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SPECTRAL STUDY OF THE TWO-DIMENSIONAL MIT BAG MODEL IN A SECTOR

LOÏC LE TREUST AND THOMAS OURMIÈRES-BONAFOS

ABSTRACT. This paper deals with the study of the two-dimensional Dirac operator with *infinite mass* boundary condition in a sector. We investigate the question of self-adjointness depending on the aperture of the sector: when the sector is convex it is self-adjoint on a usual Sobolev space whereas when the sector is non-convex it has a family of self-adjoint extensions parametrized by a complex number of the unit circle. As a byproduct of this analysis we are able to give self-adjointness results on polygones. We also discuss the question of *distinguished* self-adjoint extensions and study basic spectral properties of the operator in the sector.

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1. INTRODUCTION

1.1. Motivations and state of the art. In order to describe the confinement of quarks inside hadrons a phenomenological model was proposed by physicists in the MIT in the mid-70's (see [8, 7, 6, 14, 15]). This model, called the *MIT bag model*, involves a Hamiltonian given by the Dirac operator in a bounded domain of the Euclidean space \mathbb{R}^3 with suitable boundary conditions.

From a mathematical point of view, the first challenge studying these Hamiltonians is to prove their self-adjointness on a suitable functional space. Because the Dirac operator is an elliptic operator of order one, we expect this functional space to be contained into the usual Sobolev space H^1 and this question has been tackled for sufficiently smooth domains of \mathbb{R}^3 , for instance in [3, Thm. 4.11], or in [5] for C^∞ -smooth domains using pseudo-differential techniques and Calderón projectors (see also [19] for a direct application to the MIT bag model) and in [2] for $C^{2,1}$ domains constructing extension operators.

Our goal in this paper is to understand how the regularity of the domain plays a role in the question of self-adjointness. In a first attempt to handle this question, we focus on the influence of a corner for a two-dimensional problem set in a sector with the so-called *infinite mass* boundary conditions which can be seen as the two-dimensional analogue of the MIT bag boundary conditions.

Note that because of their importance in modelling low-energy excitations in graphenes, two-dimensional Dirac operators are also of interest. We refer to [4], for the study of their self-adjointness in C^2 domains of the Euclidean plane \mathbb{R}^2 for a large class of boundary conditions.

The techniques that we use to study the self-adjointness of the MIT bag Dirac operator is mainly inspired by the study of the 3-dimensional Dirac-Coulomb operator [22, Section 4.6] and of the Schrödinger operator with radial potential $-\Delta + V(r)$ in \mathbb{R}^n where $n \in \mathbb{N} \setminus \{0\}$ [20, Theorem X.11] (see also [11] and the references therein as well as [10] for the case of a radial δ -shell interaction). The main idea of these works amount to look at the restriction of the operator to stable subspaces obtained fixing the angular momentum of the wavefunctions. The restricted operators only act on the radial variable and their self-adjoint extensions can be studied using standard ODE techniques [23].

1.2. The MIT bag operator in a sector. Let $\omega \in (0, \pi)$. We study the Dirac operator with MIT bag boundary condition in the following two-dimensional sector

$$(1.1) \quad \Omega_\omega = \{r(\cos(\theta), \sin(\theta)) \in \mathbb{R}^2 : r > 0, |\theta| < \omega\}.$$

The MIT bag Dirac operator $(D, \mathcal{D}(D))$ of mass $m \in \mathbb{R}$ is defined as follows

$$(1.2) \quad \begin{aligned} \mathcal{D}(D) &= \{u \in H^1(\Omega_\omega; \mathbb{C}^2) : \mathcal{B}_n u = u \text{ on } \partial\Omega_\omega \setminus \{0\}\} \\ Du &= -i\sigma \cdot \nabla u + m\sigma_3 u, \text{ for all } u \in \mathcal{D}(D). \end{aligned}$$

The Pauli matrices $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are 2×2 hermitian matrices defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

which satisfy

$$(1.3) \quad (\sigma \cdot a)(\sigma \cdot b) = 1_2(a \cdot b) + i\sigma \cdot (a \times b)$$

for any $a, b \in \mathbb{R}^3$. Here, we denote

$$\sigma \cdot a = \sum_{k=1}^d \sigma_k a_k$$

for any $a \in \mathbb{R}^d$ with $d = 2, 3$.

For any $s \in \partial\Omega_\omega \setminus \{0\}$, $\mathbf{n}(s)$ is the outward-pointing normal to Ω_ω at the point s and the matrix $\mathcal{B}_\mathbf{v}$ is defined for any unit vector \mathbf{v} in \mathbb{R}^2 by

$$(1.4) \quad \mathcal{B}_\mathbf{v} = -i\sigma_3\sigma \cdot \mathbf{v}.$$

Let us remark that $\mathcal{B}_\mathbf{v}$ satisfies :

$$(1.5) \quad \mathcal{B}_\mathbf{v}^* = \mathcal{B}_\mathbf{v}, \quad \mathcal{B}_\mathbf{v}^2 = 1_2, \quad \text{Sp}(\mathcal{B}_\mathbf{v}) = \{\pm 1\}$$

where 1_2 denotes the identity matrix of $\mathbb{C}^{2 \times 2}$.

Remark 1.1. The operator $(D, \mathcal{D}(D))$ is symmetric and densely defined (see Lemma 3.2).

1.3. Main results.

1.3.1. *Self-adjointness results on a corner domain.* In the following theorem, we give all the self-adjoint extensions of the MIT bag Dirac operator in a sector.

Theorem 1.2. *The following hold true.*

(i) **[Convex corners]**

If $\omega \in (0, \pi/2]$, $(D, \mathcal{D}(D))$ is self-adjoint.

(ii) **[Non-Convex corners]**

If $\omega \in (\pi/2, \pi)$, $(D, \mathcal{D}(D))$ is symmetric and closed but not self-adjoint. It has several self-adjoint extensions $(D^\gamma, \mathcal{D}(D^\gamma))$ defined for $\gamma \in \mathbb{C}$ such that $|\gamma| = 1$ by

$$\mathcal{D}(D^\gamma) = \mathcal{D}(D) + \text{span}(v_+ + \gamma v_-)$$

$$D^\gamma v = Dv$$

$$D^\gamma(v_+ + \gamma v_-) = i(v_+ - \gamma v_-) + m\sigma_3(v_+ + \gamma v_-)$$

for $v \in \mathcal{D}(D)$ where

$$v_+(r \cos(\theta), r \sin(\theta)) = K_{\nu_0}(r)u_0(\theta) - iK_{\nu_0+1}(r)u_{-1}(\theta),$$

$$v_-(r \cos(\theta), r \sin(\theta)) = K_{\nu_0}(r)u_0(\theta) + iK_{\nu_0+1}(r)u_{-1}(\theta),$$

$$u_0(\theta) := \frac{1}{2\sqrt{\omega}} \begin{pmatrix} e^{i\theta\nu_0} \\ -ie^{-i\theta\nu_0} \end{pmatrix}, \quad u_{-1}(\theta) := \frac{1}{2\sqrt{\omega}} \begin{pmatrix} e^{-i\theta(\nu_0+1)} \\ ie^{i\theta(\nu_0+1)} \end{pmatrix}.$$

Here, $r > 0$, $\theta \in (-\omega, \omega)$, $\nu_0 = \frac{\pi-2\omega}{4\omega}$, and K_ν denotes the modified Bessel function of the second kind of parameter ν .

