
f(\theta_{n_k})\phi^{\rightarrow 1} & \text{if } m_k = n_k + 1 \\
1 - f(\theta_{n_k}) & \text{if } m = n \text{ \& } 1 < m < N \\
1 - f(\theta_{n_k})\phi^{\rightarrow 1} & \text{if } m_k = n_k = 1 \\
1 - f(\theta_{n_k})\phi^{\rightarrow 0} & \text{if } m_k = n_k = N
\end{cases}
\end{aligned} \tag{13}$$

for all $k \in C$, with $\phi^{\rightarrow 1}, \phi^{\rightarrow 0}$ evaluated with $q_k(t) = a_k$. Define by $\underline{\phi}(t) = (\phi_1(t), \phi_2(t), \dots, \phi_N(t))^T$, and

$$\Phi(\phi^{\rightarrow 1}) := \begin{bmatrix}
1 - f(\theta_1)\phi^{\rightarrow 1} & f(\theta_2)\phi^{\rightarrow 0} & 0 & \dots \\
f(\theta_1)\phi^{\rightarrow 1} & 1 - f(\theta_2) & f(\theta_3)\phi^{\rightarrow 0} & \dots \\
\vdots & & &
\end{bmatrix} \tag{14}$$

where $\phi^{\rightarrow 0} = 1 - \phi^{\rightarrow 1}$ and Φ is similar in structure to \mathbf{P} . The dynamics of population with common opinions in the cluster can be expressed as

$$\underline{\phi}(t+1) = \Phi(\phi^{\rightarrow 1})\underline{\phi}(t)$$

in matrix form or as quadratic dynamics:

$$\begin{aligned} \phi_1(t+1) &= \phi_1(t) - f(\theta_1)\phi_1(t)\phi^{\rightarrow 1} + \phi_2(t)f(\theta_2)\phi^{\rightarrow 0} \\ \phi_m(t+1) &= (1 - f(\theta_m))\phi_m(t) + \\ &\quad f(\theta_{m-1})\phi_{m-1}(t)\phi^{\rightarrow 1} + f(\theta_{m+1})\phi_{m+1}(t)\phi^{\rightarrow 0} \\ \phi_N(t+1) &= \phi_N(t) - f(\theta_N)\phi_N(t)\phi^{\rightarrow 0} + \\ &\quad \phi_{N-1}(t)f(\theta_{N-1})\phi^{\rightarrow 1} \end{aligned} \quad (15)$$

This allows us to study the behavior of agents in a well-connected cluster. In particular, when $\lambda \rightarrow 1$, we can show that

Proposition 2 (Preservation with $\lambda \rightarrow 1$): If C is a cluster as per Definition 5, with an interconnection coefficient $\lambda \rightarrow 1$ and $|C| \rightarrow \infty$, the population shares of agents in C denoted by $\underline{\phi}$ allows for three distinct equilibrium points irrespective of the actions of agents in $\mathcal{K} \setminus C$. Two of these are stable equilibrium points, and are given by $\underline{\phi} = (1, 0, \dots, 0)$ and $\underline{\phi} = (0, \dots, 0, 1)$.

Proof:

For any configuration of actions for agents in $\mathcal{K} \setminus C$, we can write

$$\phi^{\rightarrow 1} = (1 - \lambda)z^+(t) + \lambda\phi^+(t)$$

and

$$\phi^{\rightarrow 0} = (1 - \lambda)z^-(t) + \lambda\phi^-(t)$$

where $\phi^+(t) = \sum_{n=N/2+1}^N \phi_n(t)$, $\phi^-(t) = \sum_{n=1}^{N/2} \phi_n(t)$ and $z^+(t), z^-(t) \in [0, 1]$ denote the perturbation caused by external agents outside the cluster. This follows from the property of λ in Definition 5.

