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Christophe Leuridan. Complementability and maximality in different contexts: ergodic theory, Brownian and poly-adic iterations. 2017. hal-01560477v1

**HAL Id: hal-01560477**

**<https://hal.science/hal-01560477v1>**

Preprint submitted on 11 Jul 2017 (v1), last revised 6 Sep 2019 (v5)

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# Complementability and maximality in different contexts: ergodic theory, Brownian and poly-adic filtrations

Christophe Leuridan

July 11, 2017

## Abstract

The notions of complementability and maximality were introduced in 1974 by Ornstein and Weiss in the context of the automorphisms of a probability space, in 2008 by Brossard and Leuridan in the context of the Brownian filtrations, and in 2017 by Leuridan in the context of the poly-adic filtrations indexed by the non-positive integers. We present here some striking analogies and also some differences existing between these three contexts.

*Mathematics Subject Classification* : 37A05, 60J05.

*Keywords* : Automorphisms of Lebesgue spaces, factors, entropy, filtrations indexed by the non-positive integers, poly-adic filtrations, Brownian filtrations, immersed filtrations, complementability, maximality.

## 1 Introduction

### 1.1 General context

In the present paper, we will work with three types of objects: automorphisms of Lebesgue spaces, Brownian filtrations and filtrations indexed by  $\mathbf{Z}$  or  $\mathbf{Z}_-$ ; the reason for choosing  $\mathbf{Z}$  or  $\mathbf{Z}_-$  and to rule out  $\mathbf{Z}_-$  is that for discrete-time filtrations, the interesting phenomena occur near time  $-\infty$ .

Among the invertible measure-preserving maps, Bernoulli shifts form a remarkable class. Similarly, the product-type filtrations (namely, the filtrations generated by sequences of independent random variables) are considered as a well-understood class. The Brownian filtrations (namely, the filtrations generated by Brownian motions) form a natural and widely studied class of continuous-time filtrations, although less simple.

Measure-preserving maps considered here will be taken on diffuse Lebesgue spaces. Various equivalent definitions of Lebesgue spaces exist. A possible definition of a Lebesgue space is a probability space which is isomorphic modulo the null sets to the union of some sub-interval of  $[0, 1]$ , endowed with the Lebesgue  $\sigma$ -field and the Lebesgue measure, and a countable set of atoms. Most of the time, the Lebesgue space considered is non-atomic, so the sub-interval is  $[0, 1]$  itself. The class of Lebesgue spaces includes the completion of every Polish space. See [14] to find the main properties of Lebesgue spaces or [10] to get equivalent definitions. Working on Lebesgue spaces provides non-trivial measurability results, existence of generators... We recall in section 6 the definitions and the main properties of partitions, generators, entropy used in the present paper.

Similarly, the filtrations considered here will be defined on a standard Borel probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , namely  $(\Omega, \mathcal{F})$  is the Borel space associated to some Polish space, to ensure the existence of regular conditional probabilities. We say that sub- $\sigma$ -fields  $\mathcal{A}$  and  $\mathcal{B}$  are equal modulo the null events (or modulo  $\mathbb{P}$ ) and we write  $\mathcal{A} = \mathcal{B} \pmod{\mathbb{P}}$  if for every  $A \in \mathcal{A}$ , there exists  $B \in \mathcal{B}$  such that  $\mathbb{P}(A \Delta B) = 0$ , and conversely. We do not complete systematically the  $\sigma$ -fields since it would cause troubles when working with conditional probabilities.

## 1.2 Reminders on filtrations indexed by $\mathbf{Z}$ or $\mathbf{Z}_-$

We now recall some classical but less known definitions and facts on filtrations. Given a filtration  $(\mathcal{F}_n)_n$  indexed by  $\mathbf{Z}$  or  $\mathbf{Z}_-$ , one says that  $(\mathcal{F}_n)_n$  is *product-type* if  $(\mathcal{F}_n)_n$  can be generated modulo  $\mathbb{P}$  by some sequence  $(I_n)_n$  of (independent) random variables.

One says that  $(\mathcal{F}_n)_n$  has *independent increments* if there exists a sequence  $(I_n)_n$  of random variables such that for every  $n$  in  $\mathbf{Z}$  or  $\mathbf{Z}_-$ ,

$$\mathcal{F}_n = \mathcal{F}_{n-1} \vee \sigma(I_n) \pmod{\mathbb{P}} \text{ and } I_n \text{ is independent of } \mathcal{F}_{n-1}.$$

Such a sequence  $(I_n)_n$  is called a *sequence of innovations* and is necessarily a sequence of independent random variables.

One says that  $(\mathcal{F}_n)_n$  is  $(a_n)_n$ -*adic* when it admits some sequence  $(I_n)_n$  of innovations such that each  $I_n$  is uniformly distributed on some finite set which size  $a_n$ . One says that  $(\mathcal{F}_n)_n$  is *poly-adic* when  $(\mathcal{F}_n)_n$  is  $(a_n)_n$ -adic for some sequence  $(a_n)_n$  of positive integers, called *adicity*.

One says that  $(\mathcal{F}_n)_n$  is *Kolmogorovian* if the tail  $\sigma$ -field

$$\mathcal{F}_{-\infty} := \bigcap_n \mathcal{F}_n$$

is trivial (namely, contains only events with probability 0 or 1).

By the definition and by Kolmogorov's zero-one law, any filtration indexed by  $\mathbf{Z}$  or  $\mathbf{Z}_-$  must have independent increments and must be Kolmogorovian to be product-type. But Vershik showed in [28] that the converse is not true. Actually Vershik worked with decreasing sequences of measurable partitions indexed by  $\mathbf{Z}_+$  and this frame was translated into filtrations indexed by  $\mathbf{Z}_-$  by M. Emery and W. Scharchermayer [12].

## 1.3 $K$ -automorphisms

The Kolmogorov property for filtrations indexed by  $\mathbf{Z}$  or  $\mathbf{Z}_-$  has an analogue for dynamical systems, although the definition is less simple in this frame : given an invertible measure preserving map  $T$  of a Lebesgue space  $(Z, \mathcal{Z}, \pi)$ , one says that  $T$  is a  *$K$ -automorphism* (or that  $T$  has completely positive entropy) if for every  $A \in \mathcal{Z}$  such that  $h(T, \{A, A^c\}) > 0$  whenever  $0 < \pi(A) < 1$ . This condition is the triviality of *Pinsker's factor*, namely of the  $\sigma$ -field

$$\Pi(T) := \{A \in \mathcal{Z} : h(T, \{A, A^c\}) = 0\}.$$

Actually, the 'events' of Pinsker's factor can be seen as the 'asymptotic events'. Indeed, if  $\gamma$  is a countable generator of  $(Z, \mathcal{Z}, \pi, T)$ , then

$$\Pi(T) = \bigcap_{n \geq 0} \bigvee_{k \geq n} T^{-k} \gamma = \bigcap_{n \geq 0} \bigvee_{k \geq n} T^k \gamma \pmod{\pi}.$$

To make the analogy clearer, set  $\gamma = \{A_\lambda, \lambda \in \Lambda\}$ . To each  $x \in Z$ , call  $f(x) \in \Lambda$  the only index  $\lambda$  such that  $x \in A_\lambda$ . For every  $k \in \mathbf{Z}$ , the  $\sigma$ -field generated by  $T^{-k}\gamma$  is the  $\sigma$ -field associated  $f \circ T^k$  viewed as a  $\Lambda$ -valued random variable on  $(Z, \mathcal{Z}, \pi)$ . Therefore,  $\Pi(T)$  is the asymptotic  $\sigma$ -field generated by the sequence  $(f \circ T^k)_{k \geq 0}$ .

## 1.4 Content of the paper

We have just viewed the analogy between the ‘Kolmogorovianity’ of a filtration indexed by  $\mathbf{Z}_-$  and the  $K$ -property of an automorphism of a Lebesgue space.

The next section is devoted to a parallel presentation of analogue notions and results in the three following contexts: automorphisms of Lebesgue spaces, filtrations indexed by  $\mathbf{Z}_-$  and Brownian filtrations. We investigate two notions - complementability and maximality - involving factors or poly-adic immersed filtrations or Brownian immersed filtrations according to the context. The results presented are essentially due to Ornstein and Weiss [22], Ornstein [21], and Thouvenot [26] for automorphisms of Lebesgue spaces; They come from [19] for filtrations indexed by  $\mathbf{Z}_-$ . They are due to Brossard, Émery and Leuridan [6, 4, 5] for Brownian filtrations.

With some restrictions on the nature of the complement, complementability implies maximality. Section 3 is devoted to the proof of this implication. The converse was already known to be false for factors of automorphisms of Lebesgue spaces and for poly-adic immersed filtrations. In section 4, we provide a counter-example in the context of Brownian filtrations. The construction relies on a counter-example for poly-adic immersed filtrations which is inspired by non-published notes of Tsirelson [27].

In spite of the similitude of the notions regardless the context, some differences exist. In section 5, we provide a non-complementable filtration (associated to a stationary process) yielding a complementable factor. This example is inspired by Vershik’s decimation process (example 2 in [28]).

In section 6, we recall the definitions and the main properties of partitions, generators, entropy used in the present paper.

## 2 Parallel notions and results

### 2.1 Factors and Immersed filtrations

Given an invertible measure preserving map  $T$  of a Lebesgue space  $(Z, \mathcal{Z}, \pi)$ , we call *factor of  $T$* , or more rigorously a factor of the dynamical system  $(Z, \mathcal{Z}, \pi, T)$ , any sub- $\sigma$ -field  $\mathcal{B}$  of  $\mathcal{Z}$  such that  $T^{-1}\mathcal{B} = \mathcal{B} = T\mathcal{B} \pmod{\pi}$ . Actually, the factor is the dynamical system  $(Z, \mathcal{B}, \pi|_{\mathcal{B}}, T)$ , which will be abbreviated in  $(T, \mathcal{B})$  in the present paper. This definition of factor is equivalent to the usual one.<sup>1</sup>

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<sup>1</sup>Actually, Rokhlin’s theory ensures that if  $\mathcal{B}$  is a factor of a Lebesgue space  $(Z, \mathcal{Z}, \pi, T)$ , then there exists a map  $f$  from  $Z$  to some Polish space  $E$  such that  $\mathcal{B}$  is generated up to the negligible events by the map  $\Phi : x \mapsto (f(T^k(x)))_{k \in \mathbf{Z}}$  from  $Z$  to the product space  $E^{\mathbf{Z}}$ . Let  $\nu = \Phi(\pi)$ . Then the completion  $(E^{\mathbf{Z}}, \mathcal{B}(E^{\mathbf{Z}}), \nu)$  is a Lebesgue space, the shift operator  $S : (y_k)_{k \in \mathbf{Z}} \mapsto (y_{k+1})_{k \in \mathbf{Z}}$  is an automorphism of  $E^{\mathbf{Z}}$ , and  $S \circ \Phi = \Phi \circ T$ .

Conversely, if  $(Y, \mathcal{Y}, \nu, S)$  is a dynamical system and  $\Phi$  a measurable map from  $Z$  to  $Y$  such that  $\Phi(\pi) = \nu$  and  $S \circ \Phi = \Phi \circ T$ , then the  $\sigma$ -field  $\Phi^{-1}(\mathcal{Y})$  is a factor of  $(Z, \mathcal{Z}, \pi, T)$ .

Given two filtrations  $(\mathcal{U}_t)_{t \in T}$  and  $(\mathcal{Z}_t)_{t \in T}$  on some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , indexed by a same subset  $T$  of  $\mathbb{R}$ , one says that  $(\mathcal{U}_t)_{t \in T}$  is *immersed in*  $(\mathcal{Z}_t)_{t \in T}$  if every martingale in  $(\mathcal{U}_t)_{t \in T}$  is still a martingale in  $(\mathcal{Z}_t)_{t \in T}$ . The notion of immersion is stronger than the inclusion. Actually,  $(\mathcal{U}_t)_{t \in T}$  is immersed in  $(\mathcal{Z}_t)_{t \in T}$  if and only if the two conditions below hold:

1. for every  $t \in T$ ,  $\mathcal{U}_t \subset \mathcal{Z}_t$ .
2. for every  $s < t$  in  $T$ ,  $\mathcal{U}_t$  and  $\mathcal{Z}_s$  are independent conditionally on  $\mathcal{U}_s$ .

The additional condition means that the largest filtration does not give information in advance on the the smallest one. We also make the useful following observation.

**Lemma 1.** *Assume that  $(\mathcal{U}_t)_{t \in T}$  is immersed in  $(\mathcal{Z}_t)_{t \in T}$ . Then  $(\mathcal{U}_t)_{t \in T}$  is completely determined (up to null sets) by its final  $\sigma$ -field*

$$\mathcal{U}_\infty := \bigvee_{t \in T} \mathcal{U}_t.$$

*More precisely,  $\mathcal{U}_t = \mathcal{U}_\infty \cap \mathcal{Z}_t \text{ mod } \mathbb{P}$  for every  $t \in T$ . In particular, if  $\mathcal{U}_\infty = \mathcal{Z}_\infty \text{ mod } \mathbb{P}$ , then  $\mathcal{U}_t = \mathcal{U}_\infty \cap \mathcal{Z}_t \text{ mod } \mathbb{P}$  for every  $t \in T$ .*

*Proof.* The inclusion  $\mathcal{U}_t \subset \mathcal{U}_\infty \cap \mathcal{Z}_t$  is immediate. To prove the converse, let  $A \in \mathcal{U}_\infty \cap \mathcal{Z}_t$ . Since  $\mathcal{U}_\infty$  and  $\mathcal{Z}_t$  are independent conditionally on  $\mathcal{U}_t$ , we get  $\mathbb{P}[A|\mathcal{U}_t] = \mathbb{P}[A|\mathcal{Z}_t] = \mathbf{1}_A$  a.s., so  $A \in \mathcal{U}_t \text{ mod } \mathbb{P}$ .  $\square$

When one works with Brownian filtrations, namely with filtrations generated by Brownian motions, then the immersion has many equivalent translations. The next statements are very classical (close statements are proved in [1]) and they rely on the stochastic calculus and the predictable representation property of Brownian filtrations.

**Proposition 2.** *Let  $(B_t)_{t \geq 0}$  be a finite-dimensional Brownian motion adapted to some filtration  $(\mathcal{Z}_t)_{t \geq 0}$ , and  $(\mathcal{B}_t)_{t \geq 0}$  its natural filtration. The following statements are equivalent.*

1.  $(B_t)_{t \geq 0}$  is a martingale in  $(\mathcal{Z}_t)_{t \geq 0}$ .
2.  $(\mathcal{B}_t)_{t \geq 0}$  is immersed in  $(\mathcal{Z}_t)_{t \geq 0}$ .
3. For every  $t \geq 0$ , the process  $B_{t+} - B_t$  is independent of  $\mathcal{Z}_t$ .
4.  $(B_t)_{t \geq 0}$  is a Markov process in  $(\mathcal{Z}_t)_{t \geq 0}$ .

**Definition 3.** *When these statements hold, we say that  $(B_t)_{t \geq 0}$  is a Brownian motion in the filtration  $(\mathcal{Z}_t)_{t \geq 0}$ .*

Note the analogy between the following two results.

**Theorem 4** (Ornstein [20]). *Every factor of a Bernoulli shift is equivalent to a Bernoulli shift.*

**Theorem 5** (Vershik [28]). *If  $(\mathcal{Z}_n)_{n \leq 0}$  is a product-type filtration such that the final sigma-field  $\mathcal{Z}_0$  is essentially separable, then every poly-adic filtration immersed in  $(\mathcal{Z}_n)_{n \leq 0}$  is product-type.*

One-dimensional Brownian filtrations can be viewed as a continuous-time versions of dyadic product-type filtrations. In this analogy, the predictable representation property of the continuous-time filtration corresponds to the diadicity of the discrete-time filtration. Yet, the situation is much more involved when one works with Brownian filtrations, and the following question remains open.

If a filtration  $(\mathcal{F}_t)_{t \geq 0}$  is immersed in some (possibly infinite-dimensional) Brownian filtration and has the predictable representation property with regard to some one-dimensional Brownian motion  $\beta$  (namely, each martingale in  $(\mathcal{F}_t)_{t \geq 0}$  can be obtained as a stochastic integral with regard to  $\beta$ ), then is  $(\mathcal{F}_t)_{t \geq 0}$  necessarily a Brownian filtration?

A partial answer was given by Émery in [11].

**Theorem 6** (Émery[11]). *Assume that the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is Brownian after 0, namely, there exists a finite-dimensional Brownian motion  $(B_t)_{t \geq 0}$  in  $(\mathcal{F}_t)_{t \geq 0}$  such that for every  $t \geq \varepsilon > 0$ ,  $\mathcal{F}_t$  is generated by  $\mathcal{F}_\varepsilon$  and the increments  $(B_s - B_\varepsilon)_{\varepsilon \leq s \leq t}$ . If  $(B_t)_{t \geq 0}$  is immersed in some (possibly infinite-dimensional) Brownian filtration, then  $(\mathcal{F}_t)_{t \geq 0}$  is a Brownian filtration.*

In the statement, the role of the stronger hypothesis that  $(\mathcal{F}_t)_{t \geq 0}$  is Brownian after 0 is to guarantee that the difficulties arise only at time 0+, so the situation gets closer to filtrations indexed by  $\mathbf{Z}$  or  $\mathbf{Z}_-$ , for which the difficulties arise only at time  $-\infty$ .