Remark 1.3. The distinction between convex and non-convex corners in Theorem 1.2 is not surprising: It is reminiscent of [16] where the study of the so-called corner singularities for elliptic operators of even order are dealt with. We also refer to the books [13, 12] where the Laplacian in polygonal domains with various boundary conditions is studied.

Remark 1.4. For $\theta_0 \in [0, 2\pi]$, let us consider the rotated corner

$$\Omega_{\omega, \theta_0} := \{r(\cos(\theta), \sin(\theta)) \in \mathbb{R}^2 : r > 0, |\theta - \theta_0| < \omega\}.$$

Remark that $e^{-i\sigma_2\theta_0}$ is a rotation matrix of angle θ_0 which gives $\Omega_{\omega, \theta_0} = e^{-i\sigma_2\theta_0}\Omega_\omega$. The unitary transformation

$$\begin{aligned} \mathcal{U}_{\theta_0} : L^2(\Omega_{\omega, \theta_0}, \mathbb{C}^2) &\longrightarrow L^2(\Omega_\omega, \mathbb{C}^2) \\ v &\longmapsto e^{i(\theta_0/2)\sigma_3} v(e^{-i\sigma_2\theta_0} \cdot) \end{aligned}$$

satisfies

$$\begin{aligned} \mathcal{U}_{\theta_0}^{-1}(-i\sigma \cdot \nabla + m\sigma_3)\mathcal{U}_{\theta_0} &= -i\sigma \cdot \nabla + m\sigma_3 \\ \mathcal{U}_{\theta_0}^{-1}(\sigma \cdot \mathbf{n})\mathcal{U}_{\theta_0} &= \sigma \cdot (e^{-i\sigma_2\theta_0} \mathbf{n}) \end{aligned}$$

for any unitary vector $\mathbf{n} \in \mathbb{R}^2$ (see [22, Sections 2 and 3]). This ensures that Theorem 1.2 essentially covers every corner cases.

Remark 1.5. Let us recall some of the properties of the modified Bessel functions K_ν of the second kind of parameter ν that will be used in this paper (see [18, Chapter 7 Section 8 and Chapter 12 Section 1] or [1]). Let $\nu \in \mathbb{R}$.

- (i) The functions $r \in (0, +\infty) \mapsto K_\nu(r) \in \mathbb{R}$ are positive and decreasing.
- (ii) For $r > 0$, we have

$$K_\nu(r) = K_{-\nu}(r).$$

- (iii) For $r > 0$, we have that

$$(1.6) \quad K_\nu(r) \sim_{r \rightarrow 0} \begin{cases} \frac{\Gamma(\nu)}{2} \left(\frac{r}{2}\right)^{-\nu} & \text{if } \nu > 0 \\ -\log(r) & \text{if } \nu = 0. \end{cases}$$

and

$$K_\nu(r) \sim_{r \rightarrow +\infty} \left(\frac{\pi}{2r}\right)^{1/2} e^{-r}.$$

In particular, the domain of D^γ in Point (ii) of Theorem 1.2 can be rewritten using $r^{-|\nu_0|}\chi(r)$ resp. $r^{-(1-|\nu_0|)}\chi(r)$ instead of K_{ν_0} resp. K_{ν_0+1} where $\chi : \mathbb{R}_+ \mapsto [0, 1]$ is a regular function which is equal to 1 in a neighborhood of 0 and 1 for $r > 0$ large enough.

1.3.2. Physical remarks on the self-adjoint extensions on a corner domain. For non-convex corners, a natural question is to know if some of the self-adjoint extensions given in Point (ii) of Theorem 1.2 are more relevant than others from the physical point of view. The following propositions try to shed some light on this question.

The charge conjugation symmetry.

The Dirac operator D anti-commutes with the charge conjugation operator C defined for $u \in \mathbb{C}^2$ by

$$Cu = \sigma_1 \bar{u}.$$

In particular, for any $\omega \in (0, \pi)$, the operator C is an anti-unitary transformations that leave $\mathcal{D}(D)$ invariant and satisfies $C^2 = 1_2$ and

$$DC = -CD.$$

This property is strongly related with the particles/anti-particles interpretation of the spectrum of the Dirac operator (see [22, Section 1.4.6]). Studying the relations between C and the extensions of D , we get the following result.

Proposition 1.6. *Let $\omega \in (\pi/2, \pi)$. The only self-adjoint extensions of $(D, \mathcal{D}(D))$ such that*

$$C\mathcal{D}(D^\gamma) = \mathcal{D}(D^\gamma)$$

are the extensions $(D^\gamma, \mathcal{D}(D^\gamma))$ for $\gamma = \pm 1$. In these cases, we have the anticommutation relation

$$\{C, D^\gamma\} = CD^\gamma + D^\gamma C = 0.$$

Scale invariance.

Since Ω_ω is invariant by dilations, we immediately get that $\mathcal{D}(D)$ is stable by change of scale. In the case of non-convex corner, we get the following proposition.

Proposition 1.7. *Let $\omega \in (\pi/2, \pi)$. The only self-adjoint extensions of $(D, \mathcal{D}(D))$ such that for any $u \in \mathcal{D}(D^\gamma)$ and $\alpha > 0$,*

$$[\mathbf{x} \in \Omega_\omega \mapsto u(\alpha\mathbf{x}) \in \mathbb{C}^2] \in \mathcal{D}(D^\gamma),$$

are the extensions $(D^\gamma, \mathcal{D}(D^\gamma))$ for $\gamma = \pm 1$.

This property is used, for instance, in the proofs using Virial identities (see Remark 1.13).

Kinetic energy.

From a physical point of view, it is reasonable to impose that the domain of the MIT bag Dirac operator is included in the *formal* form domain $H^{1/2}(\Omega)$ of the Dirac operator on Ω . This additional constraint allows us to pick a single self-adjoint extension. In particular, we have the following result.

Proposition 1.8. *Let $\omega \in (\pi/2, \pi)$. The only self-adjoint extension of $(D, \mathcal{D}(D))$ satisfying $\mathcal{D}(D^\gamma) \subset H^{1/2}(\Omega_\omega)$ is $(D^1, \mathcal{D}(D^1))$.*

Remark 1.9. The proof of proposition 1.8 shows a stronger statement. If $\gamma = 1$, we have $\mathcal{D}(D^\gamma) \subset H^{3/4-\varepsilon}(\Omega_\omega)$ for any $\varepsilon \in (0, 1/4)$.

1.3.3. *Some results on polygonal domains.* Using Point (i) of Theorem 1.2, Remark 1.4 and partitions of the unity, we immediately obtain the following result.

Corollary 1.10. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal open set. The Dirac operator $(D_\Omega, \mathcal{D}(D_\Omega))$ defined by*

$$\begin{aligned} \mathcal{D}(D_\Omega) &= \{u \in H^1(\Omega, \mathbb{C}^2), \mathcal{B}_{\mathbf{n}}u = u \text{ on } \partial\Omega\} \\ Du &= -i\sigma \cdot \nabla u + m\sigma_3 u \text{ for any } u \in \mathcal{D}(D_\Omega), \end{aligned}$$

is self-adjoint.