Now, the result of the proposition directly follows by applying the proof of theorem 1 since, if $\lambda \rightarrow 1$,

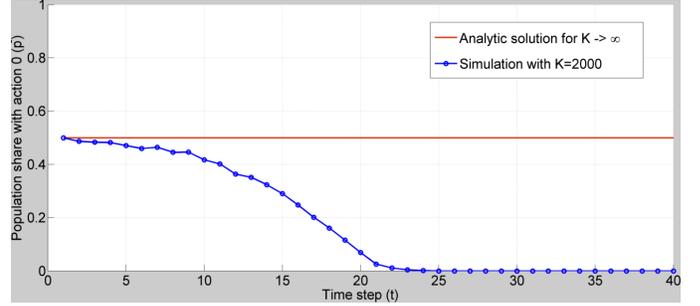
$$\lim_{\lambda \rightarrow 1} \phi^{\rightarrow 1} = \phi^+(t)$$

Therefore, the opinion dynamics of agents in a "strongly connected cluster" ($\lambda \rightarrow 1$) are identical to that of agents in an all-to-all connected network. ■

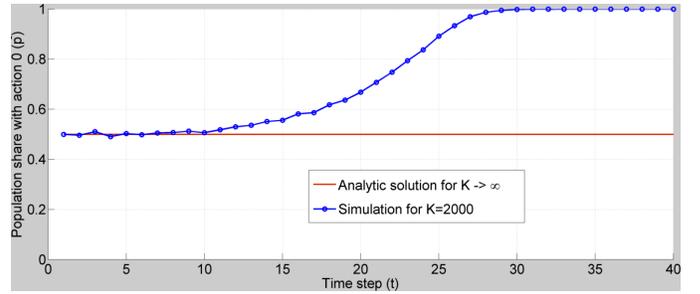
This result implies that regardless of external opinion, agents inside C will have their own equilibria, and if all agents in C start with action 1, they will all continue to have action 1 regardless of the opinions of external agents, i.e., the agents inside the cluster *preserve* their opinion.

Practically, $\lambda \rightarrow 1$ implies that the number of connections from $\mathcal{K} \setminus C$ to C is finite, while $|C|$ and K are arbitrarily large and all agents inside C are connected to each other. However, if $|\mathcal{K}|, |C|$ are finite, but large, and $\lambda \approx 1$, then $\underline{\phi} \approx (1, 0, \dots, 0)$ is still an equilibrium. This is stable when $|C| \rightarrow \infty$. For $|C|$ finite, the only absorbing state of the finite Markov chain is with $\phi_N = 1$ (all agents in \mathcal{K} have identical opinion, not just agents in C). The equilibrium point with $\phi_1 \approx 1$ is referred to as a *metastable state*, and this kind of phenomenon is often seen in epidemic dynamics [16]. The metastable persists for an arbitrarily long duration

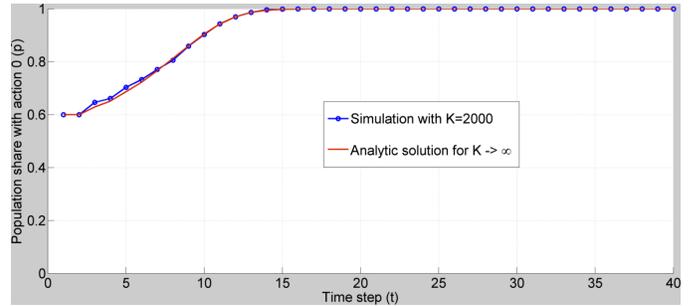
with an arbitrarily large number of agents. In the finite (but large) number of agents case, the study of metastable states is still highly relevant, as this can approximate the transient behavior well, and we study this behavior in our simulations.



(a) Symmetric starting conditions with $p_1(1) = 0.5$ and $p_4(1) = 0.5$. The simulation with finite agents breaks away from the unstable equilibrium for the infinite K case and achieves one of the two stable equilibria.



(b) Symmetric starting conditions with $p_1(1) = 0.5$ and $p_4(1) = 0.5$. Here, the simulation achieves the other stable equilibrium.



