

A conjecture about prime numbers assuming the Riemann hypothesis

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Abstract

In this paper we propose a conjecture about prime numbers. Based on the result of Pierre Dusart stating that the n^{th} prime number is smaller than $n(\ln n + \ln \ln n - 0.9484)$ for $n \geq 39017$ we propose that the n^{th} prime number is smaller than $n(\ln n + \ln \ln n - 1^+)$ when $n \rightarrow +\infty$.

Keywords: Prime numbers, Dusart, Riemann hypothesis, conjecture

Conjecture. The n^{th} prime number is smaller than $n(\ln n + \ln \ln n - 0.999\dots)$ when $n \rightarrow +\infty$

We write $\ln_2 n$ instead of $\ln \ln n$.

Let $p(n)$ denote the n^{th} prime number. In this work we try to improve the result of Pierre Dusart, assuming the Riemann hypothesis and stating that for $n \geq 39017$: $p(n) \leq n(\ln n + \ln \ln n - 0.9484)$ [1].

Theorem. For $39017 \leq n \leq 2.10^{17}$,

$$p(n) \leq n(\ln n + \ln_2 n - 0.9484)$$

Proof in [1]. We deduce that:

$$n(\ln n + \ln_2 n - 0.9484) = n(\ln n + \ln_2 n - 1 + 0.0516)$$

$$n(\ln n + \ln_2 n - 1 + 0.0516) = n(\ln n + \ln_2 n - 1) + 0.0516n$$

$$n(\ln n + \ln_2 n - 1) + 0.0516n = n(\ln n + \ln_2 n - 1) + \frac{129n}{2500}$$

If $p(n) \leq n(\ln n + \ln_2 n - 0.9484)$ we have:

$$p(n) \leq n(\ln n + \ln_2 n - 1) + \frac{129n}{2500}$$

.

Consequently we have

$$p(n) - (n(\ln n + \ln_2 n - 1)) \leq n(\ln n + \ln_2 n - 1) + \frac{129n}{2500} - (n(\ln n + \ln_2 n - 1)) \quad (A)$$

According to (A) if $p(n) \leq n(\ln n + \ln_2 n - 0.9484)$ we have:

$$\frac{n}{n(\ln n + \ln_2 n - 1) + \frac{129n}{2500} - (n(\ln n + \ln_2 n - 1))} \leq \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))}$$

$$\frac{\frac{129n}{2500}}{n} \leq \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))}$$

$$19.37984496 \leq \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \quad (B)$$

We confirmed the result (B) for $39017 \leq n \leq 2.10^{17}$ by using a statistical approach and we observe that $\frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))}$ increases when n increases.

The n^{th} prime numbers were found using a list of prime numbers and with a program (see Tools).

Let x and y be two positive real numbers, we deduce:

$$p(n) \leq n(\ln n + \ln_2 n - 1) + \frac{x.n}{y} \leftrightarrow \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \geq \frac{y}{x} \quad (C)$$

Examples. For $n = 10^5$ we have $p(10^5) = 1299709$.

$$\frac{10^5}{1299709 - (10^5(\ln 10^5 + \ln_2 10^5 - 1))} = 24.57354013 \geq 19.37984496$$

Consequently $1299709 \leq 10^5(\ln 10^5 + \ln_2 10^5 - 0.9484)$

In this first example we have $x = 129$ and $y = 2500$ but we can choose other values if

$$\frac{y}{x} \leq 24.57354013.$$

For $n = 2.10^{17}$ we have $p(2.10^{17}) = 8512677386048191063$

$$\frac{2 \cdot 10^{17}}{8512677386048191063 - (2 \cdot 10^{17} (\ln 2 \cdot 10^{17} + \ln_2 2 \cdot 10^{17} - 1))} = 24.099471 \geq 19.37984496$$

Consequently $8512677386048191063 \leq 2 \cdot 10^{17} (\ln 2 \cdot 10^{17} + \ln_2 2 \cdot 10^{17} - 0.9484)$

But if we choose $x = 2$ and $y = 48$ we have $\frac{48}{2} = 24 \leq 24.099471$ and

$$8512677386048191063 \leq 2 \cdot 10^{17} (\ln 2 \cdot 10^{17} + \ln_2 2 \cdot 10^{17} - 1 + \frac{2}{48})$$

$$8512677386048191063 \leq 2 \cdot 10^{17} (\ln 2 \cdot 10^{17} + \ln_2 2 \cdot 10^{17} - \frac{23}{24})$$

Conjecture. Based on our previous statistical approach we conjecture that $\frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \rightarrow +\infty$ increases when $n \rightarrow +\infty$. More precisely we conjecture that $\frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \rightarrow +\infty$ when $n \rightarrow +\infty$. For this reason we have, according to (C): $\frac{y}{x}$ that can be very high, consequently $\frac{x \cdot n}{y} \rightarrow 0^+ n$

When $n \rightarrow +\infty$ we deduce that: $p(n) \leq n(\ln n + \ln_2 n - 1) + 0^+ n \leftrightarrow p(n) \leq n(\ln n + \ln_2 n - 1^+)$

Finally we conjecture that, when $n \rightarrow +\infty$:

$$p(n) \leq n(\ln n + \ln_2 n - 1^+) \leftrightarrow \frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \geq \frac{y}{x}$$

It remains to prove that $\frac{n}{p(n) - (n(\ln n + \ln_2 n - 1))} \rightarrow +\infty$ when $n \rightarrow +\infty$.

Tools

Statistics. Statistics were performed using Microsoft Excel 2016 and with a program.

The list of prime numbers used in this study. http://compoasso.free.fr/primelistweb/page/prime/liste_online.php

Reference

1. PIERRE DUSART, The k^{th} prime is greater than $k(\ln k + \ln \ln k - 1)$ for $k \geq 2$, Math. Comp. 68 (1999), 411-415