Remark 1.11. A similar result can be stated in the case of non-convex polygonal open set using Point (ii) of Theorem 1.2. We choose not to write it down here for the sake of readability.

1.3.4. *Spectral properties on Corner domains.* Let us study now the spectral properties of the self-adjoint operators on corner domains. We restrict ourselves to the *physical case* $\gamma = 1$. In the next proposition, we use the following unified notation:

$$D^{sa} = \begin{cases} D & \text{if } \omega \in (0, \pi/2], \\ D^1 & \text{if } \omega \in (\pi/2, \pi). \end{cases}$$

where D and D^1 are defined in (1.2) and Point (ii) of Theorem 1.2 respectively. As defined D^{sa} is self-adjoint.

Proposition 1.12. *For any $\omega \in (0, \pi)$, we have*

$$\mathrm{Sp}(D^{sa}) = \mathrm{Sp}_{ess}(D^{sa}) = \begin{cases} \mathbb{R} & \text{if } m \leq 0, \\ \mathbb{R} \setminus (-m, m) & \text{if } m \geq 0. \end{cases}$$

Remark 1.13. Using the Virial identity (see in particular [22, Section 4.7.2] and Section 5.2), we get that there is no point spectrum in $\mathbb{R} \setminus (-|m|, |m|)$. Nevertheless, the proof gives no information on the point spectrum in $(-|m|, |m|)$ for negative m . We choose not to adress this problem in this work to focus on the physical case $m \geq 0$.

1.4. Organisation of the paper. In Section 2, we prove Theorem 1.2 and give the main lemmas that we use. Their proofs are gathered in Section 3. In Section 4, we discuss the physically relevant self-adjoint extensions. Finally, the spectral properties of Proposition 1.12 are proved in Section 5.

2. STUDY OF THE SELF-ADJOINT EXTENSIONS OF D

In this section, we give the main lemmas on which rely the proofs of Points (i) and (ii) of Theorem 1.2. Their proofs are detailed in Section 3. Note that without loss of generality, we can assume that $m = 0$ since $m\sigma_3$ is a bounded self-adjoint operator.

2.1. Notations. We denote by $\langle \cdot, \cdot \rangle$ resp. $\langle \cdot, \cdot \rangle_{L^2}$ the scalar products on \mathbb{C}^2 resp. L^2 which are anti-linear in the first variable.

2.2. The operator in polar coordinates. On $\Omega_\omega \setminus \{0\}$, we introduce the polar coordinates

$$(2.1) \quad x(r, \theta) = \begin{pmatrix} x_1(r, \theta) \\ x_2(r, \theta) \end{pmatrix} = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} = r e_{rad}(\theta),$$

for $r > 0$ and $\theta \in (-\omega, \omega)$ where

$$(2.2) \quad e_{rad}(\theta) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \quad e_{ang}(\theta) = \frac{d}{d\theta} e_{rad}(\theta) = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}.$$

In what follows, the following basic relation will be usefull

$$(2.3) \quad i\sigma_3 \sigma \cdot e_{ang} = \sigma \cdot e_{rad}.$$

For all $\Psi \in L^2(\Omega_\omega, \mathbb{C}^2)$, we get that

$$\psi(r, \theta) = \Psi(x(r, \theta))$$

is an element of $L^2((0, +\infty), r dr) \otimes L^2((-\omega, \omega), \mathbb{C}^2)$. Using this system of coordinates, we rewrite the Dirac operator as

$$(2.4) \quad \begin{aligned} D &= -i\sigma \cdot e_{rad} \partial_r - \frac{i\sigma \cdot e_{ang}}{r} \partial_\theta = -i\sigma \cdot e_{rad} \left(\partial_r + i \frac{\sigma_3}{r} \partial_\theta \right) \\ &= -i\sigma \cdot e_{rad} \left(\partial_r + \frac{1_2 - K}{2r} \right) \end{aligned}$$

where

$$(2.5) \quad K = \sigma_3 (-2i\partial_\theta) + 1_2.$$

Remark 2.1. In the following, we will rely on the properties of K to build subspaces that D leaves stable. Our choice for K is commented in Remark 2.4.

2.3. Study of the operator K . Let us first remark that for any $r > 0$, we have on the boundary $\partial\Omega_\omega \setminus \{0\}$ that

$$(2.6) \quad \begin{aligned} \mathcal{B}_{\mathbf{n}(re^{i\omega})} &= -i\sigma_3\sigma \cdot e_{ang}(\omega) =: \mathcal{B}_+ \\ \mathcal{B}_{\mathbf{n}(re^{-i\omega})} &= i\sigma_3\sigma \cdot e_{ang}(-\omega) =: \mathcal{B}_- \end{aligned}$$

where $\mathcal{B}_{\mathbf{n}}$ is defined in (1.4). Now, let us study the spectral properties of K .

Lemma 2.2. *We have that*

(i) *The operator $(K, \mathcal{D}(K))$ on $L^2((-\omega, \omega), \mathbb{C}^2)$ where K is defined in (2.5) and*

$$\mathcal{D}(K) = \{u \in H^1((-\omega, \omega), \mathbb{C}^2) : \mathcal{B}_+u(\omega) = u(\omega) \text{ and } \mathcal{B}_-u(-\omega) = u(-\omega)\}$$

is self-adjoint and of compact resolvent.

(ii) *Its spectrum is*

$$\text{Sp}(K) = \{\lambda_\kappa, \kappa \in \mathbb{Z}\}$$

where $\lambda_\kappa := \frac{\pi(1+2\kappa)}{2\omega}$. For $\kappa \in \mathbb{Z}$, we have $\ker(K - \lambda_\kappa) = \text{span}(u_\kappa)$ where

$$u_\kappa := \frac{1}{2\sqrt{\omega}} \begin{pmatrix} e^{i\theta \frac{\lambda_\kappa - 1}{2}} \\ (-1)^{\kappa+1} i e^{-i\theta \frac{\lambda_\kappa - 1}{2}} \end{pmatrix},$$

and $(u_\kappa)_{\kappa \in \mathbb{Z}}$ is an orthonormal basis of $L^2((-\omega, \omega), \mathbb{C}^2)$.

(iii) *We have that $(\sigma \cdot e_{rad}) \mathcal{D}(K) \subset \mathcal{D}(K)$, $\{K, \sigma \cdot e_{rad}\} = 0$ and*

$$u_{-(\kappa+1)} = (-1)^\kappa i (\sigma \cdot e_{rad}) u_\kappa.$$

Remark 2.3. Thanks to Point (iii), we recover the fact that $\text{Sp}(K)$ is symmetric with respect to 0.

Remark 2.4. The decomposition of the wavefunctions in angular harmonics for the Dirac operator on \mathbb{R}^2 has been a major inspiration for this work. In that case, the following operator acting on the angular variable

$$\begin{aligned} \tilde{K} &= -2i\partial_\theta + \sigma_3 = \sigma_3 K \\ \mathcal{D}(\tilde{K}) &= H^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{C}^2) \end{aligned}$$

is called the spin-orbit operator. It is self-adjoint and commutes with the Dirac operator D . Hence, one can restrict the study of D to the eigenspaces of \tilde{K} which are left stable by D . We refer, in particular, to [22, Section 4.6] where the spherical symmetry in \mathbb{R}^3 is extensively studied.