## 2.2 Complementability

By complementability, we will mean the existence of existence of some independent complement, although we will have to precise the nature of the complement.

The following definition is abridged from [22].

**Definition 7.** *Let  $(Z, \mathcal{Z}, \pi, T)$  be a Lebesgue dynamical system and  $\mathcal{B}$  be a factor of  $T$ . One says that  $\mathcal{B}$  is complementable if  $\mathcal{B}$  possesses an independent complement in  $(Z, \mathcal{Z}, \pi, T)$ , namely a factor  $\mathcal{C}$  of  $T$  which is independent of  $\mathcal{B}$  (with regard to  $\pi$ ) such that  $\mathcal{B} \vee \mathcal{C} = \mathcal{Z} \text{ mod } \pi$ .*

If  $(Z, \mathcal{Z}, \pi, T)$  is the direct product of two dynamical systems  $(Z_1, \mathcal{Z}_1, \pi_1, T_1)$  and  $(Z_2, \mathcal{Z}_2, \pi_2, T_2)$ , then  $\mathcal{Z}_1 \otimes \{\emptyset, Z_2\}$  and  $\{\emptyset, Z_1\} \otimes \mathcal{Z}_2$  are factors of  $(Z, \mathcal{Z}, \pi, T)$  and each of them is a complement of the other one.

**Example 8.** *Let us give a counterexample: Let  $T$  be the Bernoulli shift on  $Z = \{-1, 1\}^{\mathbf{Z}}$  endowed with product  $\sigma$ -field  $\mathcal{Z}$  and the uniform law. The map  $\Phi : Z \rightarrow Z$  defined by  $\Phi((x_n)_{n \in \mathbf{Z}}) = (x_{n-1}x_n)_{n \in \mathbf{Z}}$  commutes with  $T$ , so  $\Phi^{-1}(\mathcal{Z})$  is a factor of  $T$ . One checks that this factor is not complementable. Yet, if  $p_0 : Z \rightarrow \{-1, 1\}$  is the canonical projection defined by  $p_0((x_n)_{n \in \mathbf{Z}}) = x_0$ , the  $\sigma$ -field  $p_0^{-1}(\mathcal{Z})$  is an independent complement of  $\Phi^{-1}(\mathcal{Z})$ , but this complement is not a factor.*

We now define the notion of complementability in the world of filtrations.

**Definition 9.** *Consider two filtrations  $(\mathcal{U}_t)_{t \in T}$  and  $(\mathcal{Z}_t)_{t \in T}$  on some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , indexed by a same subset  $T$  of  $\mathbb{R}$ , on some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . One says that  $(\mathcal{U}_t)_{t \in T}$  is complementable in  $(\mathcal{Z}_t)_{t \in T}$  if there exists a filtration  $(\mathcal{V}_t)_{t \in T}$  such that for every  $t \in T$ ,  $\mathcal{U}_t$  and  $\mathcal{V}_t$  are independent and  $\mathcal{U}_t \vee \mathcal{V}_t = \mathcal{Z}_t \text{ mod } \mathbb{P}$ .*

Since independent enlargements of a filtration always produce filtrations in which the initial filtration is immersed,  $(\mathcal{U}_t)_{t \in T}$  needs to be immersed in  $(\mathcal{Z}_t)_{t \leq T}$  to possess an independent complement.

We will use many times the next result, abridged from [19].

**Proposition 10.** *Keep the notations of the last definition. Let  $U$  be a random variable valued in some measurable space  $(E, \mathcal{E})$ , such that  $\sigma(U) = \bigvee_{t \in T} \mathcal{U}_t$ , and  $(\mathbb{P}_u)_{u \in E}$  a regular version of the conditional probability  $\mathbb{P}$  given  $U$ . Assume that  $(\mathcal{U}_t)_{t \in T}$  is complementable in  $(\mathcal{Z}_t)_{t \in T}$  by a filtration  $(\mathcal{V}_t)_{t \in T}$ . Then for  $U(\mathbb{P})$ -almost every  $u \in E$ , the filtered probability space  $(\Omega, \mathcal{A}, \mathbb{P}_u, (\mathcal{Z}_t)_{t \leq T})$  is isomorphic to the filtered probability space  $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{V}_t)_{t \leq T})$ .*

Let us give applications of the last result, that will be used in the present paper.

**Corollary 11.** *(Particular cases)*

- *If a filtration  $(\mathcal{U}_n)_{n \leq 0}$  is complementable in  $(\mathcal{Z}_n)_{n \leq 0}$  by some product-type filtration, then  $(\mathcal{Z}_n)_{n \leq 0}$  is product-type under almost every  $\mathbb{P}_u$ .*
- *If a filtration  $(\mathcal{U}_t)_{t \geq 0}$  is complementable in  $(\mathcal{Z}_t)_{t \geq 0}$  by some Brownian filtration, then  $(\mathcal{Z}_t)_{t \geq 0}$  is a Brownian filtration under almost every  $\mathbb{P}_u$ .*

Determining whether a 1-dimensional Brownian filtration immersed in a 2-dimensional Brownian filtration is complementable or not is often difficult. Except trivial cases, the only known cases are related to skew-product decomposition of the planar Brownian motion, see [5].

## 2.3 Maximality

The definition of the maximality requires a tool to measure the quantity of information. When one works with factors of an automorphism of a Lebesgue space, the quantity of information is the entropy. When one works with poly-adic filtrations, the quantity of information is the sequence of positive integers giving the adicity. When one works with poly-adic filtrations, the quantity of information is the dimension of any generating Brownian motion. The classical statements below show how these quantities vary when one considers a factor, a poly-adic immersed filtration, or a Brownian immersed filtration.

**Remark 12.** *(Quantity of information in subsystems)*

1. *If  $\mathcal{B}$  is a factor of  $(Z, \mathcal{Z}, \pi, T)$ , then  $h(T, \mathcal{B}) \leq h(T)$ .*
2. *If a  $(b_n)_{n \leq 0}$ -adic filtration is immersed in an  $(r_n)_{n \leq 0}$ -adic filtration, then  $b_n$  divides  $r_n$  for every  $n$ .*
3. *If a  $m$ -dimensional Brownian filtration is immersed in a  $n$ -dimensional Brownian filtration, then  $m \leq n$ .*

The first statement is very classical. The second one is proved in [19]. The last one shows that the dimension of a Brownian filtration makes sense and is proved in the footnote. <sup>2</sup>

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<sup>2</sup>Let  $Z$  be a  $n$ -dimensional Brownian motion and  $B$  be a  $m$ -dimensional Brownian motion in  $\mathcal{F}^Z$ . Then one can find an  $\mathcal{F}^Z$ -predictable process  $M$  taking values in the set of all  $p \times n$  real matrices whose lines form an orthonormal family, such that  $B = \int_0^\cdot M_s dZ_s$ . In particular, the  $m$  lines of each matrix  $M_s$  are independent and lie in a  $n$ -dimensional vector space, so  $m \leq n$ .

Let us give precise definitions, respectively abridged from [21], [19] and [6] or [4].

**Definition 13.** Let  $(Z, \mathcal{Z}, \pi, T)$  be a Lebesgue dynamical system and  $\mathcal{B}$  be a factor of  $T$ . One says that  $\mathcal{B}$  is maximal if  $(T, \mathcal{B})$  has a finite entropy and if for any factor  $\mathcal{A}$ , the conditions  $\mathcal{A} \supset \mathcal{B}$  and  $h(T, \mathcal{A}) = h(T, \mathcal{B})$  entail  $\mathcal{A} = \mathcal{B}$  modulo null sets.

**Definition 14.** Let  $(\mathcal{B}_n)_{n \leq 0}$  be a  $(b_n)_{n \leq 0}$ -adic filtration immersed in  $(\mathcal{Z}_n)_{n \leq 0}$ . One says that  $(\mathcal{B}_n)_{n \leq 0}$  is maximal in  $(\mathcal{Z}_n)_{n \leq 0}$  if every  $(b_n)_{n \leq 0}$ -adic filtration immersed in  $(\mathcal{Z}_n)_{n \leq 0}$  and containing  $(\mathcal{B}_n)_{n \leq 0}$  is equal to  $(\mathcal{B}_n)_{n \leq 0}$  modulo null events.

**Definition 15.** Let  $(\mathcal{B}_t)_{t \geq 0}$  be a  $d$ -dimensional Brownian filtration immersed in a Brownian filtration  $(\mathcal{Z}_t)_{t \geq 0}$ . One says that  $(\mathcal{B}_t)_{t \geq 0}$  is maximal in  $(\mathcal{Z}_t)_{t \geq 0}$  if every  $d$ -dimensional Brownian filtration immersed in a  $(\mathcal{Z}_t)_{t \geq 0}$  and containing  $(\mathcal{B}_t)_{t \geq 0}$  is equal to  $(\mathcal{B}_t)_{t \geq 0}$  modulo null events.

Let us come back to example 8 in which  $T$  be the Bernoulli shift on  $Z = \{-1, 1\}^{\mathbf{Z}}$  endowed with product  $\sigma$ -field  $\mathcal{Z}$  and the uniform law. Since the map  $\Phi : Z \rightarrow Z$  defined by  $\Phi((x_n)_{n \in \mathbf{Z}}) = (x_{n-1}x_n)_{n \in \mathbf{Z}}$  commutes with  $T$  and preserves the uniform law on  $Z$ , the factor  $(T, \Phi^{-1}(\mathcal{Z}))$  is a Bernoulli  $(1/2, 1/2)$  shift, like  $T$  itself. Hence  $(T, \Phi^{-1}(\mathcal{Z}))$  has the same (finite) entropy as  $T$ , so the factor  $\Phi^{-1}(\mathcal{Z})$  is not maximal.

This example above can be abridged in the context of filtrations indexed by the relative integers : consider a sequence  $(\xi_n)_{n \in \mathbf{N}}$  of independent uniform random variables taking values in  $\{-1, 1\}$ . Then the sequence  $(\eta_n)_{n \in \mathbf{N}}$  defined by  $\eta_n = \xi_{n-1}\xi_n$  has the same law as  $(\xi_n)_{n \in \mathbf{N}}$ . One checks that the filtration  $(\mathcal{F}_n^\eta)_{n \in \mathbf{N}}$  is immersed in  $(\mathcal{F}_n^\xi)_{n \in \mathbf{N}}$ , that these filtrations differ although both of them have a trivial tail  $\sigma$ -field at  $-\infty$  and  $(\eta_n)_{n \in \mathbf{N}}$  is an innovation sequence for  $(\mathcal{F}_n^\xi)_{n \in \mathbf{N}}$  as for  $(\mathcal{F}_n^\eta)_{n \in \mathbf{N}}$ . Actually, one bit of information is lost when one transforms  $(\xi_n)_{n \in \mathbf{N}}$  into  $(\eta_n)_{n \in \mathbf{N}}$ : for each  $n_0 \in \mathbf{Z}$ , the value  $\xi_{n_0}$  is independent of  $(\eta_n)_{n \in \mathbf{N}}$ , and the knowledge of  $\xi_{n_0}$  and  $(\eta_n)_{n \in \mathbf{N}}$  is sufficient to recover  $(\xi_n)_{n \in \mathbf{N}}$ . The paradox is that this loss of information is asymptotic at time  $-\infty$  but invisible when one looks at  $\mathcal{F}_{-\infty}^\xi$  and  $\mathcal{F}_{-\infty}^\eta$ .

The situation can be much more complex when one works with Brownian filtrations. For example, consider a linear Brownian motion  $W$ . Since  $W$  spends a null-time at 0, the stochastic integral

$$W' = \int_0^\cdot \text{sgn}(W_s) dW_s = |W| - L$$

(where  $L$  denotes the local time of  $W$  at 0) is still a linear Brownian motion. The natural filtration  $\mathcal{F}^{W'}$  is immersed and strictly included in  $\mathcal{F}^W$ , therefore it is not maximal in  $\mathcal{F}^W$ . Actually,  $W'$  generates the same filtration as  $|W|$ , so the Lévy transformation - which transforms the sample paths of  $W$  into the sample paths of  $W'$  - forgets the signs of all excursions of  $W$ , which are independent of  $|W|$ . Here, the loss of information occurs at each beginning of excursion of  $W$ , and not at time  $0+$ .

## 2.4 Necessary or sufficient conditions for maximality

Given a finite-entropy factor, a poly-adic immersed filtration or a Brownian immersed filtration, one wishes to enlarge it to get a maximal one having the same entropy, adicity or dimension. This leads to the following constructions, abridged from [21, 19, 4].



**Proposition 16.** *Let  $(Z, \mathcal{Z}, \pi, T)$  be a Lebesgue dynamical system and  $\mathcal{B}$  be a factor of  $T$  with finite entropy. Let  $\gamma$  is a generator of  $T$ . Then*

$$\mathcal{B}' := \bigcap_{n \geq 0} \left( \mathcal{B} \vee \bigvee_{k \geq n} T^{-k} \gamma \right).$$

*is a maximal factor containing  $\mathcal{B}$ , having the same entropy as  $\mathcal{B}$ . Moreover,  $\mathcal{B}'$  does not depend on the choice of the generator  $\gamma$ , up to null sets.*

**Proposition 17.** *Let  $(\mathcal{B}_n)_{n \leq 0}$  be a  $(b_n)_{n \leq 0}$ -adic filtration immersed in some filtration  $(\mathcal{Z}_n)_{n \leq 0}$ . Then  $(\mathcal{B}_n)_{n \leq 0}$  is immersed in the filtration  $(\mathcal{B}'_n)_{n \leq 0}$  defined by*

$$\mathcal{B}'_n := \bigcap_{s \leq 0} (\mathcal{B}_n \vee \mathcal{Z}_s).$$

*Moreover  $(\mathcal{B}'_n)_{n \leq 0}$  is a poly-adic filtration, immersed and maximal in  $(\mathcal{Z}_n)_{n \leq 0}$ , having the same adicity as  $(\mathcal{B}_n)_{n \leq 0}$ .*

**Proposition 18.** *Let  $(\mathcal{B}_t)_{t \geq 0}$  be a  $d$ -dimensional Brownian filtration immersed in some Brownian filtration  $(\mathcal{Z}_t)_{t \geq 0}$ . Then  $(\mathcal{B}_t)_{t \geq 0}$  is immersed in the filtration  $(\mathcal{B}'_t)_{t \geq 0}$  defined by*

$$\mathcal{B}'_t := \bigcap_{s > 0} (\mathcal{B}_t \vee \mathcal{Z}_s).$$

*Moreover  $(\mathcal{B}'_t)_{t \geq 0}$  is a  $d$ -dimensional Brownian filtration immersed in  $(\mathcal{Z}_t)_{t \geq 0}$ .*

The analogy between these three constructions is striking. In the three situations above, we must have  $\mathcal{B}' = \mathcal{B}$  up to null sets for  $\mathcal{B}$  to be maximal. Moreover, applying the same procedure to  $\mathcal{B}'$  leads to  $\mathcal{B}'' = \mathcal{B}'$ . Hence the condition  $\mathcal{B}' = \mathcal{B}$  up to null sets is also sufficient for  $\mathcal{B}$  to be maximal in the first two cases (factors and poly-adic filtrations).

Yet, once again, the situation is more complex when one works with Brownian filtrations, since the filtration  $\mathcal{B}'$  may be non-maximal. Here is a counterexample (the proof will be given in section 3).

**Example 19.** *Let  $Z = (X, Y)$  be a planar Brownian motion, set*

$$B = \int_0^\cdot \text{sgn}(X_s) dX_s,$$

*and call  $\mathcal{Z}$  and  $\mathcal{B}$  the natural filtration of  $Z$  and  $B$ . Then the filtration  $\mathcal{B}'$  defined by proposition 18 coincides with  $\mathcal{B}$  up to null events, so it is not maximal.*

Actually, the maximality of Brownian filtrations is not an asymptotic property at  $0+$ , unlike the almost sure equality  $\mathcal{B}' = \mathcal{B}$ . To try to produce a maximal Brownian filtration containing a given Brownian filtration, one should perform the infinitesimal enlargement above at every time, but we do not see how to do that.

Yet, the equality  $\mathcal{B}' = \mathcal{B} \pmod{\mathbb{P}}$  ensures the maximality of  $\mathcal{B}$  if we make the (strong) additional hypothesis that  $\mathcal{B}$  is complementable after 0, namely there exists a Brownian filtration  $\mathcal{C}$  immersed in  $\mathcal{Z}$  and independent of  $\mathcal{B}$  such that, up to null events,

$$\forall t \geq 0, \mathcal{Z}_t = \bigcap_{s > 0} (\mathcal{B}_t \vee \mathcal{C}_t \vee \mathcal{Z}_s) \pmod{\pi}.$$

See [4] for a proof of this statement.

The next sufficient condition for the maximality of a poly-adic immersed filtration comes from [19].