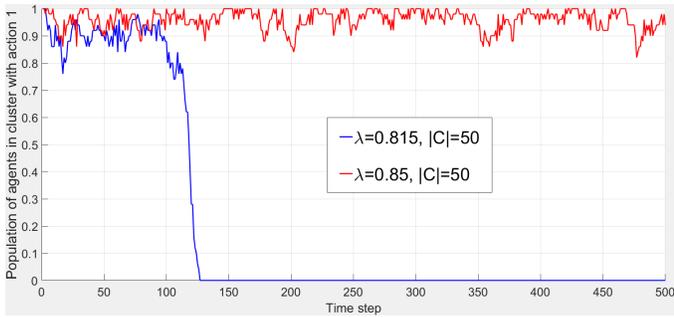
(c) Asymmetric starting conditions with $p_1(1) = 0.6$ and $p_4(1) = 0.4$, here the simulation and the analytic prediction are quite close and both lead the same stable equilibrium.

Fig. 1: Population share with the action 0, i.e., $p^-(t)$ plotted against t for a graph with all to all identical connections with $\rho_{i,j} = 1 \forall i, j \in \mathcal{K}$.

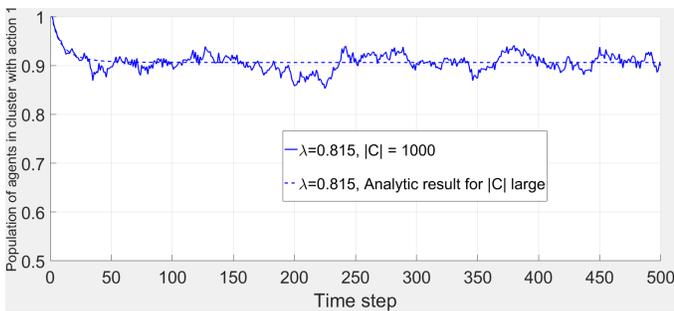
V. NUMERICAL RESULTS

For our simulations we take $\Theta = \{0.2, 0.4, 0.6, 0.8\}$ and $f(\theta) = 4\theta(1 - \theta)$.

In Fig. 1, we consider an all-to-all connected network, with the connections being of same degree of influence $\rho_{i,j} = 1 \forall i, j \in \mathcal{K}$. Therefore, Fig. 1 represents the case as treated in Section IV-A. We set the initial configuration of the network to be of two opinion type sets, one set with $x_i(1) = 0.2$



(a) Population of agents with action 1 in a cluster with 50 agents.



(b) Population of agents with action 1 in a cluster with 1000 agents.

Fig. 2: In both figures, there are 1000 agents outside the cluster with $x_i(t) = 0.2$ for all $i \in \mathcal{K} \setminus C$ at all t , i.e., these agents always pull the agents in the cluster to action 0. With a large λ , even a small cluster can preserve its opinion for a large period of time, but with a large cluster, preservation of opinion can hold even for smaller, but sufficiently high λ regardless of external opinion.

and the second set with opinion $x_i(1) = 0.8$. The initial population of each type can be inferred based on the figure. Fig. 1a and 1b shows that even for large K , the system reaches consensus at either $q = 0$ or $q = 1$ for all agents. On the other hand, if we use the large K approximation as in Section IV, we predict the system to converge to an unstable equilibrium. As the simulations are always with finite K (in this case 2000), the approximation of large K causes perturbations (from the randomness) that result in the system collapsing to global consensus. (Note that when $p^+ = 0.5$, there is no global consensus.) Note that when we start with asymmetric initial conditions as in Fig. 1c, the simulation results for $K = 2000$ and the analytic results for $K \rightarrow \infty$ match.

For the next simulation (Fig. 2), we look at the case where the agents are divided into a clusters C and many external agents with a weaker influence on the cluster, specified by λ . Here, we do simulations with $|C|=50$ and $|C|=1000$ with all agents in the cluster starting with opinion 0.2, and all other agents (i.e. agents in $\mathcal{K} \setminus C$) having opinion 0.8 for all time. Fig. 2a shows that for small $|C|$, the system quickly converges to an absorbing state when the cluster coefficient is not large enough. However, Fig. 2b shows that when a large number of agents are taken, we see that the system stays in the special meta-stable equilibrium as predicted using the

large number approximation in Section IV-B and does not end up converging to an absorbing state (in a reasonably large enough window of time) even for $\lambda = 0.815$. From Fig. 2 it can be seen that the random perturbations caused due to finite K does not destabilize the system and the system stays at this equilibrium even at $t = 500$.