In our case, \tilde{K} does not behave well with respect to the MIT bag boundary condition. Nevertheless, the slight change we made allows to build subspaces that are left stable by D .

We list here the properties of K motivating its introduction :

- (a) it is a first order operator in the θ variable.
- (b) its domain takes into account the MIT bag boundary condition and made him self-adjoint.
- (c) it has good (anti)-commutation relations with D .

2.4. Study of D on stable subspaces. We are now in a good position to introduce subspaces that are left stable by D and then to study the operators built by restriction of D .

Lemma 2.5. *We have that*

$$L^2((0, +\infty), r dr) \otimes L^2((-\omega, \omega), \mathbb{C}^2) = \bigoplus_{\kappa \geq 0} E_\kappa$$

where $E_\kappa = L^2((0, +\infty), r dr) \otimes \text{span}(u_\kappa, u_{-(\kappa+1)})$. Moreover, the following holds true.

(i) For any $\kappa \in \mathbb{N}$, the operator $(d^\kappa, \mathcal{D}(d^\kappa))$ defined by

$$\begin{aligned} \mathcal{D}(d^\kappa) &= \mathcal{D}(D) \cap E_\kappa \\ d^\kappa &= D|_{E_\kappa} \end{aligned}$$

is a well-defined unbounded operator of the Hilbert space E_κ .

(ii) For any $\kappa \in \mathbb{N}$, the operator $(d^\kappa, \mathcal{D}(d^\kappa))$ is unitarily equivalent to the operator $(\mathbf{d}_\omega^\kappa, \mathcal{D}(\mathbf{d}_\omega^\kappa))$ defined by

$$\begin{aligned} \mathcal{D}(\mathbf{d}_\omega^\kappa) &= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in L^2((0, +\infty), r dr, \mathbb{C}^2) : \right. \\ &\quad \left. \int_0^\infty \left(|\dot{a}|^2 + |\dot{b}|^2 + \frac{|\lambda_\kappa - 1|^2}{4r^2} |a|^2 + \frac{|\lambda_\kappa + 1|^2}{4r^2} |b|^2 \right) r dr < +\infty \right\}, \\ \mathbf{d}_\omega^\kappa &= (-1)^\kappa \left(i\sigma_2 \left(\partial_r + \frac{1}{2r} \right) + \sigma_1 \frac{\lambda_\kappa}{2r} \right) = (-1)^\kappa \begin{pmatrix} 0 & \partial_r + \frac{\lambda_\kappa + 1}{2r} \\ -\partial_r + \frac{\lambda_\kappa - 1}{2r} & 0 \end{pmatrix}. \end{aligned}$$

(iii) Let

$$v = \sum_{\kappa \in \mathbb{Z}} a_\kappa u_\kappa$$

be any element of $\mathcal{D}(D)$, we have that

$$\|Dv\|_{L^2}^2 = \sum_{\kappa \in \mathbb{Z}} \int_0^\infty \left(|\dot{a}_\kappa|^2 + |\lambda_\kappa - 1|^2 \frac{|a_\kappa(r)|^2}{4r^2} \right) r dr = \|\nabla v\|_{L^2}^2.$$

(iv) The operators $(D, \mathcal{D}(D))$ and $(\mathbf{d}_\omega^\kappa, \mathcal{D}(\mathbf{d}_\omega^\kappa))$ for any $\kappa \in \mathbb{N}$ are symmetric and closed.

(v) The operator $(D, \mathcal{D}(D))$ is self-adjoint (possibly with a larger domain) if and only if the operators $(\mathbf{d}_\omega^\kappa, \mathcal{D}(\mathbf{d}_\omega^\kappa))$ are self-adjoint (possibly with larger domains). Moreover, if this is the case, the spectrum of D is equal to the union of the spectrum of the operators \mathbf{d}_ω^κ .

The following lemma allows us to conclude our study. Its proof relies on [21, Theorem VIII.3], [20, Theorem X.2] and some properties of modified Bessel functions [18, 1].

Lemma 2.6. *The following holds true.*

(i) For any $\kappa \geq 1$ and any $\omega \in (0, \pi)$, or $\kappa = 0$ and any $\omega \in (0, \pi/2)$, the operator $(\mathbf{d}_\omega^\kappa, \mathcal{D}(\mathbf{d}_\omega^\kappa))$ is self-adjoint.

(ii) For any $\omega \in (\pi/2, \pi)$, $(\mathbf{d}_\omega^0, \mathcal{D}(\mathbf{d}_\omega^0))$ is not self-adjoint but has several self-adjoint extensions $(\mathbf{d}_\omega^{0,\gamma}, \mathcal{D}(\mathbf{d}_\omega^{0,\gamma}))$ defined by

$$\begin{aligned} \mathcal{D}(\mathbf{d}_\omega^{0,\gamma}) &= \mathcal{D}(\mathbf{d}_\omega^0) + \text{span}(a_+ + \gamma\sigma_3 a_+) \\ \mathbf{d}_\omega^{0,\gamma}(a + c_0(a_+ + \gamma\sigma_3 a_+)) &= \mathbf{d}_\omega^0 a + c_0 i(a_+ - \gamma\sigma_3 a_+) \end{aligned}$$

where $a \in \mathcal{D}(d_\omega^0)$, $c_0 \in \mathbb{C}$,

$$a_+ = \begin{pmatrix} K_{\frac{\lambda_0-1}{2}}(r) \\ -iK_{\frac{\lambda_0+1}{2}}(r) \end{pmatrix},$$

$\gamma \in \mathbb{C}$ is such that $|\gamma| = 1$.

Points (i) and (ii) of Theorem 1.2 follow from Lemmas 2.2, 2.5 and 2.6.

3. PROOFS OF LEMMAS 2.2, 2.5 AND 2.6

3.1. Preliminary study.

3.1.1. *Some algebraic properties of the boundary matrices $\mathcal{B}_\mathbf{v}$.*

Lemma 3.1. *For any unit vector $\mathbf{v} \in \mathbb{R}^2$, the matrix $\mathcal{B}_\mathbf{v}$ satisfies*

$$\ker(\mathcal{B}_\mathbf{v} \pm 1_2) = \sigma_3 \ker(\mathcal{B}_\mathbf{v} \pm 1_2)^\perp = \sigma \cdot \mathbf{v} \ker(\mathcal{B}_\mathbf{v} \pm 1_2)^\perp.$$

Proof. Since $\{\sigma_3, \mathcal{B}_\mathbf{v}\} = 0$, we have that

$$\sigma_3 \ker(\mathcal{B}_\mathbf{v} \pm 1_2)^\perp = \sigma_3 \operatorname{ran}(\mathcal{B}_\mathbf{v} \pm 1_2) = \operatorname{ran}(\mathcal{B}_\mathbf{v} \mp 1_2) = \ker(\mathcal{B}_\mathbf{v} \pm 1_2).$$

Since $\{\sigma \cdot \mathbf{v}, \mathcal{B}_\mathbf{v}\} = 0$, the same algebraic proof works for the other equality. \square

3.1.2. *Symmetry of D .* For the sake of completeness, we recall the following standard result on MIT bag Dirac operator.