**Proposition 20.** *Let  $(\mathcal{B}_n)_{n \leq 0}$  be a  $(b_n)_{n \leq 0}$ -adic filtration immersed in  $(\mathcal{Z}_n)_{n \leq 0}$ . Let  $U$  is a random variable valued in some measurable space  $(E, \mathcal{E})$ , such that  $\sigma(U) = \mathcal{B}_0$  and  $(\mathbb{P}_u)_{u \in E}$  a regular version of the conditional probability  $\mathbb{P}$  given  $U$ . If for  $U(\mathbb{P})$ -almost every  $u \in E$ , the filtered probability space  $(\Omega, \mathcal{A}, \mathbb{P}_u, (\mathcal{Z}_n)_{n \leq 0})$  is Kolmogorovian, then the filtration  $(\mathcal{B}_n)_{n \leq 0}$  is maximal in  $(\mathcal{Z}_n)_{n \leq 0}$ .*

The assumption that  $(\mathcal{Z}_n)_{n \leq 0}$  is Kolmogorovian under almost every conditional probability  $\mathbb{P}_u$  echoes to the alternative terminology of conditional  $K$ -automorphisms used by Thouvenot in [26]. Indeed, up to null sets, the factor  $\mathcal{B}'$  defined in proposition 16 coincides with the  $\mathcal{B}$ -conditional Pinsker factor, which can be defined as

$$\Pi(T, \mathcal{B}) := \{A \in \mathcal{Z} : h(T, \{A, A^c\} | \mathcal{B}) = 0\},$$

where

$$h(T, \{A, A^c\} | \mathcal{B}) = \lim_{n \rightarrow +\infty} \frac{1}{n} H \left( \bigvee_{k=0}^{n-1} \{T^{-k} A, T^{-k} A^c\} \middle| \mathcal{B} \right).$$

When  $\Pi(T, \mathcal{B}) = \mathcal{B}$  up to null sets, Thouvenot says that  $T$  has the property  $K$  conditionally on  $\mathcal{B}$ ; this condition is equivalent to the maximality of  $\mathcal{B}$ .

## 2.5 Complementability and maximality

In the three contexts (factors of an automorphism of a Lebesgue space, poly-adic filtrations immersed in a filtration indexed by  $\mathbf{Z}_-$ , Brownian filtrations immersed in a Brownian filtration), we get very similar results.

The first one is stated by Ornstein in [21] as a direct consequence of lemma stated in [22].

**Proposition 21.** *Let  $\mathcal{B}$  be a factor of  $T$  with finite entropy. If  $\mathcal{B}$  is complementable by some  $K$ -automorphism, then  $\mathcal{B}$  is maximal.*

The second one comes from [19].

**Proposition 22.** *Let  $(\mathcal{B}_n)_{n \leq 0}$  be an  $(b_n)_{n \leq 0}$ -adic filtration immersed in  $(\mathcal{Z}_n)_{n \leq 0}$ . If  $(\mathcal{B}_n)_{n \leq 0}$  can be complemented by some Kolmogorovian filtration, then  $(\mathcal{B}_n)_{n \leq 0}$  is maximal in  $(\mathcal{Z}_n)_{n \leq 0}$ .*

A particular case of the third one (in which the dimension of the Brownian filtrations are 1 and 2) can be found in [6] or [4].

**Proposition 23.** *Let  $(\mathcal{B}_t)_{t \geq 0}$  be a Brownian filtration immersed in a Brownian filtration  $(\mathcal{Z}_t)_{t \geq 0}$  with larger dimension. If  $(\mathcal{B}_t)_{t \leq 0}$  can be complemented by some Brownian filtration, then  $(\mathcal{B}_t)_{t \leq 0}$  is maximal in  $(\mathcal{Z}_t)_{t \leq 0}$ .*

The proofs of these three statements are rather simple, and present some similarities, although they are different. In the next section, we provide two different proofs of proposition 21. The first one relies on Ornstein and Weiss' lemma (lemma 2 in [22]). The second one is a bit simpler but requires that  $T$  has finite entropy, and relies on Berg's lemma (lemma 2.3 in [3]). We also provide a proof of proposition 23. Proposition 22 follows from proposition 20 and corollary 11.

The converses of the three implications above are false, but providing counterexamples is not simple. Ornstein constructed in [21] an example of maximal but non-complementable factor in [21], but the proof is difficult to read. Two counter-examples of a maximal but non-complementable poly-adic filtration are given in [19]. In the present paper, we use a third example to construct a maximal but non-complementable Brownian filtration.

### 3 Complementability implies maximality: proofs

#### 3.1 Key lemma for factors of a Lebesgue automorphism

Proposition 21 follows from the next lemma, which can be derived from lemma in [22] or from lemma 2.3 in [3].

**Lemma 24.** *Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  be three factors of  $T$ . Assume that:*

1.  $\mathcal{A} \supset \mathcal{B}$ ;
2.  $h(T, \mathcal{A}) = h(T, \mathcal{B}) < +\infty$ ;
3.  $(T, \mathcal{C})$  has the property  $K$ .
4.  $\mathcal{B}$  and  $\mathcal{C}$  are independent.

*Then  $\mathcal{A}$  and  $\mathcal{C}$  are independent.*

First, we show how to deduce proposition 21 from lemma 24.

*Proof.* (Proof of proposition 21) Let  $\mathcal{C}$  be an independent complement of  $\mathcal{B}$  having the property  $K$ . Let  $\mathcal{A}$  be a factor of  $T$  such that  $\mathcal{A} \supset \mathcal{B}$  and  $h(T, \mathcal{A}) = h(T, \mathcal{B})$ . Then lemma 24 yields that  $\mathcal{A}$  and  $\mathcal{C}$  are independent. But  $\mathcal{Z} = \mathcal{B} \vee \mathcal{C} \subset \mathcal{A} \vee \mathcal{C} \subset \mathcal{Z}$ , so  $\mathcal{A} \vee \mathcal{C} = \mathcal{B} \vee \mathcal{C}$ . Hence  $\mathcal{A} = \mathcal{B}$ .  $\square$

At the end of the proof, we have used the following general statement, which will also help us in the context of Brownian filtrations.

**Lemma 25.** *Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  be three sub- $\sigma$ -fields of any probability space  $(Z, \mathcal{Z}, \pi)$  such that*

- $\mathcal{A} \supset \mathcal{B}$ ;
- $\mathcal{A}$  and  $\mathcal{C}$  are independent;
- $\mathcal{A} \vee \mathcal{C} = \mathcal{B} \vee \mathcal{C}$ .

*Then  $\mathcal{A} = \mathcal{B} \pmod{\pi}$ .*

*Proof.* Let  $A \in \mathcal{A}$ . Then  $\sigma(A) \vee \mathcal{B} \subset \mathcal{A}$ , so  $\sigma(A) \vee \mathcal{B}$  is independent of  $\mathcal{C}$ , and

$$\pi[A|\mathcal{B}] = \pi[A|\mathcal{B} \vee \mathcal{C}] = \pi[A|\mathcal{A} \vee \mathcal{C}] = \mathbf{1}_A \quad \pi - \text{almost surely.}$$

Hence  $A \in \mathcal{B} \pmod{\pi}$ .  $\square$

We now give two different proofs of lemma 24. The second one is a bit simpler, but it requires that  $(T, \mathcal{C})$  has a also finite entropy.

### 3.2 Proof of lemma 24

The proof below is a reformulation of the proof given in 24.

Assume that the assumptions hold. Let  $\alpha, \beta, \gamma$  be countable partitions generating  $(T, \mathcal{A}), (T, \mathcal{B}), (T, \mathcal{C})$ , respectively. Given  $n \geq 1$ , set

$$\mathcal{C}_n = \bigvee_{q \in \mathbf{Z}} T^{-qn} \gamma \text{ and } \mathcal{D}_n = \bigvee_{k \geq n} T^{-k} \gamma.$$

Then  $\mathcal{C}_n$  is a factor of  $T^n$  and  $\alpha \vee \dots \vee T^{-n+1} \alpha \vee \gamma$  is a generator of  $(T^n, \mathcal{A} \vee \mathcal{C}_n)$  whereas  $\beta \vee \dots \vee T^{-n+1} \beta \vee \gamma$  is a generator of  $(T^n, \mathcal{B} \vee \mathcal{C}_n)$ .

Therefore, on the one hand,

$$\begin{aligned} h(T^n, \mathcal{A} \vee \mathcal{C}_n) &= H\left(\alpha \vee \dots \vee T^{-n+1} \alpha \vee \gamma \middle| \bigvee_{q \geq 1} T^{-qn} (\alpha \vee \dots \vee T^{-n+1} \alpha \vee \gamma)\right) \\ &= H\left(\alpha \vee \dots \vee T^{-n+1} \alpha \vee \gamma \middle| \bigvee_{k \geq n} T^{-k} \alpha \vee \bigvee_{q \geq 1} T^{-qn} \gamma\right) \\ &= H\left(\alpha \vee \dots \vee T^{-n+1} \alpha \middle| \bigvee_{k \geq n} T^{-k} \alpha \vee \bigvee_{q \geq 1} T^{-qn} \gamma\right) \\ &\quad + H\left(\gamma \middle| \bigvee_{k \geq 0} T^{-k} \alpha \vee \bigvee_{q \geq 1} T^{-qn} \gamma\right) \\ &\leq H\left(\alpha \vee \dots \vee T^{-n+1} \alpha \middle| \bigvee_{k \geq n} T^{-k} \alpha\right) + H(\gamma | \alpha) \\ &= h(T^n, \mathcal{A}) + H(\gamma | \alpha) = nh(T, \mathcal{A}) + H(\gamma | \alpha). \end{aligned}$$

On the other hand, by independence of  $\mathcal{B}$  and  $\mathcal{C}$ ,

$$\begin{aligned} h(T^n, \mathcal{B} \vee \mathcal{C}_n) &= H\left(\beta \vee \dots \vee T^{-n+1} \beta \middle| \bigvee_{k \geq n} T^{-k} \beta \vee \bigvee_{q \geq 1} T^{-qn} \gamma\right) \\ &\quad + H\left(\gamma \middle| \bigvee_{k \geq 0} T^{-k} \beta \vee \bigvee_{q \geq 1} T^{-qn} \gamma\right) \\ &= H\left(\beta \vee \dots \vee T^{-n+1} \beta \middle| \bigvee_{k \geq n} T^{-k} \beta\right) + H\left(\gamma \middle| \bigvee_{q \geq 1} T^{-qn} \gamma\right) \\ &\geq h(T^n, \mathcal{B}) + H(\gamma | \mathcal{D}_n) = nh(T, \mathcal{B}) + H(\gamma | \mathcal{D}_n). \end{aligned}$$

But  $h(T^n, \mathcal{B} \vee \mathcal{C}_n) \leq h(T^n, \mathcal{A} \vee \mathcal{C}_n)$  since  $\mathcal{B} \subset \mathcal{A}$ . Putting things together and using the assumption  $h(T, \mathcal{A}) = h(T, \mathcal{B}) < +\infty$  yields  $H(\gamma | \mathcal{D}_n) \leq H(\gamma | \alpha)$ .

But  $(\mathcal{D}_n)_{n \geq 1}$  is a decreasing sequence of  $\sigma$ -fields with trivial intersection since  $(T, \mathcal{C})$  has the property  $K$ , so  $H(\gamma | \mathcal{D}_n) \rightarrow H(\gamma)$  as  $n \rightarrow +\infty$ . Hence,  $H(\gamma) \leq H(\gamma | \alpha)$ , so  $\alpha$  and  $\gamma$  are independent. The same conclusion holds if one replaces the generators  $\alpha$  and  $\gamma$  by the supremum of  $T^{-k} \alpha$  and  $T^{-k} \gamma$  over all  $k \in \llbracket -n, n \rrbracket$ . Letting  $n$  go to the infinity yields the independence of  $\mathcal{A}$  and  $\mathcal{C}$ .

### 3.3 Alternative proof of lemma 24

Here, we make the additional hypothesis that  $(T, \mathcal{C})$  has a finite entropy. The inclusion  $\mathcal{A} \supset \mathcal{B}$  and the independence of  $\mathcal{B}$  and  $\mathcal{C}$  yield

$$h(T, \mathcal{A}) + h(T, \mathcal{C}) \geq h(T, \mathcal{A} \vee \mathcal{C}) \geq h(T, \mathcal{B} \vee \mathcal{C}) \leq h(T, \mathcal{B}) + h(T, \mathcal{C}).$$

But  $h(T, \mathcal{A}) = h(T, \mathcal{B})$ , hence  $h(T, \mathcal{A} \vee \mathcal{C}) = h(T, \mathcal{A}) + h(T, \mathcal{C})$ . Since  $(T, \mathcal{C})$  is a  $K$ -automorphism with finite entropy, Berg's lemma below shows that  $\mathcal{A}$  and  $\mathcal{C}$  independent.

**Lemma 26.** (lemma 2.3 in [3]) *Let  $\mathcal{A}$  and  $\mathcal{C}$  be two factors of the dynamical system  $(Z, \mathcal{Z}, \pi, T)$ , such that  $h(T, \mathcal{A} \vee \mathcal{C}) = h(T, \mathcal{A}) + h(T, \mathcal{C}) < +\infty$  and  $(T, \mathcal{C})$  is a  $K$ -automorphism. Then  $\mathcal{A}$  and  $\mathcal{C}$  independent.*

*Proof.* Let  $\alpha$  and  $\gamma$  be countable generating partitions of  $(T, \mathcal{A})$  and  $(T, \mathcal{C})$ , respectively. Set

$$\alpha^- = \bigvee_{k \geq 1} T^{-k} \alpha \text{ and } \gamma^- = \bigvee_{k \geq 1} T^{-k} \gamma$$

Then  $h(T, \mathcal{A}) = H(\alpha | \alpha^-)$ ,  $h(T, \mathcal{C}) = H(\gamma | \gamma^-)$  and  $h(T, \mathcal{A} \vee \mathcal{C}) = H(\alpha \vee \gamma | \alpha^- \vee \gamma^-)$ . By Pinsker's formula (theorem 6.3 in [23]) :

$$H(\alpha \vee \gamma | \alpha^- \vee \gamma^-) = H(\alpha | \alpha^-) + H(\gamma | \mathcal{A} \vee \gamma^-).$$

Therefore, the assumption  $h(T, \mathcal{A} \vee \mathcal{C}) = h(T, \mathcal{A}) + h(T, \mathcal{C}) < +\infty$  yields  $H(\gamma | \gamma^-) = H(\gamma | \mathcal{A} \vee \gamma^-)$ .

For any partition  $\delta \subset \mathcal{A}$  with finite entropy, we derive  $H(\gamma | \delta \vee \gamma^-) = H(\gamma | \gamma^-)$ , so

$$H(\delta \vee \gamma | \gamma^-) = H(\delta | \gamma^-) + H(\gamma | \delta \vee \gamma^-) = H(\delta | \gamma^-) + H(\gamma | \gamma^-).$$

But we have also

$$H(\delta \vee \gamma | \gamma^-) = H(\gamma | \gamma^-) + H(\delta | \gamma \vee \gamma^-).$$

Hence  $H(\delta | \gamma^-) = H(\delta | \gamma \vee \gamma^-)$

Let  $m \geq 0$  and  $n$  be integers. Applying the last equality to  $\delta := \bigvee_{|k| \leq m} T^{n-k} \alpha$  yields

$$H\left(\bigvee_{|k| \leq m} T^{n-k} \alpha \mid \bigvee_{k \geq 1} T^{-k} \gamma\right) = H\left(\bigvee_{|k| \leq m} T^{n-k} \alpha \mid \bigvee_{k \geq 0} T^{-k} \gamma\right).$$

Since  $T$  preserves  $\pi$ , thais is equivalent to

$$H\left(\bigvee_{|k| \leq m} T^{-k} \alpha \mid \bigvee_{k \geq n+1} T^{-k} \gamma\right) = H\left(\bigvee_{|k| \leq m} T^{-k} \alpha \mid \bigvee_{k \geq n} T^{-k} \gamma\right).$$

As a result, the entropy above does not depend on  $n$ . Letting  $n$  go to  $-\infty$  and to  $+\infty$ , and using the fact that  $(T, \mathcal{C})$  is a  $K$ -automorphism, we get at the limit

$$H\left(\bigvee_{|k| \leq m} T^{-k} \alpha \mid \mathcal{C}\right) = H\left(\bigvee_{|k| \leq m} T^{-k} \alpha\right),$$

so the partition  $\bigvee_{|k| \leq m} T^{-k} \alpha$  is independent of  $\mathcal{C}$ . Letting  $m$  go to  $+\infty$  yields the independence of  $\mathcal{A}$  and  $\mathcal{C}$ .  $\square$

### 3.4 Proof in the context of Brownian filtrations

The proof below may look suspiciously simple, but actually, it relies on non-trivial theorem of stochastic integration, like the predictable representation property and the bracket characterization of multi-dimensional Brownian motions among local martingales. The immersion of a filtration into another one is a strong property, as shown for example the characterizations for a Brownian filtration recalled in the introduction (proposition 2).

The key step is very similar to lemma 24.

**Lemma 27.** *Let  $Z$  be a finite-dimensional Brownian motion. Let  $A, B, C$  be three Brownian motions in the filtration  $\mathcal{F}^Z$ . Assume that:*

1.  $\sigma(A) \supset \sigma(B)$ ;
2.  $A$  and  $B$  have the same dimension;
3.  $B$  and  $C$  are independent.

