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# THE THREE-DIMENSIONAL CAHN-HILLIARD-BRINKMAN SYSTEM WITH UNMATCHED VISCOSITIES

MONICA CONTI<sup>1</sup> & ANDREA GIORGINI<sup>2</sup>

**ABSTRACT.** This paper is focused on a diffuse interface model for the motion of binary fluids with different viscosities. The system consists of the Brinkman–Darcy equations governing the fluid velocity, nonlinearly coupled with a convective Cahn–Hilliard equation for the difference of the fluid concentrations. For the three-dimensional Cahn–Hilliard–Brinkman system with free energy density of logarithmic type we prove the well-posedness of weak solutions and we establish the global-in-time existence of strong solutions. Furthermore, we discuss the validity of the separation property from the pure states, which occurs instantaneously in dimension two and asymptotically in dimension three.

**Keywords:** diffuse interface models, Darcy’s law, logarithmic potential, uniqueness, strong solutions, separation property.

## 1. INTRODUCTION

The Cahn–Hilliard–Brinkman (CHB) system is a diffuse interface model describing the motion and interaction of two incompressible and viscous fluids (see, e.g., [14, 33, 35]). The CHB model couples a modified Darcy’s law introduced by Brinkman in [10], which governs the volume-averaged fluid velocity  $\mathbf{u}$ , with a convective Cahn–Hilliard equation for the difference of the fluid concentrations  $\varphi$  (order parameter). By definition, the latter does take value between  $-1$  and  $1$ , the extremals  $\varphi = \pm 1$  representing the pure phases. Assuming that the binary mixture has uniform density (Boussinesq approximation) in a bounded domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , the system reads as follows

$$\begin{cases} -\operatorname{div}(\nu(\varphi)D\mathbf{u}) + \eta(\varphi)\mathbf{u} + \nabla\pi = \mu\nabla\varphi, \\ \operatorname{div}\mathbf{u} = 0, \\ \partial_t\varphi + \operatorname{div}(\varphi\mathbf{u}) = \Delta\mu, \\ \mu = -\Delta\varphi + \Psi'(\varphi), \end{cases} \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

completed with the boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (1.2)$$

being  $\mathbf{n}$  the exterior normal on  $\partial\Omega$ , and the initial condition

$$\varphi(0) = \varphi_0, \quad \text{in } \Omega. \quad (1.3)$$

Here,  $D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\operatorname{tr}})$  is the symmetric gradient,  $\pi$  denotes the fluid pressure,  $\Psi'$  is the first derivative of a double well potential  $\Psi$ , and  $\mu$  denotes the chemical potential, which is the variational derivative of the Ginzburg–Landau free energy

$$\mathcal{E}(\varphi) = \int_{\Omega} \frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi) \, dx.$$

The thermodynamically relevant energy density  $\Psi$  introduced in the Cahn-Hilliard theory (see [11]) is the logarithmic function

$$\Psi(s) = \frac{\theta}{2} \left( (1+s) \ln(1+s) + (1-s) \ln(1-s) \right) - \frac{\theta_0}{2} s^2, \quad \forall s \in (-1, 1). \quad (1.4)$$

In this context, the parameters  $\theta$  and  $\theta_0$  are related to the absolute temperature of the mixture and the critical temperature, respectively, satisfying the physical relation  $0 < \theta < \theta_0$ . The concentration dependent term  $\nu$  describes the viscosity of the mixture. When the two fluids have the same viscosity,  $\nu$  is a positive constant (the so-called matched viscosity case). Instead, a significant form for  $\nu$  in the unmatched viscosity case is the linear combinations of the components (see, e.g., [18, 24])

$$\nu(s) = \nu_1 \frac{1+s}{2} + \nu_2 \frac{1-s}{2}, \quad \forall s \in (-1, 1), \quad (1.5)$$

where  $\nu_1, \nu_2 > 0$  are the (different) viscosities of the two fluids. The term  $\eta$  is related to the permeability of the system, which can be modelled analogously by either a constant or a linear combination of the fluid components.

Diffuse interface models play nowadays an important role in Fluid dynamics as analytical and numerical methods to describe the behavior of multi-component (or multi-phase) fluids flows, see [3, 24] for a general overview. In this framework, the CHB system has been proposed in [14, 33, 35] as an efficient model for phase separation phenomena in porous media. A further interest in studying the CHB system comes from its deep connections with other well-known diffuse interface models:

- (i) **Model H for low Reynolds number flows.** The classical diffuse interface system for the dynamic of two incompressible Newtonian fluids is the well-known model H (see, e.g., [8, 22, 27]). This is a Cahn–Hilliard–Navier–Stokes (CHNS) system which couples (1.1)<sub>3,4</sub> with the (adimensionalized) Navier–Stokes equations

$$\text{Re} \left( \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = \text{div} (\nu(\varphi) D\mathbf{u}) - \nabla \pi + \mu \nabla \varphi, \quad (1.6)$$

being  $\text{Re}$  the Reynolds number. In this context, the Brinkman–Darcy’s law in (1.1)<sub>1</sub> (with  $\eta = 0$ ) corresponds to the Stokes inertialess approximation of (1.6) for low Reynolds values. In particular, such an approximation has provided efficient numerical simulations for the mixing of fluids in a driven cavity in [12].

- (ii) **A model for Hele–Shaw flows.** The flow confined in a Hele–Shaw cell is described by a simplification of the model H. This is the Cahn–Hilliard–Hele–Shaw (CHHS) system, where the velocity field satisfies the Darcy’s law

$$\eta(\varphi) \mathbf{u} + \nabla \pi = \mu \nabla \varphi. \quad (1.7)$$

The CHHS system has been employed for the pinchoff and reconnection of interfaces in binary fluids (see [25]) and the Saffman–Taylor instability, also known as viscous fingering (see, e.g., [13]). More recently, it has been used for global impact issues such as tumor growth dynamics (see, e.g., [19, 28]). The Brinkman’s law represents the relaxation of (1.7) via the term  $\text{div} (\nu(\varphi) D\mathbf{u})$  successfully introduced in [10]. In the same direction, it arises from the description of thermocapillary or Marangoni flow [38] accounting for the friction force of the flow due to the plates.

This paper is focused on the CHB system in presence of logarithmic potential and unmatched viscosities and permeabilities. Before presenting our results, we briefly discuss the existing literature relative to the diffuse interface models introduced above. The mathematical analysis of such systems, in particular the uniqueness issue and the existence of global-in-time strong solutions, is quite challenging. This is due to the intrinsic difficulty of handling the equations for the velocity field, especially when  $\nu$  is non-constant, and the singular behavior of  $\Psi'(s)$  and its derivatives as  $s$  approaches  $\pm 1$ . For this reason, to simplify the analysis, most of the papers addressed the case of constant viscosity and regular potentials, namely, the logarithmic potential  $\Psi$  is replaced with polynomial functions. It is worth mentioning that, although the latter approximation is broadly justified in the regime  $\theta$  close to  $\theta_0$ , a polynomial potential cannot ensure the existence of solutions such that  $\varphi$  ranges in the physical interval  $[-1, 1]$ . Under such restrictions, the CHB system has been investigated in [6] (see also [26] and [42]). In the same framework, among a vast literature, we refer the reader to [7, 16, 27, 41] for the CHNS system and to [18, 29, 39, 40] for the CHHS system. On the other hand, there are only few papers concerning the above-mentioned systems with the physically relevant logarithmic potential. The sole contribution on the CHNS model is [1] (see also [17, 30] for related models). Notably, in the case with unmatched viscosities, the well-posedness of strong solutions is established: the order parameter  $\varphi$  is global-in-time in any dimensions, while the velocity field  $\mathbf{u}$  is global-in-time in dimension two and local-in-time in dimension three. However, the uniqueness of weak solutions in dimension two remains a relevant open problem. The CHHS model with logarithmic potential and matched viscosities has been studied in the recent paper [20]. In particular, the uniqueness of weak solutions and their instantaneous regularization in time have been shown in dimension two. Besides, in dimension three the existence of global-in-time strong solution is established provided that the initial state  $\varphi_0$  is sufficiently close to any local minimizer of the Ginzburg–Landau free energy.

The result of our investigation is a comprehensive mathematical theory of global-in-time well-posedness of weak and strong solutions in dimension three. More precisely, our main results for the CHB system in a smooth domain  $\Omega \subset \mathbb{R}^3$  are the following:

- ◇ uniqueness of weak solutions,
- ◇ global well-posedness of strong solutions,
- ◇ further regularity and separation property.

In accordance with the previous discussion, the completeness of our results is a validation of the CHB system as a robust diffuse interface model for the description of three dimensional two-component flows.

Let us conclude the Introduction with a brief explanation of some technical points in our analysis. We start by discussing the existence of physical weak (finite energy) solutions such that

$$\varphi \in L^\infty(\Omega \times (0, T)) \text{ with } |\varphi(x, t)| < 1 \text{ a.e. } (x, t) \in \Omega \times (0, T).$$

The proof of their uniqueness turns out to be delicate due to the non-constant term  $\nu(\varphi)$ . Indeed, the natural control  $\mathbf{u} \in L^2(0, T; \mathbf{V}_\sigma)$  (cf. (2.4)) obtained by the energy method is not enough. Nonetheless, the enhanced regularity  $\varphi \in L^4(0, T; H^2(\Omega))$  (in the class of weak solutions) together with the elliptic structure of the Brinkman/Stokes equations yields the stronger integrability  $\mathbf{u} \in L^4(0, T; \mathbf{V}_\sigma)$ . Then, making use of the inverse of the Stokes operator with

variable coefficients (cf. Appendix B), we find a sharp estimate for the difference of two velocity fields in  $\mathbf{L}^2(\Omega)$ . Combining these ingredients, we manage to control the extra term due to  $\nu$  and we find a continuous dependence estimate, implying the uniqueness of weak solutions (cf. Theorem 4.1).

Next, investigating the existence of global-in-time strong solutions turns out to be more complicated. Aiming to prove higher-order estimates for  $\partial_t \varphi$ , we are able to obtain a suitable differential inequality for  $\Delta_h \varphi(t) = \varphi(t+h) - \varphi(t)$ . However, the low integrability-in-time  $\mathbf{u} \in L^4(0, T; \mathbf{L}^2(\Omega))$  is not sufficient to deduce directly a proper control of  $\Delta_h \varphi(0)$ . The key idea is then showing fractional properties of regularity for  $\varphi$  in Besov spaces. This allows us to improve the summability of the velocity field up to  $\mathbf{u} \in L^\infty(0, T; \mathbf{L}^2(\Omega))$ , implying as a byproduct the desired regularity  $\partial_t \varphi \in L^\infty(0, T; V') \cap L^2(0, T; V)$ . At this point, exploiting the interplay between the elliptic problem for  $\varphi$  (given by the very definition of  $\mu$ , cf. Appendix A) and the Brinkman/Stokes equations (cf. Appendix B), we recover the full regularity of a strong solution (cf. Theorem 5.1).

Further relevant aspects regarding global regularity issues depend on the integrability of  $\Psi''(\varphi)$  in  $L^p(\Omega \times [0, T])$ , for some  $p > 2$ . Being a strong solution such that  $\Psi'(\varphi)$  belongs to  $L^\infty(0, T; L^6(\Omega))$  (cf. Theorem 5.1), the desired control is not straightforward due to the relative growth condition

$$\Psi''(s) \leq e^{C(1+|\Psi'(s)|)}, \quad \forall s \in (-1, 1), \quad (1.8)$$

which prevents the possibility to estimate  $\Psi''(\varphi)$  simply by  $\Psi'(\varphi)$  in  $L^p$ -spaces. Actually, this problem is deeply connected with the so-called *separation property*, namely whether there exist  $\delta > 0$  and an interval  $I \subset [0, \infty)$  such that

$$\sup_{t \in I} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta.$$

Beyond its physical meaning (uniform mixing of fluids), such a property has an important consequence from the mathematical viewpoint. Indeed, on the time interval  $I$ , the singularities of  $\Psi'$  and its derivatives play no longer any role and the original system can be treated as a model with regular potential. On this topic we have a different picture depending on the space dimension. If  $d = 2$ , we are able to handle (1.8) thanks to the Trudinger–Moser inequality (cf. Lemma A.6). This is the basic step to show that the separation property is instantaneous, namely it holds on any time interval  $[\sigma, \infty)$ , for  $\sigma > 0$ . Instead, this is not the case if  $d = 3$ . Nevertheless, with a different argument based on the regularity of stationary states, the separation occurs after an eventually large time which is not explicitly computable. Accordingly, we are able to improve the Sobolev regularity of the solutions on proper time intervals, depending on the space dimension.

**Plan of the paper.** In Section 2 we introduce the notation and the mathematical tools which will be used throughout the analysis. In Section 3 we discuss the existence of weak solutions and their basic properties. Section 4 is devoted to the uniqueness of weak solutions. The existence of global-in-time strong solutions is established in Section 5. In Section 6 we discuss the separation property and the long-time behavior of solutions. In Appendix A we collect several elliptic estimates for a Laplace-Neumann problem with logarithmic nonlinearity. In Appendix B we recall the elliptic regularity theory for a Stokes problem with non-constant coefficients. In Appendix C we report the proof of the existence of a weak solution.

