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Traffic flow within a two-dimensional continuum anisotropic network

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Abstract

Network flow computing based on macroscopic traffic flow models for large and dense networks involves a large number of parameters and variables, and significant computational efforts. We aim at reducing these and introduce a modelling framework at two-dimensional scale in order to model traffic flow of transportation systems of large-scale networks. We present a network flow pattern corresponding to network flows modelling with a few network sensors of traffic count locations. We manage and evaluate traffics on wide and dense networks with a minimum of available measurements and data, through modelling of global behaviours based on local behaviours. We find that the traffic at this scale is governed by multidimensional hyperbolic conservations laws. Godunov-type method has been proposed to compute the network flow flux across computational domains. We validate the approach by dealing with an example for the simplified road network of the city of Paris as a case study of the proposed model, and provide an example of road network modelling with a two-dimensional dynamical traffic flow pattern.

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Keywords: dense network of large surface, continuum anisotropic network, two-dimensional traffic flow, multidimensional hyperbolic PDEs system, finite volume method.

1. Introduction and Motivation

We often observe phenomena of congestion and mobile jams in areas where users' transport demand is very important with respect to the supply of existing networks, especially at peak hours. To ensure greater safety and reduce the risk of traffic incidents, a new trend has emerged in the field of transportation networks modelling for traffic management of large-scale networks. That is the two-dimensional modelling of network flows. It aims at managing and evaluating traffic on wide and dense networks with a minimum of available measurements and traffic data, through

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modelling of global behaviour based on local behaviours. The technique allows aggregating the network links as a continuous medium, where road traffic flows as a fluid on a surface. This modelling approach intends to reduce the large number of parameters, the unknown variables and the significant computational efforts involved in macroscopic flow models for large and dense urban networks. The approach aims to be suitable for network modelling with scarce traffic data. It considers the area of the large network as a continuum anisotropic media where vehicles behave like a two-dimensional fluid with four preferred directions of propagation (Saumtally, 2013; Wong, 1998, and references therein). Roads and vehicles are both aggregated, and the anisotropy of the network is due to the directions of propagation of vehicular flows in any position of the network. Few researches on this approach for network flows management and control have been performed, mainly in the static case (Wong, 1998; Prez and Benitez, 2010; Saumtally, 2013) and recently based on the existence and the concept of a network fundamental diagram (Keyvan-Ekbatani et al., 2012, 2015b,a).

The outline of the paper is the following. First we recall the definition of the terms "dense urban network" and "continuum anisotropic network" and give examples of cities which road network may be considered as dense, continuum and anisotropic. With intersection model of (Costeseque and Lebacque, 2012) we describe cars-flow at any point of an urban network, and express a non-negativity of flows. We describe at the same time, in details, existing movements, in any network point. We build thereafter multidimensional hyperbolic conservation laws that describes the dynamic of vehicles in all the computational domains. A Godunov-like scheme is deduced, which allows easy numerical calculation of the two-dimensional model.

2. Two-dimensional Anisotropic continuum network

2.1. Anisotropy of network

Definition 2.1.

In traffic theory, a network is said anisotropic when there are many possible interactions within a cutoff location.

One derives from local interactions global interactions which may be predetermined in a network. In the paper we reduce the local interactions in four different preferred directions of propagation of vehicles-flow in any location. Let us denote by \mathcal{U} the area of a urban road network. \mathcal{U} is supposed be a bounded and open subspace of the Euclidean space \mathbb{R}^2 , since any city has a frontier and then its urban road network is limited by its frontier too. Clearly $\mathcal{U} \subset \mathbb{R}^2$ and $meas(\mathcal{U}) < +\infty$, with $meas$ the Lebesgue measure in two dimensions. At any point P , of coordinates (x, y) , of the network, we assume four preferred directions of movement of vehicles, depicted by the Figure 1(a).

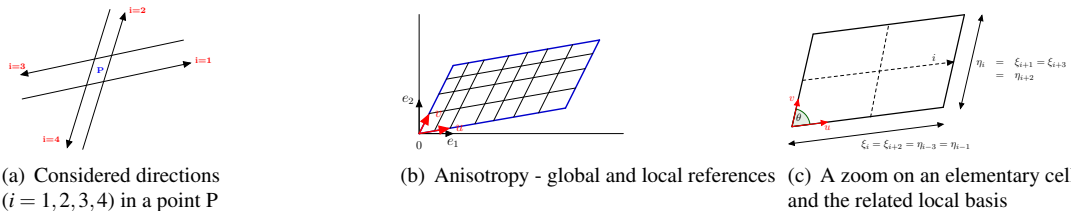


Fig. 1: Anisotropy - Structure of the network - Local and global basis

We assumed that \mathcal{U} is decomposed in zones or sub-areas \mathcal{U}_m , $m = 1, \dots, M$ respecting the below criteria:

1. $\mathcal{U} = \bigcup_{p=1}^P \mathcal{U}_m$
2. \mathcal{U}_m is a polygonal domain of \mathbb{R}^2 , which frontier is of Lebesgue measure strictly positive.
3. Each zone \mathcal{U}_m is subdivided in elementary cells or grids, providing an admissible mesh in the sense of (Eymard et al., 2000, Definition 5.1).
4. $\forall m, m' \in M, m \neq m' \Rightarrow \mathcal{U}_m \cap \mathcal{U}_{m'} \in \{\emptyset, \text{point, polygonal segment}\}$.

Keeping the anisotropy property of such a large network allows, in an easy way, the dynamical modelling of the traffic flows, ranging from dynamic assignment to control strategies.

2.2. Dense urban area as a continuum

We are interested in large dense urban networks. That is the urban areas of dense roads network. The density of the network in traffic, refers to the plurality of road sections of short length and their closeness's. A continuum approach then comes out by an aggregation of the roads network (as it is showed in (Prez and Benitez, 2010; Wong, 1998)), which makes vehicles behave on these networks like a two-dimensional fluid. One may consider the road networks of cities like Paris, Atlanta, Tokyo as dense road networks. According to the definition 2.1 the road networks of these cities may be classified as anisotropic when the traffic flow on these roads are observed from a significant height.

2.3. Considered flows functions and turning movements

Let $t > 0$ be the time variable, and $P(x, y) \in \mathcal{U}$ the position variable. The index $i \in \{1, 2, 3, 4\}$ denotes the direction of propagation of the traffic flow. Let us denote by $\tilde{q}_i(P, t)$ the car-flow section and $\tilde{\rho}_i(P, t)$ the car-density section in the direction $i \in \{1, 2, 3, 4\}$, at the time t and at the point P . These quantities can be considered as the flows and densities on individual links of the network in the direction i . We denote by λ_i the average number of sections per unit length (lane density) and by v_i the average number of lanes per link in the direction i .

According to the plain traffic conservation laws, the flow $\tilde{q}_j(P, t)$ in the direction $j \in \{1, 2, 3, 4\}$, satisfies:

$$\forall P(x, y), \forall t, \quad \tilde{q}_j(P, t) = \sum_{i=1}^4 \Gamma_{ij}(P, t) \tilde{q}_i(P, t), \quad (1)$$

$(\Gamma_{ij})_{i,j=1}^4$ being the turning movement coefficients. The matrix form of this equation is

$$\forall P(x, y), \forall t, \quad \tilde{q}(P, t) = (\tilde{q} \cdot \Gamma)(P, t), \quad (2)$$

where $\tilde{q} = (\tilde{q}_1 \ \tilde{q}_2 \ \tilde{q}_3 \ \tilde{q}_4)^T$ is the flow-vector of all directions. Besides, there is a relationship between the lane-traffic states $\tilde{\rho}$ and \tilde{q} and the traffic states ρ and q respectively as follows:

$$\tilde{\rho} = \frac{\rho}{\lambda v} \quad \text{and} \quad \tilde{q} = \frac{q}{\lambda v}. \quad (3)$$

Remark 2.1. The functions $(P, t) \mapsto (\Gamma_{ij}(P, t))_{i,j}$ are matrix functions of turning movement coefficients. Clearly $\Gamma_{ij}(P, t)$ is the fraction of turning movement coming from lanes of density λ_i in the directions i , reaching the point P and then going to direction j . The matrix Γ expresses the assignment of traffic in the network. We have two obvious relations:

$$\forall i, j \in \{1, 2, 3, 4\}, \quad \forall P(x, y), \forall t, \quad \Gamma_{ij}(P, t) \geq 0, \quad (4)$$

and

$$\forall i \in \{1, 2, 3, 4\}, \quad \forall P(x, y), \forall t, \quad \sum_{j=1}^4 \Gamma_{ij}(P, t) = 1. \quad (5)$$

The equation (5) is interpreted as a conservation law, meaning that cars do not appear or disappear on the lanes.

Remark 2.2. The two properties (4) and (5) show that $\Gamma(P, t) = (\Gamma_{ij}(P, t))_{i,j}$ is a non-negative and positive stochastic matrix.

Corollary 2.1. Let us assume that the matrix $\Gamma(P, t)$ is irreducible.