*Then  $A$  and  $C$  are independent.*

*Proof.* Call  $n$  the dimension of  $Z$ ,  $p$  the dimension of  $A$  and  $B$ ,  $q$  the dimension of  $C$ .

Since  $B$  is a Brownian motion in  $\mathcal{F}^Z$  and its own filtration, it is also a Brownian motion in the intermediate filtration  $\mathcal{F}^A$ . Here, one can find an  $\mathcal{F}^A$ -predictable process  $H$  with values in  $O_p(\mathbb{R})$  such that

$$B = \int_0^\cdot H_s dA_s.$$

Since  $H_s^* H_s = I_p$  for every  $s \geq 0$ , we have also

$$A = \int_0^\cdot H_s^* dB_s.$$

Looking at the components, we get for every  $i \in \llbracket 1, p \rrbracket$ ,

$$A^{(i)} = \sum_{j=1}^p \int_0^\cdot H_s(j, i) dB_s^{(j)}.$$

Since  $B$  (respectively  $C$ ) is a Brownian motion in  $\mathcal{F}^Z$ , one can find an  $\mathcal{F}^Z$ -predictable process  $M$  (respectively  $N$ ) taking values in the set of all  $p \times n$  (respectively  $q \times n$ ) real matrices whose lines form an orthonormal family, such that

$$B = \int_0^\cdot M_s dZ_s \quad \left( \text{respectively } B = \int_0^\cdot N_s dZ_s \right).$$

For every  $i \in \llbracket 1, p \rrbracket$  and  $k \in \llbracket 1, q \rrbracket$ , we get

$$\langle A^{(i)}, C^{(k)} \rangle = \sum_{j=1}^p \int_0^\cdot H_s(j, i) d\langle B^{(j)}, C^{(k)} \rangle_s = 0,$$

since  $\langle B^{(j)}, C^{(k)} \rangle = 0$  by independence of  $B$  et  $C$ . We derive that  $(A, C)$  is a  $p + q$ -Brownian motion in  $\mathcal{F}^Z$ , so  $A$  and  $C$  are independent.  $\square$

Deducing proposition 23 from last lemma uses almost the same arguments as deducing proposition 21 from lemma 24.

*Proof.* Proof of proposition 23. Let  $\mathcal{Z}$  be a finite Brownian filtration, and  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  be three Brownian filtrations in  $\mathcal{Z}$  such that  $\mathcal{A}_t \supset \mathcal{B}_t$  for every  $t \geq 0$ ,  $\mathcal{A}$  and  $\mathcal{B}$  have the same dimension, and  $\mathcal{C}$  is an independent complement of  $\mathcal{B}$  in  $\mathcal{Z}$ .

Let  $A, B, C$  be Brownian motions generating  $\mathcal{A}, \mathcal{B}, \mathcal{C}$ . Then lemma 27 applies, so  $A$  is independent of  $C$ . Then lemma 25 applies, so  $\sigma(A) = \sigma(B) \pmod{\mathbb{P}}$ ,

But the filtration  $\mathcal{F}^B$  is immersed in  $\mathcal{F}^A$ . Since the final  $\sigma$ -fields  $\mathcal{F}_\infty^A$  and  $\mathcal{F}_\infty^B$  coincide almost surely, we get  $\mathcal{F}_t^A$  and  $\mathcal{F}_t^B \pmod{\mathbb{P}}$  for every  $t \geq 0$  by lemma 1.  $\square$

### 3.5 Proof of the statements of example 19

Let  $Z = (X, Y)$  be a planar Brownian motion, set

$$B = \int_0^\cdot \operatorname{sgn}(X_s) dX_s,$$

and call  $\mathcal{Z}$  and  $\mathcal{B}$  the natural filtration of  $Z$  and  $B$ . We have to prove that the larger filtration  $\mathcal{B}'$  defined by

$$\mathcal{B}'_t = \bigcap_{s>0} (\mathcal{B}_t \vee \mathcal{Z}_s)$$

coincides with  $\mathcal{B}$  up to null events.

**First step.** We use a description of the conditional law of  $X$  given  $B$  established in [7].

Let  $\mathbb{W}$  the set of all continuous paths  $w$  from  $\mathbb{R}_+$  to  $\mathbb{R}$  such that  $w(0) = 0$ . For every  $w \in \mathbb{W}$ , we define the map  $\underline{w} \in \mathbb{W}$  by  $\underline{w}(t) = \min\{w(s) : s \in [0, t]\}$ .

By Tanaka's formula  $|X| = B + L$ , where  $L$  is the local time of  $X$  at 0. Since (almost surely),  $L$  is continuous, non-decreasing, and increases only on the set of all zeroes of  $X$ , Skorohod lemma shows that that  $L_t = -\underline{B}_t$ . Hence, the knowledge of  $B$  completely determines  $|X| = B - \underline{B}$ . Informally, given  $X$ , the signs of the excursions of  $X$  are independent and uniform on  $\{-1, 1\}$ .

The difficulty is that there is no canonical way to number the excursions. So we fix an enumeration  $(q_n)_{n \geq 1}$  of the positive rational numbers, and we number each non-trivial excursion of a path  $w \in \mathbb{W}$ , by we the first rational number in the sequence  $(q_n)_{n \geq 1}$  which falls in the corresponding open excursion interval. This method of numerotation is increasing (posterior excursions receive larger numbers) but it omits infinitely many rational numbers.

We now define an action of the multiplicative group  $\Gamma = \{-1, 1\}^{\mathbb{Q}_+^*}$  on  $\mathbb{W}$ . Given  $w \in \mathbb{W}$  and a family of signs  $e = (e(q))_{q \in \mathbb{Q}_+^*} \in \Gamma$ , we obtain the path  $e \cdot w$  from  $w$  as follows: for each  $q \in \mathbb{Q}_+^*$ , we multiply each excursion number  $q$  of  $w$  (if it exists) by the number  $e(q)$ .

Fix a uniform random variable  $\varepsilon$  on  $\Gamma$ , independent of  $Z$  and denote by  $K(w, \cdot)$  the law of  $G \cdot (w - \underline{w})$ . Then  $(K(w, \cdot))_{w \in \mathbb{W}}$  is a regular version of the conditional law of  $X$  given  $B$ . Call  $P_W$  the Wiener measure on  $\mathbb{W}$ . Since  $X$  and  $Y$  are independent,  $(K(w, \cdot) \otimes P_W)_{w \in \mathbb{W}}$  is a regular version of the conditional law of  $(X, Y)$  given  $B$ .

**Second step.** We prove that the sigma-field

$$\mathcal{Z}_{0+} = \bigcap_{s>0} \mathcal{Z}_s$$

is trivial under almost every probability  $\mathbb{P}[\cdot|B = w]$ .

To do this, we construct a random variable  $\eta$  from  $\varepsilon$  by modifying some signs as follows: if the excursion number  $q$  of  $X$  exists,  $\eta(q)$  is its sign; otherwise  $\eta(q) = \varepsilon(q)$ . By construction,  $X = \eta \cdot |X| = \eta \cdot (B - \underline{B})$ . From the independence of  $X$ ,  $Y$  and  $\varepsilon$  and from step 1, we deduce that  $|X|$  (or  $B$ ),  $Y$  and  $\eta$  are independent and that  $\eta$  is uniform on  $\Gamma$ .

For  $P_W$ -almost every  $w \in \mathbb{W}$ , the following properties hold.

1. The law of  $(Y, \eta)$  under  $\mathbb{P}[\cdot|B = w]$  is the same as under  $\mathbb{P}$ .
2. Equality  $B = w$  holds  $\mathbb{P}[\cdot|B = w]$ -almost surely;
3. The zeros of  $w$  accumulate at 0,  $\mathbb{P}[\cdot|B = w]$ -almost surely.

We now fix such a path  $w$ .

Let  $R$  be any  $\mathcal{F}_{0+}$ -measurable real random variable. For every  $s > 0$ , we can find a measurable function  $f_s$  such that  $R = f_s(X|_{[0,s]}, Y|_{[0,s]})$ .

Set  $R_{s,w} = f_s((\eta \cdot (w - \underline{w}))|_{[0,s]}, Y|_{[0,s]})$  and denote by  $D_s(w)$  the first zero of  $w$  in the time interval  $[s, +\infty[$ . By assumption,  $D_s(w) \rightarrow 0$  as  $s \rightarrow 0$ . Then  $R_{s,w}$  is a function of  $(\eta(q))_{q \in \mathbb{Q}_+^* \cap [0, D_s(w)]}$  and  $Y|_{[0,s]}$ .

Set  $L_w = \limsup_{n \rightarrow +\infty} R_{1/n, w}$ . The independence of  $Y$  and  $\eta$  under  $\mathbb{P}[\cdot|B = w]$  enables us to apply lemma 30 stated in section 4 (in the particular case  $\mathcal{A} = \mathcal{C} = \{\emptyset, \Omega\}$ ), so Kolmogorov's 0-1 law and the triviality of  $\mathcal{F}_{0+}^Y$  under  $\mathbb{P}[\cdot|B = w]$  yield that  $L$  is almost surely constant under  $\mathbb{P}[\cdot|B = w]$ .

But under  $\mathbb{P}[\cdot|B = w]$ , one has  $R = R_{s,w}$  almost surely since  $X = \eta \cdot (w - \underline{w})$  almost surely. Thus  $R = L_w$  almost surely, hence  $R$  is almost surely constant. Step 2 follows.

**Third step.** We prove the desired result. Since the filtration  $\mathcal{B}$  is immersed in the filtration  $\mathcal{B}'$ , one only needs to check that  $\mathcal{B}'_\infty \subset \mathcal{B}_\infty \text{ mod } \mathbb{P}$ .

Let  $A \in \mathcal{B}'_\infty$ . For every  $s > 0$ , we can find a measurable function  $g_s$  such that  $\mathbf{1}_A = g_s(X|_{[0,s]}, Y|_{[0,s]}, B)$ .

Fix a generic  $w \in \mathbb{W}$  like in step 2, set  $I_{s,w} = g_s((\eta \cdot (w - \underline{w}))|_{[0,s]}, Y|_{[0,s]}, w)$  and  $I_w = \limsup_{n \rightarrow +\infty} I_{1/n, w}$ .

Since the random variable  $I_w = \limsup_{n \rightarrow +\infty} I_{1/n, w}$  is  $\mathcal{Z}_{0+}$ -measurable, it is almost surely constant under  $\mathbb{P}[\cdot|B = w]$  by step 2.

But under  $\mathbb{P}[\cdot|B = w]$ , we get like in step 2 that  $\mathbf{1}_A = I_w$  almost surely, hence  $\mathbf{1}_A$  is almost surely constant, equal to  $\mathbb{E}[\mathbf{1}_A|B = w]$ .

As a result,  $\mathbf{1}_A = \mathbb{E}[\mathbf{1}_A|\sigma(B)]$  almost surely, so  $A \in \mathcal{B}_\infty \text{ mod } \mathbb{P}$ . The proof is complete.



## 4 A maximal but not-complementable Brownian filtration

### 4.1 A maximal but not-complementable filtration in a dyadic product-type filtration

This subsection is devoted to the proof of the following lemma, which is the first step of the construction of a maximal but not-complementable Brownian filtration.

**Lemma 28.** *One can construct :*

- a dyadic product-type filtration  $(\mathcal{Z}_n)_{n \leq 0}$ ,
- a filtration  $(\mathcal{U}_n)_{n \leq 0}$  immersed in  $(\mathcal{Z}_n)_{n \leq 0}$ ,
- a random variable  $U$  with values in some Polish space  $(E, \mathcal{E})$  and generating  $\mathcal{U}_0$ ,

such that for  $U(\mathbb{P})$ -almost every  $u \in E$ ,  $(\mathcal{Z}_n)$  is Kolmogorovian but not product-type under  $\mathbb{P}_u = \mathbb{P}[\cdot | U = u]$ . Therefore, the filtration  $(\mathcal{U}_n)_{n \leq 0}$  is maximal but non complementable in  $(\mathcal{Z}_n)$ .

*Proof.* We begin with a variant of an example given in [9], which was itself inspired from an unpublished note of Tsirelson [27].

For every  $n \leq 0$ , call  $K_n$  the finite field with  $q_n = 2^{2^{|n|}}$  elements. Start with a sequence of independent random variables  $(Z_n)_{n \leq 0}$  such that for every  $n \leq 0$ ,  $Z_{2n} = (X_n, Y_n)$  is uniform on  $K_n^4 \times K_n^4$  and  $Z_{2n-1} = B_n$  is uniform on  $K_n^4$ . By construction, the filtration  $(\mathcal{F}_n^Z)_{n \leq 0}$  is product-type and  $(r_n)_{n \leq 0}$ -adic, with  $r_{2n-1} = q_n^4$  and  $r_{2n} = q_n^8$  for every  $n \leq 0$ .

Since  $|K_{n-1}| = 2^{2^{|n|+1}} = |K_n|^2$ , one can fix a bijection between  $K_{n-1}^4 \times K_{n-1}^4$  and the set  $\mathcal{M}_4(K_n)$  of all  $4 \times 4$  matrices with entries in  $K_n$ . Call  $A_n$  the uniform random variable on  $\mathcal{M}_4(K_n)$  corresponding to  $Z_{2n-2}$  through this bijection, and set  $U_{2n-1} = 0$  and  $U_{2n} = Y_n - A_n X_n - B_n$ .

For every  $n \leq 0$ ,  $(X_n, Y_n)$  is independent of  $\mathcal{F}_{2n-1}^Z$  and uniform on  $K_n^4 \times K_n^4$ . Since the random map  $(x, y) \mapsto (x, y - A_n x - B_n)$  from  $K_n^4 \times K_n^4$  to itself is  $\mathcal{F}_{2n-1}^Z$ -measurable and bijective,  $(X_n, U_{2n})$  is also independent of  $\mathcal{F}_{2n-1}^Z$  and uniform on  $K_n^4 \times K_n^4$  and is still an innovation at time  $2n$  of the filtration  $\mathcal{F}^Z$ . Therefore, the filtration  $(\mathcal{F}_n^U)_{n \leq 0}$  is immersed in  $(\mathcal{F}_n^Z)_{n \leq 0}$ , product-type and  $(r_n/q_n^4)_{n \leq 0}$ -adic.

As the integers  $(r_n)_{n \leq 0}$  and  $(r_n/q_n^4)_{n \leq 0}$  are powers of 2, one can interpolate the filtrations  $(\mathcal{F}_n^Z)_{n \leq 0}$  and  $(\mathcal{F}_n^U)_{n \leq 0}$  to get two filtrations  $\mathcal{Z} = (\mathcal{Z}_n)_{n \leq 0}$  and  $\mathcal{U} = (\mathcal{U}_n)_{n \leq 0}$  such that

- $\mathcal{Z}$  is a dyadic product-type filtration,
- $\mathcal{U}$  is immersed in  $\mathcal{Z}$ ,
- $(\mathcal{F}_n^Z)_{n \leq 0} = (\mathcal{Z}_{t_n})_{n \leq 0}$  and  $(\mathcal{F}_n^U)_{n \leq 0} = (\mathcal{U}_{t_n})_{n \leq 0}$  for some sequence  $0 = t_0 > t_{-1} > t_{-2} > \dots$  of integers.

To get such filtrations, it suffices to split the random variables  $(B_n)_{n \leq 0}$ ,  $(X_n)_{n \leq 0}$  and  $(U_{2n})_{n \leq 0}$  into independent random bits. The bits coming from the  $B_n$  and the  $X_n$  provide innovations of the filtration  $\mathcal{Z}$  at the times when the filtration  $\mathcal{Z}$  only increases.

The bits coming from the  $U_{2n}$  provide innovations of the two filtrations  $\mathcal{Z}$  and  $\mathcal{U}$  at the times when both filtrations  $\mathcal{Z}$  increase.

The random variable  $U = (U_n)_{n \leq 0}$  generates  $\mathcal{U}_0 = \mathcal{F}_0^U$ . Let us check that for  $U(\mathbb{P})$ -almost every  $u \in E$ ,  $(\mathcal{Z}_n)$  for  $U(\mathbb{P})$ -almost every  $u \in E$ ,  $(\mathcal{Z}_n)$  is Kolmogorovian but not product-type under  $\mathbb{P}_u = \mathbb{P}[\cdot | U = u]$ . By proposition \*\* of [19], the last two statements will follow.

First, we note that for every  $n \leq 0$ ,  $\mathcal{F}_{2n}^Z = \mathcal{F}_{2n}^U \vee \mathcal{F}_n^{B,X}$ , where  $\mathcal{F}_{2n}^U$  and  $\mathcal{F}_n^{B,X}$  are independent. Therefore, for  $U(\mathbb{P})$ -almost every  $u \in E$ , the filtration  $(\mathcal{F}_{2n}^Z)_{n \leq 0}$  seen under  $\mathbb{P}_u$  is isomorphic to the filtration  $(\mathcal{F}_n^{B,X})_{n \leq 0}$  seen under  $\mathbb{P}$ , which is product-type. Thus,  $\mathcal{Z}_{-\infty} = \mathcal{F}_{-\infty}^Z$  is trivial under  $\mathbb{P}_u$ .

