

# Computing a front-facing ellipse that subtends the same solid angle as an arbitrarily oriented ellipse

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## Abstract

A front-facing ellipse, with respect to an observation point, is an ellipse that is perpendicular to the line that goes through the origin and the center of the ellipse. Front-facing ellipses are more convenient than arbitrarily oriented ellipses for solid-angle computations.

In this report, we present an algebraic method for computing a front-facing ellipse that covers the same solid angle as an arbitrarily oriented ellipse. A previous approach was described by Conway [Con10] but it is involved, in terms of trigonometric derivations and computations. Our formulation shows that this problem is equivalent to computing the eigenvectors of a  $3 \times 3$  matrix, which is simpler to derive and compute.

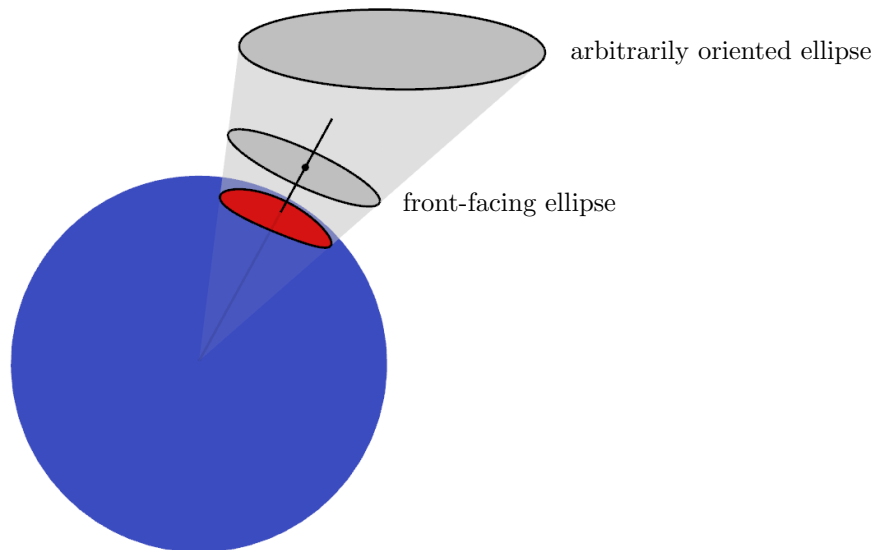


Figure 1: We compute a front-facing ellipse that covers the same solid angle as an arbitrarily oriented ellipse.

## References

- [Con10] John T. Conway. Analytical solution for the solid angle subtended at any point by an ellipse via a point source radiation vector potential. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, 614(1):17 – 27, 2010. ([document](#))

# 1 Arbitrarily oriented ellipse

We define the observation point as the origin:  $O = (0,0,0)$ . We consider an arbitrarily oriented ellipse of center  $C$  and of orthonormal axes of normalized directions  $V_x$  and  $V_y$  and of respective lengths  $l_x$  and  $l_y$ . The normal of the ellipse is  $V_z = V_x \times V_y$  such that  $(V_x, V_y, V_z)$  is an orthonormal basis. In this basis, the coordinates of the ellipse center are

$$x_c = C \cdot V_x, \quad (1)$$

$$y_c = C \cdot V_y, \quad (2)$$

$$z_c = C \cdot V_z. \quad (3)$$

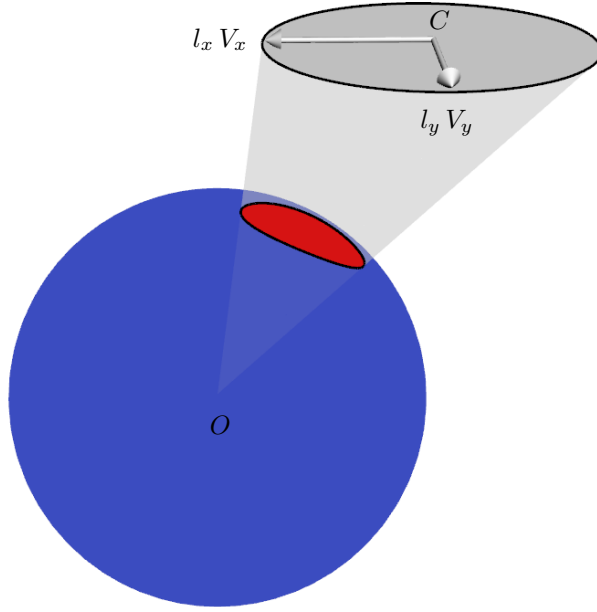


Figure 2: Arbitrarily oriented ellipse.