Considering the remark 2.2, and by applying the Theorem of Perron-Frobenius or the theorem of Brouwer of fixed point, we show that Γ has a real maximum positive eigenvalue equal to 1, and at left, $\mu(P, t)$ such that: $\tilde{q} = \mu U^T$, with U^T the eigenvector at left of Γ associated to the eigenvalue 1. This eigenvector has all its components positive. It can be viewed as yielding the dominant directional components of traffic at any point.

2.4. Motion equation at a network point - Node model

Besides, since every car-passenger aims at maximizing its own time travel, this leads to a car-flows optimization problem, stated as below, according to (Costeseque and Lebacque, 2012) optimization model for intersections:

$$\begin{aligned} & \max_{\tilde{q}} \sum_{j=1}^4 \Phi_j(\tilde{q}_j) \\ & \text{under constraints} \begin{cases} \forall j \in \{1, 2, 3, 4\}, 0 \leq \tilde{q}_j \leq \min(\delta_j, \sigma_j), \\ \forall j \in \{1, 2, 3, 4\}, \tilde{q}_j - \sum_{i=1}^4 \tilde{q}_i \Gamma_{ij} = 0. \end{cases} \end{aligned} \quad (6)$$

The functions Φ_j are assumed increasing and strictly concave with the flow \tilde{q}_j for a system where there is no regulation at intersection which sounds with a two-dimensional continuum transportation network (see the Lebacque's Intersection model (Costeseque and Lebacque, 2012; Lebacque and Khoshyaran, 2013)). The $(\Phi_j)_{j=1,2,3,4}$ are the attributes of the intersection model.

Under the Karush-Kuhn-Tucker conditions of optimality, one easily find for each i , the implicit analytic expression of the flow function \tilde{q}_i as follows:

$$\forall i \in \{1, 2, 3, 4\}, \tilde{q}_i = \min \left(\max \left(0, \Phi_i'^{-1} \left(l_i - \sum_{j=1}^4 l_j \Gamma_{ij} \right) \right), \min(\delta_i, \sigma_i) \right), \quad (7)$$

with $l = (l_i)_{i=1,2,3,4}$ being the Lagrange multiplier vector, that is related to the last constraint listed in (6), and $\Phi_i'^{-1}$ the inverse function of the attribute Φ_i . The Equation (7) expresses an implicit intersection charge for the link i . Let us specify the functions δ_i and σ_i . $\delta_i(P, t)$ is the users' transport demand function to direction i upstream the point P at time t , and $\sigma_i(P, t)$ is the network supply downstream to the same point P at the same time t . We have analytic expressions of these variables functions stated in (8). They are depicted by the well-known flow fundamental diagram in Figure 2 emphasize the equilibrium between the users' transport demand and the network supply. We have:

$$\delta_i = v_i \Delta_i \left(\frac{\tilde{\rho}_i}{v_i} \right) \quad \text{and} \quad \sigma_i = v_i \Omega_i \left(\frac{\tilde{\rho}_i}{v_i} \right), \quad (8)$$

where $\frac{\tilde{\rho}_i}{v_i}$ is the lineic density per lane, in the direction i .

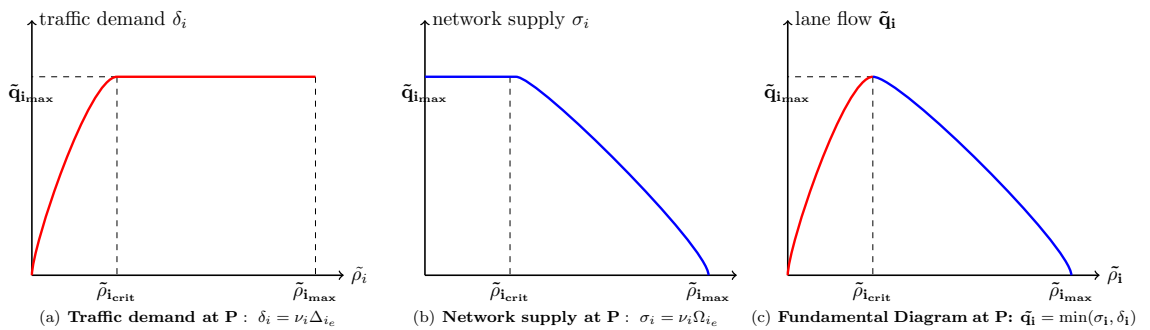


Fig. 2: Cars-flow fundamental diagram at a point P

Lemma 2.1. Under above assumptions and hypothesis, the following relation holds:

$$\forall i \in \{1, 2, 3, 4\}, \tilde{q}_i = \mu U_i^F, \quad \text{with} \quad \mu = \min_{1 \leq j \leq 4} \left\{ \frac{\min(\delta_j, \sigma_j)}{U_j^F} \right\}. \quad (9)$$

Proof. One has: $\mu U_i^\Gamma = \tilde{q}_i \leq \min(\delta_i, \sigma_i)$. Hence, $\mu \leq \min_{1 \leq i \leq 4} \left\{ \frac{\min(\delta_i, \sigma_i)}{U_i^\Gamma} \right\}$. We assume that μ have to attend its upper bound for responding the optimality condition of the flows, verifying by the physics of the traffics. \square

The relations (3) and (9) constitute the behavioural equation of the model.