## 2. THE MATHEMATICAL SETTING

Let  $\Omega$  be a connected bounded domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ , with smooth boundary  $\partial\Omega$ . Let  $X$  be a Banach space. We denote by  $\|\cdot\|_X$  its norm, by  $X'$  the dual space and by  $\langle \cdot, \cdot \rangle$  the corresponding duality product. With  $\mathbf{X}$  we indicate the vectorial space  $X^d$  endowed with the product structure. Given  $1 \leq p \leq \infty$  and an interval  $I \subseteq [0, \infty)$ , the set  $L^p(I; X)$  consists of all Bochner measurable  $p$ -integrable functions defined on  $I$  with values in  $X$ . We denote by  $W^{1,p}(0, T; X)$  the space of functions  $f \in L^p(0, T; X)$  with the vector-valued distributional derivative  $\partial_t f$  in  $L^p(0, T; X)$ . In particular, we set  $H^1(0, T; X) = W^{1,2}(0, T; X)$ . The family of continuous functions  $f : I \rightarrow X$  is denoted by  $\mathcal{C}(I, X)$ . The set of Hölder continuous functions of exponent  $s \in (0, 1)$  is denoted by  $\mathcal{C}^s(I, X)$ , with norm

$$\|f\|_{\mathcal{C}^s(I, X)} = \sup_{t \in I} \|f(t)\|_X + \sup_{t, \tau \in I} \frac{\|f(t) - f(\tau)\|_X}{|t - \tau|^s}.$$

We will use the Besov spaces  $B_{p, \infty}^s(I, X)$ , with  $s \in (0, 1)$ . They consist of the sets of functions  $f \in L^p(I; X)$  with finite norm

$$\|f\|_{B_{p, \infty}^s(I, X)} = \|f\|_{L^p(I, X)} + \sup_{0 < h \leq 1} h^{-s} \|\Delta_h f(t)\|_{L^p(I_h; X)},$$

where  $\Delta_h f(t) = f(t+h) - f(t)$  and  $I_h = \{t \in I : t+h \in I\}$ . We recall that  $B_{\infty, \infty}^s(I, X) = \mathcal{C}^s(I, X)$ . We report the following embedding result (cf. [36, Corollary 26,28])

$$B_{p, \infty}^s(I, X) \hookrightarrow \mathcal{C}^{s-\frac{1}{p}}(I, X), \quad \forall s > \frac{1}{p}. \quad (2.1)$$

Precisely, there exists  $C = C(s, p)$ , independent of  $I$ , such that

$$\|f\|_{\mathcal{C}^{s-\frac{1}{p}}(I, X)} \leq C \|f\|_{B_{p, \infty}^s(I, X)}, \quad \forall f \in B_{p, \infty}^s(I, X).$$

For any positive integer  $k$ , let  $W^{k,p}(\Omega)$  be the Sobolev space of functions in  $L^p(\Omega)$  with distributional derivative of order less or equal to  $k$  in  $L^p(\Omega)$ . In particular, the Hilbert space  $W^{k,2}(\Omega)$  is denoted by  $H^k(\Omega)$  with norm  $\|\cdot\|_{H^k(\Omega)}$ . We denote by  $\mathcal{C}^s(\bar{\Omega})$  the space of Hölder continuous functions with norm  $\|f\|_{\mathcal{C}^s(\bar{\Omega})} = \sup_{x \in \bar{\Omega}} |f(x)| + \sup_{x, y \in \bar{\Omega}} \frac{|f(x) - f(y)|}{|x - y|^s}$ . As customary, we set  $H = L^2(\Omega)$  with inner product denoted by  $(\cdot, \cdot)$  and corresponding norm  $\|\cdot\|$ . We also set  $V = H^1(\Omega)$ . We denote by  $\bar{f}$  the average of  $f$  over  $\Omega$ , that is  $\bar{f} = |\Omega|^{-1} \langle f, 1 \rangle$ , for all  $f \in V'$ . We recall the Hilbert interpolation inequality

$$\|f\| \leq C \|f\|_{V'}^{\frac{1}{2}} \|f\|_V^{\frac{1}{2}}, \quad \forall f \in V, \quad (2.2)$$

and the Poincaré–Wirtinger inequality

$$\|f - \bar{f}\|_V \leq C \|\nabla f\|, \quad \forall f \in V. \quad (2.3)$$

As a byproduct, we have that  $f \rightarrow (\|\nabla f\|^2 + |\bar{f}|^2)^{\frac{1}{2}}$  is an equivalent norm on  $V$ . We introduce the space of zero-mean functions  $V_0 = \{f \in V : \bar{f} = 0\}$  and its dual space  $V'_0 = \{g \in V' : \bar{g} = 0\}$ . We then consider the operator  $A \in \mathcal{L}(V, V')$  defined by

$$\langle Af, v \rangle = \int_{\Omega} \nabla f \cdot \nabla v \, dx, \quad \forall f, v \in V.$$

Since the restriction of  $A$  in  $V_0$  is an isomorphism from  $V_0$  onto  $V'_0$ , we define the inverse map  $\mathcal{N} : V'_0 \rightarrow V_0$ . It is well-known that for all  $g \in V'_0$ ,  $\mathcal{N}g$  is the unique  $f \in V_0$  such that  $\langle Af, v \rangle = \langle g, v \rangle$ , for all  $v \in V$ . On account of the above definitions, the following properties hold

$$\begin{aligned} \langle Af, \mathcal{N}g \rangle &= \langle f, g \rangle, \quad \forall f \in V, \forall g \in V'_0, \\ \langle f, \mathcal{N}g \rangle &= \langle g, \mathcal{N}f \rangle = \int_{\Omega} \nabla(\mathcal{N}f) \cdot \nabla(\mathcal{N}g) dx, \quad \forall f, g \in V'_0. \end{aligned}$$

It is straightforward to prove that  $f \rightarrow \|f\|_* = \|\nabla \mathcal{N}f\|$  and  $f \rightarrow \|f\|_{-1}^2 = \|f - \bar{f}\|_*^2 + |\bar{f}|^2$  are equivalent norms in  $V'_0$  and  $V'$ , respectively. In addition, it follows the chain rule

$$\frac{1}{2} \frac{d}{dt} \|f(t)\|_*^2 = \langle \partial_t f(t), \mathcal{N}f(t) \rangle, \quad \text{for a.e. } t \in (0, T), \forall f \in H^1(0, T; V'_0).$$

Next, to handle the velocity field  $\mathbf{u}$ , we introduce the Hilbert space of solenoidal vector fields

$$\mathbf{H}_\sigma = \{ \mathbf{u} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \}.$$

In the sequel, we denote by  $(\cdot, \cdot)$  and  $\|\cdot\|$  also the norm and the inner product, respectively, in  $\mathbf{H}_\sigma$ . Then, we define the Hilbert space

$$\mathbf{V}_\sigma = \{ \mathbf{u} \in \mathbf{H}^1(\Omega) : \operatorname{div} \mathbf{u} = 0, \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega \} \quad (2.4)$$

with inner product  $(\mathbf{u}, \mathbf{v})_{\mathbf{V}_\sigma} = (\nabla \mathbf{u}, \nabla \mathbf{v})$  and norm  $\|\mathbf{u}\|_{\mathbf{V}_\sigma} = \|\nabla \mathbf{u}\|$ . We recall that the Korn inequality entails

$$\|\nabla \mathbf{u}\|^2 \leq 2 \|D\mathbf{u}\|^2 \leq 2 \|\nabla \mathbf{u}\|^2, \quad \forall \mathbf{u} \in \mathbf{V}_\sigma,$$

which, in turn, gives that  $\mathbf{u} \rightarrow \|D\mathbf{u}\|$  is an equivalent norm in  $\mathbf{V}_\sigma$ .

We report some embeddings and interpolation inequalities within the theory of Sobolev spaces. We refer the reader to classical references, see, e.g., [5, 32].

◇ Sobolev embeddings:

$$W^{1,p}(\Omega) \hookrightarrow C^\alpha(\bar{\Omega}), \quad \alpha = 1 - \frac{d}{p}, \quad \forall p > d. \quad (2.5)$$

◇ Gagliardo–Nirenberg inequalities ( $d = 3$ ):

$$\|f\|_{L^3(\Omega)} \leq C \|f\|^{1/2} \|f\|_V^{1/2}, \quad \forall f \in V, \quad (2.6)$$

$$\|f\|_{L^\infty(\Omega)} \leq C \|f\|^{1/4} \|f\|_{W^{1,6}(\Omega)}^{3/4}, \quad \forall f \in W^{1,6}(\Omega), \quad (2.7)$$

$$\|f\|_{W^{1,4}(\Omega)} \leq C \|f\|_{L^\infty(\Omega)}^{1/2} \|f\|_{H^2(\Omega)}^{1/2}, \quad \forall f \in H^2(\Omega), \quad (2.8)$$

$$\|f\|_{W^{1,47/5}(\Omega)} \leq C \|f\|_V^{47/68} \|f\|_{W^{2,6}(\Omega)}^{21/68}, \quad \forall f \in W^{2,6}(\Omega). \quad (2.9)$$

◇ Trudinger–Moser inequality in  $d = 2$ :

$$\int_{\Omega} e^{|f|} dx \leq C e^{C \|f\|_V^2}, \quad \forall f \in V. \quad (2.10)$$

Throughout the paper,  $C > 0$  will stand for a generic constant which may be estimated by the parameters of the system and whose value may change even within the same line of a given equation. Further dependencies will be specified at occurrence.

### 3. EXISTENCE OF WEAK SOLUTIONS

Let us state the main assumptions of this work, that we will always assume to be in place. We require that the viscosity  $\nu = \nu(s)$  and the permeability  $\eta = \eta(s)$  belong to  $\mathcal{C}^2(\mathbb{R})$  and satisfy

$$0 < 2\nu_* \leq \nu(s) \leq C, \quad 0 \leq \eta(s) \leq C, \quad \forall s \in \mathbb{R}. \quad (3.1)$$

Next, we assume that  $\Psi$  is a quadratic perturbation of a singular (strictly) convex function in  $[-1, 1]$ . This is

$$\Psi(s) = F(s) - \frac{\theta_0}{2}s^2$$

where the convex part  $F$  belongs to  $\mathcal{C}([-1, 1]) \cap \mathcal{C}^2(-1, 1)$  and fulfils

$$\lim_{s \rightarrow -1} F'(s) = -\infty, \quad \lim_{s \rightarrow 1} F'(s) = +\infty, \quad F''(s) \geq \theta, \quad \forall s \in (-1, 1), \quad (3.2)$$

for some  $\theta > 0$ . Here, we study the physical case of double well (singular) potentials, namely we assume

$$\alpha = \theta_0 - \theta > 0.$$

We also extend  $F(s) = +\infty$  for any  $s \notin [-1, 1]$ . Note that the above assumptions imply that there exists  $s_0 \in (-1, 1)$  such that  $F'(s_0) = 0$ . Without loss of generality, we assume that  $s_0 = 0$  and that  $F(s_0) = 0$  as well. In particular, this entails that  $F(s) \geq 0$  for all  $s \in [-1, 1]$ .

**Remark 3.1.** The assumptions are satisfied and motivated by the logarithmic potential (1.4)  $F(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s))$ . Besides, the viscosity (1.5) complies with (3.1) on the interval  $[-1, 1]$ , being  $\nu_* = \frac{1}{2}\min\{\nu_1, \nu_2\}$ . Note that our assumptions on the permeability term allow the case  $\eta \equiv 0$ , which is important in connection with the CHNS system mentioned in the Introduction.

**Definition 3.2.** Let  $\varphi_0 \in V \cap L^\infty(\Omega)$  with  $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$  and  $|\bar{\varphi}_0| < 1$ . Given  $T > 0$ , a pair  $(\varphi, \mathbf{u})$  is a weak solution to the CHB system (1.1)-(1.3) on  $[0, T]$  if

$$\begin{aligned} \mathbf{u} &\in L^2(0, T; \mathbf{V}_\sigma), \\ \varphi &\in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \cap H^1(0, T; V'), \\ \varphi &\in L^\infty(\Omega \times (0, T)) \text{ with } |\varphi(x, t)| < 1 \text{ a.e. } (x, t) \in \Omega \times (0, T), \end{aligned}$$

and

$$(\nu(\varphi)D\mathbf{u}, D\mathbf{v}) + (\eta(\varphi)\mathbf{u}, \mathbf{v}) = (\mu\nabla\varphi, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_\sigma, \quad (3.3)$$

$$\langle \partial_t \varphi, v \rangle + (\mathbf{u} \cdot \nabla \varphi, v) + (\nabla \mu, \nabla v) = 0, \quad \forall v \in V, \quad (3.4)$$

for almost every  $t \in (0, T)$ , where  $\mu \in L^2(0, T; V)$  is given by

$$\mu = -\Delta\varphi + \Psi'(\varphi). \quad (3.5)$$

Moreover,  $\partial_n \varphi = 0$  almost everywhere on  $\partial\Omega \times (0, T)$  and  $\varphi(0) = \varphi_0$  almost everywhere on  $\Omega$ .

**Remark 3.3.** Notice that any  $\varphi_0$  in the class of admissible initial conditions has finite energy  $\mathcal{E}(\varphi_0) < \infty$ . Indeed, by  $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$  we easily infer  $\Psi(\varphi_0) \in L^1(\Omega)$ . The assumption on the total mass however prevents the admissibility of the pure phases (i.e.  $\varphi \equiv 1$  or  $\varphi \equiv -1$ ) as initial conditions. Besides, it is straightforward to observe that any solution satisfies the mass conservation property, namely

$$\bar{\varphi}(t) = \bar{\varphi}_0, \quad \forall t \geq 0. \quad (3.6)$$

**Remark 3.4.** The equation (3.3) is equivalent to

$$(\nu(\varphi)D\mathbf{u}, D\mathbf{v}) + (\eta(\varphi)\mathbf{u}, \mathbf{v}) = -(\nabla\varphi \otimes \nabla\varphi, \nabla\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_\sigma,$$

in light of the equality

$$\mu\nabla\varphi = \nabla\left(\frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi)\right) - \operatorname{div}(\nabla\varphi \otimes \nabla\varphi).$$

**Remark 3.5.** As customary, the pressure term is dropped in the weak formulation of the Brinkman's law. The pressure  $\pi$  can be recovered (up to a constant) thanks to the classical de Rham's theorem (see, for instance, [9]).

We state and prove our result on the existence of weak solutions.

**Theorem 3.6.** *Let  $\varphi_0 \in V \cap L^\infty(\Omega)$  with  $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$  and  $|\bar{\varphi}_0| < 1$ . Then, there exists a global in time weak solution  $(\varphi, \mathbf{u})$  to the CHB system such that*

$$\mathbf{u} \in L^4(0, T; \mathbf{V}_\sigma), \quad \varphi \in L^4(0, T; H^2(\Omega)) \cap \mathcal{C}([0, T], V), \quad \forall T > 0. \quad (3.7)$$

Moreover, any weak solution satisfies the energy identity

$$\mathcal{E}(\varphi(t)) + \int_s^t \|\nabla\mu(\tau)\|^2 + \|\sqrt{\nu(\varphi(\tau))}D\mathbf{u}(\tau)\|^2 + \|\sqrt{\eta(\varphi(\tau))}\mathbf{u}(\tau)\|^2 d\tau = \mathcal{E}(\varphi(s))$$

for all  $0 \leq s < t < \infty$ . Assuming that  $|\bar{\varphi}_0| \leq m \in (0, 1)$ , we have the dissipative estimates

$$\mathcal{E}(\varphi(t)) + \int_t^{t+1} \|\mu(\tau)\|_{\mathbf{V}}^2 d\tau \leq C\mathcal{E}(\varphi_0)e^{-t} + C, \quad (3.8)$$

and

$$\int_t^{t+1} \|\varphi(\tau)\|_{H^2(\Omega)}^4 + \|\mathbf{u}(\tau)\|_{\mathbf{V}_\sigma}^4 d\tau \leq C\mathcal{E}(\varphi_0)^2e^{-t} + C, \quad (3.9)$$

for every  $t \geq 0$ , where  $C$  is a positive constant depending on  $m$  but independent of the specific initial datum. In addition, for any  $p \geq 2$ , there exists  $C = C(m, p)$  such that,

$$\int_t^{t+1} \|\varphi(\tau)\|_{W^{2,p}(\Omega)}^2 + \|F'(\varphi(\tau))\|_{L^p(\Omega)}^2 d\tau \leq C\mathcal{E}(\varphi_0)e^{-t} + C, \quad (3.10)$$

for every  $t \geq 0$ .

*Proof.* The existence of a global weak solution is obtained via a standard technique which is based on an approximation procedure and energy estimates. For the readers' convenience the proof is contained in Appendix C. We now proceed by proving the regularity properties contained in (3.7), the energy equality and the dissipative estimates (3.8)-(3.10).