Lemma 3.2. *The operator $(D, \mathcal{D}(D))$ is symmetric and densely defined.*

Proof. Let $u, v \in \mathcal{D}(D)$. Since Ω_ω is a Lipschitz domain, we have by an integration by parts that

$$\langle Du, v \rangle_{L^2} - \langle u, Dv \rangle_{L^2} = \int_{\partial\Omega_\omega} \langle -i\sigma \cdot \mathbf{n}u, v \rangle$$

(see [17, Section 3.1.2]). By Lemma 3.1, we have, almost everywhere on the boundary, that

$$-i(\sigma \cdot \mathbf{n})u \in \sigma \cdot \mathbf{n} \ker(\mathcal{B}_\mathbf{n} - 1_2) = \ker(\mathcal{B}_\mathbf{n} - 1_2)^\perp$$

so that $\langle -i\sigma \cdot \mathbf{n}u, v \rangle = 0$ and

$$\langle Du, v \rangle_{L^2} = \langle u, Dv \rangle_{L^2}.$$

\square

3.2. **Study of the angular part : proof of Lemma 2.2.** We divide the proof in several steps.

Step 1: Symmetry of K . Let $u, v \in \mathcal{D}(K)$, we have by an integration by parts and Lemma 3.1 that

$$(3.1) \quad \begin{aligned} \langle Ku, v \rangle_{L^2} - \langle u, Kv \rangle_{L^2} &= \int_{-\omega}^{\omega} \partial_\theta \langle -2i\sigma_3 u, v \rangle d\theta \\ &= \langle -2i\sigma_3 u(\omega), v(\omega) \rangle - \langle -2i\sigma_3 u(-\omega), v(-\omega) \rangle = 0 \end{aligned}$$

so that K is symmetric.

Step 2 : Self-adjointness of K . Let $u \in \mathcal{D}(K^*)$. Taking the test functions in $C_c^\infty((-\omega, \omega), \mathbb{C}^2) \subset \mathcal{D}(K)$, we get that the distribution Ku belongs to $L^2((-\omega, \omega), \mathbb{C}^2)$ so that $u \in H^1((-\omega, \omega), \mathbb{C}^2)$. Using again the integration by parts performed in (3.1), we obtain that

$$\begin{aligned}\sigma_3 u(\omega) &\in \ker(\mathcal{B}_+ - 1_2)^\perp \\ \sigma_3 u(-\omega) &\in \ker(\mathcal{B}_- - 1_2)^\perp.\end{aligned}$$

By Lemma 3.1, we get $u \in \mathcal{D}(K)$ and then K is self-adjoint. Using the compact Sobolev embedding

$$H^1((-\omega, \omega), \mathbb{C}^2) \hookrightarrow L^2((-\omega, \omega), \mathbb{C}^2),$$

we get that K has compact resolvent and that its spectrum is discrete. This ends the proof of Point (i).

Step 3 : Study of its spectrum. Let $\lambda \in \mathbb{R}$, we look for the solutions of

$$(3.2) \quad Ku = \lambda u$$

that belongs to $\mathcal{D}(K)$. Let us remark that without any boundary conditions assumed, the set of solutions of (3.2) is

$$E_\lambda^1 := \text{span} \left(\begin{pmatrix} e^{i\theta \frac{\lambda-1}{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^{-i\theta \frac{\lambda-1}{2}} \end{pmatrix} \right)$$

Let $u \in E_\lambda^1 \cap \mathcal{D}(K)$, we have that there exist $a, b \in \mathbb{C}$ such that

$$u = \begin{pmatrix} ae^{i\theta \frac{\lambda-1}{2}} \\ be^{-i\theta \frac{\lambda-1}{2}} \end{pmatrix},$$

and, by (2.3), the boundary condition reads

$$\begin{aligned}u(\omega) &= \mathcal{B}_+ u(\omega) = -\sigma \cdot e_{rad}(\omega) u(\omega) = - \begin{pmatrix} 0 & e^{-i\omega} \\ e^{i\omega} & 0 \end{pmatrix} u(\omega) = \begin{pmatrix} -be^{-i\omega \frac{\lambda+1}{2}} \\ -ae^{i\omega \frac{\lambda+1}{2}} \end{pmatrix} \\ u(-\omega) &= \mathcal{B}_- u(-\omega) = \sigma \cdot e_{rad}(-\omega) u(-\omega) = \begin{pmatrix} 0 & e^{i\omega} \\ e^{-i\omega} & 0 \end{pmatrix} u(-\omega) = \begin{pmatrix} be^{i\omega \frac{\lambda+1}{2}} \\ ae^{-i\omega \frac{\lambda+1}{2}} \end{pmatrix}\end{aligned}$$

so that

$$a = be^{i\omega\lambda} = -be^{-i\omega\lambda}.$$

Hence, there is a nontrivial solution of (3.2) that belongs to $\mathcal{D}(K)$ if and only if $e^{2i\omega\lambda} = -1$. We deduce that the spectrum of K is

$$\text{Sp}(K) = \left\{ \frac{\pi(1+2\kappa)}{2\omega}, \kappa \in \mathbb{Z} \right\}.$$

We define $\lambda_\kappa := \frac{\pi(1+2\kappa)}{2\omega}$ for any $\kappa \in \mathbb{Z}$. If κ is even, we have $a = ib$ and

$$\ker(K - \lambda_\kappa) = \text{span} \begin{pmatrix} e^{i\theta \frac{\lambda_\kappa-1}{2}} \\ -ie^{-i\theta \frac{\lambda_\kappa-1}{2}} \end{pmatrix},$$

if κ is odd, we have $a = -ib$ so that

$$\ker(K - \lambda_\kappa) = \text{span} \begin{pmatrix} e^{i\theta \frac{\lambda_\kappa-1}{2}} \\ ie^{-i\theta \frac{\lambda_\kappa-1}{2}} \end{pmatrix}.$$

This proves Point (ii).

Step 4 : Study of the commutation relation. Since $\sigma \cdot e_{rad}$ commutes with \mathcal{B} , we get that

$$(\sigma \cdot e_{rad})\mathcal{D}(K) \subset \mathcal{D}(K).$$

We also have

$$\begin{aligned} K\sigma \cdot e_{rad} &= \sigma_3 (\sigma \cdot e_{rad}(-2i\partial_\theta) - 2i\sigma \cdot e_{ang}) + \sigma \cdot e_{rad} \\ &= -\sigma \cdot e_{rad}\sigma_3(-2i\partial_\theta) - \sigma \cdot e_{rad} = -\sigma \cdot e_{rad}K. \end{aligned}$$

This ends the proof of Point (iii).

3.3. Study of the stable subspaces : proof of Lemma 2.5. Let us remark that the direct sum decomposition is an immediate consequence of Point (ii) of Lemma 2.2. We divide the remaining of our proof into several steps.

Proof of Points (i) and (ii). These points follow from identity (2.4) :

$$D = -i\sigma \cdot e_{rad} \left(\partial_r + \frac{1_2 - K}{2r} \right)$$

and Point (iii) of Lemma 2.2. Indeed, for any $\kappa \in \mathbb{N}$, any $v \in E_\kappa$, there exist $a, b \in L^2((0, +\infty), r dr)$ such that for all $r > 0$ and all $\theta \in (-\omega, \omega)$

$$v(r, \theta) = a(r)u_\kappa(\theta) + b(r)u_{-(\kappa+1)}(\theta).$$

If $v \in H^1(\Omega_\omega, \mathbb{C}^2)$, since $-i\sigma_3\partial_\theta = \frac{K-1}{2}$, we have

$$\|\nabla v\|_{L^2}^2 = \int_0^\infty \left(|\dot{a}|^2 + |\dot{b}|^2 + \frac{|\lambda_\kappa - 1|^2}{4r^2} |a|^2 + \frac{|\lambda_{-(\kappa+1)} - 1|^2}{4r^2} |b|^2 \right) r dr$$

and

$$Dv = d^\kappa v = (-1)^{\kappa+1} u_{-(\kappa+1)} \left(\dot{a} + \frac{1 - \lambda_\kappa}{2r} a \right) + (-1)^\kappa u_\kappa \left(\dot{b} + \frac{1 + \lambda_\kappa}{2r} b \right).$$

This ends this part of the proof.