3. Dynamic of car-flows within anisotropic continuum media

This section is devoted to the description of car-flows over network, whose area \mathcal{U} is considered as an open subset of \mathbb{R}^2 , which is my be decomposed on M traffic-zones $\mathcal{U}_m, m = 1, \dots, M$. We shall see that the dynamic of car-flows within each U_m anisotropic continuum media is governed by system of hyperbolic conservations laws.

3.1. Dynamic of number of vehicles in a zone

Therefore, there is internal conservation of the mass flow in any direction i of propagation. The two flows \tilde{q}_i and q_i functions are related with $q_i = \lambda_i \tilde{q}_i$ formula. As mentioned above, each zone U_m is meshed by a family of cells $C_m^{\alpha,\beta}$ which satisfies (Eymard et al., 2000, Definition 5.1). We assume that $\lambda_3 = \lambda_1$ and $\lambda_4 = \lambda_2$, and that $\Gamma_{13} = \Gamma_{31} = \Gamma_{24} = \Gamma_{42} = 0$. The first relationship implies that at any point P there is always four directions which are pairwise side by side. The second expresses that there is no U -turning movement at any point. Under these assumptions, the dynamical traffic flow model in (Saumtally et al., 2013) is valid for the topological structure of the network, presented in this study. The model in (Saumtally et al., 2013) boils down to a system of conservation laws of cars-density within an elementary cell (10). For any fixed zone-index $m = 1, 2, \dots, M$, the same system shall be valid on \mathcal{U}_m , since any cell of \mathcal{U}_m has the same values of lane-densities $\lambda_i, i = 1, 2, 3, 4$ (which may differ from one zone to another). Below, (10) is the multidimensional hyperbolic conservations laws that describes the variation of car-density vector ρ in all directions. This holds in the local basis (u, v) related to each elementary cell, and then for the zone containing the cell. We have:

$$\partial_t \rho + \partial_x Q^x(\rho) + \partial_y Q^y(\rho) = 0 \quad t \in \mathbb{R}^+, (x, y) \in U_m \subset \mathcal{U} \subset \mathbb{R}^2, m \in M, \tag{10}$$

with $\rho = (\rho_1 \ \rho_2 \ \rho_3 \ \rho_4)^T$ taking values in \mathbb{R}^4 , where ρ_i is the car-density in the preferred direction of propagation $i, i \in \{1, 2, 3, 4\}$, $Q^x(\rho) = (Q_1^x(\rho) \ Q_2^x(\rho) \ Q_3^x(\rho) \ Q_4^x(\rho))^T$ and $Q^y(\rho) = (Q_1^y(\rho) \ Q_2^y(\rho) \ Q_3^y(\rho) \ Q_4^y(\rho))^T$. The expression of the given vectorial function $Q : \mathbb{R}^4 \mapsto \mathbb{R}^8$ is quite complicated. It takes into account a hydrodynamical law and velocities profile in road traffic theory. The system (10) is hyperbolic: for any $\rho \in \mathbb{R}^4$, the Jacobian matrix $DQ(\rho)$ is diagonalizable in \mathbb{R} .

$$Q^x(\rho) = \begin{pmatrix} q_1 \\ \frac{1}{2} \left(\Gamma_{12} q_1 - \Gamma_{32} q_3 + \frac{\lambda_1}{\lambda_2} (\Gamma_{21} - \Gamma_{23}) q_2 \right) \\ -q_3 \\ \frac{1}{2} \left(\Gamma_{14} q_1 - \Gamma_{34} q_3 + \frac{\lambda_1}{\lambda_2} (\Gamma_{41} - \Gamma_{43}) q_4 \right) \end{pmatrix}, \quad Q^y(\rho) = \begin{pmatrix} \frac{1}{2} \left(\Gamma_{21} q_2 - \Gamma_{41} q_4 + \frac{\lambda_2}{\lambda_1} (\Gamma_{12} - \Gamma_{14}) q_1 \right) \\ q_2 \\ \frac{1}{2} \left(\Gamma_{23} q_2 - \Gamma_{43} q_4 + \frac{\lambda_2}{\lambda_1} (\Gamma_{32} - \Gamma_{34}) q_3 \right) \\ -q_4 \end{pmatrix}. \tag{11}$$

$Q^x(\rho)$ and $Q^y(\rho)$ are the flow vector functions for measuring the change in flow in the x -direction and y -direction. By using an affine transformation, the system can be transformed regarding the global basis (e_1, e_2) (Figure 1(b)).