**Energy equality and dissipativity.** Given a weak solution  $(\varphi, \mathbf{u})$  to the CHB system, let us define the functional  $\mathcal{J} : H \rightarrow H$  given by

$$\mathcal{J}(\varphi) = \frac{1}{2} \|\nabla \varphi\|^2 + \int_{\Omega} F(\varphi) \, dx.$$

It is clear that  $\mathcal{J}$  is proper, convex and lower-semicontinuous. Hence, appealing to [34, Lemma 4.1], we infer that  $t \mapsto \mathcal{J}(\varphi)$  is absolutely continuous on  $[0, T]$  and

$$\frac{d}{dt} \mathcal{J}(\varphi) = \langle \partial_t \varphi, \mu - \theta_0 \varphi \rangle, \quad \text{a.e. } t \in [0, T].$$

In particular, as a byproduct of the boundedness of  $F$  and  $\varphi \in \mathcal{C}([0, T], H)$ , it follows from the Lebesgue theorem that  $\int_{\Omega} F(\varphi(\cdot)) \, dx \in \mathcal{C}([0, T])$ , which in turn gives  $\varphi \in \mathcal{C}([0, T], V)$ . Now, taking  $v = \mu$  in (3.4) and exploiting the standard chain rule, we obtain

$$\frac{d}{dt} \mathcal{E}(\varphi) + \|\nabla \mu\|^2 + (\mathbf{u} \cdot \nabla \varphi, \mu) = 0, \quad \text{a.e. } t \in [0, T].$$

At this point, taking  $\mathbf{v} = \mathbf{u}$  in (3.3) and summing up to the last equality, we find

$$\frac{d}{dt} \mathcal{E}(\varphi) + \|\nabla \mu\|^2 + (\nu(\varphi) D\mathbf{u}, D\mathbf{u}) + (\eta(\varphi) \mathbf{u}, \mathbf{u}) = 0, \quad \text{a.e. } t \in [0, T], \quad (3.11)$$

which implies the energy equality. Let us now show the dissipative estimate (3.8). In the sequel the generic constant  $C$  depends on  $m$ , but is independent of the initial condition and  $T$ . We multiply  $\mu$  by  $\varphi - \bar{\varphi}$ , getting

$$\|\nabla \varphi\|^2 + (F'(\varphi), \varphi - \bar{\varphi}) = \theta_0(\varphi, \varphi - \bar{\varphi}) + (\mu - \bar{\mu}, \varphi - \bar{\varphi}). \quad (3.12)$$

Since  $F$  is convex, we know that

$$\int_{\Omega} F(\varphi) \, dx \leq \int_{\Omega} F'(\varphi)(\varphi - \bar{\varphi}) \, dx + \int_{\Omega} F(\bar{\varphi}) \, dx,$$

while, recalling that  $\|\varphi\|_{L^\infty(\Omega)} \leq 1$ ,

$$\theta_0(\varphi, \varphi - \bar{\varphi}) + (\mu - \bar{\mu}, \varphi - \bar{\varphi}) \leq C(1 + \|\nabla \mu\|). \quad (3.13)$$

Then, we arrive at

$$\|\nabla \varphi\|^2 + \int_{\Omega} F(\varphi) \, dx \leq C\|\nabla \mu\| + CF(\bar{\varphi}),$$

and, using again the boundedness of  $\varphi$  and the mass conservation, we find

$$\mathcal{E}(\varphi) \leq \frac{1}{2} \|\nabla \mu\|^2 + C.$$

Summing up with (3.11), in light of (3.1) and the Korn inequality, we obtain

$$\frac{d}{dt} \mathcal{E}(\varphi) + \mathcal{E}(\varphi) + \frac{1}{2} \|\nabla \mu\|^2 + \nu_* \|\nabla \mathbf{u}\|^2 \leq C. \quad (3.14)$$

An application of the Gronwall lemma yields

$$\mathcal{E}(\varphi(t)) \leq \mathcal{E}(\varphi_0) e^{-t} + C, \quad \forall t \geq 0.$$

A subsequent integration of (3.14) on  $[t, t+1]$  gives us

$$\int_t^{t+1} \|\nabla \mu(\tau)\|^2 + \|\nabla \mathbf{u}(\tau)\|^2 \, d\tau \leq C \mathcal{E}(\varphi_0) e^{-t} + C.$$

We are left to control the total mass of  $\mu$ . To this aim, we recall that there exists  $C = C(m) > 0$  such that

$$\int_{\Omega} |F'(\varphi)| \, dx \leq C \left| \int_{\Omega} F'(\varphi)(\varphi - \bar{\varphi}) \, dx \right| + C,$$

where  $C$  diverges to  $+\infty$  as  $|m| \rightarrow 1$  (see [15, 31] for the proof). By (3.12) and (3.13), we observe that

$$(F'(\varphi), \varphi - \bar{\varphi}) \leq C(1 + \|\nabla\mu\|),$$

and

$$|\bar{\mu}| \leq C(1 + \|F'(\varphi)\|_{L^1(\Omega)}).$$

Thus, we arrive at

$$|\bar{\mu}| \leq C(1 + \|\nabla\mu\|). \quad (3.15)$$

Collecting (3.15) with the above controls, the dissipative estimate (3.8) follows.

**Further Regularity of Weak Solutions.** We read the definition of  $\mu$  as the Neumann problem

$$\begin{cases} -\Delta\varphi + F'(\varphi) = \mu^*, & \text{in } \Omega, \\ \partial_{\mathbf{n}}\varphi = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $\mu^* = \mu + \theta_0\varphi$ . Since  $\mu^* \in V$ , an application of Lemma A.3 immediately entails (3.10). Besides, by Lemma A.4 we obtain

$$\|\Delta\varphi\|^4 \leq \|\nabla\varphi\|^2 \|\mu^*\|_V^2.$$

Thus, exploiting (3.8), we have

$$\int_t^{t+1} \|\varphi(\tau)\|_{H^2(\Omega)}^4 \, d\tau \leq C\mathcal{E}(\varphi_0)^2 e^{-t} + C, \quad \forall t \geq 0, \quad (3.16)$$

which proves that  $\varphi \in L^4(0, T; H^2(\Omega))$  and the first part of (3.9). We are left to show that  $\mathbf{u} \in L^4(t, t+1, \mathbf{V}_\sigma)$ , for all  $t \geq 0$ . To this aim, we take  $\mathbf{v} = \mathbf{u}$  in (3.3) (cf. Remark 3.4) yielding

$$(\nu(\varphi)D\mathbf{u}, D\mathbf{u}) + (\eta(\varphi)\mathbf{u}, \mathbf{u}) = (\nabla\varphi \otimes \nabla\varphi, \nabla\mathbf{u}).$$

Hence, exploiting (3.1) and the Korn inequality, we have

$$\nu_* \|\nabla\mathbf{u}\|^2 \leq (\nabla\varphi \otimes \nabla\varphi, \nabla\mathbf{u}).$$

By (2.8), we deduce

$$\begin{aligned} (\nabla\varphi \otimes \nabla\varphi, \nabla\mathbf{u}) &\leq \|\nabla\mathbf{u}\| \|\nabla\varphi\|_{L^4(\Omega)}^2 \\ &\leq C \|\nabla\mathbf{u}\| \|\varphi\|_{L^\infty(\Omega)} \|\varphi\|_{H^2(\Omega)} \\ &\leq \frac{\nu_*}{2} \|\nabla\mathbf{u}\|^2 + C \|\varphi\|_{H^2(\Omega)}^2, \end{aligned}$$

so we end up with the control

$$\|\nabla\mathbf{u}\| \leq C \|\varphi\|_{H^2(\Omega)}.$$

The desired control follows by (3.16).  $\square$

## 4. CONTINUOUS DEPENDENCE AND UNIQUENESS

In this section we prove the uniqueness of the weak solution.

**Theorem 4.1.** *Let  $\varphi_{01}, \varphi_{02}$  be such that  $\varphi_{0i} \in V$ ,  $\|\varphi_{0i}\|_{L^\infty(\Omega)} \leq 1$  and  $|\bar{\varphi}_{0i}| < 1$ ,  $i = 1, 2$ . Then, any pair of weak solutions  $(\varphi_1, \mathbf{u}_1)$  and  $(\varphi_2, \mathbf{u}_2)$  of the CHB system on  $[0, T]$  with initial data  $\varphi_{01}$  and  $\varphi_{02}$ , respectively, satisfies*

$$\|\varphi_1(t) - \varphi_2(t)\|_{V'} \leq C\|\varphi_{01} - \varphi_{02}\|_{V'} + C|\bar{\varphi}_{01} - \bar{\varphi}_{02}|^{\frac{1}{2}},$$

for any  $t \in [0, T]$ , where  $C > 0$  depends on  $\mathcal{E}(\varphi_{0i})$ ,  $\bar{\varphi}_{0i}$ ,  $i = 1, 2$  and  $T$ . In particular, the weak solution to CHB is unique.

*Proof.* Let us consider  $(\varphi_1, \mathbf{u}_1)$  and  $(\varphi_2, \mathbf{u}_2)$  two weak solutions to the CHB system with total mass  $\bar{\varphi}_1(0)$  and  $\bar{\varphi}_2(0)$ . Their differences  $\varphi = \varphi_1 - \varphi_2$ ,  $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$  solve

$$\begin{aligned} (\nu(\varphi_1)D\mathbf{u}, D\mathbf{v}) + (\eta(\varphi_1)\mathbf{u}, \mathbf{v}) + (\nu(\varphi_1) - \nu(\varphi_2)D\mathbf{u}_2, D\mathbf{v}) + (\eta(\varphi_1) - \eta(\varphi_2)\mathbf{u}_2, \mathbf{v}) \quad (4.1) \\ = (\nabla\varphi_1 \otimes \nabla\varphi, \nabla\mathbf{v}) + (\nabla\varphi \otimes \nabla\varphi_2, \nabla\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_\sigma, \end{aligned}$$

and

$$\langle \partial_t \varphi, v \rangle + (\mathbf{u}_1 \cdot \nabla\varphi, v) + (\mathbf{u} \cdot \nabla\varphi_2, v) + (\nabla\mu, \nabla v) = 0, \quad \forall v \in V, \quad (4.2)$$

for almost every  $t \in (0, T)$ , where

$$\mu = -\Delta\varphi + \Psi'(\varphi_1) - \Psi'(\varphi_2).$$

Notice that  $\bar{\varphi}(t) = \bar{\varphi}_1(0) - \bar{\varphi}_2(0)$  for all  $t \geq 0$  (cf. (3.6)) and

$$\|\varphi_i(t)\|_V \leq C, \quad \|\varphi_i(t)\|_{L^\infty(\Omega)} \leq 1, \quad \forall t \geq 0. \quad (4.3)$$

Taking  $v = \mathcal{N}(\varphi - \bar{\varphi})$  in (4.2), we get

$$\frac{1}{2} \frac{d}{dt} \|\varphi - \bar{\varphi}\|_*^2 + (\mu, \varphi - \bar{\varphi}) = I_1 + I_2,$$

having set

$$I_1 = (\varphi\mathbf{u}_1, \nabla\mathcal{N}(\varphi - \bar{\varphi})), \quad I_2 = (\varphi_2\mathbf{u}, \nabla\mathcal{N}(\varphi - \bar{\varphi})).$$

By the assumptions on  $\Psi$ , we deduce that

$$\begin{aligned} (\mu, \varphi - \bar{\varphi}) &= \|\nabla\varphi\|^2 + (\Psi'(\varphi_1) - \Psi'(\varphi_2), \varphi_1 - \varphi_2) + (\Psi'(\varphi_1) - \Psi'(\varphi_2), \bar{\varphi}) \\ &\geq \|\nabla\varphi\|^2 - \alpha\|\varphi\|^2 - |(\Psi'(\varphi_1) - \Psi'(\varphi_2), \bar{\varphi})| \\ &\geq \|\varphi\|_V^2 - (\alpha + 1)\|\varphi\|^2 - (\|\Psi'(\varphi_1)\|_{L^1(\Omega)} + \|\Psi'(\varphi_2)\|_{L^1(\Omega)})|\bar{\varphi}|. \end{aligned}$$

By (2.2) we have

$$\begin{aligned} (\alpha + 1)\|\varphi\|^2 &\leq C\|\varphi\|_V\|\varphi\|_{V'} \\ &\leq \frac{1}{2}\|\varphi\|_V^2 + C\|\varphi\|_{-1}^2. \end{aligned}$$

Setting

$$\Upsilon(t) = C(\|\Psi'(\varphi_1(t))\|_{L^1(\Omega)} + \|\Psi'(\varphi_2(t))\|_{L^1(\Omega)}),$$

and owing to the mass conservation, we thus obtain

$$\frac{1}{2} \frac{d}{dt} \|\varphi\|_{-1}^2 + \frac{1}{2} \|\varphi\|_V^2 \leq C\|\varphi\|_{-1}^2 + \Upsilon|\bar{\varphi}| + I_1 + I_2. \quad (4.4)$$

We proceed by estimating  $I_1$  and  $I_2$ . By (4.3), we get

$$\begin{aligned} I_1 &\leq \|\mathbf{u}_1\|_{\mathbf{L}^3(\Omega)} \|\varphi\|_{L^6(\Omega)} \|\varphi - \bar{\varphi}\|_* \\ &\leq \frac{1}{4} \|\varphi\|_V^2 + C \|\mathbf{u}_1\|_{\mathbf{L}^3(\Omega)}^2 \|\varphi\|_{-1}^2, \end{aligned}$$

and

$$\begin{aligned} I_2 &\leq \|\mathbf{u}\| \|\varphi_2\|_{L^\infty(\Omega)} \|\varphi - \bar{\varphi}\|_* \\ &\leq \|\mathbf{u}\| \|\varphi\|_{-1}. \end{aligned}$$

In order to find a control for  $\|\mathbf{u}\|$ , we take  $\mathbf{v} = \mathbf{N}_{\varphi_1} \mathbf{u}$  in (4.1), with  $\mathbf{N}_{\varphi_1} \mathbf{u}$  as defined in Appendix B. We find

$$(\nu(\varphi_1) D\mathbf{u}, D\mathbf{N}_{\varphi_1} \mathbf{u}) + (\eta(\varphi_1) \mathbf{u}, \mathbf{N}_{\varphi_1} \mathbf{u}) = I_3 + I_4 + I_5, \quad (4.5)$$

where

$$\begin{aligned} I_3 &= -((\nu(\varphi_1) - \nu(\varphi_2)) D\mathbf{u}_2, D\mathbf{N}_{\varphi_1} \mathbf{u}), \\ I_4 &= -((\eta(\varphi_1) - \eta(\varphi_2)) \mathbf{u}_2, \mathbf{N}_{\varphi_1} \mathbf{u}), \\ I_5 &= (\nabla \varphi_1 \otimes \nabla \varphi, \nabla \mathbf{N}_{\varphi_1} \mathbf{u}) + (\nabla \varphi \otimes \nabla \varphi_2, \nabla \mathbf{N}_{\varphi_1} \mathbf{u}). \end{aligned}$$

By (B.3), we have

$$(\nu(\varphi_1) D\mathbf{u}, D\mathbf{N}_{\varphi_1} \mathbf{u}) + (\eta(\varphi_1) \mathbf{u}, \mathbf{N}_{\varphi_1} \mathbf{u}) = \|\mathbf{u}\|^2.$$

By the assumptions on  $\nu$  and  $\eta$  (cf. (3.1)), exploiting (2.2), (2.6), and (B.5) (with  $p = 2$  and  $r = \infty$ ), we find the control

$$\begin{aligned} I_3 &\leq C \|\varphi\|_{L^3(\Omega)} \|\nabla \mathbf{u}_2\| \|\nabla \mathbf{N}_{\varphi_1} \mathbf{u}\|_{\mathbf{L}^6(\Omega)} \\ &\leq C \|\varphi\|_{-1}^{\frac{1}{2}} \|\varphi\|_V^{\frac{1}{2}} \|\nabla \mathbf{u}_2\| \|\mathbf{N}_{\varphi_1} \mathbf{u}\|_{\mathbf{H}^2(\Omega)} \\ &\leq C \|\varphi\|_{-1}^{\frac{1}{4}} \|\varphi\|_V^{\frac{3}{4}} \|\nabla \mathbf{u}_2\| \|\mathbf{u}\| (1 + \|\nabla \varphi_1\|_{\mathbf{L}^\infty(\Omega)}) \\ &\leq \frac{1}{8} \|\mathbf{u}\|^2 + C \|\varphi\|_{-1}^{\frac{1}{2}} \|\varphi\|_V^{\frac{3}{2}} \|\nabla \mathbf{u}_2\|^2 (1 + \|\nabla \varphi_1\|_{\mathbf{L}^\infty(\Omega)}^2). \end{aligned}$$

