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# A conjecture about prime numbers assuming the Riemann hypothesis

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## Abstract

In this paper we propose a conjecture about prime numbers. Based on the result of Pierre Dusart stating that the  $n^{\text{th}}$  prime number is smaller than  $n(\ln n + \ln \ln n - 0.9484)$  for  $n \geq 39017$  we propose that the  $n^{\text{th}}$  prime number is smaller than  $n(\ln n + \ln \ln n - 1^+)$  when  $n \rightarrow +\infty$ .

**Keywords:** Prime numbers, Dusart, Riemann hypothesis, conjecture

**Conjecture.** The  $n^{\text{th}}$  prime number is smaller than  $n(\ln n + \ln \ln n - 0.999\dots)$  when  $n \rightarrow +\infty$

We write  $\ln_2 n$  instead of  $\ln \ln n$ .

Let  $p(n)$  denote the  $n^{\text{th}}$  prime number. In this work we try to improve the result of Pierre Dusart, assuming the Riemann hypothesis and stating that for  $n \geq 39017$  :  $p(n) \leq n(\ln n + \ln \ln n - 0.9484)$  [1].

**Theorem.** For  $39017 \leq n \leq 2.10^{17}$ ,

$$p(n) \leq n(\ln n + \ln_2 n - 0.9484)$$

*Proof in [1].* We deduce that:

$$n(\ln n + \ln_2 n - 0.9484) = n(\ln n + \ln_2 n - 1 + 0.0516)$$

$$n(\ln n + \ln_2 n - 1 + 0.0516) = n(\ln n + \ln_2 n - 1) + 0.0516n$$

$$n(\ln n + \ln_2 n - 1) + 0.0516n = n(\ln n + \ln_2 n - 1) + \frac{129n}{2500}$$

If  $p(n) \leq n(\ln n + \ln_2 n - 0.9484)$  we have:

$$p(n) \leq n(\ln n + \ln_2 n - 1) + \frac{129n}{2500}$$

.

Consequently we have

$$p(n) - (n(\ln n + \ln_2 n - 1)) \leq n(\ln n + \ln_2 n - 1) + \frac{129n}{2500} - (n(\ln n + \ln_2 n - 1)) \quad (A)$$

According to (A) if  $p(n) \leq n(\ln n + \ln_2 n - 0.9484)$  we have:

$$\frac{n}{n(\ln n + \ln_2 n - 1) + \frac{129n}{2500} - (n(\ln n + \ln_2 n - 1))} \leq \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))}$$

$$\frac{n}{\frac{129n}{2500}} \leq \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))}$$

$$19.37984496 \leq \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \quad (B)$$

We confirmed the result (B) for  $39017 \leq n \leq 2.10^{17}$  by using a statistical approach and we observe that  $\frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))}$  increases when n increases.

The  $n^{th}$  prime numbers were found using a list of prime numbers and with a program (see Tools).

Let x and y be two positive real numbers, we deduce:

$$p(n) \leq n(\ln n + \ln_2 n - 1) + \frac{x.n}{y} \Leftrightarrow \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \geq \frac{y}{x} \quad (C)$$

*Examples.* For  $n = 10^5$  we have  $p(10^5) = 1299709$ .

$$\frac{10^5}{1299709 - (10^5(\ln 10^5 + \ln_2 10^5 - 1))} = 24.57354013 \geq 19.37984496$$

Consequently  $1299709 \leq 10^5(\ln 10^5 + \ln_2 10^5 - 0.9484)$

In this first example we have  $x = 129$  and  $y = 2500$  but we can choose other values if

$$\frac{y}{x} \leq 24.57354013.$$

For  $n = 2.10^{17}$  we have  $p(2.10^{17}) = 8512677386048191063$

$$\frac{2 \cdot 10^{17}}{8512677386048191063 - (2 \cdot 10^{17} (\ln 2 \cdot 10^{17} + \ln_2 2 \cdot 10^{17} - 1))} = 24.099471 \geq 19.37984496$$

Consequently  $8512677386048191063 \leq 2 \cdot 10^{17} (\ln 2 \cdot 10^{17} + \ln_2 2 \cdot 10^{17} - 0.9484)$

But if we choose  $x = 2$  and  $y = 48$  we have  $\frac{48}{2} = 24 \leq 24.099471$  and

$$8512677386048191063 \leq 2 \cdot 10^{17} (\ln 2 \cdot 10^{17} + \ln_2 2 \cdot 10^{17} - 1 + \frac{2}{48})$$

$$8512677386048191063 \leq 2 \cdot 10^{17} (\ln 2 \cdot 10^{17} + \ln_2 2 \cdot 10^{17} - \frac{23}{24})$$

**Conjecture.** Based on our previous statistical approach we conjecture that  $\frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))}$  increases when  $n \rightarrow +\infty$ . More precisely we conjecture that  $\frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \rightarrow +\infty$  when  $n \rightarrow +\infty$ . For this reason we have, according to (C):  $\frac{y}{x}$  that can be very high, consequently  $\frac{x \cdot n}{y} \rightarrow 0^+ n$

When  $n \rightarrow +\infty$  we deduce that:  $p(n) \leq n(\ln n + \ln_2 n - 1) + 0^+ n \leftrightarrow p(n) \leq n(\ln n + \ln_2 n - 1^+)$

Finally we conjecture that, when  $n \rightarrow +\infty$ :

$$p(n) \leq n(\ln n + \ln_2 n - 1^+) \leftrightarrow \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \geq \frac{y}{x}$$

It remains to prove that  $\frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \rightarrow +\infty$  when  $n \rightarrow +\infty$ .

## Tools

**Statistics.** Statistics were performed using Microsoft Excel 2016 and with a program.

**The list of prime numbers used in this study.** [http://compoasso.free.fr/primelistweb/page/prime/liste\\_online.php](http://compoasso.free.fr/primelistweb/page/prime/liste_online.php)

## Reference

1. PIERRE DUSART, The  $k^{\text{th}}$  prime is greater than  $k(\ln k + \ln \ln k - 1)$  for  $k \geq 2$ , Math. Comp. 68 (1999), 411-415