3.2. Users' transport demand in a cell and network cell supply

Let m fixed. $C_m^{\alpha,\beta}$ will denote an elementary cell of \mathcal{U}_m . Networks supply and users demands in a cell (α, β) to direction $i \in \{1, 2, 3, 4\}$ are defined as follows:

$$\delta_i^{\alpha,\beta} = \lambda_i v_i \Delta_i^{\alpha,\beta} \left(\frac{\rho_i^{\alpha,\beta}}{v_i} \right) \quad \text{and} \quad \sigma_i^{\alpha,\beta} = \lambda_i v_i \Omega_i^{\alpha,\beta} \left(\frac{\rho_i^{\alpha,\beta}}{v_i} \right). \tag{12}$$

where $\Omega_i^{\alpha,\beta}$ and $\Delta_i^{\alpha,\beta}$ are supply and demand at a point (and hence of a point of a cell) illustrated by the Figure 2.

3.3. Inflow and outflow of an elementary cell

Remark 3.1. Within the same zone U_m for m fixed, values of λ_i from one cell to the other are the same. Let us mention that, in the two dimensional approach, there is no buffer between cells. Hence, cells outflow rate do not depend upon cells inflow rates.

The outflow and inflow rates in the four privileged directions with their constraints are expressed as follows (the outflows in the first column and the inflows in the second one of (13)).

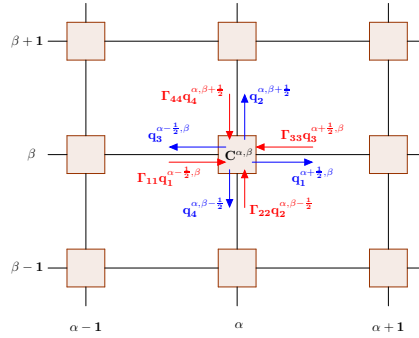


Fig. 3: Inflows and outflows of computing cells

Other turning flows are: $q_1^{\alpha,\beta-\frac{1}{2}}\Gamma_{21}$ and $q_4^{\alpha,\beta+\frac{1}{2}}\Gamma_{41}$ to direction $i = 1$, $q_1^{\alpha-\frac{1}{2},\beta}\Gamma_{14}$ and $q_3^{\alpha+\frac{1}{2},\beta}\Gamma_{34}$ to direction $i = 2$, $q_2^{\alpha,\beta-\frac{1}{2}}\Gamma_{23}$ and $q_4^{\alpha,\beta+\frac{1}{2}}\Gamma_{43}$ to direction $i = 3$, and $q_1^{\alpha-\frac{1}{2},\beta}\Gamma_{14}$ and $q_3^{\alpha+\frac{1}{2},\beta}\Gamma_{34}$ to direction $i = 4$.

$$\begin{aligned}
 & \bullet (\alpha, \beta) \longleftarrow (\alpha + 1, \beta) : q_3^{\alpha+\frac{1}{2},\beta} \leq \sigma_3^{\alpha,\beta} & \bullet (\alpha, \beta) \longrightarrow (\alpha + 1, \beta) : q_1^{\alpha+\frac{1}{2},\beta} \leq \delta_1^{\alpha,\beta} \\
 & \bullet (\alpha, \beta) \longleftarrow (\alpha - 1, \beta) : q_1^{\alpha-\frac{1}{2},\beta} \leq \sigma_1^{\alpha,\beta} & \bullet (\alpha, \beta) \longrightarrow (\alpha - 1, \beta) : q_3^{\alpha-\frac{1}{2},\beta} \leq \delta_3^{\alpha,\beta} \\
 & \bullet (\alpha, \beta) \longleftarrow (\alpha, \beta - 1) : q_4^{\alpha,\beta+\frac{1}{2}} \leq \sigma_4^{\alpha,\beta} & \bullet (\alpha, \beta) \longrightarrow (\alpha, \beta - 1) : q_2^{\alpha,\beta+\frac{1}{2}} \leq \delta_2^{\alpha,\beta} \\
 & \bullet (\alpha, \beta) \longleftarrow (\alpha, \beta + 1) : q_2^{\alpha,\beta-\frac{1}{2}} \leq \sigma_2^{\alpha,\beta} & \bullet (\alpha, \beta) \longrightarrow (\alpha, \beta + 1) : q_4^{\alpha,\beta-\frac{1}{2}} \leq \delta_4^{\alpha,\beta}
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 & \bullet q_1^{\alpha-\frac{1}{2},\beta} = \min(\delta_1^{\alpha-1,\beta}, \sigma_1^{\alpha,\beta}) & \bullet q_3^{\alpha+\frac{1}{2},\beta} = \min(\delta_3^{\alpha+1,\beta}, \sigma_3^{\alpha,\beta}) \\
 & \bullet q_2^{\alpha,\beta-\frac{1}{2}} = \min(\delta_2^{\alpha,\beta-1}, \sigma_2^{\alpha,\beta}) & \bullet q_4^{\alpha,\beta+\frac{1}{2}} = \min(\delta_4^{\alpha,\beta+1}, \sigma_4^{\alpha,\beta})
 \end{aligned} \tag{14}$$