Proof of Points (iii) and (iv). Let

$$v = \sum_{\kappa \in \mathbb{Z}} a_\kappa u_\kappa$$

be any element of $\mathcal{D}(D)$. By Lemma A.1, we have that

$$\begin{aligned} \|Dv\|_{L^2}^2 &= \sum_{\kappa \in \mathbb{Z}} \int_0^\infty \left| \dot{a}_\kappa + \frac{1 - \lambda_\kappa}{2r} a_\kappa \right|^2 r dr = \sum_{\kappa \in \mathbb{Z}} \int_0^\infty \left(|\dot{a}_\kappa|^2 + |\lambda_\kappa - 1|^2 \frac{|a_\kappa(r)|^2}{4r^2} \right) r dr \\ &= \|\nabla v\|_{L^2}^2. \end{aligned}$$

Point (iv) follows immediatly.

Proof of Point (v). This last point is proven as in [22, Lemma 4.15] using [21, Theorem VIII.3].

3.4. Proof of Lemma 2.6. Let us fix $\kappa \in \mathbb{N}$. In this proof, we apply the basic criterion for self-adjointness [21, Theorem VIII.3]. Hence, we have to study the sets

$$\ker((\mathbf{d}_\omega^\kappa)^* \pm i1_2).$$

Let us first remark that

$$\mathcal{D}((\mathbf{d}_\omega^\kappa)^*) \subset \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in L^2((0, \infty), r dr)^2 : \mathbf{d}_\omega^\kappa \begin{pmatrix} a \\ b \end{pmatrix} \in L^2((0, \infty), r dr)^2 \right\}.$$

Since $\{\mathbf{d}_\omega^\kappa, \sigma_3\} = 0$, we get that

$$\ker((\mathbf{d}_\omega^\kappa)^* - i1_2) = \sigma_3 \ker((\mathbf{d}_\omega^\kappa)^* + i1_2).$$

Hence, it remains to look if solutions of

$$(3.3) \quad (\mathbf{d}_\omega^\kappa - i1_2) \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

that belongs to $L^2((0, \infty), r dr)^2$ exist. This is a linear system of differential equations of first order. The set of solutions is a vector space of dimension 2 and the solutions are regular on $(0, \infty)$. We have

$$(\mathbf{d}_\omega^\kappa + i1_2)(\mathbf{d}_\omega^\kappa - i1_2) = (\mathbf{d}_\omega^\kappa)^2 + 1_2$$

so that

$$\begin{aligned} 0 &= ((\mathbf{d}_\omega^\kappa)^2 + 1_2) \begin{pmatrix} a \\ b \end{pmatrix} \\ &= -\frac{1}{r^2} \begin{pmatrix} r^2 \partial_r^2 + r \partial_r - \left(r^2 + \frac{(\lambda_\kappa - 1)^2}{4}\right) & 0 \\ 0 & r^2 \partial_r^2 + r \partial_r - \left(r^2 + \frac{(\lambda_\kappa + 1)^2}{4}\right) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \end{aligned}$$

The functions a and b are modified Bessel functions (see [18, Chapter 12, Section 1] and [1]) of parameters $\frac{\lambda_\kappa - 1}{2}$ and $\frac{\lambda_\kappa + 1}{2}$. The modified Bessel functions of the first kind do not belong to $L^2((1, \infty), r dr)$. Hence, the only option for (a, b) to be in $L^2((1, \infty), r dr)^2$ is that $a = a_0 K_{\frac{\lambda_\kappa - 1}{2}}$ and $b = b_0 K_{\frac{\lambda_\kappa + 1}{2}}$ with $a_0, b_0 \in \mathbb{C}$. We recall that K_ν are the modified Bessel function of the second kind of parameter $\nu \in \mathbb{R}$. They satisfy $K_\nu = K_{-\nu}$ and

$$(3.4) \quad K_{|\nu|} + \frac{|\nu|}{r} K_{|\nu|} = -K_{|\nu|-1}.$$

By remark 1.5, for b to belong to $L^2((0, 1), r dr)$, one has to have $\lambda_\kappa < 1$. We have

- (a) $\lambda_\kappa \geq 3/2$ for any $\kappa \geq 1$ and any $\omega \in (0, \pi)$,
- (b) $\lambda_0 \geq 1$ for any $\omega \in (0, \pi/2)$,
- (c) $\lambda_0 < 1$ for any $\omega \in (\pi/2, \pi)$.

Hence, in Cases (a) and (b), we necessarily get $b_0 = 0$. Taking into account (3.3) and (3.4) we also get $a_0 = 0$ which implies

$$\ker((\mathbf{d}_\omega^\kappa)^* \pm i1_2) = \{0\}$$

so that [21, Theorem VIII.3] ensures that $(\mathbf{d}_\omega^\kappa, \mathcal{D}(\mathbf{d}_\omega^\kappa))$ is a self-adjoint operator. In Case (c), we get that

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \text{span}(a_+^0), \text{ with } a_+^0 = \begin{pmatrix} K_{\frac{\lambda_0 - 1}{2}}(r) \\ -iK_{\frac{\lambda_0 + 1}{2}}(r) \end{pmatrix}.$$

we easily prove that $a_+^0 \in \mathcal{D}((\mathbf{d}_\omega^\kappa)^*)$ which yields

$$\ker((\mathbf{d}_\omega^\kappa)^* - i1_2) = \text{span}(a_+^0) \text{ and } \ker((\mathbf{d}_\omega^\kappa)^* + i1_2) = \text{span}(\sigma_3 a_+^0)$$

We conclude by [20, Theorem X.2].

4. DISTINGUISHED SELF-ADJOINT EXTENSIONS OF D

The goal of this section is to prove Propositions 1.6 and 1.8 about the distinguished extensions of $(D, \mathcal{D}(D))$ when $\omega \in (\pi/2, \pi)$.

4.1. Proof of Proposition 1.6. The anticommutation of C with D is straightforward. The only thing left to prove is the following lemma.

Lemma 4.1. *Let $\omega \in (\pi/2, \pi)$ and $\gamma \in \mathbb{C}$ such that $|\gamma| = 1$. The following statements are equivalent:*

- (a) $\gamma = \pm 1$,
- (b) $\mathcal{D}(D^\gamma)$ is left invariant by C and $D^\gamma C = -CD^\gamma$.