By (2.2) and (B.4), we obtain

$$\begin{aligned} I_4 &\leq C \|\varphi\| \|\mathbf{u}_2\|_{\mathbf{L}^3(\Omega)} \|\mathbf{N}_{\varphi_1} \mathbf{u}\|_{\mathbf{L}^6(\Omega)} \\ &\leq \frac{1}{8} \|\mathbf{u}\|^2 + C \|\mathbf{u}_2\|_{\mathbf{L}^3(\Omega)}^2 \|\varphi\|_{-1} \|\varphi\|_V. \end{aligned}$$

By the embedding  $W^{2,6}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$  (cf. (2.5)), and using (B.4), we have

$$\begin{aligned} I_5 &\leq \|\nabla \varphi\| (\|\nabla \varphi_1\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla \varphi_2\|_{\mathbf{L}^\infty(\Omega)}) \|\nabla \mathbf{N}_{\varphi_1} \mathbf{u}\| \\ &\leq \frac{1}{4} \|\mathbf{u}\|^2 + C (\|\varphi_1\|_{W^{2,6}(\Omega)}^2 + \|\varphi_2\|_{W^{2,6}(\Omega)}^2) \|\varphi\|_V^2. \end{aligned}$$

Thus, we learn by the above inequalities that

$$\begin{aligned} \|\mathbf{u}\| &\leq C \|\nabla \mathbf{u}_2\| (1 + \|\nabla \varphi_1\|_{\mathbf{L}^\infty(\Omega)}) \|\varphi\|_{-1}^{\frac{1}{4}} \|\varphi\|_V^{\frac{3}{4}} \\ &\quad + C \|\mathbf{u}_2\|_{\mathbf{L}^3(\Omega)} \|\varphi\|_{-1}^{\frac{1}{2}} \|\varphi\|_V^{\frac{1}{2}} + C (\|\varphi_1\|_{W^{2,6}(\Omega)} + \|\varphi_2\|_{W^{2,6}(\Omega)}) \|\varphi\|_V. \end{aligned}$$

Exploiting this in  $I_2$ , we find

$$\begin{aligned}
I_2 &\leq C \|\nabla \mathbf{u}_2\| \left(1 + \|\nabla \varphi_1\|_{\mathbf{L}^\infty(\Omega)}\right) \|\varphi\|_{-1}^{\frac{5}{4}} \|\varphi\|_V^{\frac{3}{4}} + \|\mathbf{u}_2\|_{\mathbf{L}^3(\Omega)} \|\varphi\|_{-1}^{\frac{3}{2}} \|\varphi\|_V^{\frac{1}{2}} \\
&\quad + C \left(\|\varphi_1\|_{W^{2,6}(\Omega)} + \|\varphi_2\|_{W^{2,6}(\Omega)}\right) \|\varphi\|_{-1} \|\varphi\|_V \\
&\leq \frac{1}{4} \|\varphi\|_V^2 + C \left(\|\nabla \mathbf{u}_2\|_{\frac{8}{5}}^{\frac{8}{5}} + \|\nabla \mathbf{u}_2\|_{\frac{8}{5}}^{\frac{8}{5}} \|\nabla \varphi_1\|_{\mathbf{L}^\infty(\Omega)}^{\frac{8}{5}}\right) \|\varphi\|_{-1}^2 + C \|\mathbf{u}_2\|_{\mathbf{L}^3(\Omega)}^{\frac{4}{3}} \|\varphi\|_{-1}^2 \\
&\quad + C \left(\|\varphi_1\|_{W^{2,6}(\Omega)}^2 + \|\varphi_2\|_{W^{2,6}(\Omega)}^2\right) \|\varphi\|_{-1}^2.
\end{aligned}$$

By the interpolation inequality (2.7), together with the controls (4.3), we get

$$\begin{aligned}
\|\nabla \mathbf{u}_2\|_{\frac{8}{5}}^{\frac{8}{5}} \|\nabla \varphi_1\|_{\mathbf{L}^\infty(\Omega)}^{\frac{8}{5}} &\leq C \|\nabla \mathbf{u}_2\|^4 + C \|\nabla \varphi_1\|_{\mathbf{L}^\infty(\Omega)}^{\frac{8}{3}} \\
&\leq C \|\nabla \mathbf{u}_2\|^4 + C \|\nabla \varphi_1\|_{\frac{2}{3}}^{\frac{8}{3}} \|\varphi_1\|_{W^{2,6}(\Omega)}^2 \\
&\leq C \|\nabla \mathbf{u}_2\|^4 + C \|\varphi_1\|_{W^{2,6}(\Omega)}^2.
\end{aligned}$$

Collecting the above estimates for  $I_1$  and  $I_2$  in (4.4), we obtain the differential inequality

$$\frac{d}{dt} \|\varphi\|_{-1}^2 \leq \Gamma \|\varphi\|_{-1}^2 + \Upsilon |\bar{\varphi}|,$$

having set

$$\Gamma(t) = C \left(1 + \|\varphi_1(t)\|_{W^{2,6}(\Omega)}^2 + \|\varphi_2(t)\|_{W^{2,6}(\Omega)}^2 + \|\mathbf{u}_1(t)\|_{\mathbf{L}^3(\Omega)}^2 + \|\nabla \mathbf{u}_2(t)\|^4\right),$$

which belongs to  $L^1(0, T)$  in light of (3.9)-(3.10). We observe that  $\Upsilon$  is summable as well. Therefore, an application of the Gronwall lemma gives

$$\|\varphi(t)\|_{-1}^2 \leq C \|\varphi(0)\|_{-1}^2 + C |\bar{\varphi}(0)|, \quad \forall t \in [0, T].$$

In particular, if  $\varphi_1(0) = \varphi_2(0)$ , then  $\varphi_1 \equiv \varphi_2$ . By (4.5) with  $J_1 = J_2 = J_3 = 0$ , we easily deduce that  $\mathbf{u}_1 \equiv \mathbf{u}_2$ , thus uniqueness follows.  $\square$

## 5. GLOBAL WELL-POSEDNESS OF STRONG SOLUTIONS

In this section we prove the existence of global-in-time strong solutions. As a consequence, thanks to the parabolic dissipative nature of the model, we deduce that any weak solution becomes instantaneously a strong solution.

**Theorem 5.1.** *Let  $\varphi_0 \in V \cap L^\infty(\Omega)$  with  $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$ ,  $|\bar{\varphi}_0| < 1$ . Assume, in addition that,  $\varphi_0 \in H^2(\Omega)$  with  $\partial_{\mathbf{n}}\varphi_0 = 0$  on  $\partial\Omega$  and  $\nabla\mu_0 \in \mathbf{H}$ , where  $\mu_0 = -\Delta\varphi_0 + \Psi'(\varphi_0)$ . Then, for every  $T > 0$ , the weak solution is a strong solution to problem (1.1)-(1.2) on  $[0, T]$ , in the following sense:*

$$\begin{aligned}
\mathbf{u} &\in L^\infty(0, T; \mathbf{V}_\sigma \cap \mathbf{W}^{2,p}(\Omega)) \cap L^2(0, T; \mathbf{H}^3(\Omega)) \cap H^1(0, T; \mathbf{H}_\sigma), \\
\varphi &\in L^\infty(0, T; W^{2,p}(\Omega)), \quad \partial_t \varphi \in L^\infty(0, T; V') \cap L^2(0, T; V), \\
F'(\varphi) &\in L^\infty(0, T; L^p(\Omega)), \\
\mu &\in L^\infty(0, T; V) \cap L^2(0, T; H^3(\Omega)),
\end{aligned}$$

where  $p = 6$  if  $d = 3$  and any  $2 \leq p < \infty$  if  $d = 2$ . Moreover, the solution satisfies

$$\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi = \Delta \mu \tag{5.1}$$

almost everywhere on  $\Omega \times (0, \infty)$ , and  $\partial_{\mathbf{n}}\mu = 0$  almost everywhere on  $\partial\Omega \times (0, \infty)$ .

Let us briefly explain the strategy of the proof of Theorem 5.1. First, relying on the same ideas employed in the proof of uniqueness, we obtain a differential inequality (cf. (5.6)) that is the core of the regularization issues. However, the low regularity in time of  $\mathbf{u}$  is not sufficient to prove directly higher-order estimates on  $\partial_t \varphi$  (cf. (5.8)). To overcome this difficulty, we exploit the framework of Besov spaces in time in order to get the intermediate step of regularity  $\varphi \in L^\infty([0, T]; W^{1,q}(\Omega))$ , for some  $q > 3$ . Then, it follows a uniform-in-time estimate of  $\mathbf{u}$ , which in turn entails the needed regularity for  $\partial_t \varphi$ . Once this is achieved, the regularity theory of the Neumann problem with logarithmic nonlinearity in Appendix A and the corresponding theory for the Stokes problem in Appendix B allow us to recover the higher-order regularity of the strong solution.

*Proof.* The proof is divided in several steps. In the sequel the generic constant  $C > 0$  may depend on  $\mathcal{E}(\varphi_0)$ ,  $\bar{\varphi}_0$  and  $\|\nabla \mu_0\|$ .

**1. A differential inequality.** Given  $h > 0$ , let us introduce the difference in time of a function  $v$  by

$$\Delta_h v(t) = v(t+h) - v(t).$$

Owing to (3.3)-(3.4), for any  $h > 0$  the weak solution satisfies

$$\begin{aligned} (\nu(\varphi) D \Delta_h \mathbf{u}, D \mathbf{v}) + (\eta(\varphi) \Delta_h \mathbf{u}, \mathbf{v}) + (\Delta_h \nu(\varphi) D \mathbf{u}(\cdot+h), D \mathbf{v}) + (\Delta_h \eta(\varphi) \mathbf{u}(\cdot+h), \mathbf{v}) \\ = (\nabla \varphi \otimes \nabla \Delta_h \varphi, \nabla \mathbf{v}) + (\nabla \Delta_h \varphi \otimes \nabla \varphi(\cdot+h), \nabla \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_\sigma, \end{aligned} \quad (5.2)$$

and

$$\langle \partial_t \Delta_h \varphi, v \rangle + (\mathbf{u} \cdot \nabla \Delta_h \varphi, v) + (\Delta_h \mathbf{u} \cdot \nabla \varphi(\cdot+h), v) + (\nabla \Delta_h \mu, \nabla v) = 0, \quad \forall v \in V, \quad (5.3)$$

for almost every  $t \in (0, T)$ , where

$$\Delta_h \mu = -\Delta \Delta_h \varphi + \Delta_h \Psi'(\varphi).$$

Taking  $v = \mathcal{N} \Delta_h \varphi$  in (5.3), we find

$$\frac{1}{2} \frac{d}{dt} \|\Delta_h \varphi\|_*^2 + (\Delta_h \mu, \Delta_h \varphi) = J_1 + J_2, \quad (5.4)$$

where

$$J_1 = (\Delta_h \varphi \mathbf{u}, \nabla \mathcal{N} \Delta_h \varphi), \quad J_2 = (\varphi(\cdot+h) \Delta_h \mathbf{u}, \nabla \mathcal{N} \Delta_h \varphi).$$

By definition of  $\Delta_h \mu$  and making use of the homogeneous Neumann boundary condition for  $\Delta_h \varphi$ , the assumptions on  $\Psi'$  and (2.2), we obtain

$$\begin{aligned} (\Delta_h \mu, \Delta_h \varphi) &\geq \|\nabla \Delta_h \varphi\|^2 - \alpha \|\Delta_h \varphi\|^2 \\ &\geq \frac{1}{2} \|\Delta_h \varphi\|_V^2 - C \|\Delta_h \varphi\|_*^2. \end{aligned}$$

In order to control  $J_1$  and  $J_2$ , we argue similarly to the proof of Theorem 4.1. We estimate  $J_1$  as follows

$$\begin{aligned} J_1 &\leq \|\mathbf{u}\|_{\mathbf{L}^3(\Omega)} \|\Delta_h \varphi\|_{L^6(\Omega)} \|\nabla \mathcal{N} \Delta_h \varphi\| \\ &\leq \frac{1}{8} \|\nabla \Delta_h \varphi\|^2 + C \|\mathbf{u}\|_{\mathbf{L}^3(\Omega)}^2 \|\Delta_h \varphi\|_*^2. \end{aligned}$$

Regarding  $J_2$ , we first have

$$J_2 \leq \|\Delta_h \mathbf{u}\| \|\Delta_h \varphi\|_*.$$

Hence, in order to control  $\Delta_h \mathbf{u}$  in terms of  $\Delta_h \varphi$ , we take  $\mathbf{v} = \mathbf{N}_\varphi \Delta_h \mathbf{u}$  in (5.3) getting

$$\|\Delta_h \mathbf{u}\|^2 = J_3 + J_4 + J_5,$$

having set

$$\begin{aligned} J_3 &= (\Delta_h \nu(\varphi) D\mathbf{u}(\cdot + h), D\mathbf{N}_\varphi \Delta_h \mathbf{u}), \\ J_4 &= (\Delta_h \eta(\varphi) \mathbf{u}(\cdot + h), \mathbf{N}_\varphi \Delta_h \mathbf{u}), \\ J_5 &= (\nabla \varphi \otimes \nabla \Delta_h \varphi, \nabla \mathbf{N}_\varphi \Delta_h \mathbf{u}) + (\nabla \Delta_h \varphi \otimes \nabla \varphi(\cdot + h), \nabla \mathbf{N}_\varphi \Delta_h \mathbf{u}). \end{aligned}$$

Here we have used again (B.3). Exploiting again (2.2), (2.6), (B.5) (with  $p = 2$  and  $r = \infty$ ) and (B.4), we control  $J_i$ ,  $i = 3, 4, 5$ , as follows

$$\begin{aligned} J_3 &\leq C \|\Delta_h \varphi\|_{L^3(\Omega)} \|\nabla \mathbf{u}(\cdot + h)\| \|\nabla \mathbf{N}_\varphi \Delta_h \mathbf{u}\|_{\mathbf{L}^6(\Omega)} \\ &\leq C \|\Delta_h \varphi\|_{L^3(\Omega)}^{\frac{1}{2}} \|\Delta_h \varphi\|_{V}^{\frac{1}{2}} \|\nabla \mathbf{u}(\cdot + h)\| \|\mathbf{N}_\varphi \Delta_h \mathbf{u}\|_{\mathbf{H}^2(\Omega)} \\ &\leq C \|\Delta_h \varphi\|_{*}^{\frac{1}{4}} \|\Delta_h \varphi\|_{V}^{\frac{3}{4}} \|\nabla \mathbf{u}(\cdot + h)\| \|\Delta_h \mathbf{u}\| (1 + \|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)}), \\ J_4 &\leq C \|\Delta_h \varphi\| \|\mathbf{u}(\cdot + h)\|_{\mathbf{L}^3(\Omega)} \|\mathbf{N}_\varphi \Delta_h \mathbf{u}\|_{\mathbf{L}^6(\Omega)} \\ &\leq C \|\Delta_h \varphi\|_{*}^{\frac{1}{2}} \|\Delta_h \varphi\|_{V}^{\frac{1}{2}} \|\mathbf{u}(\cdot + h)\|_{\mathbf{L}^3(\Omega)} \|\Delta_h \mathbf{u}\|, \end{aligned}$$

and

$$\begin{aligned} J_5 &\leq C (\|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla \varphi(\cdot + h)\|_{\mathbf{L}^\infty(\Omega)}) \|\nabla \Delta_h \varphi\| \|\nabla \mathbf{N}_\varphi \Delta_h \mathbf{u}\| \\ &\leq C (\|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla \varphi(\cdot + h)\|_{\mathbf{L}^\infty(\Omega)}) \|\Delta_h \varphi\|_V \|\Delta_h \mathbf{u}\|. \end{aligned}$$