4. Numerical method

We provide in this section a Godunov-type numerical scheme construction for the system (10) - (14) and a particular case study of road network dynamics. Let us mentioned that Riemann problems for this two-dimensional system (10) are not addressed in this paper.

4.1. Discretized model : Case of rectangular mesh of $(\mathcal{U}_m)_{m=1,\dots,M}$

We assume a construction $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_M)$ of rectangular meshes of $(\mathcal{U}_1, \dots, \mathcal{U}_M)$ in the sense of (Eymard et al., 2000, Definition 5.1). The average value of the two-dimensional dynamic traffic state ρ , at time level t over a particular

cell $C^{\alpha,\beta}$ is defined as

$$\rho^{\alpha,\beta}(t) = \frac{1}{|C^{\alpha,\beta}|} \int_{C^{\alpha,\beta}} \rho(x,y,t) dx dy. \tag{15}$$

The solution at each time step t is described as a piecewise constant function:

$$\rho(x,y,t) = \sum_{m \in M} \sum_{\substack{\alpha_m \in N_{\xi_m} \\ \beta_m \in N_{\eta_m}}} \rho^{\alpha_m, \beta_m}(t) \chi_{\alpha_m, \beta_m} \quad \text{with} \quad \chi_{\alpha_m, \beta_m} = \begin{cases} 1 & \text{if } (x,y) \in C^{\alpha_m, \beta_m}, \\ 0 & \text{if } (x,y) \notin C^{\alpha_m, \beta_m}. \end{cases} \tag{16}$$

For a particular cell $C^{\alpha,\beta}$ the density vector at the next time level $(t + 1)$, is computed as follows:

$$\left\{ \begin{aligned} \rho_1^{\alpha,\beta}(t+1) &= \rho_1^{\alpha,\beta}(t) + \frac{\Delta t}{\xi_1 \eta_1} \left(\eta_1 \Gamma_{11} q_1^{\alpha-\frac{1}{2}, \beta} + \eta_2 \Gamma_{21} \frac{\lambda_1 v_1}{\lambda_2 v_2} q_2^{\alpha, \beta-\frac{1}{2}} + \eta_4 \Gamma_{41} \frac{\lambda_1 v_1}{\lambda_4 v_4} q_4^{\alpha, \beta+\frac{1}{2}} + \eta_1 q_1^{\alpha+\frac{1}{2}, \beta} \right) \\ \rho_2^{\alpha,\beta}(t+1) &= \rho_2^{\alpha,\beta}(t) + \frac{\Delta t}{\xi_2 \eta_2} \left(\eta_2 \Gamma_{22} q_2^{\alpha, \beta-\frac{1}{2}} + \eta_3 \Gamma_{32} \frac{\lambda_2 v_2}{\lambda_3 v_3} q_3^{\alpha-\frac{1}{2}, \beta} + \eta_1 \Gamma_{12} \frac{\lambda_2 v_2}{\lambda_1 v_1} q_1^{\alpha+\frac{1}{2}, \beta} - \eta_2 q_2^{\alpha, \beta+\frac{1}{2}} \right) \\ \rho_3^{\alpha,\beta}(t+1) &= \rho_3^{\alpha,\beta}(t) + \frac{\Delta t}{\xi_3 \eta_3} \left(\eta_3 \Gamma_{33} q_3^{\alpha+\frac{1}{2}, \beta} - \eta_4 \Gamma_{43} \frac{\lambda_3 v_3}{\lambda_4 v_4} q_4^{\alpha, \beta+\frac{1}{2}} + \eta_2 \Gamma_{23} \frac{\lambda_2 v_2}{\lambda_2 v_2} q_2^{\alpha, \beta-\frac{1}{2}} + \eta_3 q_3^{\alpha-\frac{1}{2}, \beta} \right) \\ \rho_4^{\alpha,\beta}(t+1) &= \rho_4^{\alpha,\beta}(t) + \frac{\Delta t}{\xi_4 \eta_4} \left(\eta_4 \Gamma_{44} q_4^{\alpha, \beta+\frac{1}{2}} + \eta_3 \Gamma_{34} \frac{\lambda_4 v_4}{\lambda_3 v_3} q_3^{\alpha-\frac{1}{2}, \beta} + \eta_1 \Gamma_{14} \frac{\lambda_4 v_4}{\lambda_1 v_1} q_1^{\alpha-\frac{1}{2}, \beta} - \eta_4 q_4^{\alpha, \beta-\frac{1}{2}} \right) \end{aligned} \right. \tag{17}$$