Proof. Let us consider $u \in \mathcal{D}(D^\gamma)$. By Point (ii) Theorem 1.2, we know that there exists $v \in \mathcal{D}(D)$ and $c_0 \in \mathbb{C}$ such that $u = v + c_0(v_+ + \gamma v_-)$. Remark that we have

$$Cv_+ = iv_+, \quad Cv_- = iv_-.$$

Thus, we have $Cu = Cv + i\bar{c}_0(v_+ + \bar{\gamma}v_-)$ so that $Cu \in \mathcal{D}(D^\gamma)$ if and only if $\gamma \in \mathbb{R}$, that is $\gamma = \pm 1$. Moreover, assume that $\gamma = \pm 1$, we have:

$$D^\gamma Cu = DCv - \bar{c}_0(v_+ - \gamma v_-) = -CDv - \bar{c}_0(v_+ - \gamma v_-).$$

As $D^\gamma(v_+ + \gamma v_-) = i(v_+ - \gamma v_-)$, we get $CD^\gamma(v_+ + \gamma v_-) = (v_+ - \gamma v_-)$ which yields $D^\gamma Cu = -CD^\gamma u$. \square

4.2. Proof of Proposition 1.7. Proposition 1.7 follows from the following lemma.

Lemma 4.2. *Let $\alpha > 0$ and $\gamma = e^{is} \in \mathbb{C}$ for $s \in [0, 2\pi)$. The unitary application*

$$\begin{aligned} \mathcal{V}_\alpha : L^2(\Omega_\omega, \mathbb{C}^2) &\longrightarrow L^2(\Omega_\omega, \mathbb{C}^2) \\ u &\longmapsto \alpha u(\alpha \cdot) \end{aligned}$$

satisfies

$$\begin{aligned} \mathcal{V}_\alpha^{-1}(-i\sigma \cdot \nabla) \mathcal{V}_\alpha &= \alpha(-i\sigma \cdot \nabla) \\ \mathcal{V}_\alpha \mathcal{D}(D^\gamma) &= \begin{cases} \mathcal{D}(D^\gamma) & \text{if } \gamma = \pm 1 \text{ (i.e. if } s \in \{0, \pi\}), \\ \mathcal{D}(D^{\tilde{\gamma}}) & \text{otherwise} \end{cases} \end{aligned}$$

where $\tilde{\gamma} = e^{2i \arctan(\frac{\tan(s/2)}{\alpha^{\lambda_0}})}$.

Proof. Let $\alpha > 0$ and $\gamma = e^{is} \in \mathbb{C}$. By Lemma 2.5, we are reduced to study the change of scale of the operator $\mathbf{d}_\omega^{0,\gamma}$ because we easily get that $\mathcal{D}(\mathbf{d}_\omega^\kappa)$ is scaling invariant. By Remark 1.5, we have

$$\mathcal{V}_\alpha \mathcal{D}(D^{\pm 1}) = \mathcal{D}(D^{\pm 1})$$

and that for $\gamma \neq -1$ and $r > 0$,

$$(1 + \gamma) \begin{pmatrix} K_{\frac{1-\lambda_0}{2}}(\alpha r) \\ -i \frac{1-\gamma}{1+\gamma} K_{\frac{1+\lambda_0}{2}}(\alpha r) \end{pmatrix} \sim_{r \rightarrow 0} \frac{1+\gamma}{\alpha^{\frac{1-\lambda_0}{2}}} \begin{pmatrix} K_{\frac{1-\lambda_0}{2}}(r) \\ -i \alpha^{-\lambda_0} \frac{1-\gamma}{1+\gamma} K_{\frac{1+\lambda_0}{2}}(r) \end{pmatrix}.$$

We have

$$-i \frac{1-\gamma}{1+\gamma} = -\tan(s/2)$$

so that

$$-i\alpha^{-\lambda_0} \frac{1-\gamma}{1+\gamma} = -\alpha^{-\lambda_0} \tan(s/2) = -i \frac{1-\tilde{\gamma}}{1+\tilde{\gamma}}$$

for $\tilde{\gamma} = e^{2i \arctan\left(\frac{\tan(s/2)}{\alpha^{\lambda_0}}\right)}$. This ensures that $\mathcal{V}_\alpha \mathcal{D}(D^\gamma) = \mathcal{D}(D^{\tilde{\gamma}})$ and the result follows. \square

4.3. Proof of Proposition 1.8. To prove Proposition 1.8 it is enough to prove the following lemma.

Lemma 4.3. *Let $\omega \in (\pi/2, \pi)$ and ν_0 as defined in Theorem 1.2. The following holds true.*

(i) *The function*

$$(r \cos(\theta), r \sin(\theta)) \in \Omega_\omega \mapsto K_{\nu_0}(r)u_0(\theta)$$

belongs to $H^{1/2}(\Omega_\omega)$.

(ii) *The function*

$$(r \cos(\theta), r \sin(\theta)) \in \Omega_\omega \mapsto K_{\nu_0+1}(r)u_{-1}(\theta)$$

does not belong to $H^{1/2}(\Omega_\omega)$.

Proof. Using [9, Cor. 4.53.], we have $H^{1/2}(\Omega_\omega) \hookrightarrow L^4(\Omega_\omega)$ and by Remark 1.5

$$|K_{\nu_0+1}(r)u_{-1}(\theta)|^4 r = \frac{1}{2^3 \omega^2} |K_{\nu_0+1}(r)|^4 r \sim_{r \rightarrow 0} \frac{C}{r^{4(\nu_0+1)-1}}$$

for any $r > 0$ and $\theta \in (-\omega, \omega)$. Since

$$\nu_0 = \frac{\pi - 2\omega}{4\omega} > -1/2,$$

this function does not belong to $L^4(\Omega_\omega)$ and Point (ii) holds true. Let us prove Point (i). We have for $r > 0$ and $\theta \in (-\omega, \omega)$ that

$$|\nabla K_{\nu_0} u_0|^2(r, \theta) = \frac{1}{4\omega} |\partial_r K_{\nu_0}(r)|^2 + \frac{|K_{\nu_0}(r)|^2}{4\omega r^2} 2|\nu_0|^2.$$

By (3.4) and Remark 1.5, we have that $K_{\nu_0} u_0$ belongs to $W^{1,p}(\Omega_\omega)$ as soon as

$$1 \leq p < \frac{2}{|\nu_0| + 1}.$$

Since

$$\min_{\omega \in (\pi/2, \pi)} \frac{2}{|\nu_0| + 1} = \frac{8}{5} > \frac{4}{3}$$

and $W^{1,4/3}(\Omega_\omega) \hookrightarrow H^{1/2}(\Omega_\omega)$, we get Point (i). \square

5. SPECTRUM OF D^{sa}

5.1. Proof of Proposition 1.12. The proof is divided into two steps.

Step 1 : Study of the spectrum for the Dirac operator on \mathbb{R}^2 and $\mathbb{R}_+^2 = \Omega_{\pi/2}$ and consequences. Let us remark that

$$\begin{aligned} \text{Sp}(D_1) &= \mathbb{R} \setminus (-|m|, |m|), \\ \text{Sp}(D_2) &= \begin{cases} \mathbb{R} \setminus (-m, m) & \text{if } m \geq 0, \\ \mathbb{R} & \text{if } m < 0, \end{cases} \end{aligned}$$

where D_1 is the Dirac operator on \mathbb{R}^2 and D_2 is the Dirac operator on the half-plane $\Omega_{\pi/2}$. This second point can be easily obtained using the test functions of the form

$$u(x_1, x_2) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{mx_1} v(x_2) \in \mathcal{D}(D_2)$$

where v is a scalar function. Up to send Weyl sequences for D_2 to infinity and to truncate them, we have that

$$\text{Sp}(D^{sa}) \supset \text{Sp}(D_2) \supset \text{Sp}(D_1).$$

In particular, this gives the essential spectrum in the case $m < 0$.