To show that  $(\mathcal{Z}_n)$  is not product-type under  $\mathbb{P}_u$ , it suffices to show that the extracted filtration  $(\mathcal{F}_n^Z)_{n \leq 0}$  is not product-type under  $\mathbb{P}_u$ . To do this, we check that the random variable  $Z_0$  does not satisfy the I-cosiness criterion.

Let  $Z' = (X', Y')$  and  $Z'' = (X'', Y'')$  be two copies of the process  $Z$  under  $\mathbb{P}_u$ , defined on some probability space  $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}}_u)$ , such that both natural filtrations  $\mathcal{F}^{Z'}$  and  $\mathcal{F}^{Z''}$  are immersed in some filtration  $\mathcal{G}$ .

For every  $n \leq 0$ , define the copies  $A'_n, A''_n$  and  $B'_n, B''_n$  of the random variables  $A_n$  and  $B_n$  by the obvious way, and set  $S_n = \{x \in K_n^4 : A'_n x + B'_n = A''_n x + B''_n\}$ .

For  $U(\mathbb{P})$ -almost every  $u \in E$ , and for every  $n \leq 0$ , the equalities  $Y'_n = A'_n X'_n + B'_n + u_{2n}$  and  $Y''_n = A''_n X''_n + B''_n + u_{2n}$  hold  $\bar{\mathbb{P}}_u$ -almost surely. Therefore,

$$\mathbf{1}_{[Z'_{2n} = Z''_{2n}]} = \mathbf{1}_{[X'_n = X''_n \in S_n]} \leq \mathbf{1}_{[X'_n \in S_n]} \bar{\mathbb{P}}_u\text{-almost surely.}$$

But the random variable  $S_n$  is  $\mathcal{G}_{2n-1}$ -measurable whereas  $X'_n$  is uniform on  $K_n^4$  conditionally on  $\mathcal{G}_{2n-1}$  since  $\mathcal{F}^{Z'}$  is immersed in  $\mathcal{G}$ . Thus

$$\bar{\mathbb{P}}_u[Z'_{2n} = Z''_{2n} | \mathcal{G}_{2n-1}] = \frac{|S_n|}{q_n^4} \leq \mathbf{1}_{[A'_n = A''_n]} + \frac{1}{q_n} \mathbf{1}_{[A'_n \neq A''_n]} \bar{\mathbb{P}}_u\text{-almost surely,}$$

since  $S_n$  contains at most  $q_n^3$  points when  $A'_n \neq A''_n$ . Passing to the complements and taking the expectations yields

$$\bar{\mathbb{P}}_u[Z'_{2n} \neq Z''_{2n}] \geq \left(1 - \frac{1}{q_n}\right) \bar{\mathbb{P}}_u[A'_n \neq A''_n] = \left(1 - \frac{1}{q_n}\right) \bar{\mathbb{P}}_u[Z'_{2n-2} \neq Z''_{2n-2}].$$

By induction, one gets that for every  $n \leq 0$

$$\bar{\mathbb{P}}_u[Z'_0 \neq Z''_0] \geq \prod_{k=n+1}^0 \left(1 - \frac{1}{q_k}\right) \times \bar{\mathbb{P}}_u[Z'_{2n} \neq Z''_{2n}].$$

If, for some  $N > -\infty$ , the sigma-fields  $\mathcal{F}_N^{Z'}$  and  $\mathcal{F}_N^{Z''}$  are independent, then  $\bar{\mathbb{P}}_u[Z'_{2n} \neq Z''_{2n}] \rightarrow 1$  as  $n \rightarrow -\infty$ , so

$$\bar{\mathbb{P}}_u[Z'_0 \neq Z''_0] \geq \prod_{k \leq 0} \left(1 - \frac{1}{q_k}\right) > 0.$$

The proof is complete. □

## 4.2 Embedding dyadic filtrations in Brownian filtrations

We start with the two filtrations provided by lemma 28. By construction, the filtration  $(\mathcal{Z}_n)_{n \leq 0}$  can be generated by some heads or tails sequence  $(\varepsilon_n)_{n \leq 0}$ .

Fix an increasing sequence  $(t_n)_{n \leq 0}$  of positive real numbers such that  $t_0 = 1$  and  $t_n \rightarrow 0$  as  $n \rightarrow -\infty$  (e.g.  $t_n = 2^n$  for every  $n \leq 0$ ). By symmetry and independence of Brownian increments, one may construct a Brownian motion  $X$  such that for every  $n \leq 0$ ,  $\varepsilon_n = \text{sign}(X_{t_n} - X_{t_{n-1}})$ . Let  $Y$  be any another Brownian motion, independent of  $X$ .

The filtration  $(\mathcal{U}_n)_{n \leq 0}$  is  $(a_n)_{n \leq 0}$ -adic for some sequence  $(a_n)_{n \leq 0}$  taking values 1 and 2 only. Call  $D \subset \mathbf{Z}_-$  the set of all  $n \leq 0$  such that  $a_n = 2$ . Since  $(\mathcal{U}_n)_{n \leq 0}$  is poly-adic and immersed in a product-type filtration,  $(\mathcal{U}_n)_{n \leq 0}$  is also product-type. Therefore, the filtration  $(\mathcal{U}_n)_{n \leq 0}$  can be generated by some sequence  $(\eta_n)_{n \leq 0}$  of independent random variables with  $\eta_n$  uniform on  $\{-1, 1\}$  if  $n \in D$ ,  $\eta_n = 0$  if  $n \notin D$ .

By immersion of  $(\mathcal{U}_n)_{n \leq 0}$  in  $(\mathcal{Z}_n)_{n \leq 0}$ , each  $\eta_n$  is  $\mathcal{Z}_n$ -measurable and independent of  $\mathcal{Z}_{n-1}$ . So when  $n \in D$ ,  $\eta_n$  can be written  $\eta_n = H_n \varepsilon_n$ , where  $H_n$  is some  $\mathcal{Z}_{n-1}$ -random variable taking values in  $\{-1, 1\}$ .

Since  $\mathcal{Z}_{n-1} \subset \mathcal{F}_{t_{n-1}}^{X,Y}$  for every  $n \leq 0$ , one gets a predictable process  $(A_t)_{0 < t \leq 1}$  with values in  $O_2(\mathbb{R})$  by setting for every  $t \in ]t_{n-1}, t_n]$ ,

$$A_t = \begin{pmatrix} H_n & 0 \\ 0 & 1 \end{pmatrix} \text{ if } n \in D, \quad A_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ if } n \notin D.$$

So one gets two independent Brownian motions in  $\mathcal{F}^{X,Y}$  on the time-interval  $[0, 1]$  by setting

$$\begin{pmatrix} dU_t \\ dV_t \end{pmatrix} = A_t \begin{pmatrix} dX_t \\ dY_t \end{pmatrix}.$$

**Theorem 29.** *The Brownian motion  $U$  thus defined is complementable after 0, maximal, but not complementable in  $\mathcal{F}^{X,Y}$ .*

*Proof.* **Complementability after 0**

Let us check that  $V$  is a complement after 0 of  $U$ , namely

$$\forall s \in ]0, 1], \quad \mathcal{F}_1^{U,V} \vee \mathcal{F}_s^Z = \mathcal{F}_1^{X,Y}.$$

Since  $t_m \rightarrow 0+$  as  $m \rightarrow -\infty$ , it is sufficient to consider instants  $s$  which are some subdivision time  $t_m$  with  $m \leq 0$ . Since for every  $n \geq m$ , the process  $A$  coincides on each time-interval  $]t_n, t_{n+1}]$  with an  $\mathcal{F}_{t_n}^{X,Y}$ -measurable random variable, the formula

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = A_t^{-1} \begin{pmatrix} dU_t \\ dV_t \end{pmatrix}$$

enables us to recover  $(X, Y)$  from the knowledge of  $(Z_s)_{0 \leq s \leq t_m}$  and  $(U, V)$ .

**Maximality**

By the sufficient condition for given in [4], the maximality of  $U$  can be deduced from its complementability after 0 and from the equality

$$\mathcal{F}_1^U = \bigcap_{s \in ]0, 1]} (\mathcal{F}_1^U \vee \mathcal{F}_s^Z).$$

The intersection above, over all  $s \in ]0, 1]$  can be restricted to the instants  $t_m$  with  $m \leq 0$ .

It is now convenient to introduce the notations

$$\Delta X_n = (X_t - X_{t_{n-1}})_{t_{n-1} \leq t \leq t_n} \text{ and } \varepsilon_n \Delta X_n = (\varepsilon_n (X_t - X_{t_{n-1}}))_{t_{n-1} \leq t \leq t_n}.$$

Recall that  $\varepsilon_n = \text{sign}(X_{t_n} - X_{t_{n-1}})$ . Therefore,  $\sigma(\Delta X_n) = \sigma(\varepsilon_n) \vee \sigma(\varepsilon_n \Delta X_n)$ , with  $\sigma(\varepsilon_n)$  and  $\sigma(\varepsilon_n \Delta X_n)$  independent by symmetry of Brownian increments.

We define by the same way the processes  $\Delta Y_n$ ,  $\Delta U_n$ ,  $\Delta V_n$  and  $\eta_n \Delta U_n$ . By definition of  $U$  and  $V$ ,

$$\Delta U_n = H_n \Delta X_n = \eta_n \varepsilon_n \Delta X_n \text{ and } \Delta V_n = \Delta Y_n \text{ if } n \in D,$$

$$\Delta U_n = \Delta Y_n \text{ and } \Delta V_n = \Delta X_n \text{ if } n \in D^c.$$

Moreover, when  $n \in D$ ,  $\eta_n = \text{sign}(U_{t_n} - U_{t_{n-1}})$  is independent of  $\eta_n \Delta U_n = \varepsilon_n \Delta X_n$ .

Therefore,  $\mathcal{F}_1^U = \mathcal{A} \vee \mathcal{B}$ , with

$$\mathcal{A} = \sigma((\eta_n)_{n \in D}), \quad \mathcal{B} = \sigma((\varepsilon_n \Delta X_n)_{n \in D}) \vee \sigma((\Delta Y_n)_{n \in D^c}).$$

For  $m \in \mathbf{Z}_-$ , set  $D_m = D \cap ]-\infty, m]$ ,  $D_m^c = ]-\infty, m] \setminus D$ , and

$$\mathcal{C}_m = \mathcal{F}_m^\varepsilon, \quad \mathcal{D}_m = \sigma((\varepsilon_n \Delta X_n)_{n \in D_m^c}) \vee \sigma((\Delta Y_n)_{n \in D_m}).$$

Then  $\mathcal{F}_m^\varepsilon \vee \mathcal{F}_{t_m}^V = \mathcal{C}_m \vee \mathcal{D}_m$ .

The maximality of  $\mathcal{F}^\eta$  in  $\mathcal{F}^\varepsilon$  yields the equality

$$\mathcal{A} = \bigcap_{m \leq 0} (\mathcal{A} \vee \mathcal{C}_m).$$

By independence of  $U$  and  $V$ , the sigma-fields  $\mathcal{B}$  and  $\mathcal{D}_0$  are independent, so the following exchange property holds

$$\mathcal{B} = \mathcal{B} \vee \mathcal{D}_{-\infty} = \bigcap_{m \leq 0} (\mathcal{B} \vee \mathcal{D}_m).$$

Since the three sequences  $(\varepsilon_n)_{n \leq 0}$ ,  $(\varepsilon_n \Delta X_n)_{n \leq 0}$  and  $(\Delta Y_n)_{n \leq 0}$  are independent, the  $\sigma$ -fields  $\mathcal{A} \vee \mathcal{C}_0 = \mathcal{F}_0^\varepsilon$  and  $\mathcal{B} \vee \mathcal{D}_0 = \mathcal{F}_0^{\varepsilon \Delta X, Y}$  are independent. Therefore, by lemma 30 below, one has

$$\mathcal{F}_1^U = \mathcal{A} \vee \mathcal{B} = \bigcap_{m \leq 0} (\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}_m \vee \mathcal{D}_m) = \bigcap_{m \leq 0} (\mathcal{F}_1^U \vee \mathcal{F}_m^\varepsilon \vee \mathcal{F}_{t_m}^V) = \bigcap_{m \leq 0} (\mathcal{F}_1^U \vee \mathcal{F}_{t_m}^Z).$$

This proves the maximality of  $U$ .

### Non-complementability

We keep the notations introduced in the proof of the maximality.

First, we note that for  $U(\mathbb{P})$ -almost every  $u \in C([0, 1], \mathbb{R})$ , the filtration  $(\mathcal{F}_n^\varepsilon)_{n \leq 0}$  is immersed in  $(\mathcal{F}_{t_n}^{X, Y})_{n \leq 0}$  under  $\mathbb{P}_u = \mathbb{P}[\cdot | U = u]$ . Indeed, the filtration  $(\mathcal{F}_{t_n}^{X, Y})_{n \leq 0}$  is the supremum of the three natural filtrations associated to the sequences  $\varepsilon = (\varepsilon_n)_{n \leq 0}$ ,  $\eta \Delta U = (\eta_n \Delta U_n)_{n \leq 0}$  and  $\Delta V = (\Delta V_n)_{n \leq 0}$ , whereas  $\mathcal{U}_0 = \sigma(U)$  is the  $\sigma$ -field generated by the two sequences  $\eta = (\eta_n)_{n \leq 0}$  and  $\eta \Delta U$ . The immersion follows from the independence of  $\varepsilon$ ,  $\eta \Delta U$ ,  $\Delta V$  and from the inclusion  $\sigma(\eta) \subset \sigma(\varepsilon)$ .

Moreover, since  $\eta\Delta U$  is independent of  $\varepsilon$  whereas the sequence  $\eta = (\eta_n)_{n \leq 0}$  is some measurable function  $\Phi$  of  $U$ , the law of  $\varepsilon$  under  $\mathbb{P}_u$  coincides with the law of  $\varepsilon = (\varepsilon_n)_{n \leq 0}$  under  $\mathbb{P}[\cdot | \eta = \Phi(u)]$ . Therefore, for  $U(\mathbb{P})$ -almost every  $u \in C([0, 1], \mathbb{R})$ , the filtration  $(\mathcal{F}_n^\varepsilon)_{n \leq 0}$  is  $(2/a_n)$ -adic but not product-type under  $\mathbb{P}_u$ .

If the Brownian motion  $U$  were complementable in  $\mathcal{F}^{X,Y}$ , then for  $U(\mathbb{P})$ -almost every  $u \in C([0, 1], \mathbb{R})$ , the filtration  $(\mathcal{F}_t^Z)_{0 \leq t \leq 1}$  would be Brownian under  $\mathbb{P}_u$ , hence by independence of Brownian increments,  $(\mathcal{F}_{t_n}^{X,Y})_{n \leq 0}$  would be product-type under  $\mathbb{P}_u$ . By immersion, the  $(2/a_n)$ -adic filtration  $(\mathcal{F}_n^\varepsilon)_{n \leq 0}$  would also be product-type, leading to a contradiction. □

**Lemma 30.** *Let  $\mathcal{A}, \mathcal{B}$  be two  $\sigma$ -fields and  $(\mathcal{C}_n)_{n \leq 0}, (\mathcal{D}_n)_{n \leq 0}$  be two filtrations of the probability space  $(\Omega, \mathcal{T}, \mathbb{P})$ . If*

$$\mathcal{A} = \bigcap_{m \leq 0} (\mathcal{A} \vee \mathcal{C}_m) \quad \text{mod } \mathbb{P}, \quad \mathcal{B} = \bigcap_{m \leq 0} (\mathcal{B} \vee \mathcal{D}_m) \quad \text{mod } \mathbb{P},$$

and if  $\mathcal{A} \vee \mathcal{C}_0$  and  $\mathcal{B} \vee \mathcal{D}_0$  are independent, then

$$\mathcal{A} \vee \mathcal{B} = \bigcap_{m \leq 0} (\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}_m \vee \mathcal{D}_m) \quad \text{mod } \mathbb{P}.$$

*Proof.* Since both sides of the equality to be proved are sub- $\sigma$ -fields of  $\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}_0 \vee \mathcal{D}_0$ , it is sufficient to prove that for every  $Z \in L^1(\mathcal{A} \vee \mathcal{C}_0 \vee \mathcal{B} \vee \mathcal{D}_0)$ , one has

$$\mathbb{E}[Z | \mathcal{A} \vee \mathcal{B}] = \mathbb{E}\left[Z \mid \bigcap_{m \leq 0} (\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}_m \vee \mathcal{D}_m)\right].$$

By density, one may consider only random variables  $Z = XY$  with  $X \in L^\infty(\mathcal{A} \vee \mathcal{C}_0)$  and  $Y \in L^\infty(\mathcal{B} \vee \mathcal{D}_0)$ .