where the partial flows, especially the flow flux across cell, appearing in the above formulation (17) are determined by the traffic equilibrium principle. That is the traffic equilibrium between users’ transport demand in a cell and network supply of these neighbor cells, depicted by Figure 2 and expressed by (14).

Remark 4.1. For any admissible mesh of quadrangular cells, one shall apply the method presented in (Vides et al., 2014) where a simple two-dimensional HLL Riemann Solver has been proposed to find approximate solutions to the above system of nonlinear hyperbolic conservations laws. The resulting numerical scheme takes into account all conflicts across the cell interfaces by the resolution of the associated two-dimensional HLL Riemann problems, both on rectangular meshes and non-rectangular meshes. In the case to apply a high resolution method to the system (10), we indicate the works of (Leveque and Shyue, 1996; Leveque, 1996).

4.2. Algorithms

With regard to the two-dimensional system equations we present a snippet of pseudo-code to compute the values of cell demands and cell supplies, followed by the flux across cell interfaces, and then the density per cell; and all to the directions $i = 1, 2, 3, 4$. The obtained algorithm is an extension of (Saumtally et al., 2013, Algorithm 2).

Large anisotropic/orthotropic continuous networks algorithm

Input: * the mesh of the considered domain (in this paper we use .msh file generated by *GMSH* software).
* lane density λ_i
* average number of lanes v_i per link in the direction i
* turning movement coefficients Γ_{ij}
* boundary conditions: cell supplies $\sigma_i^{t=0}$ and users' demands $\delta_i^{t=0}$ at the network entries and exits
* ending time T

* empty network initial conditions (at time $t = 0$): $\rho_i^{\alpha,\beta,0} = 0, i = 1, 2, 3, 4$

for $t = 0$ to T **do**

 for $\alpha = 1$ to N do (N being the total number of the cells)

 for $\beta = 1$ to N do

 * calculate users' demand $\delta_i^{\alpha,\beta,t}$ and cell supply $\sigma_i^{\alpha,\beta,t}$ with (12)

 end for

 end for

 for $\alpha = 1$ to N do

 for $\beta = 1$ to N do

 * calculate the flows $q_i^{\alpha,\beta,t}$ across cell interfaces with the min-formula (14)

 * calculate densities $\rho_i^{\alpha,\beta,t}$ within each cell with the expressions of (17)

 end for

 end for

end for

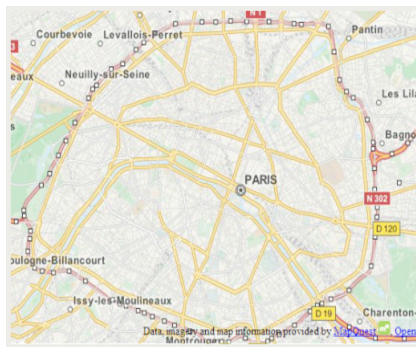
Output: * the flows $q_i^{\alpha,\beta,t}, t = 0 \dots T$ and the densities $\rho_i^{\alpha,\beta,t}, t = 0 \dots T$

4.3. Numerical aspect

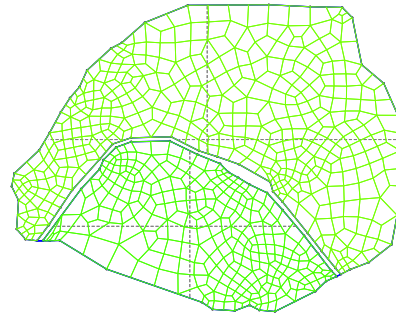
Let us consider a large urban transportation network like road networks of the city of Paris as shown with the map in the Figure 4(a). And let us extract specific geodesic coordinates of the boundary of the city of Paris and of the Seine. These coordinates have been converted in Euclidean coordinates which allowing get geometric subspace of the Euclidean space \mathbb{R}^2 for the real road network map. The geometric domain is meshed with *GMSH* software for the Godunov numerical scheme. The result is depicted by the Figure 4(b). We get information about the input topological parameters of the model presented in this paper. These are the vectors lane density $\lambda = (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^T$, the average number of lanes per link in the directions $v = (v_1 v_2 v_3 v_4)^T$, the cell capacity $\kappa = (\kappa_1 \kappa_2 \kappa_3 \kappa_4)^T$ (its maximal capacity to absorb transport demand in a cell, with respect to the directions of the flow). And then apply the algorithm 4.2 with artificial data for the input of the model.