Accordingly, we arrive at

$$\begin{aligned} \|\Delta_h \mathbf{u}\| &\leq C \|\nabla \mathbf{u}(\cdot + h)\| (1 + \|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)}) \|\Delta_h \varphi\|_{*}^{\frac{1}{4}} \|\Delta_h \varphi\|_{V}^{\frac{3}{4}} \\ &\quad + C \|\mathbf{u}(\cdot + h)\|_{\mathbf{L}^3(\Omega)} \|\Delta_h \varphi\|_{*}^{\frac{1}{2}} \|\Delta_h \varphi\|_{V}^{\frac{1}{2}} \\ &\quad + C (\|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla \varphi(\cdot + h)\|_{\mathbf{L}^\infty(\Omega)}) \|\Delta_h \varphi\|_V. \end{aligned} \tag{5.5}$$

Now, we deduce that

$$\begin{aligned} J_2 &\leq C \|\nabla \mathbf{u}(\cdot + h)\| (1 + \|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)}) \|\Delta_h \varphi\|_{*}^{\frac{5}{4}} \|\Delta_h \varphi\|_{V}^{\frac{3}{4}} \\ &\quad + C \|\mathbf{u}(\cdot + h)\|_{\mathbf{L}^3(\Omega)} \|\Delta_h \varphi\|_{*}^{\frac{3}{2}} \|\Delta_h \varphi\|_{V}^{\frac{1}{2}} \\ &\quad + C (\|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla \varphi(\cdot + h)\|_{\mathbf{L}^\infty(\Omega)}) \|\Delta_h \varphi\|_{*} \|\Delta_h \varphi\|_V \\ &\leq \frac{1}{8} \|\Delta_h \varphi\|_V^2 + C \|\nabla \mathbf{u}(\cdot + h)\|^{\frac{8}{5}} (1 + \|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)}^{\frac{8}{5}}) \|\Delta_h \varphi\|_{*}^2 \\ &\quad + C \|\mathbf{u}(\cdot + h)\|_{\mathbf{L}^3(\Omega)}^{\frac{4}{3}} \|\Delta_h \varphi\|_{*}^2 + C (\|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)}^2 + \|\nabla \varphi(\cdot + h)\|_{\mathbf{L}^\infty(\Omega)}^2) \|\Delta_h \varphi\|_{*}^2 \\ &\leq \frac{1}{8} \|\Delta_h \varphi\|_V^2 + C (1 + \|\nabla \mathbf{u}(\cdot + h)\|^4 + \|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)}^{\frac{8}{3}}) \|\Delta_h \varphi\|_{*}^2 \\ &\quad + C (\|\mathbf{u}(\cdot + h)\|_{\mathbf{L}^3(\Omega)}^{\frac{4}{3}} + \|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)}^2 + \|\nabla \varphi(\cdot + h)\|_{\mathbf{L}^\infty(\Omega)}^2) \|\Delta_h \varphi\|_{*}^2. \end{aligned}$$

Exploiting (2.7) and the embedding  $W^{2,6}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ , we thus obtain the control

$$J_2 \leq \frac{1}{8} \|\Delta_h \varphi\|_V^2 + C(1 + \|\varphi\|_{W^{2,6}(\Omega)}^2 + \|\varphi(\cdot + h)\|_{W^{2,6}(\Omega)}^2 + \|\nabla \mathbf{u}(\cdot + h)\|^4) \|\Delta_h \varphi\|_*^2.$$

Collecting the above inequalities in (5.4), we finally find the differential inequality

$$\frac{d}{dt} \|\Delta_h \varphi(t)\|_*^2 + \|\Delta_h \varphi(t)\|_V^2 \leq \Lambda \|\Delta_h \varphi(t)\|_*^2, \quad (5.6)$$

where

$$\Lambda(t) = C(1 + \|\varphi(t)\|_{W^{2,6}(\Omega)}^2 + \|\varphi(t+h)\|_{W^{2,6}(\Omega)}^2 + \|\mathbf{u}(t)\|_{\mathbf{L}^3(\Omega)}^2 + \|\nabla \mathbf{u}(t+h)\|^4).$$

In light of Theorem 3.6, the function  $\Lambda$  belongs to  $L^1(t, t+1)$ , for all  $t \geq 0$ , and there exists a constant  $C$  independent of  $t$  such that

$$\|\Lambda\|_{L^1(t, t+1)} \leq C, \quad \forall t \geq 0. \quad (5.7)$$

**2. A control of the initial condition.** By the assumptions on  $\Psi$ , the fact that  $\partial_{\mathbf{n}}(\varphi - \varphi_0) = 0$  on  $\partial\Omega$  and  $\bar{\varphi} = \bar{\varphi}_0$ , we have

$$\begin{aligned} (\mu - \mu_0, \varphi - \varphi_0) &= (-\Delta(\varphi - \varphi_0), \varphi - \varphi_0) + (\Psi'(\varphi) - \Psi'(\varphi_0), \varphi - \varphi_0) \\ &\geq \|\nabla(\varphi - \varphi_0)\|^2 - \alpha \|\varphi - \varphi_0\|^2 \\ &\geq \frac{1}{2} \|\nabla(\varphi - \varphi_0)\|^2 - C \|\varphi - \varphi_0\|_*^2. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varphi - \varphi_0\|_*^2 &= \langle \partial_t \varphi, \mathcal{N}(\varphi - \varphi_0) \rangle \\ &= -(\mu, \varphi - \varphi_0) + (\varphi \mathbf{u}, \nabla \mathcal{N}(\varphi - \varphi_0)) \\ &= -(\mu - \mu_0, \varphi - \varphi_0) - (\nabla \mu_0, \nabla \mathcal{N}(\varphi - \varphi_0)) + (\varphi \mathbf{u}, \nabla \mathcal{N}(\varphi - \varphi_0)) \\ &\leq -\frac{1}{2} \|\nabla(\varphi - \varphi_0)\|^2 + C \|\varphi - \varphi_0\|_*^2 + \|\nabla \mu_0\| \|\varphi - \varphi_0\|_* + \|\mathbf{u}\| \|\varphi - \varphi_0\|_*, \end{aligned}$$

which gives the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\varphi - \varphi_0\|_*^2 \leq C(1 + \|\nabla \mu_0\| + \|\mathbf{u}\|) \|\varphi - \varphi_0\|_*.$$

An application of the Gronwall lemma stated in [4, Lemma A.5] yields

$$\|\varphi(t) - \varphi_0\|_* \leq C(1 + \|\nabla \mu_0\|)t + \int_0^t \|\mathbf{u}(\tau)\| d\tau, \quad \forall t \geq 0,$$

which, in turn, entails

$$\|\Delta_h \varphi(0)\|_* \leq C(1 + \|\nabla \mu_0\|)h + \int_0^h \|\mathbf{u}(\tau)\| d\tau, \quad \forall h > 0. \quad (5.8)$$

**3. An intermediate step of regularity.** Let us set  $I = [0, 1]$ . Owing to (5.8), the fact that  $\mathbf{u}$  belongs to  $L^4(I; \mathbf{H}_\sigma)$  allows us to improve the regularity in time of  $\varphi$  in Besov spaces. Indeed, by the Hölder inequality, we have

$$h^{-\frac{3}{4}} \|\Delta_h \varphi(0)\|_* \leq Ch^{\frac{1}{4}} + \left( \int_0^h \|\mathbf{u}(\tau)\|^4 d\tau \right)^{\frac{1}{4}},$$

hence

$$\sup_{0 < h \leq 1} h^{-\frac{3}{4}} \|\Delta_h \varphi(0)\|_* \leq C. \quad (5.9)$$

An application of the Gronwall inequality to (5.6) gives us

$$\|\Delta_h \varphi(t)\|_*^2 \leq \|\Delta_h \varphi(0)\|_*^2 e^{\int_I \Lambda(\tau) d\tau}, \quad \forall t \in I.$$

Recalling that  $\Lambda$  satisfies (5.7) and using (5.9), we infer

$$\sup_{0 < h \leq 1} \sup_{t \in I} h^{-\frac{3}{4}} \|\Delta_h \varphi(t)\|_* \leq C,$$

that is  $\varphi \in B_{\infty, \infty}^{\frac{3}{4}}(I; V')$ . After an integration in time of (5.6) on  $I$ , we get

$$\int_I \|\Delta_h \varphi(\tau)\|_V^2 d\tau \leq \|\Delta_h \varphi(0)\|_*^2 + \|\Delta_h \varphi\|_{L^\infty(I; V')}^2 \int_I \Lambda(\tau) d\tau.$$

Hence, we deduce that

$$\sup_{0 < h \leq 1} h^{-\frac{3}{4}} \left( \int_I \|\Delta_h \varphi(\tau)\|_V^2 d\tau \right)^{\frac{1}{2}} \leq C \|\varphi\|_{B_{\infty, \infty}^{\frac{3}{4}}(I; V')},$$

meaning that  $\varphi \in B_{2, \infty}^{\frac{3}{4}}(I; V)$ . Based on this, we are in a position to show that

$$\varphi \in B_{2, \infty}^s(I; W^{1, \frac{17}{5}}(\Omega)) \quad \text{where} \quad s = \frac{47}{68} \cdot \frac{3}{4}.$$

Indeed, exploiting (2.9), we have

$$\begin{aligned} & \sup_{0 < h \leq 1} h^{-s} \left( \int_I \|\Delta_h \varphi(\tau)\|_{W^{1, \frac{17}{5}}(\Omega)}^2 d\tau \right)^{\frac{1}{2}} \\ & \leq C \sup_{0 < h \leq 1} h^{-s} \left( \int_I \|\Delta_h \varphi(\tau)\|_{V}^{\frac{47}{34}} \|\Delta_h \varphi(\tau)\|_{W^{2,6}(\Omega)}^{\frac{21}{34}} d\tau \right)^{\frac{1}{2}} \\ & \leq C \left[ \left( \int_0^2 \|\varphi(\tau)\|_{W^{2,6}(\Omega)}^2 d\tau \right)^{\frac{1}{2}} \right]^{\frac{21}{68}} \sup_{0 < h \leq 1} h^{-s} \left[ \left( \int_I \|\Delta_h \varphi(\tau)\|_V^2 d\tau \right)^{\frac{1}{2}} \right]^{\frac{47}{68}} \\ & \leq C \|\varphi\|_{L^2(0,2; W^{2,6}(\Omega))}^{\frac{21}{68}} \left[ \sup_{0 < h \leq 1} h^{-\frac{68}{47}s} \|\Delta_h \varphi\|_{L^2(I; V)} \right]^{\frac{47}{68}} \\ & \leq C \|\varphi\|_{L^2(0,2; W^{2,6}(\Omega))}^{\frac{21}{68}} \|\varphi\|_{B_{2, \infty}^{\frac{3}{4}}(I; V)}^{\frac{47}{68}}. \end{aligned}$$

In light of (3.10), this implies that

$$\|\varphi\|_{B_{2, \infty}^s(I; W^{1, \frac{17}{5}}(\Omega))} \leq C. \quad (5.10)$$

Since  $s > \frac{1}{2}$ , by the classical embedding (2.1) we finally obtain that  $\varphi \in \mathcal{C}(I, W^{1, \frac{17}{5}}(\Omega))$  and

$$\|\varphi\|_{\mathcal{C}(I, W^{1, \frac{17}{5}}(\Omega))} \leq C. \quad (5.11)$$

Thanks to this, we also have a uniform-in-time control of the velocity field. Indeed, taking  $\mathbf{v} = \mathbf{N}_\varphi \mathbf{u}$  in (3.3), and using (B.3), we have

$$\|\mathbf{u}\|^2 = (\nabla \varphi \otimes \nabla \varphi, \nabla \mathbf{N}_\varphi \mathbf{u}).$$

By (5.11) and (B.6), we obtain

$$\begin{aligned}\|\mathbf{u}\|^2 &\leq C\|\nabla\varphi\|_{\mathbf{L}^{\frac{12}{5}}(\Omega)}^2\|\nabla\mathbf{N}_\varphi\mathbf{u}\|_{\mathbf{L}^6(\Omega)} \\ &\leq C\|\varphi\|_{W^{1,\frac{17}{5}}(\Omega)}^2\|\mathbf{N}_\varphi\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \\ &\leq C\|\mathbf{u}\|.\end{aligned}$$

Therefore, we learn that  $\mathbf{u} \in L^\infty(I; \mathbf{H}_\sigma)$  and

$$\|\mathbf{u}\|_{L^\infty(I; \mathbf{H}_\sigma)} \leq C. \quad (5.12)$$

**4. Regularity of the time derivative.** Let us introduce the difference quotient of a function  $v$  by

$$\partial_t^h v = \frac{v(t+h) - v(t)}{h}, \quad h > 0.$$

In light of (5.6),  $\partial_t^h v$  satisfies the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\partial_t^h \varphi(t)\|_*^2 + \frac{1}{2} \|\partial_t^h \varphi(t)\|_V^2 \leq \Lambda \|\partial_t^h \varphi(t)\|_*^2. \quad (5.13)$$

An application of the Gronwall Lemma gives

$$\|\partial_t^h \varphi(t)\|_*^2 \leq \|\partial_h \varphi(0)\|_*^2 e^{\int_I \Lambda(\tau) d\tau}, \quad \forall t \in I.$$

At this point, using (5.12) in order to estimate  $\|\partial_h \varphi(0)\|_*$  in (5.8), we learn that

$$\begin{aligned}\|\partial_t^h \varphi(0)\|_* &\leq C(1 + \|\nabla\mu_0\|) + \frac{1}{h} \int_0^h \|\mathbf{u}(\tau)\| d\tau \\ &\leq C(1 + \|\nabla\mu_0\|).\end{aligned}$$

Therefore, recalling that  $\Lambda \in L^1(I)$ , after a further integration in time of (5.13) we end up with

$$\sup_{t \in I} \|\partial_t^h \varphi(t)\|_* + \|\partial_t^h \varphi(\tau)\|_{L^2(I; V)}^2 \leq C.$$

Since  $C$  is independent of  $h$  and  $\partial_t^h \varphi$  converges to  $\partial_t \varphi$  weakly in  $L^2(I; V')$  as  $h \rightarrow 0$ , it is easily seen that

$$\|\partial_t \varphi\|_{L^\infty(I; V')} + \|\partial_t \varphi\|_{L^2(I; V)} \leq C. \quad (5.14)$$

**5. Higher-order estimates.** Let us prove the claimed regularities for  $\varphi$ ,  $\mu$  and  $\mathbf{u}$ . Arguing by comparison in (3.4), we have that

$$\|\nabla\mu\| \leq C\|\partial_t \varphi\|_{V'} + C\|\mathbf{u}\| \|\nabla\varphi\|_{\mathbf{L}^3(\Omega)}.$$