Remark: Note that in Fig. 2, global consensus is not achieved when λ is sufficiently large as agents outside the cluster have a contradicting opinion. Even within the cluster, for $\lambda = 0.815$, about 90% of the agents in C have action 1 but the remaining 10% are with action 0.

Finally, we study the opinion dynamics when the graph has a very specific structure, i.e., a lattice graph. Agents are arranged into a 20×20 lattice, such that if i is at coordinates (m, n) in the lattice, where $m, n \in \{1, 2, \dots, 20\}$, its neighbors are the agents with the coordinates $(m+1, n)$, $(m-1, n)$, $(m, n+1)$ or $(m, n-1)$. The connection $\rho_{i,j} = 1$ if i and j are neighbors in the lattice, and $\rho_{i,j} = 0$ otherwise. For such a graph, we run simulations to observe the transient behavior, and this is shown in Fig. 3. Fig. 3a shows the initial opinion levels at each agent coordinate in the lattice, while Fig. 3b shows the opinions after a short amount of time has passed. Fig. 3c and 3d show the population shares for each opinion evolving in time (short term and long term respectively). We observe that although global consensus is eventually achieved, the structure of the graph results in a large time to achieve consensus.

VI. CONCLUSION

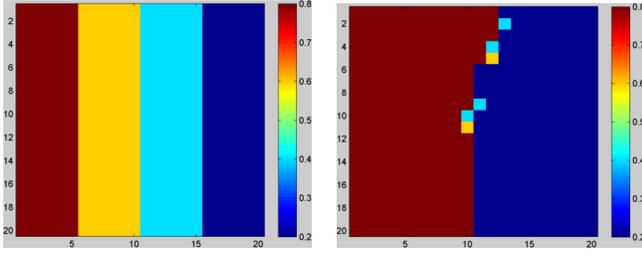
We have proposed and studied a stochastic multi-agent opinion dynamics model. This model features multi-leveled opinions of each agent that are influenced by the binary action of a random neighboring agent. We show that for a finite number of connected agents, global consensus is asymptotically achieved at either all agents with the lowest opinion θ_1 or at all agents with the highest opinion θ_N . On the other hand, when the number of agents is infinitely large, the time to achieve consensus may become exponentially high as illustrated in both our analytic results and numerical simulations with a large number of agents. Of particular interest is the case case of multiple clusters where we identify stable equilibrium points that are *not* at global consensus as seen in Fig 2b. As most social networks have very large number of agents, these results are very relevant. Although we observe that some of the equilibrium points are stable and unstable via numerical studies, proving such properties analytically is a relevant extension to this work.

APPENDIX

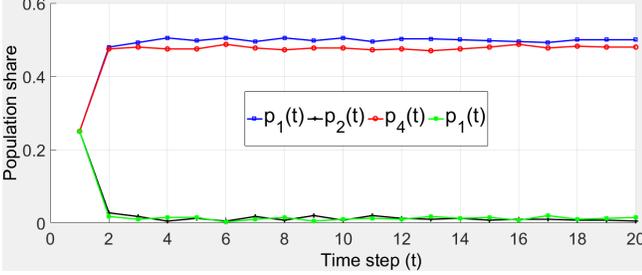
A. Proof of Proposition 1

Proof: We can verify that α_1 and α_2 are absorbing states by evaluating $\Pr(x_k(t+1) = \theta_{m_k} | X(t) = \alpha_n) = \alpha_n$ through equation (6). We know $Q_k^-(\alpha_1) = 1$, $Q_k^+(\alpha_1) = 0$, $Q_k^-(\alpha_2) = 0$ and $Q_k^+(\alpha_2) = 1$ for any k and for any graph G . This results in