5. Conclusion and perspectives

We presented a dynamic traffic model of large urban transportation networks (in the area of network flow modelling) that takes into account the multi-commodity flows and the anisotropy of the networks. The model reduces well the significant computational efforts and the large number of parameters and variables involved, in the case of the description of network traffic states by macroscopic models. Since there may exist main arteries road on urban network, the model shall be connected with macroscopic traffic models, mainly with the generic second order family models, namely GSOM, on networks. In that case of network, the dense area shall be modelled by a two-dimensional medium as we do in this paper, and the main arteries shall be modelled by the family GSOM models. This can be done naturally using the network supply and the users' demand framework. Further issues are the assimilation of Lagrangian data (the floating cars etc) and the modelling of dynamic assignment. An example of dynamic assignment is given in (Jiang et al., 2011; Yang et al., 1994). The problem is to calculate the matrix of turning-movements Γ



(a) Maps of Paris road network



(b) Mesh of Paris road network

Fig. 4: From geographic map to a mesh for finite volume method

which is given in this paper. The matrix Γ may significantly and locally vary in some sub-areas or cells depending on the users' paths choice which depend on the state of the system.

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References

- Costeseque, G., Lebacque, J., 2012. Intersection modeling using a convergent scheme based on hamilton-jacobi equation. *Procedia - Social and Behavioral Sciences* 2012 , 343–363.
- Eymard, R., Gallouet, T., Herbin, R., 2000. Finite Volume Methods. North-Holland, Amsterdam. volume 7 of *Handbook of Numerical Analysis*. chapter 5.1. pp. 713–1020.
- Jiang, Y., Wong, S., Ho, H., Zhang, P., Liu, R., Sumalee, A., 2011. A dynamic traffic assignment model for a continuum transportation system. *Transportation Research Part B* 45, 343–363.
- Keyvan-Ekbatani, M., Kouvelas, A., Papamichail, I., Papageorgiou, M., 2012. Exploiting the fundamental diagram of urban networks for feedback-based gating. *Transp. Res. Part B: Methodol.* 46, 1393–1403.
- Keyvan-Ekbatani, M., Papageorgiou, M., Knoop, V.L., 2015a. Controller design for gating traffic control in presence of time-delay in urban road networks. *Transport. Res. Part C* .
- Keyvan-Ekbatani, M., Yildirimoglu, M., Geroliminis, N., Papageorgiou, M., 2015b. Multiple concentric gating traffic control in large-scale urban networks. *IEEE Trans. Intell. Transp. Syst.* 99, 1–14.
- Lebacque, J.P., Khoshyaran, M., 2013. A variational formulation for higher order macroscopic traffic flow models of the gsom family. *Procedia-Social and Behavioral Sciences* 83, 370–394.
- Leveque, R.J., 1996. Wave propagation algorithms for multidimensional hyperbolic systems. *Journal of Computational Physics* Volume 131, Issue 2, 327–353.
- Leveque, R.J., Shyue, K.M., 1996. Two-dimensional front tracking based on high resolution wave propagation methods. *Journal of Computational Physics* Volume 123, Issue 2, 354–368.
- Prez, L.M.R., Benitez, F.G., 2010. Traffic flow continuum modeling by hypersingular boundary integral equations. *International journal for numerical methods in engineering* 82, 47–63.
- Saumtally, T., 2013. Modèles bidimensionnels de trafic. PhD dissertation. Université Paris-Est.
- Saumtally, T., Lebacque, J.P., Haj-Salem, H., 2013. A dynamical two-dimensional traffic model in an anisotropic network. *Networks and Heterogeneous Media* , 663–684.
- Vides, J., Nkonga, B., Audit, E., 2014. A Simple Two-Dimensional Extension of the HLL Riemann Solver for Gas Dynamics. [Research Report] RR-8540, hal-00998235v2. Project-Team Castor.
- Wong, S., 1998. Multi-commodity traffic assignment by continuum approximation of network flow with variable demand. *Transportation Research B* 32, 567–581.
- Yang, H., Yagar, S., Iida, Y., 1994. Traffic assignment in congested discrete/ continuous transportation system. *Transportation Research B* 28B, 161–174.