Hence, observing that  $\frac{17}{5} > 3$ , by (5.11), (5.12) and (5.14) we obtain

$$\|\nabla\mu\|_{L^\infty(I; \mathbf{H})} \leq C.$$

Recalling (3.15) we end up with

$$\|\mu\|_{L^\infty(I; V)} \leq C.$$

By reading the definition of the chemical potential as the elliptic equation  $-\Delta\varphi + \Psi'(\varphi) = f$ , where  $f = \mu + \theta_c \varphi$ , and applying Lemma A.3, we find that

$$\|\varphi\|_{L^\infty(I; W^{2,p}(\Omega))} + \|\Psi'(\varphi)\|_{L^\infty(I; L^p(\Omega))} \leq C,$$

where  $p = 6$  if  $d = 3$  and any  $2 \leq p < \infty$  if  $d = 2$ . Noticing that  $\mu \nabla \varphi \in L^\infty(I; \mathbf{L}^6(\Omega))$ , in light of Lemma B.1 we arrive at

$$\|\mathbf{u}\|_{L^\infty(I; \mathbf{W}^{2,p}(\Omega))} \leq C,$$

with  $p$  as above. It is now easy to deduce that  $\mathbf{u} \cdot \nabla \varphi \in L^2(I; V)$  since

$$\|\mathbf{u} \cdot \nabla \varphi\|_V \leq C \|\mathbf{u}\|_{\mathbf{L}^\infty(\Omega)} \|\varphi\|_{H^2(\Omega)} + C \|\mathbf{u}\|_V \|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)}.$$

Thus, by the standard elliptic regularity theory of the Neumann problem applied to (3.4), we learn that  $\mu \in L^2(I; H^3(\Omega))$  and  $\partial_n \mu = 0$  on  $\partial\Omega$  for almost every  $t \in (0, 1)$ , and (5.1) holds. By the above regularity for  $\varphi$  and  $\mu$  we learn that  $\mu \nabla \varphi \in L^2(I; \mathbf{V})$ , hence (B.7) yields  $\mathbf{u} \in L^2(I; \mathbf{H}^3(\Omega))$ . Finally, we have

$$\|\partial_t^h \mathbf{u}\| \leq C(1 + \|\partial_t^h \varphi\|_V),$$

which, in turn, entails  $\partial_t \mathbf{u} \in L^2(I; \mathbf{H}_\sigma)$ . In order to conclude the proof it is now sufficient to repeat all the arguments above by replacing  $I = [0, 1]$  with any time interval of the form  $I = [n, n + 1]$  for any given  $n \in \mathbb{N}$ .  $\square$

Let us complete this section by proving the instantaneous propagation of regularity for weak solutions. To this aim, we prove the validity of regularity results on any time interval of the form  $[\sigma, \infty)$ ,  $\sigma > 0$ , which are uniform for bundles of trajectories departing from initial data with the same total mass and bounded energy. More precisely, we have the following result.

**Theorem 5.2.** *Let  $R > 0$ ,  $m \in (-1, 1)$  and  $\sigma > 0$  be given. Let  $\varphi_0$  be any initial datum such that  $\mathcal{E}(\varphi_0) \leq R$  and  $\bar{\varphi}_0 = m$ , and let  $(\varphi, \mathbf{u})$  be the weak solution departing from  $\varphi_0$ . Then, there exists  $C > 0$  depending on  $R$ ,  $m$  and  $\sigma$ , but independent of  $\varphi_0$ , such that*

$$\|\partial_t \varphi\|_{L^\infty(\sigma, \infty; V')} + \|\mu\|_{L^\infty(\sigma, \infty; V)} + \|\mathbf{u}\|_{L^\infty(\sigma, \infty; \mathbf{V}_\sigma \cap \mathbf{W}^{2,6}(\Omega))} \leq C,$$

and

$$\|\mathbf{u}\|_{L^2(t, t+1; \mathbf{H}^3(\Omega))} + \|\partial_t \mathbf{u}\|_{L^2(t, t+1; \mathbf{H}_\sigma)} + \|\partial_t \varphi\|_{L^2(t, t+1; V)} \leq C, \quad \forall t \geq \sigma.$$

Moreover, for every  $p \geq 2$ , we have

$$\|\varphi\|_{L^\infty(\sigma, \infty; W^{2,p}(\Omega))} + \|F'(\varphi)\|_{L^\infty(\sigma, \infty; L^p(\Omega))} \leq C, \quad (5.15)$$

where  $C$  also depends on  $p$ .

*Proof.* Let us go back to the differential inequality (5.13) satisfied by  $\partial_t^h \varphi$ . Owing to the control

$$\|\partial_t^h \varphi\|_{L^2(t, t+1; V')} \leq \|\partial_t \varphi\|_{L^2(t, t+1+h; V')}, \quad \forall t \geq 0,$$

and the uniform integrability of  $\Lambda$  as in (5.7), we are allowed to apply the uniform Gronwall Lemma to (5.13) on  $[\sigma, \infty)$ , for any  $\sigma > 0$ . In this way we find the estimate

$$\|\partial_t^h \varphi\|_{L^\infty(\sigma, \infty; V')} \leq C, \quad \forall t \geq \sigma,$$

for some  $C > 0$  depending on  $\sigma$ , but independent of  $h$ . Besides, a further integration in time of (5.13) yields

$$\|\partial_t^h \varphi\|_{L^2(t, t+1; V)} \leq C, \quad \forall t \geq \sigma.$$

A final passage to the limit as  $h \rightarrow 0$  (cf. (5.14)) yields

$$\|\partial_t \varphi\|_{L^\infty(\sigma, \infty; V')} + \sup_{t \geq \sigma} \|\partial_t \varphi\|_{L^2(t, t+1; V)} \leq C. \quad (5.16)$$

We proceed by proving the claimed estimates for  $\varphi$ ,  $\mu$  and  $\mathbf{u}$ . Note first that (5.16) gives

$$\sup_{t \geq \sigma} \|\varphi\|_{B_{2,\infty}^{\frac{3}{4}}(t,t+1;V)} \leq C,$$

in light of the embedding  $H^1(t, t+1; V) \hookrightarrow B_{2,\infty}^{\frac{3}{4}}(t, t+1; V)$ . Since  $\varphi$  also belongs to  $L^2(t, t+1; W^{2,6}(\Omega))$ , we can repeat the argument in the proof of Theorem 5.1 (cf. step 3) on each time interval  $I = [\sigma + n, \sigma + n + 1]$ , for any non-negative integer  $n \in \mathbb{N}_0$ . Recalling that in those computations all the constants depend on  $I$  only through its length, we obtain

$$\sup_{t \geq \sigma} \|\varphi(t)\|_{W^{1,\frac{17}{5}}(\Omega)} \leq C.$$

As a byproduct, we also get

$$\|\mathbf{u}\|_{L^\infty(\sigma,\infty;\mathbf{H}_\sigma)} \leq C.$$

At this point, repeating line by line the last part of the proof of Theorem 5.1 (cf. step 5) we reach all the desired conclusions.  $\square$

**Remark 5.3.** Observe that, if an initial datum  $\varphi_0$  satisfies the assumptions of Theorem 5.1, then  $\varphi_0 \in W^{2,6}(\Omega)$  in light of Lemma A.3. In turn this means that  $\varphi_0 \in \mathcal{C}^{\frac{1}{2}}(\overline{\Omega})$ . Furthermore, it is easily seen (cf., e.g., [9, Theorem II.5.16]) that  $\varphi \in \mathcal{C}(\overline{\Omega} \times I)$ , where  $I = [0, \infty)$  for a strong solution and  $I = (0, \infty)$  for a weak solution. Also, by the regularity proved for the velocity, one can deduce that  $\mathbf{u} \in \mathcal{C}(I, \mathbf{H}_\sigma)$ . As a byproduct, we derive that

$$\sup_{t \in I} \|\varphi(t)\|_{W^{2,6}(\Omega)} + \sup_{t \in I} \|\mathbf{u}(t)\|_{\mathbf{W}^{2,6}(\Omega)} \leq C, \quad (5.17)$$

where  $C$  depends only on  $\mathcal{E}(\varphi_0)$  and  $m$ .

## 6. SEPARATION PROPERTY AND ITS CONSEQUENCES

In this section, we investigate the so-called separation property, which is a relevant issue from both the physical and mathematical viewpoints. Such a property means that the order parameter  $\varphi$  stays eventually within a suitable closed subset of  $(-1, 1)$ . More precisely, we investigate whether there exist  $\delta > 0$  and an interval  $I \subset [0, \infty)$  such that

$$\sup_{t \in I} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta.$$

As mentioned in the Introduction, we show the validity of the *instantaneous* separation property in dimension two. Our proofs rely on the techniques introduced in [21]. Instead, reasoning as in [1], in dimension three we can prove the *asymptotic* separation property, meaning that there exists a certain time  $t^* > 0$ , depending on the initial datum and eventually large, such that the solution is bounded away from the pure phases when  $t$  is larger than  $t^*$ . It is worth mentioning that  $t^*$  can not be exactly estimated. Next, we discuss the question whether a solution departing from an initial state  $\varphi_0$  such that  $\|\varphi_0\|_{L^\infty(\Omega)} < 1$  remains uniformly away from the pure states over time. We conclude our analysis with some remarks on the longtime behavior.

**6.1. Two-dimensional case: instantaneous separation property.** We prove the validity of the instantaneous separation property for a class of singular potentials which includes, in particular, the physically relevant logarithmic free energy (1.4). As above, let us fix  $R > 0$  and  $m \in (-1, 1)$ , and let  $(\varphi, \mathbf{u})$  be the solution to the CHB system departing from  $\varphi_0$  satisfying

$$\mathcal{E}(\varphi_0) \leq R \quad \text{and} \quad \bar{\varphi}_0 = m.$$

Therefore, in the sequel, the generic constant  $C > 0$  may depend on  $R$  and  $m$ .

**Theorem 6.1.** *Let  $d = 2$  and  $\sigma > 0$ . Assume that  $F''$  is convex and*

$$F''(s) \leq e^{C|F'(s)|+C}, \quad \forall s \in (-1, 1),$$

*for some  $C > 0$ . Then, there exists  $\delta = \delta(\sigma, R, m) > 0$  such that*

$$\sup_{t \geq 2\sigma} \|\varphi(t)\|_{C(\bar{\Omega})} \leq 1 - \delta. \quad (6.1)$$

*Proof.* First, in light of the extra assumption on  $F$ , we have a key control on  $F''(\varphi)$ . Indeed, by Lemma A.6, for any  $p \geq 2$ , there exists  $C = C(\sigma, p)$ , such that

$$\|F''(\varphi)\|_{L^\infty(\sigma, \infty; L^p(\Omega))} \leq C. \quad (6.2)$$

We are now in a position to prove higher order Sobolev estimates on the time interval  $[2\sigma, \infty)$ . Given  $h > 0$ , we recall that  $\partial_t^h v$  is the difference quotient of  $v$ . Owing to (5.1), the quotient  $\partial_t^h \varphi$  solves

$$\partial_t \partial_t^h \varphi + \mathbf{u} \cdot \nabla \partial_t^h \varphi + \partial_t^h \mathbf{u} \cdot \nabla \varphi(\cdot + h) = \Delta \partial_t^h \mu.$$

Multiplying the above equation by  $\partial_t^h \varphi$ , we have

$$\frac{1}{2} \frac{d}{dt} \|\partial_t^h \varphi\|^2 = (\Delta \partial_t^h \mu, \partial_t^h \varphi) + W_1 + W_2, \quad (6.3)$$

having set

$$W_1 = (\varphi(\cdot + h) \partial_t^h \mathbf{u}, \nabla \partial_t^h \varphi), \quad W_2 = (\partial_t^h \varphi \mathbf{u}, \nabla \partial_t^h \varphi).$$

Integrating by parts and exploiting the boundary conditions, we get

$$\begin{aligned} (\Delta \partial_t^h \mu, \partial_t^h \varphi) &= (\partial_t^h \mu, \Delta \partial_t^h \varphi) \\ &= -\|\Delta \partial_t^h \varphi\|^2 + \theta_0 \|\nabla \partial_t^h \varphi\|^2 + (\partial_t^h F'(\varphi), \Delta \partial_t^h \varphi). \end{aligned}$$

Since  $F''$  is convex, we find the control

$$\begin{aligned} \frac{1}{h} \left| F'(\varphi(\cdot + h)) - F'(\varphi(\cdot)) \right| &\leq \int_0^1 F''(\tau \varphi(\cdot + h) + (1 - \tau) \varphi(\cdot)) |\partial_t^h \varphi| \, d\tau \\ &\leq \int_0^1 (\tau F''(\varphi(\cdot + h)) + (1 - \tau) F''(\varphi(\cdot))) |\partial_t^h \varphi| \, d\tau \\ &\leq (F''(\varphi(\cdot + h)) + F''(\varphi(\cdot))) |\partial_t^h \varphi|. \end{aligned}$$

Hence, using (6.2), we obtain

$$\begin{aligned} (\partial_t^h F'(\varphi), \Delta \partial_t^h \varphi) &\leq \frac{1}{2} \|\Delta \partial_t^h \varphi\|^2 + C (\|F''(\varphi(\cdot + h))\|_{L^3(\Omega)}^2 + \|F''(\varphi)\|_{L^3(\Omega)}^2) \|\partial_t^h \varphi\|_{L^6(\Omega)}^2 \\ &\leq \frac{1}{2} \|\Delta \partial_t^h \varphi\|^2 + C \|\nabla \partial_t^h \varphi\|^2. \end{aligned}$$

We thus arrive at the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\partial_t^h \varphi\|^2 + \frac{1}{2} \|\Delta \partial_t^h \varphi\|^2 \leq C \|\nabla \partial_t^h \varphi\|^2 + W_1 + W_2. \quad (6.4)$$

Let us now consider the equation for  $\partial_t^h \mathbf{u}$  as in Remark 3.4. Taking  $\mathbf{v} = \partial_t^h \mathbf{u}$ , we find

$$(\nu(\varphi) D \partial_t^h \mathbf{u}, D \partial_t^h \mathbf{u}) + (\eta(\varphi) \partial_t^h \mathbf{u}, \partial_t^h \mathbf{u}) = W_3 + W_4, \quad (6.5)$$

having set

$$\begin{aligned} W_3 &= -(\partial_t^h \nu(\varphi) D \mathbf{u}(\cdot + h), D \partial_t^h \mathbf{u}) - (\partial_t^h \eta(\varphi) \mathbf{u}(\cdot + h), \partial_t^h \mathbf{u}), \\ W_4 &= (\nabla \partial_t^h \varphi \otimes \nabla \varphi(\cdot + h), \nabla \partial_t^h \mathbf{u}) + (\nabla \varphi \otimes \nabla \partial_t^h \varphi, \nabla \partial_t^h \mathbf{u}). \end{aligned}$$

Summing up (6.4) and (6.5), and exploiting (3.1) and the Korn inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\partial_t^h \varphi\|^2 + \frac{1}{2} \|\Delta \partial_t^h \varphi\|^2 + \nu_* \|\nabla \partial_t^h \mathbf{u}\|^2 \leq C \|\nabla \partial_t^h \varphi\|^2 + \sum_{k=1}^4 W_k.$$

We estimate the right-hand side term by term as follows. By Theorem 5.2 and the embedding  $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ , we have

$$W_1 \leq \|\partial_t^h \mathbf{u}\| \|\partial_t^h \varphi\|,$$

and

$$\begin{aligned} W_2 &\leq \|\mathbf{u}\|_{L^\infty(\Omega)} \|\partial_t^h \varphi\| \|\nabla \partial_t^h \varphi\| \\ &\leq C \|\partial_t^h \varphi\| \|\nabla \partial_t^h \varphi\|. \end{aligned}$$