Given  $Z$  as above, one has  $\mathbb{E}[Z | \mathcal{A} \vee \mathcal{C}_0 \vee \mathcal{B}] = X\mathbb{E}[Y | \mathcal{B}]$ , so

$$\mathbb{E}[Z | \mathcal{A} \vee \mathcal{B}] = \mathbb{E}[X | \mathcal{A} \vee \mathcal{B}] \mathbb{E}[Y | \mathcal{B}] = \mathbb{E}[X | \mathcal{A}] \mathbb{E}[Y | \mathcal{B}],$$

since  $\sigma(X) \vee \mathcal{A}$  is independent of  $\mathcal{B}$ . By the same way, one gets that for every  $n \leq 0$ ,

$$\mathbb{E}[Z | \mathcal{A} \vee \mathcal{B} \vee \mathcal{C}_m \vee \mathcal{D}_m] = \mathbb{E}[X | \mathcal{A} \vee \mathcal{C}_m] \mathbb{E}[Y | \mathcal{B} \vee \mathcal{D}_m].$$

Thus, taking the limit as  $n \rightarrow -\infty$  yields the result by the martingale convergence theorem and the assumption. □

## 5 A complementable factor associated to a non-complementable filtration

### 5.1 Definition of a uniform randomised decimation process

We denote by  $\{a, b\}^\infty$  the set of all infinite words on the alphabet  $\{a, b\}$ , namely the set of all maps from  $\mathbf{N} = \{1, 2, \dots\}$  to  $\{a, b\}$ . We endow this set with the uniform probability measure  $\mu$ : a random infinite word  $X$  is chosen according to  $\mu$  if the successive letters  $X(1), X(2), \dots$  form a sequence of independent and uniform random variables taking values in  $\{a, b\}$ .

We denote by  $\mathcal{P}(\mathbf{N})$  the power set of  $\mathbf{N}$ , namely the set of all subsets of  $\mathbf{N}$ . Given  $p \in ]0, 1[$ , we define the probability measure  $\nu_p$  on  $\mathcal{P}(\mathbf{N})$  as follows: the law of a random subset  $I$  of  $\mathbf{N}$  is  $\nu_p$  if  $\mathbf{1}_I(1), \mathbf{1}_I(2), \dots$  form an i.i.d. sequence of Bernoulli random variables with parameter  $p$ . Equivalently, this means that  $\mathbb{P}[F \subset I] = p^{|F|}$  for every finite subset  $F$  of  $\mathbf{N}$ . In this case, we note that almost surely,  $I$  is infinite with infinite complement. The law  $\nu := \nu_{1/2}$  will be called the uniform law on  $\mathcal{P}(\mathbf{N})$ .

When  $A$  is an infinite subset of  $\mathbf{N}$ , we denote by  $\psi_A(1) < \psi_A(2) < \dots$  its elements. This defines an increasing map  $\psi_A$  from  $\mathbf{N}$  to  $\mathbf{N}$  whose range is  $A$ . Conversely, for every increasing map  $f$  from  $\mathbf{N}$  to  $\mathbf{N}$ , there is a unique infinite subset  $A$  of  $\mathbf{N}$ , namely the range of  $f$ , such that  $f = \psi_A$ . These remarks lead to the following statement.

**Lemma 31.** *Let  $I$  and  $J$  be independent random infinite subsets of  $\mathbf{N}$  with respective laws  $\nu_p$  and  $\nu_q$ , and  $R = \psi_I \circ \psi_J(\mathbf{N}) = \psi_I(J)$  be the range of  $\psi_I \circ \psi_J$ . Then  $\psi_I \circ \psi_J = \psi_R$  and the law of  $R$  is  $\nu_{pq}$ .*

*Proof.* The equality  $\psi_I \circ \psi_J = \psi_R$  follows from the remarks above. Let  $F$  be a finite subset of  $\mathbf{N}$ . By injectivity of  $\psi_I$ ,

$$[F \subset R] = [F \subset I ; \psi_I^{-1}(F) \subset J]$$

and  $[F \subset I] = [|\psi_I^{-1}(F)| = |F|]$ , therefore by independence of  $I$  and  $J$ ,

$$\mathbb{P}[F \subset R \mid \sigma(I)] = \mathbf{1}_{[F \subset I]} \mathbb{P}[\psi_I^{-1}(F) \subset J \mid \sigma(I)] = \mathbf{1}_{[F \subset I]} q^{|\psi_I^{-1}(F)|} = \mathbf{1}_{[F \subset I]} q^{|F|}.$$

Thus  $\mathbb{P}[F \subset R] = \mathbb{P}[F \subset I]q^{|F|} = (pq)^{|F|}$ .  $\square$

Here is another property that we will use to define the uniform randomised decimation process on  $\{a, b\}$ , and also later.

**Lemma 32.** *Let  $X$  be a uniform random word on  $\{a, b\}^\infty$ . Let  $I$  be a random subset of  $\mathbf{N}$  with law  $\nu_p$ , independent of  $X$ . Then*

- $I, X \circ \psi_I, X \circ \psi_{I^c}$  are independent
- $X \circ \psi_I, X \circ \psi_{I^c}$  are uniform random words on  $\{a, b\}^\infty$ .

*Proof.* Almost surely,  $I$  is infinite with infinite complement, so the random maps  $\psi_I$  and  $\psi_{I^c}$  are well-defined. The integers  $\psi_I(1), \psi_{I^c}(1), \psi_I(2), \psi_{I^c}(2) \dots$  are distinct, so conditionally on  $I$ , the random variables  $X(\psi_I(1)), X(\psi_{I^c}(1)), X(\psi_I(2)), X(\psi_{I^c}(2)), \dots$  are independent and uniform on  $\{a, b\}$ . The result follows.  $\square$

**Definition 33.** *Call  $\mathcal{P}'(\mathbf{N})$  the set of all infinite subsets of  $\mathbf{N}$ . A uniform randomised decimation process in the alphabet  $\{a, b\}$  is a stationary Markov chain defined  $(X_n, I_n)_{n \in \mathbf{Z}}$  with values in  $\{a, b\}^\infty \times \mathcal{P}'(\mathbf{N})$  defined as follows: for every  $n \in \mathbf{Z}$ ,*

1. the law of  $(X_n, I_n)$  is  $\mu \otimes \nu$ ;
2.  $I_n$  is independent of  $(X_{n-1}, I_{n-1})$  and uniform on  $\mathcal{P}(\mathbf{N})$ ;
3.  $X_n = X_{n-1} \circ \psi_{I_n}$ .

Such a process is well-defined and unique in law since the law  $\mu \otimes \nu$  is invariant by the transition kernel given by conditions 2 and 3 above, thanks to lemma 32. Moreover,  $(I_n)_{n \in \mathbf{Z}}$  is a sequence of innovations for the filtration  $\mathcal{F}^{X,I}$ . Therefore, the filtration  $\mathcal{F}^{X,I}$  has independent increments or is locally of product-type, according to Laurent's terminology [17].

This process is a kind of randomisation of Vershik's decimation process given in example 3 of [28]. Indeed, Vershik's decimation process is equivalent to the process that we would get by choosing the random sets  $I_n$  uniformly among the set of all even positive integers and the set of all odd positive integers. Although Vershik's decimation process generates a non-standard filtration, we will show that our randomised process generates a standard one.

**Theorem 34.** *The uniform randomised decimation process on the alphabet  $\{a, b\}$  generates a product-type filtration.*

## 5.2 Proof of theorem 34

We have seen that the filtration  $\mathcal{F}^{X,I}$  admits  $(I_n)_{n \in \mathbf{Z}}$  as a sequence of innovations. Each innovation has diffuse law. Therefore, to prove that the filtration  $(\mathcal{F}_n^{X,I})_{n \leq 0}$ , or equivalently, the filtration  $(\mathcal{F}_n^{X,I})_{n \in \mathbf{Z}}$  is product-type, it suffices to check Vershik's first level criterion (see reminders further and definition 2.6 and theorem 2.25 in [17]). Concretely, we have to check any random variable in  $L^1(\mathcal{F}_0^{X,I}, \mathbb{R})$  can be approached in  $L^1(\mathcal{F}_0^{X,I}, \mathbb{R})$  by measurable functions of finitely many innovations of  $(\mathcal{F}^{X,I})_{n \leq 0}$ .

The innovations  $(I_n)_{n \in \mathbf{Z}}$  are inadequate to do this, since the random variable  $X_0$  is independent of the whole sequence  $(I_n)_{n \in \mathbf{Z}}$ , so functions of the  $(I_n)_{n \in \mathbf{Z}}$  cannot approach non-trivial functions of  $X_0$ . Therefore, we will have to construct new innovations. The next lemma gives us a general procedure to get some.

**Lemma 35.** *Fix  $n \in \mathbf{Z}$ . Let  $\Phi$  be some  $\mathcal{F}_{n-1}^{X,I}$ -measurable map from  $\mathbf{N}$  to  $\mathbf{N}$ . If  $\Phi$  is almost surely bijective, then the random variable  $J_n = \Phi(I_n)$  is independent of  $\mathcal{F}_{n-1}^{X,I}$  and uniform on  $\mathcal{P}(\mathbf{N})$ .*

*Proof.* For every finite subset  $F$  of  $\mathbf{N}$ ,

$$P[F \subset J_n | \mathcal{F}_{n-1}^{X,I}] = P[\Phi^{-1}(F) \subset I_n | \mathcal{F}_{n-1}^{X,I}] = (1/2)^{|\Phi^{-1}(F)|} = (1/2)^{|F|} \text{ almost surely.}$$

The result follows. □

Actually, the proof of theorem 34 is similar to the proof of the standardness of the erased-words filtration by S. Laurent [16] and uses the same tools, namely canonical coupling and cascaded permutations.

### Definition 36. (Canonical word and canonical coupling)

*By definition, the infinite canonical word  $C$  on the alphabet  $\{a, b\}$  is the word  $abab \dots$ , namely the map from  $\mathbf{N}$  to  $\{a, b\}$  which sends the odd integers on  $a$  and the even integers on  $b$ .*

*If  $x$  is an infinite word  $x$  on the alphabet  $\{a, b\}$ , namely a map from  $\mathbf{N}$  to  $\{a, b\}$ , we set for every  $i \in \mathbf{N}$ ,*

$$\begin{aligned} \phi_x(i) &= 2q - 1 \text{ if } x(i) \text{ is the } q\text{-th occurrence of the letter } a \text{ in } x, \\ \phi_x(i) &= 2q \text{ if } x(i) \text{ is the } q\text{-th occurrence of the letter } b \text{ in } x. \end{aligned}$$

**Lemma 37.** *By definition, the map  $\phi_x$  thus defined from  $\mathbf{N}$  to  $\mathbf{N}$  is injective and satisfies the equality  $x = C \circ \phi_x$ . When each possible letter  $a$  or  $b$  appears infinitely many times in  $x$ ,  $\phi_x$  is a permutation of  $\mathbf{N}$ , (called canonical coupling by S. Laurent).*

Roughly speaking, if  $x$  is a typical word of  $\{a, b\}^\infty$  endowed with the uniform law, the asymptotic proportions of  $a$  and  $b$  are  $1/2$ , so  $\phi_x$  is asymptotically close to the identity map.

**Definition 38. (New innovations and cascaded permutations)**

Let  $\Omega'$  be the almost sure event on which

- each possible letter  $a$  or  $b$  appears infinitely many times in the infinite word  $X_0$ ;
- each subset  $I_n$  is infinite.

On  $\Omega'$ , we define by recursion a sequence  $(\Phi_n)_{n \geq 0}$  of random permutations of  $\mathbf{N}$  and a sequence  $(J_n)_{n \geq 1}$  of random infinite subsets of  $\mathbf{N}$  by setting  $\Phi_0 = \phi_{X_0}$  and, for every  $n \geq 1$ ,

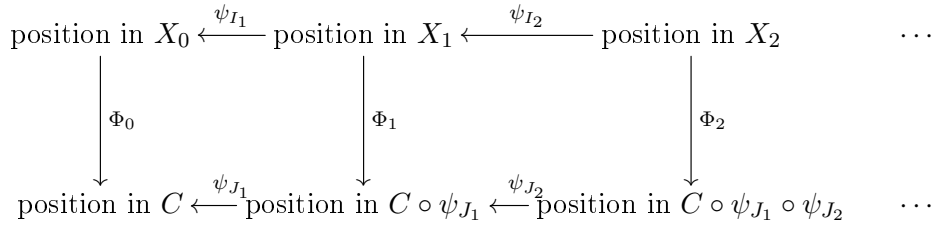
$$J_n = \Phi_{n-1}(I_n) \text{ and } \Phi_{n-1} \circ \psi_{I_n} = \psi_{J_n} \circ \Phi_n. \quad (1)$$

Let us check that the inductive construction above actually works  $\Omega'$ .

On  $\Omega'$ , the map  $\Phi_0 = \phi_{X_0}$  is bijective by lemma 37.

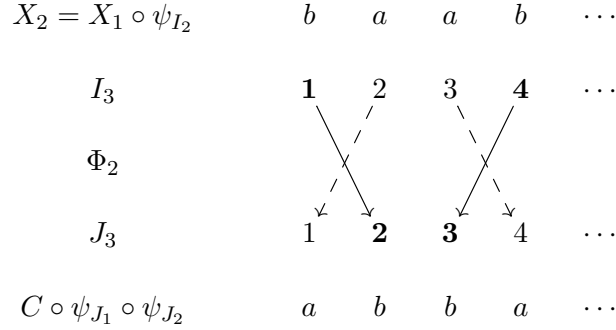
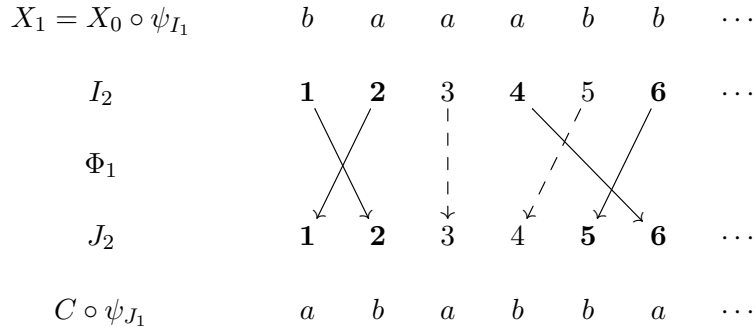
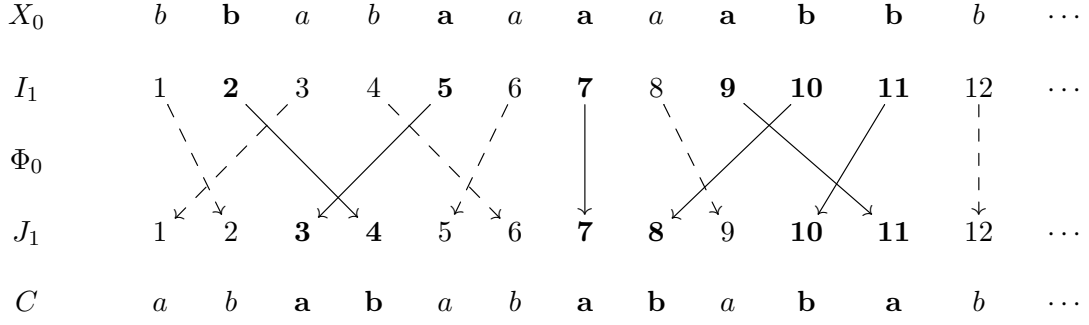
Once we know that  $\Phi_{n-1}$  is a random permutations of  $\mathbf{N}$ , the map  $\Phi_{n-1} \circ \psi_{I_n}$  is a random injective map from  $\mathbf{N}$  to  $\mathbf{N}$  with range  $\Phi_{n-1}(I_n) = J_n$ . Therefore,  $J_n$  is infinite and the map  $\Phi_n$  is uniquely and well defined by equation 1, namely for every  $k \in \mathbf{N}$ ,  $\Phi_n(k)$  is the rank of the integer  $\Phi_{n-1}(\psi_{I_n}(k))$  in the set  $J_n$ . Moreover,  $\Phi_n$  is a permutation of  $\mathbf{N}$ .

Informally, the *cascaded permutations*  $(\Phi_n)_{n \geq 0}$  are induced by  $\Phi_0 = \phi_{X_0}$  and the successive extractions. More precisely, equation 1 is represented by a commutative diagram which gives the correspondance between the positions of a same letter in different words.



Here is a realisation of the first three steps. The boldface numbers form the subsets  $I_1, J_1, I_2, J_2, \dots$ . Among the arrows representing  $\phi_{X_0}$ , the plain arrows (from elements in  $I_1$  to elements in  $J_1$ ) provide the permutation  $\phi_{X_0, I_1}$  by renumbering of the elements.





**Lemma 39.** *On the almost sure event  $\Omega'$ , the following properties hold for every  $n \geq 1$ ,*

1.  $J_n$  is independent of  $\mathcal{F}_{n-1}^{X,I}$  and is uniform on  $\mathcal{P}(\mathbf{N})$ .
2.  $\sigma(X_0, J_1, \dots, J_n) = \sigma(X_0, I_1, \dots, I_n)$ .
3. the random map  $\Phi_n$  is  $\sigma(X_0, J_1, \dots, J_n)$ -measurable.
4.  $\phi_{X_0} \circ \psi_{I_1} \circ \dots \circ \psi_{I_n} = \psi_{J_1} \circ \dots \circ \psi_{J_n} \circ \Phi_n$ .
5.  $X_n = C \circ \psi_{J_1} \circ \dots \circ \psi_{J_n} \circ \Phi_n$ .

*Proof.* Since  $\Phi_0 = \phi_{X_0}$ , properties 2, 3, 4, 5 above hold with  $n$  replaced by 0.

Let  $n \geq 1$ . Assume that properties 2, 3, 4, 5 hold with  $n$  replaced by  $n - 1$ .

Then by lemma 35, property 1 holds.