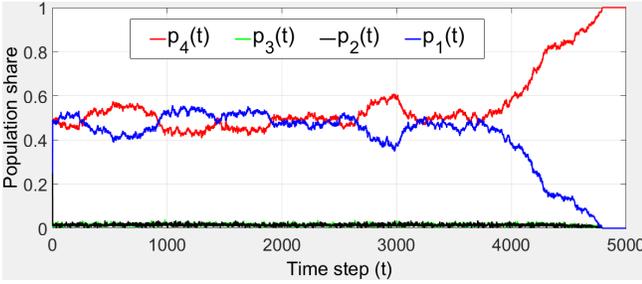
$$\Pr(x_k(t+1) = \theta_1 | X(t) = \alpha_1) = 1, \quad \forall k \in \mathcal{K}.$$



(a) Starting opinions of agents. (b) Transient opinions



(c) Population share evolution for each opinion (short term).



(d) Population share evolution for each opinion (long term).

Fig. 3: Simulations with a lattice structured graph.

Similarly

$$\Pr(x_k(t+1) = \theta_N | X(t) = \alpha_2) = 1, \quad \forall k \in \mathcal{K}.$$

Consequently, α_1, α_2 are absorbing states. Now take any state $\beta \neq \alpha_1, \alpha_2$, then;

Case 1: β is such that there exists at least one k for which $x_k(t) \neq \theta_1, \theta_N$. This agent by (6) has a non zero probability to shift its opinion (towards either θ_1 or θ_N) as the graph is connected resulting in either

$$0 < Q_k^+(\beta) < 1 \quad \text{or} \quad 0 < Q_k^-(\beta) < 1.$$

As agent k also has a non-zero probability to stay, β is therefore a non-absorbing state.

Case 2: Some of the agents have opinion θ_1 while the other agents have the opinion θ_N (any agent with a non-extreme opinion will result in Case 1). Now, since G is a strongly connected graph, there exists at least one agent with opinion θ_1 influenced by another agent with opinion θ_N resulting in a non-absorbing state.

Therefore there are exactly two absorbing states for the Markov process defined in (5) which are α_1 and α_2 . ■

B. Proof of Theorem 1

Proof: If $p^+(t) = 1$, it can be verified that

$$p_N(t+1) = p_N(t) + p_{N-1}(t)f(\theta_{N-1}),$$

which is an increasing function. Similarly, as $p^+(t) = 1$ one obtains that that $p_n(t) = 0, \forall n < N/2$. Moreover $p_n(t), \forall n < N/2$ are decreasing functions as

$$p_n(t+1) = (1 - f(\theta_n))p_n(t).$$

We can also see that $p_{N/2+1}(t')$ becomes a decreasing function when $p^+(t) = 1$ as

$$p_{N/2+1}(t+1) = (1 - f(\theta_m))p_{N/2+1}(t) + f(\theta_{m-1})p_{N/2}(t) \quad (16)$$

and $p_{N/2}(t) = 0$. Therefore, $p_{N/2+1}(t') \rightarrow 0$ and from this we can inductively show that $p_{N/2+m}(t')$ become decreasing and $p_{N/2+m}(t') \rightarrow 0$ for all $m < N/2$.

Finally, as $p_N(t')$ is the only function that indefinitely preserves its increasing property, we conclude that the population distribution will become $p_N(t') = 1$ as $t \rightarrow \infty$, and the system will asymptotically converge to it. Similarly, we can show that when $p^+(t) = 0$, the system maintains $p^+(t') = 0, \forall t' > t$ and asymptotically converges to $p_1(t') = 1$. These results are in fact the same absorbing states identified in Section III.

Next, assume that $p^+(t) = 0.5$ and $p_n f(\theta_n) = p_m f(\theta_m), \forall m, n \in \{1, 2, \dots, N\}$. Substituting these values into (9) we obtain that $p_n(t+1) = p_n(t), \forall n$, resulting in another equilibrium. This is an unstable equilibrium since the system does not converge to it while the state is in its neighborhood. This can be proven by a linear analysis of the dynamics around this equilibrium and studying the Jacobian matrix.