Next, recalling that  $\nu, \eta \in C^1(\mathbb{R})$ , by Theorem 5.2 and the embedding  $W^{2,6}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$  (cf. (2.5)), we find

$$\begin{aligned} W_3 &\leq C \|\nabla \mathbf{u}(\cdot + h)\|_{L^\infty(\Omega)} \|\partial_t^h \varphi\| \|\nabla \partial_t^h \mathbf{u}\| + C \|\mathbf{u}(\cdot + h)\|_{L^\infty(\Omega)} \|\partial_t^h \varphi\| \|\partial_t^h \mathbf{u}\| \\ &\leq C \|\partial_t^h \varphi\| \|\nabla \partial_t^h \mathbf{u}\|, \end{aligned}$$

and

$$\begin{aligned} W_4 &\leq (\|\nabla \varphi\|_{L^\infty(\Omega)} + \|\nabla \varphi(\cdot + h)\|_{L^\infty(\Omega)}) \|\nabla \partial_t^h \varphi\| \|\nabla \partial_t^h \mathbf{u}\| \\ &\leq C \|\nabla \partial_t^h \varphi\| \|\nabla \partial_t^h \mathbf{u}\|. \end{aligned}$$

Since

$$\|\nabla \partial_t^h \varphi\| \leq \|\partial_t^h \varphi\|^{\frac{1}{2}} \|\Delta \partial_t^h \varphi\|^{\frac{1}{2}},$$

collecting all the above estimates, we end up with

$$\frac{1}{2} \frac{d}{dt} \|\partial_t^h \varphi\|^2 + \frac{1}{4} \|\Delta \partial_t^h \varphi\|^2 + \frac{\nu_*}{2} \|\nabla \partial_t^h \mathbf{u}\|^2 \leq C \|\partial_t^h \varphi\|^2,$$

for almost every  $t \geq \sigma$ . Recalling that, in light of Theorem 5.2,

$$\|\partial_t^h \varphi\|_{L^2(t, t+1; H)} \leq C, \quad \forall t \geq \sigma,$$

an application of the uniform Gronwall lemma and a final passage to the limit as  $h \rightarrow 0$  yields

$$\|\partial_t \varphi\|_{L^\infty(2\sigma, \infty; H)} \leq C.$$

According to  $\mathbf{u} \cdot \nabla \varphi \in L^\infty(\sigma, \infty; H)$  (cf. Theorem 5.2), thanks to the elliptic regularity of the Neumann problem applied to (3.4), we infer

$$\|\mu\|_{L^\infty(2\sigma, \infty; H^2(\Omega))} \leq C. \quad (6.6)$$

Being  $L^\infty(2\sigma, \infty; H^2(\Omega)) \subset L^\infty(\Omega \times (2\sigma, \infty))$ , by applying Lemma A.1 with  $p = \infty$ , we deduce that

$$\|F'(\varphi)\|_{L^\infty(\Omega \times (2\sigma, \infty))} \leq C.$$

Since  $F'$  diverges at  $\pm 1$  and  $\varphi$  is continuous as established in Remark 5.3, we immediately deduce the existence of  $\delta > 0$  such that

$$|\varphi(x, t)| \leq 1 - \delta, \quad \forall (x, t) \in \overline{\Omega} \times [2\sigma, \infty).$$

The proof is completed.  $\square$

**6.2. Three-dimensional case: asymptotic separation property.** The validity of the instantaneous separation property is still an open issue in dimension three, even for the solely Cahn–Hilliard equation. Nevertheless, we are able to prove the following weaker version.

**Theorem 6.2.** *Let  $d = 3$ . There exist  $t^* > 0$  and  $\delta > 0$  such that*

$$\sup_{t \geq t^*} \|\varphi(t)\|_{C(\overline{\Omega})} \leq 1 - \delta.$$

*Proof.* Let  $(\varphi, \mathbf{u})$  be a weak solution departing from an admissible initial datum  $\varphi_0$ . Then, by Theorem 5.2 we have that  $\varphi \in L^\infty(1, \infty; W^{2,6}(\Omega))$  and  $\mathbf{u} \in L^\infty(1, \infty; \mathbf{V}_\sigma \cap \mathbf{W}^{2,6}(\Omega))$ . We define the  $\omega$ -limit set of  $\varphi_0$  as

$$\omega(\varphi_0) = \{\varphi_\infty \in W^{2,6}(\Omega) : \|\varphi_\infty\|_{L^\infty(\Omega)} \leq 1, \text{ and } \exists t_n \rightarrow \infty \text{ such that } \varphi(t_n) \rightarrow \varphi_\infty\},$$

where the convergence is  $C^s(\overline{\Omega})$ , for any  $0 < s < \frac{1}{2}$  (cf. (2.5)). Note that  $\omega(\varphi_0)$  is non-empty by (5.17). Let us now show that

$$\omega(\varphi_0) \subset \{\varphi_\infty \text{ where } \varphi_\infty \in W^{2,6}(\Omega) \text{ solves (6.7)}\}$$

where

$$\begin{cases} -\Delta \varphi_\infty + \Psi'(\varphi_\infty) = \mu_\infty, & \text{in } \Omega, \\ \partial_{\mathbf{n}} \varphi_\infty = 0, & \text{on } \partial\Omega, \end{cases} \quad (6.7)$$

where  $\mu_\infty = \overline{\Psi'(\varphi_\infty)} \in \mathbb{R}$  and  $\overline{\varphi_\infty} = \overline{\varphi_0}$ . To this aim, following [1, Lemma 11], let  $t_n \rightarrow \infty$  such that  $\lim_{n \rightarrow \infty} \varphi(t_n) = \varphi_\infty$  and set  $\varphi_n(t) = \varphi(t + t_n)$ ,  $\mathbf{u}_n(t) = \mathbf{u}(t + t_n)$ , for any  $n \in \mathbb{N}$ . Then,  $(\varphi_n, \mathbf{u}_n)$  converges to a solution  $(\varphi', \mathbf{u}')$  of (3.3)–(3.5) with initial value  $\varphi'(0) = \varphi_\infty$ . In particular, up to a subsequence

$$\lim_{n \rightarrow \infty} \mathcal{E}(\varphi_n(t)) = \mathcal{E}(\varphi'(t)), \quad \forall t \geq 0.$$

On the other hand, since the energy is a decreasing function of time,  $\lim_{t \rightarrow \infty} \mathcal{E}(\varphi(t))$  exists, and we set its value equal to  $E_\infty$ . Therefore, we learn that

$$E_\infty = \lim_{n \rightarrow \infty} \mathcal{E}(\varphi_n(t)) = \mathcal{E}(\varphi'(t)), \quad \forall t \geq 0,$$

namely  $\mathcal{E}(\varphi'(t))$  is constant. Recalling the energy identity in Theorem 3.6, we deduce that  $\nabla \mu' = 0$  and  $\mathbf{u}' = \mathbf{0}$  for almost all  $t$ . Thus  $\partial_t \varphi' = 0$  and  $\varphi'$  is constant in time. Hence,  $\varphi_\infty = \varphi'(t)$  is a solution of (6.7). Now, given any stationary solution  $\varphi_\infty$  to system (6.7), by

Remark A.2 there is a constant  $0 < \delta < 1$ , depending on  $\varphi_\infty$ , such that  $|\varphi_\infty(x)| < 1 - 2\delta$ , for all  $x \in \overline{\Omega}$ . By the compactness of  $\omega(\varphi_0)$  in  $\mathcal{C}(\overline{\Omega})$  (cf. (2.5)), we easily infer the existence of a universal constant  $0 < \delta < 1$  such that

$$|\varphi_\infty(x)| \leq 1 - \delta, \quad \forall x \in \overline{\Omega}, \forall \varphi_\infty \subset \omega(\varphi_0).$$

Since  $\lim_{t \rightarrow \infty} \text{dist}_{\mathcal{C}(\overline{\Omega})}(\varphi(t), \omega(\varphi_0)) = 0$ , we conclude that there exists  $t^* > 0$ , depending on  $\varphi$ , such that  $|\varphi(x, t)| \leq 1 - \delta$ , for all  $(x, t) \in \overline{\Omega} \times [t^*, \infty)$ .  $\square$

**6.3. Further remarks on the separation property and the regularity of solutions.** We first observe that a solution becomes more regular, as expected, when the separation property holds true. Indeed, assuming that  $\|\varphi\|_{\mathcal{C}(\overline{\Omega} \times J)} < 1 - \delta$  on some interval  $J = [t_1, t_2] \subset [0, \infty)$  and for some  $\delta > 0$ , then we have  $\Psi'(\varphi) \in L^\infty(J; H^2(\Omega))$ , being  $\Psi'$  a globally Lipschitz function with bounded derivatives on  $[-1 + \delta, 1 - \delta]$ . Therefore, recalling that  $\mu \in L^\infty(J; V)$ , the elliptic equation  $-\Delta\varphi = \mu - \Psi'(\varphi)$  with homogeneous Neumann boundary conditions yields  $\varphi \in L^\infty(J; H^3(\Omega))$ . Moreover, owing to the separation property, the proof of Theorem 6.1 can be recast in dimension  $d = 3$  in order to prove (6.6), namely  $\mu \in L^\infty([t_1 + \sigma, t_2]; H^2(\Omega))$ , for any  $\sigma > 0$ . Now, an application of Lemma A.3 provides  $\varphi \in L^\infty([t_1 + \sigma, t_2]; H^4(\Omega))$ , for any  $\sigma > 0$ , meaning that  $\varphi$  is a *classical* solution to the CHB system. At this point, it is easily seen that solution is in  $H^k(\Omega)$ ,  $k > 4$ , provided that the boundary of  $\Omega$  is sufficiently smooth.

A further interesting question is whether a solution departing from an initial datum which is strictly separated from the pure phases remains separated from them for *every time*. To this aim, let

$$\|\varphi_0\|_{\mathcal{C}(\overline{\Omega})} = 1 - \delta_0, \quad \text{for some } \delta_0 > 0,$$

and let us assume that  $\varphi_0$  complies with the requirements of Theorem 5.1. In this case, we know that the solution  $\varphi$  departing from  $\varphi_0$  satisfies, in particular,  $\varphi \in \mathcal{C}^\alpha(\overline{\Omega} \times [0, T])$  for every  $T > 0$ , where  $\alpha = \min\{s - \frac{1}{2}, \frac{2}{17}\}$  (cf. the embeddings (2.1), (2.5), the control (5.10) and Remark 5.3). This implies the existence of  $t_0 > 0$  such that

$$\sup_{t \in [0, t_0]} \|\varphi(t)\|_{\mathcal{C}(\overline{\Omega})} \leq 1 - \frac{\delta_0}{2}. \quad (6.8)$$

Note that  $t_0$  can be explicitly computed in terms of the norms of  $\varphi_0$  and  $\alpha$ . In space dimension  $d = 2$ , collecting (6.8) with Theorem 6.1 for  $\sigma = \frac{t_0}{2}$ , we obtain that

$$\sup_{t \geq 0} \|\varphi(t)\|_{\mathcal{C}(\overline{\Omega})} \leq 1 - \delta,$$

for some  $\delta$  depending only on  $t_0$  and  $\delta_0$ . Therefore, the solution is always separated from the pure phases. However, this is not the case in dimension  $d = 3$ , where the separation property might be lost for a transient interval of time  $(t_0, t^*)$ , with  $t^*$  is given by Theorem 6.2.

**6.4. Discussion of the longtime behavior.** The longtime behavior of the CHB system can be studied within the theory of infinite dimensional dynamical systems. In light of Theorem 3.6 and Theorem 4.1, for any  $m \in (-1, 1)$ , the CHB system generates a semigroup of operators via the rule

$$S_m(t)\varphi_0 = \varphi(t), \quad \forall t \geq 0,$$

being  $(\varphi, \mathbf{u})$  the unique global-in-time weak solution to (1.1)–(1.3) with initial condition

$$\varphi_0 \in \mathcal{H}_m = \{\varphi \in V \cap L^\infty(\Omega) : \|\varphi\|_{L^\infty(\Omega)} \leq 1, \bar{\varphi} = m\}.$$

The semigroup turns out to be strongly continuous, see [21, Proposition 6.1] for the proof, and dissipative due to (3.8). On account of (5.15), we deduce the existence of a (compact) absorbing set  $\mathcal{B}_m$  bounded in  $\mathcal{H}_m \cap W^{2,p}(\Omega)$ . Then, the existence of the unique global attractor follows by the classical semigroup theory (see, e.g., [37]).

**Theorem 6.3.** *Let  $d = 2, 3$ . Then, the dynamical system  $(S_m(t), \mathcal{H}_m)$  has a connected global attractor  $\mathcal{A}_m$ , which is bounded in  $\mathcal{H}_m \cap W^{2,p}(\Omega)$ , where  $p = 6$  if  $d = 3$  and any  $2 \leq p < \infty$  if  $d = 2$ .*

It is worth mentioning that  $\mathcal{A}_m$  turns out to be more regular at least in dimension two. Indeed, in light of Theorem 6.1 and the remarks in subsection 6.3, there exist  $\delta_m \in (0, 1)$  and  $R_m > 0$  such that

$$\mathcal{A}_m \subset \{\varphi \in H^4(\Omega) : \|\varphi\|_{H^4(\Omega)} \leq R_m \text{ and } \|\varphi\|_{C(\bar{\Omega})} \leq 1 - \delta_m\}.$$

A different viewpoint concerning the asymptotic behavior is the convergence of any single solution to a stationary state. As a byproduct of our analysis, we can state the following result.

**Theorem 6.4.** *Let  $d = 3$ . Given a weak solution  $(\varphi, \mathbf{u})$  to the CHB system, there exists a (unique)  $\varphi_\infty$  such that*

$$\varphi(t) \rightarrow \varphi_\infty \text{ in } H^2(\Omega) \text{ as } t \rightarrow \infty,$$

where  $\varphi_\infty$  is a solution to

$$\begin{cases} -\Delta\varphi_\infty + \Psi'(\varphi_\infty) = \mu_\infty, & \text{in } \Omega, \\ \partial_{\mathbf{n}}\varphi_\infty = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} \varphi_\infty \, dx = \int_{\Omega} \varphi \, dx, \end{cases}$$

with  $\mu_\infty \in \mathbb{R}$ .

The proof can be obtained by a classical argument (see, e.g., [1, 39, 41]) and is left to the reader. We mention that this relies on the energy equality, the separation property and the well-known Łojasiewicz-Simon inequality.

**Conflict of interest statement.** The authors declare that there is no conflict of interests in connection with this paper.

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APPENDIX A. REGULARITY THEORY FOR A NEUMANN PROBLEM  
WITH LOGARITHMIC NONLINEARITY

We consider the homogeneous Neumann problem with a logarithmic nonlinear term

$$\begin{cases} -\Delta u + F'(u) = f, & \text{in } \Omega, \\ \partial_{\mathbf{n}} u = 0, & \text{on } \partial\Omega, \end{cases} \quad (\text{A.1})$$

where  $F$  satisfies all the assumptions in Section 3. We present herein several regularity estimates to problem (A.1) within the framework of Sobolev spaces. The proof of most of these results are based on some ideas contained in [1, 21, 31]. We report them in detail (in a different form) for the sake of completeness and future reference.