By definition and by the induction hypothesis, the random set  $J_n = \Phi_{n-1}(I_n)$  is  $\sigma(X_0, I_1, \dots, I_n)$ -measurable. Conversely, since  $I_n = \Phi_{n-1}^{-1}(J_n)$ , the knowledge of  $\Phi_{n-1}$  and  $J_n$  is sufficient to recover  $I_n$ , so property 2 holds.

For every  $k \in \mathbf{N}$ ,  $\Phi_n(k)$  is the rank of the integer  $\Phi_{n-1}(\psi_{I_n}(k))$  in the set  $\Phi_{n-1}(I_n)$ . Thus the random map  $\Phi_n$  is measurable for  $\sigma(X_0, J_1, \dots, J_{n-1}, I_n) = \sigma(X_0, J_1, \dots, J_n)$ . Therefore, property 3 holds.

By induction hypothesis and by formula 1,

$$\begin{aligned} \phi_{X_0} \circ \psi_{I_1} \circ \dots \circ \psi_{I_n} &= (\phi_{X_0} \circ \psi_{I_1} \circ \dots \circ \psi_{I_{n-1}}) \circ \psi_{I_n} \\ &= (\psi_{J_1} \circ \dots \circ \psi_{J_{n-1}} \circ \Phi_{n-1}) \circ \psi_{I_n} \\ &= (\psi_{J_1} \circ \dots \circ \psi_{J_{n-1}}) \circ (\Phi_{n-1} \circ \psi_{I_n}) \\ &= (\psi_{J_1} \circ \dots \circ \psi_{J_{n-1}}) \circ (\psi_{J_n} \circ \Phi_n) \\ &= \psi_{J_1} \circ \dots \circ \psi_{J_n} \circ \Phi_n, \end{aligned}$$

so

$$\begin{aligned} X_n &= X_0 \circ \psi_{I_1} \circ \dots \circ \psi_{I_n} \\ &= C \circ \phi_{X_0} \circ \psi_{I_1} \circ \dots \circ \psi_{I_n} \\ &= C \circ \psi_{J_1} \circ \dots \circ \psi_{J_n} \circ \Phi_n, \end{aligned}$$

which yields properties 4 and 5.

Lemma 38 follows by recursion. □

The next result shows that the innovations  $(J_n)_{n \geq 1}$  constructed above provide better and better approximations of  $X_n$  as  $n \rightarrow +\infty$ .

**Lemma 40.** *Fix  $L \in \mathbf{N}$ . Then  $\mathbb{P}[X_n = C \circ \psi_{J_1} \circ \dots \circ \psi_{J_n}$  on  $[[1, L]]] \rightarrow 1$  as  $n \rightarrow +\infty$ .*

*Proof.* By equality 5, it suffices to check that,  $\mathbb{P}(E_n) \rightarrow 1$  as  $n \rightarrow +\infty$ , where  $E_n$  is the event “ $\Phi_n$  coincides on  $[[1, L]]$  with the identity map”.

By lemma 31,  $\psi_{I_1} \circ \dots \circ \psi_{I_n} = \psi_{A_n}$  and  $\psi_{J_1} \circ \dots \circ \psi_{J_n} = \psi_{B_n}$ , where  $A_n$  and  $B_n$  are random subsets of  $\mathbf{N}$  with law  $\nu_{p_n}$ , where  $p_n = 2^{-n}$ .

Therefore, by property 4 of lemma 39,  $\phi_{X_0} \circ \psi_{A_n} = \psi_{B_n} \circ \Phi_n$ , so for each  $k \in \mathbf{N}$ ,  $\Phi_n(k)$  is the rank of the integer  $\phi_{X_0}(\psi_{A_n}(k))$  in the set  $\phi_{X_0}(A_n) = B_n$ .

Thus, the event  $E_n$  holds if and only if the  $L$  first elements of the set  $\phi_{X_0}(A_n)$  in increasing order are  $\phi_{X_0}(\psi_{A_n}(1)), \dots, \phi_{X_0}(\psi_{A_n}(L))$ .

Set  $\tau_{n,k} = \psi_{A_n}(k)$  for every  $k \in \mathbf{N}$ . Since the law of  $A_n$  is  $\nu_{p_n}$ , the random variables  $\tau_{n,1}, \tau_{n,2} - \tau_{n,1}, \tau_{n,3} - \tau_{n,2}, \dots$  are independent and geometric with parameter  $p_n$ .

We have noted that

$$E_n = [\forall k \geq L + 1, \phi_{X_0}(\tau_{n,1}) < \dots < \phi_{X_0}(\tau_{n,L}) < \phi_{X_0}(\tau_{n,k})]$$

Roughly speaking, the probability of this event tends to 1 because  $\phi_{X_0}$  is close to the identity map and the set  $A_n$  gets sparser and sparser as  $n \rightarrow +\infty$ . Let us formalize this argument.

Since  $X_0$  is uniform on  $\{a, b\}^\infty$ , the random variables  $(\eta_i)_{i \geq 1} = (\mathbf{1}_{[X_0(i)=b]})_{i \geq 1}$  form an i.i.d. sequence of Bernoulli random variables with parameter  $1/2$ . For every  $t \in \mathbf{N}$ ,

the random variable  $S_t = \eta_1 + \dots + \eta_t$  counts the number of  $b$  in the subword  $X_0(\llbracket 1, t \rrbracket)$ , whereas  $t - S_t$  counts the number of  $a$  in the subword  $X_0(\llbracket 1, t \rrbracket)$ , so by definition of  $\phi_{X_0}$ ,

$$\begin{aligned}\phi_{X_0}(t) &= 2(t - S_t) - 1 && \text{if } X_0(t) = a, \\ \phi_{X_0}(t) &= 2S_t && \text{if } X_0(t) = b.\end{aligned}$$

Given  $t_1 < t_2$  in  $\mathbf{N}$ , the inequality  $\max(S_{t_1}, t_1 - S_{t_1}) < \min(S_{t_2}, t_2 - S_{t_2})$  implies  $\phi_{X_0}(t_1) < \phi_{X_0}(t_2)$  for every integer  $t \geq t_2$ . Therefore,

$$E_n \supset [\forall k \in \llbracket 1, L \rrbracket, \max(S_{\tau_{n,k}}, \tau_{n,k} - S_{\tau_{n,k}}) < \min(S_{\tau_{n,k+1}}, \tau_{n,k+1} - S_{\tau_{n,k+1}})].$$

Thus it suffices to prove that for any fixed  $k \in \mathbf{N}$ ,

$$p_{n,k} := \mathbb{P}[\max(S_{\tau_{n,k}}, \tau_{n,k} - S_{\tau_{n,k}}) \geq \min(S_{\tau_{n,k+1}}, \tau_{n,k+1} - S_{\tau_{n,k+1}})] \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Since  $X_0$  is independent of  $I_1, \dots, I_n$ , the sequence  $(S_t)_{t \geq 1}$  is independent of the sequence  $(\tau_{n,k})_{k \geq 1}$ . Moreover,  $(S_t)_{t \geq 1}$  has the same law as  $(t - S_t)_{t \geq 1}$  and  $S_{\tau_{n,k+1}} - S_{\tau_{n,k}}$  has the same law as  $S_{\tau_{n,1}}$ . Therefore, for every integer  $x \geq 1$ ,

$$\begin{aligned}p_{n,k} &\leq 2\mathbb{P}[S_{\tau_{n,k+1}} \leq \max(S_{\tau_{n,k}}, \tau_{n,k} - S_{\tau_{n,k}})] \\ &= 2\mathbb{P}[S_{\tau_{n,k+1}} - S_{\tau_{n,k}} \leq \max(0, \tau_{n,k} - 2S_{\tau_{n,k}})] \\ &\leq 2\mathbb{P}[S_{\tau_{n,k+1}} - S_{\tau_{n,k}} \leq x - 1] + 2\mathbb{P}[\tau_{n,k} - 2S_{\tau_{n,k}} \geq x] \\ &= 2\mathbb{P}[S_{\tau_{n,1}} \leq x - 1] + \mathbb{P}[|2S_{\tau_{n,k}} - \tau_{n,k}| \geq x].\end{aligned}$$

On the one hand, the random  $S_{\tau_{n,1}}$  is binomial with parameters  $\tau_{n,1}$  and  $1/2$  conditionally on  $\tau_{n,1}$ , so its generating function is given by

$$\begin{aligned}\mathbb{E}[z^{S_{\tau_{n,1}}}] &= \mathbb{E}[\mathbb{E}[z^{S_{\tau_{n,1}}} | \sigma(\tau_{n,1})]] \\ &= \mathbb{E}\left[\left(\frac{1+z}{2}\right)^{\tau_{n,1}}\right] \\ &= \frac{p_n(1+z)/2}{1 - (1-p_n)(1+z)/2} \\ &= \frac{p_n(1+z)}{1 + p_n - (1-p_n)z} \\ &= \frac{p_n(1+z)}{1 + p_n} \sum_{m=0}^{+\infty} \left(\frac{1-p_n}{1+p_n}\right)^m z^m\end{aligned}$$

for every complex number  $z$  with modulus  $< 1$ . This yields the law of  $S_{\tau_{n,1}}$ , namely

$$P[S_{\tau_{n,1}} = 0] = \frac{p_n}{1 + p_n},$$

$$P[S_{\tau_{n,1}} = s] = \frac{p_n}{1 + p_n} \left( \left(\frac{1-p_n}{1+p_n}\right)^{s-1} + \left(\frac{1-p_n}{1+p_n}\right)^s \right) = 2p_n \frac{(1-p_n)^{s-1}}{(1+p_n)^{s+1}} \text{ if } s \geq 1.$$

Therefore,  $\mathbb{P}[S_{\tau_{n,1}} = s] \leq 2p_n$  for every  $s \geq 0$ , so  $\mathbb{P}[S_{\tau_{n,1}} \leq x - 1] \leq 2p_n x$ .

On the other hand,  $(2S_t - t)_{t \geq 0}$  is a simple symmetric random walk on  $\mathbf{Z}$ , independent of  $\tau_{n,k}$  so

$$\mathbb{E}[2S_{\tau_{n,k}} - \tau_{n,k} | \sigma(\tau_{n,k})] = 0 \text{ and } \text{Var}(2S_{\tau_{n,k}} - \tau_{n,k} | \sigma(\tau_{n,k})) = \tau_{n,k}.$$

Therefore,

$$\mathbb{E}[2S_{\tau_{n,k}} - \tau_{n,k}] = 0 \text{ and } \text{Var}(2S_{\tau_{n,k}} - \tau_{n,k}) = \text{Var}(0) + \mathbb{E}[\tau_{n,k}] = k/p_n,$$

so Bienaymé-Tchebicheff's inequality yields  $\mathbb{P}[|2S_{\tau_{n,k}} - \tau_{n,k}| \geq x] \leq (k/p_n)x^{-2}$ .

Hence, for every  $n$  and  $x$  in  $\mathbf{N}$ ,  $p_{n,k} \leq 4p_n x + (k/p_n)x^{-2}$ . Choosing  $x = \lceil p_n^{-2/3} \rceil$  yields  $p_{n,k} \leq 4(p_n^{1/3} + p_n) + kp_n^{1/3}$ . The result follows.  $\square$

To finish the proof of theorem 34, we need to remind some standard facts about Vershik's first level criterion, namely definition 2.6, proposition 2.7 and proposition 2.17 of [17].

Let  $\mathcal{F} = (\mathcal{F}_n)_{n \leq 0}$  be a filtration which is locally of product-type.

Given a separable metric space  $(E, d)$ , a random variable  $R \in L^1(\mathcal{F}_0, E)$  satisfies Vershik's first level criterion (with respect to  $\mathcal{F}$ ) if for every  $\delta > 0$ , one can find an integer  $n_0 \leq 0$ , some innovations  $V_{n_0+1}, \dots, V_0$  of  $\mathcal{F}$  at times  $n_0 + 1, \dots, 0$  and some random variable  $S \in L^1(\sigma(V_{n_0+1}, \dots, V_0), E)$  such that  $\mathbb{E}[d(R, S)] < \delta$ .

The subset of random variables in  $L^1(\mathcal{F}_0, E)$  satisfying Vershik's first level criterion (with respect to  $\mathcal{F}$ ) is closed in  $L^1(\mathcal{F}_0, E)$ .

If  $R \in L^1(\mathcal{F}_0, E)$  satisfies Vershik's first level criterion, then any measurable real function of  $R$  also satisfies Vershik's first level criterion.

The first step of the proof is to check that for every  $m \leq 0$ , the random variable  $(X_m(1), \dots, X_m(L))$ , taking values in  $\{a, b\}^L$  endowed with the discrete metric, satisfies Vershik's first level criterion with respect to  $(\mathcal{F}_n^{X,I})_{n \leq 0}$ .

Indeed, by stationarity, the construction of lemma 38 can be started at any time  $n_0$  instead of time 0. Starting this construction at time  $n_0$  yields innovations  $J_{n_0+1}^{n_0}, J_{n_0+2}^{n_0}, \dots$  at times  $n_0 + 1, n_0 + 2, \dots$ . Fix two integers  $m \leq 0$  and  $L \geq 1$ . By stationarity, for every  $n_0 \leq m$ ,

$$\mathbb{P}[X_m = C \circ \psi_{J_{n_0+1}^{n_0}} \circ \dots \circ \psi_{J_m^{n_0}} \text{ on } \llbracket 1, L \rrbracket] = \mathbb{P}[X_{m-n_0} = C \circ \psi_{J_1} \circ \dots \circ \psi_{J_{m-n_0}} \text{ on } \llbracket 1, L \rrbracket].$$

Lemma 40 ensures that this probability tends to 1 as  $n \rightarrow +\infty$ .

We derive successively that the following random variables also satisfies Vershik's first level criterion:

- $X_m$ , valued in  $\{a, b\}^\infty$  endowed with the metric given by

$$d(x, y) = 2^{-\inf\{i \geq 1: x(i) \neq y(i)\}}.$$

- $(X_m, I_{m+1}, \dots, I_0)$  endowed with the product of the metrics defined as above on each factor  $\{a, b\}^\infty$  or  $\mathcal{P}(\mathbf{N})$  identified with  $\{0, 1\}^\infty$ ;
- any measurable real function of  $(X_m, I_{m+1}, \dots, I_0)$ ;
- any real random variable in  $\mathcal{F}_0^{X,I}$ .

The proof is complete.

### 5.3 A non complementable filtration yielding a complementable factor

We still work with the filtration generated by the uniform randomised decimation process  $((X_n, I_n))_{n \in \mathbf{Z}}$  on the alphabet  $\{a, b\}$ . We call  $\mathcal{P}''(\mathbf{N})$  the set of all infinite subsets of  $\mathbf{N}$  with infinite complement, and we set  $E = \{a, b\}^\infty \times \mathcal{P}''(\mathbf{N})$ . Since  $\nu(\mathcal{P}''(\mathbf{N})) = 1$ , we may assume and we do assume that the Markov chain  $((X_n, I_n))_{n \in \mathbf{Z}}$  takes values in  $E$ .

At each time  $n$ , we define the random variable  $Y_n = \psi_{I_n^c}(X_{n-1})$  coding the portion of the infinite word  $X_{n-1}$  rejected at time  $n$  to get the word  $X_n$ . Of course, the knowledge of  $I_n$ ,  $X_n$  and  $Y_n$  enables us to recover  $X_{n-1}$ : for every  $i \in \mathbf{N}$ ,  $X_{n-1}(i)$  equals  $X_n(r)$  or  $Y_n(r)$  according that  $i$  is the  $r^{\text{th}}$  element of  $I_n$  or of  $I_n^c$ . We can say more.

**Proposition 41.** *(Properties of the sequences  $(Y_n)_{n \in \mathbf{Z}}$  and  $(I_n)_{n \in \mathbf{Z}}$ )*

1. *The random variables  $Y_n$  are independent and uniform on  $\{a, b\}^\infty$ .*
2. *The sequence  $(Y_n)_{n \in \mathbf{Z}}$  is independent of the sequence  $(I_n)_{n \in \mathbf{Z}}$ .*
3. *Each  $X_n$  is almost surely a measurable function of  $I_{n+1}, Y_{n+1}, I_{n+2}, Y_{n+2}, \dots$*

Note that proposition 41 provides a constructive method to get a uniform randomised decimation process on  $\{a, b\}$ .