## 2 Planar Quadric

In the plane of the ellipse  $z = z_c$ , the equation of the ellipse is

$$\left(\frac{x - x_c}{l_x}\right)^2 + \left(\frac{y - y_c}{l_y}\right)^2 \leq 1. \quad (4)$$

## 3 Spherical Quadric

If  $(x, y, z)$  is a point on the sphere, the equation of the spherical quadric is

$$\left(\frac{\frac{x z_c}{z} - x_c}{l_x}\right)^2 + \left(\frac{\frac{y z_c}{z} - y_c}{l_y}\right)^2 \leq 1, \quad (5)$$

which can be rewritten

$$\begin{bmatrix} x & y & z \end{bmatrix} Q \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0, \quad (6)$$

with

$$Q = \begin{bmatrix} \frac{z_c^2}{l_x^2} & 0 & -\frac{z_c}{l_x^2} x_c \\ 0 & \frac{z_c^2}{l_y^2} & -\frac{z_c}{l_y^2} y_c \\ -\frac{z_c}{l_x^2} x_c & -\frac{z_c}{l_y^2} y_c & \frac{x_c^2}{l_x^2} + \frac{y_c^2}{l_y^2} - 1 \end{bmatrix}. \quad (7)$$

This quadratic equation defines an elliptic cone and the ellipse is a slice of this cone.

## 4 Eigendecomposition

Diagonalizing  $Q$  yields

$$Q = \begin{bmatrix} V_1^+ & V_2^+ & V^- \end{bmatrix} \begin{bmatrix} e_1^+ & 0 & 0 \\ 0 & e_2^+ & 0 \\ 0 & 0 & e^- \end{bmatrix} \begin{bmatrix} V_1^+ & V_2^+ & V^- \end{bmatrix}^T \quad (8)$$

where

- $V_1^+, V_2^+, V^-$  are the normalized eigenvectors,
- $e_1^+, e_2^+, e^-$  are the eigenvalues,
- $e_1^+$  and  $e_2^+$  are two positive eigenvalues,
- $e^-$  is the single negative eigenvalue.

## 5 Front-facing ellipses

A front-facing tangent ellipse with the same solid angle can be obtained from

- center  $\lambda V^-$  (or  $-\lambda V^-$  because the quadratic equation is symmetric), and
- main axes of normalized directions  $(V_1^+, V_2^+)$  and respective lengths  $\lambda \sqrt{\frac{-e^-}{e_1^+}}$  and  $\lambda \sqrt{\frac{-e^-}{e_2^+}}$ ,

where  $\lambda > 0$  is an arbitrary scaling factor.

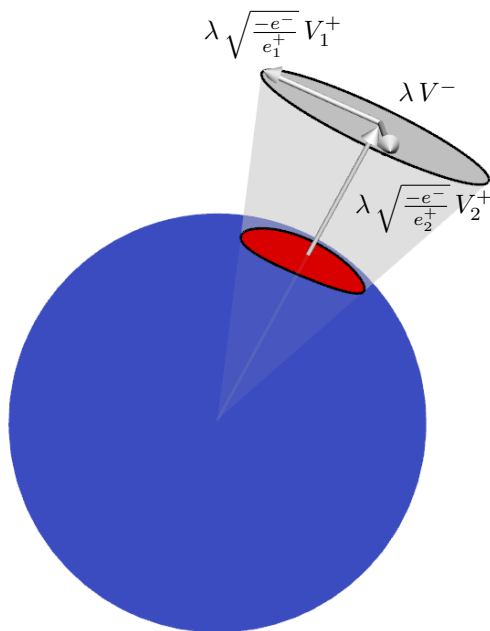


Figure 3: Front-facing ellipse.