First, we evaluate the Jacobian of the discrete dynamical system (9). Denote by $g_i(\mathbf{p})$ the dynamics of the population with opinion θ_i , i.e. $p_i(t+1) = g_i(\mathbf{p})$. If we denote the Jacobian elements by $J_{i,j}$, where $J_{i,j} = \frac{\partial g_i}{\partial p_j}$, then for all $1 < i \leq \frac{N}{2}$, and for all $\frac{N}{2} < j < N$, we have:

$$\begin{aligned} J_{11} &= 1 + f(\theta_2)p_2 - f(\theta_1) \left(\sum_{n=N/2+1}^N p_n(t) \right) \\ J_{i,i} &= 1 - f(\theta_i) + f(\theta_{i+1})p_{i+1} \\ J_{j,j} &= 1 - f(\theta_j) + f(\theta_j)p_{j-1} \\ J_{NN} &= 1 + f(\theta_{N-1})p_{N-1} - f(\theta_N) \left(\sum_{n=1}^{N/2} p_n(t) \right) \end{aligned}$$

We also have ,

$$\forall 2 < i \leq N/2, \quad J_{1i} = -f(\theta_1)p_1$$

$$\forall N/2 < i \leq N-2, \quad J_{1i} = f(\theta_2)p_2$$

and

$$\forall 2 < i \leq N/2, \quad J_{Ni} = f(\theta_{N-1})p_{N-1}$$

$$\forall N/2 < i \leq N-2, \quad J_{Ni} = -f(\theta_N)p_N.$$

For all $i, j \in \{2, 3, \dots, N-1\}$ such that $|i-j| > 1$ we have

$$J_{ij} = f(\theta_{i+1})p_{i+1}$$

when $j \leq N/2$ and

$$J_{ij} = f(\theta_{i-1})p_{i-1}$$

when $j > N/2$. Finally, we have

$$J_{12} = f(\theta_2) \left(p_2 + \sum_{n=1}^{N/2} p_n(t) \right)$$

$$J_{N-1,N} = f(\theta_{N-1}) \left(p_{N-1} + \sum_{n=N/2+1}^N p_n(t) \right)$$

and for all $i, j \in \{2, 3, \dots, N-1\}$ such that $|i-j| = 1$ we have

$$J_{i,i+1} = f(\theta_k)p_k + f(\theta_{i+1}) \left(\sum_{n=1}^{N/2} p_n(t) \right)$$

where $k = i+1$ if $i+1 \leq N/2$ and $k = i-1$ otherwise; and

$$J_{i,i-1} = f(\theta_k)p_k + f(\theta_{i-1}) \left(\sum_{n=N/2+1}^N p_n(t) \right)$$

where $k = i+1$ if $i-1 \leq N/2$ and $k = i-1$ otherwise.

We can study the stability of an equilibrium point by looking at the eigenvalues of Jacobian matrix evaluated at the equilibrium points. The Jacobian matrix at the equilibrium $p_1 = 1, p_{n>1} = 0$ writes

$$\begin{bmatrix} 1 & f(\theta_2) & 0 & \dots \\ 0 & 1 - f(\theta_2) & f(\theta_3) & \dots \\ 0 & 0 & 1 - f(\theta_3) & \dots \\ \vdots & & & \end{bmatrix}$$

which is a triangular matrix. This matrix therefore has the eigenvalues $1, 1 - f(\theta_n)$ for all $n \in \{1, 2, \dots, N\}$ (assuming $f(\theta_n) = f(\theta_{N-n})$). This can be easily verified by evaluating the determinant of the $\mathbf{J}(1, 0, \dots) - \lambda \mathbf{I}$. Therefore this equilibrium is stable. By symmetry we can also show the same for the equilibrium with $p_N = 1$.