*Proof.* The first two statements follow from a repeated application of lemma 32. Since the formulas involving the processes  $I, X, Y$  are invariant by time-translations, one needs only to check the third statement when  $n = 0$ . For every  $i \in \mathbf{N}$ , call

$$N_i = \inf\{n \geq 1 : i \notin \psi_{I_1} \circ \dots \circ \psi_{I_n}(\mathbf{N})\}.$$

the first time  $n$  at which the letter  $X_0(i)$  is rejected when forming the word  $X_n$ . For every  $n \geq 0$ ,  $[N_i > n] = [i \in \psi_{I_1} \circ \dots \circ \psi_{I_n}(\mathbf{N})]$ ; but by lemma 31, the law of the range of  $\psi_{I_1} \circ \dots \circ \psi_{I_n}$  is  $\nu_{2^{-n}}$ , so  $\mathbb{P}[N_i > n] = 2^{-n}$ . Therefore,  $N_i$  is a measurable function of  $(I_n)_{n \geq 1}$  and is almost surely finite. On the event  $[N_i < +\infty]$ ,  $X_0(i) = Y_{N_i}(R_i)$ , where  $R_i$  is the rank of  $i$  in the set  $\psi_{I_1} \circ \dots \circ \psi_{I_{N_i}} \circ \psi_{I_{N_i+1}^c}(\mathbf{N})$ . The proof is complete.  $\square$

We split each random variable  $I_n$  into two independent random variables, namely  $U_n = \{I_n, I_n^c\}$  and  $V_n = \mathbf{1}_{[1 \in I_n]}$ . The random variable  $U_n$  takes values in the set  $\Pi$  of all partitions of  $\mathbf{N}$  into two infinite blocks. Given such a partition  $u \in \Pi$ , we denote by  $u(1)$  the block containing 1 and by  $u(0)$  its complement. By construction, each one of the random variables  $U_n$ ,  $U_n(0)$  and  $U_n(1)$  carries the same information, and we have  $I_n = U_n(V_n)$ .

Call  $\mathcal{C}$  the cylindrical  $\sigma$ -field on  $E^{\mathbf{Z}}$  and  $\pi$  the law of  $((Y_n, I_n))_{n \in \mathbf{Z}}$ . By stationarity, the shift operator  $T$  is an automorphism of  $(E^{\mathbf{Z}}, \mathcal{C}, \pi)$ . The formulas defining  $U_n, V_n, Y_n$  from  $I_n$  and  $X_{n-1}$  are invariant by time-translations so the measurable maps  $\Phi$  and  $\Psi$  yielding  $(U_n)_{n \in \mathbf{Z}}$  and  $((V_n, Y_n))_{n \in \mathbf{Z}}$  from  $((Y_n, I_n))_{n \in \mathbf{Z}}$  commute with  $T$ . Therefore, the  $\sigma$ -fields  $\Phi^{-1}(\mathcal{C})$  and  $\Psi^{-1}(\mathcal{C})$  are factors of  $T$ .

**Theorem 42.** *1. The factor  $\Phi^{-1}(\mathcal{C})$  is complementable with complement  $\Psi^{-1}(\mathcal{C})$ .*

*Thus, if the Markov chain  $((Y_n, I_n))_{n \in \mathbf{Z}}$  is defined on the canonical space  $(E^{\mathbf{Z}}, \mathcal{C}, \pi)$ , then  $\mathcal{F}_\infty^U$  is a complementable factor of  $T$  with complement  $\mathcal{F}_\infty^{V, Y}$ .*

2. *Yet, the filtration  $\mathcal{F}^U$  is not complementable in the filtration  $\mathcal{F}^{X, I}$ .*

*Proof.* By proposition 41,  $\mathcal{F}_\infty^U$  and  $\mathcal{F}_\infty^{V,Y}$  are independent, and  $\mathcal{F}_\infty^U \vee \mathcal{F}_\infty^{V,Y} = \mathcal{F}_\infty^{X,I} \pmod{\mathbb{P}}$ . Therefore,  $\Phi^{-1}(\mathcal{C})$  and  $\Psi^{-1}(\mathcal{C})$  are independent in  $(E^{\mathbb{Z}}, \mathcal{C}, \pi)$  and  $\Phi^{-1}(\mathcal{C}) \vee \Psi^{-1}(\mathcal{C}) = \mathcal{C} \pmod{\pi}$ : the factor  $\sigma(\Phi)$  is complementable with complement  $\sigma(\Psi)$ .

Let  $U = (U_n)_{n \leq 0}$ . The random variable  $U$  takes values in  $\Pi^{\mathbb{Z}^-}$ .

For every  $u = (u_n)_{n \leq 0} \in \Pi^{\mathbb{Z}^-}$  and  $n \leq 0$ , call  $W_n^u$  the map from  $\{0, 1\}^{|n|}$  to  $\{a, b\}$  defined by

$$W_n^u(v_{n+1}, \dots, v_0) = X_n \circ \psi_{u_{n+1}(v_{n+1})} \circ \dots \circ \psi_{u_0(v_0)}(1).$$

By ordering the elements of  $\{0, 1\}^{|n|}$  in the lexicographic order, one identifies  $W_n^u$  with an element of  $\{a, b\}^{|n|}$ .

Since  $X_n = X_{n-1} \circ \psi_{I_n} = X_{n-1} \circ \psi_{u_n(V_n)}$   $\mathbb{P}_u$ -almost surely, we have

$$W_n^u(v_{n+1}, \dots, v_0) = W_{n-1}^u(V_n, v_{n+1}, \dots, v_0) \mathbb{P}_u\text{-almost surely,}$$

so  $W_n^u$  is the left half or the right half of  $W_{n-1}^u$  according  $V_n$  equals 0 or 1. Moreover, under  $\mathbb{P}_u$ , the random variable  $V_n$  is independent of  $\mathcal{F}_{n-1}^{W_n^u, V}$  and uniform on  $\{0, 1\}$ .

Hence, under  $\mathbb{P}_u$ , the process  $(W_n^u, V_n)_{n \leq 0}$  is a dyadic split-words process with innovations  $(V_n)_{n \leq 0}$ . The filtration of this process is known to be non-standard (see [25]). But one checks that  $(V_n)_{n \leq 0}$  is also a sequence of innovations of the larger filtration  $(\mathcal{F}^{X,I})_{n \leq 0}$  seen under  $\mathbb{P}_u$ , so  $(\mathcal{F}_n^{W_n^u, V})_{n \leq 0}$  is immersed in  $(\mathcal{F}^{X,I})_{n \leq 0}$  and  $(\mathcal{F}^{X,I})_{n \leq 0}$  is also non-standard under  $\mathbb{P}_u$ .

If  $(\mathcal{F}_n^U)_{n \leq 0}$  admitted an independent complement  $(\mathcal{G}_n)_{n \leq 0}$  in  $(\mathcal{F}_n^{X,I})_{n \leq 0}$ , this complement would be immersed in the product-type filtration  $(\mathcal{F}_n^{X,I})_{n \leq 0}$  thus standard. Therefore, for  $U(\mathbb{P})$ -almost every  $u \in \Pi^{\mathbb{Z}^-}$ , the filtration  $(\mathcal{F}_n^{X,I})_{n \leq 0}$  would be standard under the probability  $\mathbb{P}_u = \mathbb{P}[\cdot | U = u]$ , by proposition 0.1 of [19]. This leads to a contradiction.

We are done. □

## 6 Annex : reminders on partitions and entropy

We recall here classical definitions and results. Most of them can be found in [24]. See also [23].

We fix a measure-preserving map from a probability space  $(Z, \mathcal{Z}, \pi)$  to itself. In what follows,  $\alpha, \beta, \gamma$  denote measurable countable partitions of  $Z$  (here, ‘measurable’ means that the blocks belong to  $\mathcal{Z}$ ), and  $\mathcal{F}, \mathcal{G}$  denote sub- $\sigma$ -fields of  $\mathcal{Z}$ .

### 6.1 Partitions

Defining the entropy requires discretizations of the ambient probability space, that is why we introduce countable partitions (equivalently, we could use discrete random variables). We need a few basic definitions.

**Definition 43.** *The partition  $\beta$  is finer than  $\alpha$  (and we note  $\alpha \leq \beta$ ) if each block of  $\alpha$  is the union of some collection of blocks of  $\beta$ .*

**Definition 44.** *The (non-empty) intersections  $A \cap B$  with  $A \in \alpha$  and  $B \in \beta$  form a partition; this partition is the coarsest refinement of  $\alpha$  and  $\beta$  and is denoted by  $\alpha \vee \beta$ .*

**Definition 45.** The partition  $\alpha$  and  $\beta$  are independent if and only if  $\pi(A \cap B) = \pi(A)\pi(B)$  for every  $A \in \alpha$  and  $B \in \beta$ .

**Definition 46.** We denote by  $T^{-1}\alpha$  the partition defined by

$$T^{-1}\alpha = \{T^{-1}(A) : A \in \alpha\}.$$

If  $T$  is invertible (namely bimeasurable), we denote by  $T\alpha$  the partition defined by

$$T\alpha = \{T(A) : A \in \alpha\}.$$

## 6.2 Fischer information and entropy of a partition

Given  $A \in \mathcal{X}$ , we view  $-\log_2 \pi(A)$  as the quantity of information provided by the event  $A$  when  $A$  occurs, with the convention  $-\log_2 0 = +\infty$ . With this definition, the occurrence of a rare event provide much information; the information provided by two independent events  $A$  and  $B$  occurring at the same time is the sum of the informations provided by each of them separately. Taking logarithms in base 2 is an arbitrary convention which associates one unity of information to any uniform Bernoulli random variable.

**Definition 47.** The Fischer information of the partition  $\alpha$  is the random variable

$$I_{\alpha|\mathcal{F}} := - \sum_{A \in \alpha} \mathbf{1}_A \log_2 \pi(A).$$

The entropy of the partition  $\alpha$  is the quantity

$$H(\alpha) = \mathbb{E}_\pi[I_{\alpha|\mathcal{F}}] = - \sum_{A \in \alpha} \mathbf{1}_A \log_2 \pi(A).$$

Since  $I_{\alpha|\mathcal{F}}$  takes values in  $[0, +\infty]$ , the entropy  $H(\alpha)$  is well-defined and belongs to  $[0, +\infty]$ . If we view  $-\log_2 \pi(A)$  as the quantity of information provided by the event  $A$  when  $A$  occurs, the entropy of  $\alpha$  is the mean quantity of information provided by the partition  $\alpha$ .

**Proposition 48.** (First properties)

1. (invariance by  $T$ )  $H(T^{-1}\alpha) = H(\alpha)$ .
2. (monotonicity) If  $\alpha \leq \beta$ , then  $H(\alpha) \leq H(\beta)$ .
3. (sub-additivity)  $H(\alpha \vee \beta) \leq H(\alpha) + H(\beta)$ .
4. (condition for additivity) If  $H(\alpha) < +\infty$  and  $H(\beta) < +\infty$ , then  $H(\alpha \vee \beta) = H(\alpha) + H(\beta)$  if and only if  $\alpha$  is independent of  $\beta$ .

## 6.3 Conditional Fischer information and entropy of a partition

The previous definition can be generalized as follows.

**Definition 49.** The conditional Fischer information of the partition  $\alpha$  with regard to  $\mathcal{F}$  is the random variable

$$I_{\alpha|\mathcal{F}} = - \sum_{A \in \alpha} \mathbf{1}_A \log_2 \pi(A|\mathcal{F}).$$

The conditional entropy of the partition  $\alpha$  with regard to  $\mathcal{F}$  is the quantity

$$H(\alpha|\mathcal{F}) = \mathbb{E}_\pi[I_{\alpha|\mathcal{F}}] = - \sum_{A \in \alpha} \mathbb{E}_\pi[\mathbf{1}_A \log_2 \pi(A|\mathcal{F})] = - \sum_{A \in \alpha} \mathbb{E}_\pi[\pi(A|\mathcal{F}) \log_2 \pi(A|\mathcal{F})].$$

**Proposition 50.** (*First properties*)

1.  $H(\alpha|\mathcal{F}) \geq 0$ , with equality if and only if  $\alpha \in \mathcal{F} \pmod{\pi}$ .
2.  $H(\alpha|\mathcal{F}) \leq H(\alpha)$ ; when  $H(\alpha) < +\infty$ , equality holds if and only if  $\alpha$  is independent of  $\mathcal{F}$ .
3. If  $\mathcal{F} \subset \mathcal{G}$ , then  $H(\alpha|\mathcal{F}) \geq H(\alpha|\mathcal{G})$ .
4. If  $\beta$  has finite entropy, then  $H(\alpha|\beta) := H(\alpha|\sigma(\beta)) = H(\alpha \vee \beta) - H(\beta)$ .
5.  $H(\alpha \vee \beta|\mathcal{F}) = H(\alpha|\mathcal{F}) + H(\beta|\mathcal{F} \vee \sigma(\alpha))$ .

For finite partitions, the two statements below are a consequence of martingale convergence theorem. Checking uniform integrability is easy for finite partitions (the entropy of a partition into  $n$  blocks is at most  $\log_2 n$ ), but requires a technical lemma otherwise.

**Proposition 51.** (*monotone sequence of  $\sigma$ -fields*).

1. If  $(\mathcal{F}_n)_{n \geq 0}$  is a non-decreasing sequence of  $\sigma$ -fields, and  $H(\alpha) < +\infty$ , then

$$H(\alpha|\mathcal{F}_n) \rightarrow H(\alpha|\mathcal{F}_\infty) \text{ where } \mathcal{F}_\infty = \bigvee_{n \geq 0} \mathcal{F}_n.$$

2. If  $(\mathcal{F}_n)_{n \geq 0}$  is a non-increasing sequence of  $\sigma$ -fields, and  $H(\alpha) < +\infty$ , then

$$H(\alpha|\mathcal{F}_n) \rightarrow H(\alpha|\mathcal{F}_\infty) \text{ where } \mathcal{F}_\infty = \bigcap_{n \geq 0} \mathcal{F}_n.$$

## 6.4 Entropy of a measure-preserving map

First, we define quantities  $h(\alpha, T)$ .

**Proposition 52.** (*Definition and formula for  $h(\alpha, T)$* )

1. The sequence  $(H_n(\alpha, T))_{n \geq 0}$  defined by

$$H_n(\alpha, T) = H(\alpha \vee T^{-1}\alpha \vee \dots \vee T^{-(n-1)}\alpha)$$

is concave, so the limit  $h(\alpha, T) = \lim_{n \rightarrow +\infty} H_n(\alpha, T)$  exists in  $[0, +\infty]$ .

2. If  $H(\alpha) < +\infty$ , then  $h(\alpha, T) = H(\alpha|\alpha^-)$ , where  $\alpha^- = \bigvee_{k \geq 1} T^{-k}\alpha$  denotes the  $\sigma$ -field generated by the partitions  $(T^{-k}\alpha)_{k \geq 1}$ .

*Proof.* The statements follow from the equality

$$\begin{aligned} H_{n+1}(\alpha, T) - H_n(\alpha, T) &= H(\alpha \vee T^{-1}\alpha \vee \dots \vee T^{-n}\alpha) - H(T^{-1}\alpha \vee \dots \vee T^{-(n-1)}\alpha) \\ &= H(\alpha|T^{-1}\alpha \vee \dots \vee T^{-(n-1)}\alpha), \end{aligned}$$

from Cesàro's lemma and from proposition 51. □

**Definition 53.** We define the entropy of  $T$  as the quantity

$$h(T) = \sup\{h(\alpha, T) : \alpha \text{ countable measurable partition of } X\}.$$



## 6.5 Generators

Countable generating partitions help us to computing the entropy of invertible measure-preserving maps.

**Definition 54.** *Assume that  $T$  is invertible. A countable measurable partition  $\gamma$  is generating (with regard to  $T$ ) if  $\mathcal{X}$  is generated modulo the null sets by the partition  $(T^k\gamma)_{k \in \mathbf{Z}}$ .*

**Theorem 55.** *(Kolmogorov-Sinai theorem) If  $T$  is invertible and  $\gamma$  is a countable generator (with regard to  $T$ ), then  $h(T) = h(\gamma, T)$ .*

**Proposition 56.** *If  $T$  is invertible and admits a countable generator, then  $h(T^n) = |n|h(T)$  for every  $n \in \mathbf{Z}$ .*

Here is the basic example of generator.

**Example 57.** *Let  $\Lambda$  be a countable set,  $S : (y_k)_{k \in \mathbf{Z}} \mapsto (y_{k+1})_{k \in \mathbf{Z}}$  the shift operator on  $\Lambda^{\mathbf{Z}}$ , and  $\mu$  any shift-invariant probability measure on  $\Lambda^{\mathbf{Z}}$ . The partition  $p_0^{-1}\mathcal{P}(\Lambda)$  associated to the canonical projection  $p_0 : (y_k)_{k \in \mathbf{Z}} \mapsto y_0$  is a generating partition with regard to  $S$ .*

The interesting fact is that many situations can be reduced to this particular case. The proof of next theorem is outlined in [14].

**Theorem 58.** *(Rohlin's countable generator theorem) If  $T$  is invertible and  $(Z, \mathcal{Z}, \pi)$  is a Lebesgue space, then  $T$  admits a countable generating partition  $\gamma = \{C_\lambda : \lambda \in \Lambda\}$ . Moreover, the  $\gamma$ -name map  $\Phi$  from  $Z$  to  $\Lambda^{\mathbf{Z}}$ , defined by  $\Phi(z)_k = \lambda$  whenever  $T^k(z) \in C_\lambda$ , is invertible modulo the null sets, when  $\Lambda^{\mathbf{Z}}$  is endowed with the probability measure  $\Phi(\pi)$ . The measure  $\Phi(\pi)$  is shift-invariant, so  $T$  is isomorphic modulo the null sets to the shift operator on  $\Lambda^{\mathbf{Z}}$ .*

Under the additional hypothesis  $h(T) < +\infty$ , Krieger's theorem provides the existence of a finite generator with size at most  $2^{h(T)+1}$ . We do not use this refinement in the present paper.

## Acknowledgements

I thank J. Brossard, M. Émery, S. Laurent, J.P. Thouvenot for stimulating conversations.

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