Step 2 : Study of the reverse inclusion. In the following, we assume that $m > 0$. By Remark 1.9, we have $\mathcal{D}(D^{sa}) \subset H^{3/4-\varepsilon}(\Omega_\omega)$ for all $\varepsilon \in (0, 1/4)$. Hence, by an integration by parts, we obtain that for all $u \in \mathcal{D}(D^{sa})$,

$$2\text{Re} \langle -i\sigma \cdot \nabla u, \sigma_3 u \rangle_{L^2} = \|u\|_{L^2(\partial\Omega_\omega)}^2$$

so that

$$\|D^{sa}u\|_{L^2(\Omega_\omega)}^2 = \|\sigma \cdot \nabla u\|_{L^2(\Omega_\omega)}^2 + m^2 \|u\|_{L^2(\Omega_\omega)}^2 + m \|u\|_{L^2(\partial\Omega_\omega)}^2.$$

This ensures that $\text{Sp}((D^{sa})^2) \subset [m^2, +\infty)$ so that

$$\text{Sp}(D^{sa}) \subset (-\infty, -m] \cup [m, +\infty).$$

This ends the proof of Proposition 1.12.

5.2. Study of the point spectrum via the Virial Theorem. Let λ be an eigenvalue of D^{sa} associated with the eigenfunction u . We define by $u_\alpha = u(\alpha \cdot)$. By Proposition 1.7, we have that u_α belongs to $\mathcal{D}(D^{sa})$. Taking the scalar product of $-i\sigma \cdot \nabla u_\alpha$ with u and integrating by parts, we get $\langle (\lambda 1_2 - m\sigma_3)u_\alpha, u \rangle_{L^2} = 0$. Taking the limit $\alpha \rightarrow 1$, we obtain

$$\langle (\lambda 1_2 - m\sigma_3)u, u \rangle_{L^2} = 0.$$

This ensures that $|\lambda| \leq |m|$. The case $|\lambda| = |m|$ is not admissible neither. Indeed, this would imply that at least one component of the function u is 0 and the boundary condition would impose that $u = 0$. We get that there is no point spectrum in $\mathbb{R} \setminus (-|m|, |m|)$.

APPENDIX A. A RESULT ON RADIAL FUNCTIONS

Lemma A.1. *Let $a \in L^2((0, +\infty), r dr)$ be a function such that $\dot{a}, \frac{a}{r} \in L^2((0, +\infty), r dr)$. We have*

$$\operatorname{Re} \int_0^\infty \left(\overline{\dot{a}(r)} \frac{a(r)}{r} \right) r dr = 0.$$

Proof. The function $r^{1/2}a$ belongs to $H^1(0, +\infty)$ so that $a \in C^0(0, \infty)$. For $r_0 > 0$, we have that

$$\operatorname{Re} \int_{r_0}^\infty \left(\overline{\dot{a}(r)} \frac{a(r)}{r} \right) r dr = \int_{r_0}^\infty \frac{d}{dr} |a|^2(r) dr = -|a|^2(r_0)$$

so that $|a|^2$ has a finite limit at 0. Since $a/r \in L^2((0, +\infty), r dr)$, we get that $|a|^2(0) = 0$ and that

$$\operatorname{Re} \int_0^\infty \left(\overline{\dot{a}(r)} \frac{a(r)}{r} \right) r dr = 0.$$

□

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REFERENCES

- [1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions*, volume 55. Courier Corporation, 1964.
- [2] N. Arrizabalaga, L. Le Treust, and N. Raymond. Extension operator for the MIT bag model. working paper or preprint, June 2017.
- [3] C. Bär and W. Ballmann. Guide to elliptic boundary value problems for Dirac-type operators. In *Arbeitstagung Bonn 2013*, volume 319 of *Progr. Math.*, pages 43–80. Birkhäuser/Springer, Cham, 2016.
- [4] R. D. Benguria, S. Fournais, E. Stockmeyer, and H. Van Den Bosch. Self-adjointness of two-dimensional Dirac operators on domains. *Ann. Henri Poincaré*, 18(4):1371–1383, 2017.
- [5] B. Booß Bavnbeek, M. Lesch, and C. Zhu. The Calderón projection: new definition and applications. *J. Geom. Phys.*, 59(7):784–826, 2009.
- [6] A. Chodos. Field-theoretic Lagrangian with baglike solutions. *Phys. Rev. D*, 12(8):2397–2406, Oct. 1975.
- [7] A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn. Baryon structure in the bag theory. *Phys. Rev. D*, 10(8):2599–2604, Oct. 1974.
- [8] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf. New extended model of hadrons. *Phys. Rev. D*, 9(12):3471–3495, June 1974.
- [9] F. Demengel and G. Demengel. *Espaces fonctionnels*. Savoirs Actuels (Les Ulis). [Current Scholarship (Les Ulis)]. EDP Sciences, Les Ulis; CNRS Éditions, Paris, 2007. Utilisation dans la résolution des équations aux dérivées partielles. [Application to the solution of partial differential equations].
- [10] J. Dittrich, P. Exner, and P. Šeba. Dirac operators with a spherically symmetric δ -shell interaction. *J. Math. Phys.*, 30(12):2875–2882, 1989.
- [11] M. J. Esteban, M. Lewin, and E. Séré. Domains for dirac-coulomb min-max levels. *arXiv preprint arXiv:1702.04976*, 2017.
- [12] P. Grisvard. *Boundary value problems in non-smooth domains*, volume 19. University of Maryland, Dept. of Mathematics, 1980.

- [13] P. Grisvard. *Singularities in boundary value problems*, volume 22 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris; Springer-Verlag, Berlin, 1992.
- [14] A. Hosaka and H. Toki. *Quarks, Baryons and Chiral Symmetry*. World Scientific Publishing Company Incorporated, 2001.
- [15] K. Johnson. The MIT bag model. *Acta Phys. Pol.*, B6:865–892, 1975.
- [16] V. A. Kondrat'ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Trudy Moskov. Mat. Obšč.*, 16:209–292, 1967.
- [17] J. Necas. *Direct methods in the theory of elliptic equations*. Springer Science & Business Media, 2011.
- [18] F. W. Olver. *Asymptotics and special functions*. Academic press, 2014.
- [19] T. Ourmières-Bonafos and L. Vega. A strategy for self-adjointness of Dirac operators: Applications to the MIT bag model and delta-shell interactions. *ArXiv e-prints*, Dec. 2016.
- [20] M. Reed and B. Simon. *Methods of Modern Mathematical Physics, Volume 2: Fourier Analysis, Self-Adjointness*. Academic Press, New York, 1978.
- [21] M. Reed and B. Simon. *Methods of Modern Mathematical Physics, Volume 1: Functional Analysis*, volume 1. Gulf Professional Publishing, 1980.
- [22] B. Thaller. *The Dirac Equation*. Texts and Monographs in Physics. Berlin: Springer-Verlag., 1991.
- [23] J. Weidmann. *Spectral theory of ordinary differential operators*, volume 1258 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

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