Now, we study the Jacobian at the other equilibrium at $p_n f(\theta_n) = p_m f(\theta_m)$ for all m, n . Let $p_n f(\theta_n) = K$. Then, the Jacobian at this point can be evaluated, and its first column is given as

$$\left(1 + K - \frac{f(\theta_1)}{2}, K + \frac{f(\theta_1)}{2}, K, \dots, K, -K \right)^T$$

The columns j for $2 \leq j \leq N/2$ are of the form

$$\left(K, \dots, 1 + K + \frac{f(\theta_j)}{2}, K - f(\theta_j), K + \frac{f(\theta_j)}{2}, \dots, K, -K \right)^T$$

where $1 + K + \frac{f(\theta_j)}{2}$ is the diagonal term of the Jacobian. The j -th column where $N/2 < j \leq N-1$ is of the form

$$\left(-K, K, \dots, 1 + K + \frac{f(\theta_j)}{2}, K - f(\theta_j), K + \frac{f(\theta_j)}{2}, \dots, K \right)^T$$

Finally, the N -th column is given by

$$\left(-K, K, \dots, K, K + \frac{f(\theta_1)}{2}, 1 + K - \frac{f(\theta_1)}{2} \right)^T$$

The above matrix is such that each column has exactly one element which is $-K$ either at the first row (after column index is more than $N/2$) or at the last row. We can show that one of the eigenvalues of the matrix above is $1 + K(N-2)$. Using the property that adding a scalar times a row to another row does not change the determinant, we replace the first row with the sum of all rows and this results in the first row becoming all zeroes. (sum of each column is 0). Therefore, we see that this point is unstable when $N > 2$. ■

REFERENCES

- [1] C. V. Kerckhove, S. Martin, P. Gend, P. J. Rentfrow, J. M. Hendrickx, and V. D. Blondel, "Modelling influence and opinion evolution in online collective behaviour," *PLoS ONE*, vol. 11, no. 6, pp. 1–25, 2016.
- [2] E. Ising, "Contribution to the theory of ferromagnetism," Ph.D. dissertation, University of Hamburg, 1924.
- [3] P. Clifford and A. Sudbury, "A model for spatial conflict," *Biometrika*, vol. 60, no. 3, pp. 581–588, 1973.
- [4] K. Sznajd-Weron and J. Sznajd, "Opinion evolution in closed community," *International Journal of Modern Physics C*, vol. 11, no. 6, pp. 1157–1165, 2000.
- [5] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, "Mixing beliefs among interacting agents," *Advances in Complex Systems*, vol. 3, pp. 87–98, 2000.
- [6] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis, and simulation," *Journal of Artificial Societies and Social Simulation*, vol. 5, no. 3, 2002.
- [7] I.-C. Morărescu and A. Girard, "Opinion dynamics with decaying confidence: Application to community detection in graphs," *IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1862 – 1873, 2011.
- [8] A. Martins, "Continuous opinions and discrete actions in opinion dynamics problems," *International Journal of Modern Physics C*, vol. 19, no. 4, pp. 617–625, 2008.
- [9] N. R. Chowdhury, I.-C. Morărescu, S. Martin, and S. Srikant, "Continuous opinions and discrete actions in social networks: a multi-agent system approach," in *Proceedings 55th IEEE Conference on Decision and Control*, 2016.
- [10] S. Galam and S. Moscovici, "Towards a theory of collective phenomena: Consensus and attitude changes in groups," *European Journal of Social Psychology*, vol. 21, no. 1, pp. 49–74, 1991.
- [11] R. Axelrod, "The dissemination of culture: A model with local convergence and global polarization," *The Journal of Conflict Resolution*, vol. 41, no. 2, pp. 203–226, 1997.
- [12] S. Fortunato, V. Latora, A. Pluchino, and A. Rapisarda, "Vector opinion dynamics in a bounded confidence consensus model," *International Journal of Modern Physics C*, vol. 16, no. 10, pp. 1535–1551, 2005.
- [13] B. S. R. Lima, and T. Araújo, "Agent based models and opinion dynamics as markov chains," *arXiv:1108.1716v2*, 2011.
- [14] J. Lorenz, "Consensus strikes back in the hegselmann-krause model of continuous opinion dynamics under bounded confidence," *Journal of Artificial Societies and Social Simulation*, vol. 9, no. 1, p. 8, 2006.
- [15] P. G. Doyle and J. L. Snell, *Random walks and electric networks*. Mathematical Association of America, 1984.
- [16] R. Pastor-Satorras and A. Vespignani, "Epidemic dynamics and endemic states in complex networks," *Physical Review E*, vol. 63, no. 6, p. 066117, 2001.