Given  $f \in H$ , the existence of a (unique) solution to (A.1) can be proved by exploiting the convexity of  $F$ . In the sequel, we assume that  $u$  is a solution to (A.1) such that  $u \in H^2(\Omega)$  with  $F'(u) \in H$ ,  $\partial_{\mathbf{n}} u = 0$  on  $\partial\Omega$  and satisfies  $-\Delta u + F'(u) = f$  for a.e.  $x \in \Omega$ . In particular, we observe that  $\|u\|_{L^\infty(\Omega)} \leq 1$ . Next, we introduce a cutoff function which will be repeatedly used in the course of this section. For  $k \in \mathbb{N}$ , let us define the globally Lipschitz function  $h_k : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$h_k(s) = \begin{cases} -1 + \frac{1}{k}, & s < -1 + \frac{1}{k}, \\ s, & s \in [-1 + \frac{1}{k}, 1 - \frac{1}{k}], \\ 1 - \frac{1}{k}, & s > 1 - \frac{1}{k}. \end{cases} \quad (\text{A.2})$$

Then, we consider  $u_k = h_k \circ u$ . Since  $u \in V$ , the classical result on compositions in Sobolev spaces yields  $u_k \in V$  and  $\nabla u_k = \nabla u \cdot \chi_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}(u)$ , for any  $k > 0$ .

Let us start with an elliptic estimate for  $f \in L^p(\Omega)$ . A similar proof of the following result for  $p < \infty$  can be found in [1, Lemma 2].

**Lemma A.1.** *Let  $f \in L^p(\Omega)$ , where  $2 \leq p \leq \infty$ . Then, we have*

$$\|F'(u)\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

*Proof.* Let us consider  $f \in L^p(\Omega)$  with  $2 \leq p < \infty$ . We take the test function  $|F'(u_k)|^{p-2} F'(u_k)$ , which belongs to  $V$  for any  $k$ . Since  $F''(u_k)$  is well-defined and positive, we learn that

$$(-\Delta u, |F'(u_k)|^{p-2} F'(u_k)) = (p-1)(|F'(u_k)|^{p-2} F''(u_k) \nabla u \cdot \chi_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}(u), \nabla u) \geq 0.$$

Then, we deduce that

$$(F'(u), |F'(u_k)|^{p-2} F'(u_k)) \leq (f, |F'(u_k)|^{p-2} F'(u_k)).$$

Noticing that  $F'$  is increasing and  $F'(s)s \geq 0$ , we are lead to

$$\|F'(u_k)\|_{L^p(\Omega)}^p \leq (F'(u), |F'(u_k)|^{p-2} F'(u_k)).$$

By the Hölder inequality

$$(f, |F'(u_k)|^{p-2} F'(u_k)) \leq \|F'(u_k)\|_{L^p(\Omega)}^{p-1} \|f\|_{L^p(\Omega)}.$$

Therefore, we arrive at

$$\|F'(u_k)\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

By Fatou's lemma, we end up with

$$\|F'(u)\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

If  $f \in L^\infty(\Omega)$ , we infer from the above estimate that

$$\|F'(u)\|_{L^p(\Omega)} \leq C,$$

where  $C$  is independent of  $p$ . Thus, the claim follows from [2, Theorem 2.14].  $\square$

**Remark A.2.** When  $f \in L^\infty(\Omega)$ , since  $F'$  diverges at  $\pm 1$  and  $u \in C(\overline{\Omega})$ , we immediately deduce the existence of  $\delta > 0$  such that  $|u(x)| \leq 1 - \delta$ , for all  $x \in \overline{\Omega}$ .

**Lemma A.3.** *Let  $f \in V$ . Then, there exists a positive constant  $C = C(p)$  such that*

$$\|u\|_{W^{2,p}(\Omega)} + \|F'(u)\|_{L^p(\Omega)} \leq C(1 + \|f\|_V),$$

where  $p = 6$  if  $d = 3$  and for any  $p \geq 2$  if  $d = 2$ . Furthermore, if  $f \in H^k(\Omega) \cap L^\infty(\Omega)$ ,  $k = 1, 2$ , then we have

$$\|u\|_{H^{k+2}(\Omega)} \leq C(1 + \|f\|_{H^k(\Omega)}),$$

for a positive constant  $C$  depending on  $\|f\|_{L^\infty(\Omega)}$ .

*Proof.* On account of the Sobolev embeddings, an application of Lemma A.1 implies

$$\|F'(u)\|_{L^p(\Omega)} \leq C\|f\|_V,$$

where  $p = 6$  if  $d = 3$  and for any  $p \geq 2$  if  $d = 2$ . We now interpret  $u$  as the solution to  $-\Delta u + u = g$ , where  $g = f + u + F'(u)$  with homogeneous Neumann boundary condition. Then, the first estimate follows from the elliptic regularity theory of the Laplace operator. Besides, the second conclusion can be deduced in light of Remark A.2.  $\square$

**Lemma A.4.** *Let  $f \in V$ . Then, we have*

$$\|\Delta u\| \leq \|\nabla u\|^{\frac{1}{2}} \|\nabla f\|^{\frac{1}{2}}.$$

*Proof.* Testing the problem by  $-\Delta u$ , we have

$$\|\Delta u\|^2 - (F'(u_k), \Delta u) = -(f, \Delta u) + (F'(u) - F'(u_k), \Delta u).$$

Arguing as in the proof of Lemma A.1, we observe that  $-(F'(u_k), \Delta u) \geq 0$  and  $(F'(u) - F'(u_k), \Delta u) \rightarrow 0$  as  $k$  goes to  $\infty$ . Hence, by virtue of the homogeneous boundary condition, an integration by parts gives

$$\|\Delta u\|^2 \leq (\nabla f, \nabla u),$$

which, in turn, entails the claim.  $\square$

We prove a generalized version of Young's inequality.

**Lemma A.5.** *Let  $L > 0$  be given. Then, there exists  $N = N(L) > 0$  such that*

$$xye^{Ly} \leq e^{Nx-1} + \frac{1}{2}y^2e^{Ly}, \quad \forall x, y \geq 0. \quad (\text{A.3})$$

*Proof.* Let us first show that, for every  $a, b \geq 0$ ,

$$ab \leq b \ln b + e^{a-1}. \quad (\text{A.4})$$

The function  $f(b) = b \ln b + e^{a-1} - ab$  satisfies  $f(0) = e^{a-1} > 0$  and  $\lim_{b \rightarrow \infty} f(b) = \infty$ . Besides  $f'(b) = \ln b + 1 - a$ , hence  $\bar{b} = e^{a-1}$  is the absolute minimum of  $f$ . Then, we have  $f(b) \geq f(\bar{b}) = e^{a-1} \ln e^{a-1} + e^{a-1} - ae^{a-1} = 0$ , for every  $b \geq 0$ , which implies (A.4). Letting  $a = Nx$  and  $b = \frac{y}{N}e^{Ly}$  in (A.4) for any given  $N, L > 0$ , we easily find

$$\begin{aligned} xy e^{Ly} &\leq e^{Nx-1} + \frac{y}{N} e^{Ly} \left( \ln \frac{y}{N} + \ln e^{Ly} \right) \\ &\leq e^{Nx-1} + \frac{L+1}{N} y^2 e^{Ly}, \end{aligned}$$

and the claim follows with  $N > 2(L+1)$ .  $\square$

**Lemma A.6.** *Let  $d = 2$  and  $f \in V$ . Assume that there exists a positive constant  $C$  such that*

$$F''(s) \leq e^{C|F'(s)|+C}, \quad \forall s \in (-1, 1). \quad (\text{A.5})$$

*Then, for any  $p \geq 1$ , there exists a positive constant  $C = C(p)$  such that*

$$\|F''(u)\|_{L^p(\Omega)} \leq C \left( 1 + e^{C\|f\|_V^2} \right).$$

*Proof.* For  $k \in \mathbb{N}$ , let  $u_k$  be the cutoff function introduced at the beginning. Given  $L > 0$ , we consider the test function  $F'(u_k)e^{L|F'(u_k)|}$ . Since the function  $s \rightarrow F'(s)e^{L|F'(s)|}$  is monotone, by arguing as in the proof of Lemma A.1, we find

$$\int_{\Omega} |F'(u_k)|^2 e^{L|F'(u_k)|} dx \leq \int_{\Omega} |f| |F'(u_k)| e^{L|F'(u_k)|} dx.$$

We estimate the right-hand side by the generalized Young inequality (A.3) with the choice  $x = |f|$  and  $y = |F'(u_k)|$ . Accordingly, we find  $N = N(L)$  such that

$$\int_{\Omega} |f| |F'(u_k)| e^{L|F'(u_k)|} dx \leq \int_{\Omega} \frac{1}{2} |F'(u_k)|^2 e^{L|F'(u_k)|} dx + \int_{\Omega} e^{N|f|} dx,$$

and we arrive at

$$\frac{1}{2} \int_{\Omega} |F'(u_k)|^2 e^{L|F'(u_k)|} dx \leq \int_{\Omega} e^{N|f|} dx.$$

Due to the Trudinger–Moser inequality in dimension two, we have the following control

$$\frac{1}{2} \int_{\Omega} |F'(u_k)|^2 e^{L|F'(u_k)|} dx \leq C \left( 1 + e^{CN^2\|f\|_V^2} \right). \quad (\text{A.6})$$

On the other hand, by the assumption (A.5), we observe that

$$F''(s)^p \leq pC \left( 1 + |F'(s)|^2 e^{pC|F'(s)|} \right), \quad \forall s \in (-1, 1).$$

Thus, taking  $L = pC$  in (A.6), we end up with

$$\int_{\Omega} |F''(u_k)|^p dx \leq C \left( 1 + e^{C\|f\|_V^2} \right).$$

$\square$

APPENDIX B. REGULARITY THEORY FOR A STOKES PROBLEM  
WITH NONCONSTANT COEFFICIENTS

We consider the Stokes problem with nonconstant viscosity and permeability depending on a given measurable function  $\varphi$ . The system reads as

$$\begin{cases} -\operatorname{div}(\nu(\varphi)D\mathbf{u}) + \eta(\varphi)\mathbf{u} + \nabla\pi = \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \end{cases} \quad (\text{B.1})$$

where the coefficients  $\nu$  and  $\eta$  fulfil the assumptions stated in (3.1). Setting

$$B(\mathbf{u}, \mathbf{v}) = (\nu(\varphi)D\mathbf{u}, D\mathbf{v}) + (\eta(\varphi)\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}_\sigma,$$

it follows that, for every  $\mathbf{f} \in \mathbf{V}'_\sigma$ , there exists a unique solution  $\mathbf{u} \in \mathbf{V}_\sigma$  to the variational problem

$$B(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{V}_\sigma. \quad (\text{B.2})$$

We denote such a unique solution by  $\mathbf{N}_\varphi \mathbf{f}$ . Notice that

$$B(\mathbf{u}, \mathbf{N}_\varphi \mathbf{u}) = B(\mathbf{N}_\varphi \mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2, \quad \forall \mathbf{u} \in \mathbf{V}_\sigma. \quad (\text{B.3})$$

Besides, by (3.1), we have

$$2\nu_* \|\mathbf{DN}_\varphi \mathbf{u}\|^2 \leq (\nu(\varphi)D\mathbf{N}_\varphi \mathbf{u}, D\mathbf{N}_\varphi \mathbf{u}) \leq B(\mathbf{N}_\varphi \mathbf{u}, \mathbf{N}_\varphi \mathbf{u}) = (\mathbf{u}, \mathbf{N}_\varphi \mathbf{u})$$

yielding

$$\|\mathbf{DN}_\varphi \mathbf{u}\| \leq C\|\mathbf{u}\|, \quad \forall \mathbf{u} \in \mathbf{V}_\sigma. \quad (\text{B.4})$$

We report some following elliptic estimates whose proof can be easily obtained by reasoning as in [1, Sec. 4, Lemma 4].

**Lemma B.1.** *Let  $\varphi \in W^{1,r}(\Omega)$ , with  $r > d \geq 2$ , and let  $\mathbf{u} \in \mathbf{V}_\sigma$  be a weak solution to (B.1), namely  $\mathbf{u}$  solves (B.2). The following estimates hold:*

◇ Since  $\mathbf{f} \in \mathbf{V}'_\sigma$ , then

$$\|\mathbf{u}\|_{\mathbf{V}_\sigma} \leq C\|\mathbf{f}\|_{\mathbf{V}'_\sigma}.$$

◇ If  $\mathbf{f} \in \mathbf{H}$ , then there exists  $C > 0$ , depending on  $r$ , such that

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C(1 + \|\nabla\varphi\|_{\mathbf{L}^r(\Omega)})(\|\mathbf{f}\| + \|\nabla\mathbf{u}\|), \quad (\text{B.5})$$

where  $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$ . In addition, if  $\|\varphi\|_{W^{1,r}(\Omega)} \leq R$ , then there exists  $Q(R)$  such that

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leq Q(R)\|\mathbf{f}\|, \quad (\text{B.6})$$

where  $Q(\cdot)$  is a suitable positive monotone function depending on  $r$ .

◇ If  $\mathbf{f} \in \mathbf{L}^s$ , and  $\nabla\mathbf{u} \in \mathbf{L}^s(\Omega)$  with  $2 < s < \infty$ , then

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C(1 + \|\nabla\varphi\|_{\mathbf{L}^r(\Omega)})(\|\mathbf{f}\|_{\mathbf{L}^s(\Omega)} + \|\nabla\mathbf{u}\|_{\mathbf{L}^s(\Omega)}), \quad (\text{B.7})$$

where  $\frac{1}{p} = \frac{1}{s} + \frac{1}{r}$ .

◇ If  $\|\varphi\|_{W^{2,r}(\Omega)} \leq R$  and  $\mathbf{f} \in \mathbf{V}$ , then

$$\|\mathbf{u}\|_{\mathbf{H}^3(\Omega)} \leq Q(R)\|\mathbf{f}\|_{\mathbf{V}}. \quad (\text{B.8})$$

## APPENDIX C. PROOF OF THE EXISTENCE OF WEAK SOLUTIONS

This Appendix is devoted to the proof of the existence of a weak solution to problem (1.1)-(1.3) as stated in Theorem 3.6.

**The approximating problem.** Let us recall some results in [15] concerning the existence of a sequence of regular functions  $F_\lambda$  which approximate the singular potential  $F$ . More precisely, there exists a family  $F_\lambda : \mathbb{R} \rightarrow \mathbb{R}$  ( $\lambda > 0$ ) such that  $F_\lambda(0) = F'_\lambda(0) = 0$  and

- (i)  $F_\lambda$  is convex with  $F''_\lambda(s) \geq 0$  for all  $s \in \mathbb{R}$ ,
- (ii)  $F'_\lambda$  is Lipschitz on  $\mathbb{R}$  with constant  $\frac{1}{\lambda}$ ,
- (iii) there exist  $0 < \bar{\lambda} \leq 1$  and  $C > 0$  such that

$$F_\lambda(s) \geq \theta_0 s^2 - C, \quad \forall s \in \mathbb{R}, \forall \lambda \in (0, \bar{\lambda}],$$

- (iv)  $F_\lambda(s) \nearrow F(s)$ , for all  $s \in \mathbb{R}$ ,  $|F'_\lambda(s)| \nearrow |F'(s)|$  for  $s \in (-1, 1)$  and  $F'_\lambda$  converges uniformly to  $F'$  on any set  $[a, b] \subset (-1, 1)$ .

For any  $\lambda > 0$  we introduce the quadratic perturbation of  $F_\lambda$  by  $\Psi_\lambda(s) = F_\lambda(s) - \frac{\theta_0}{2}s^2$ . The corresponding regular CHB $_\lambda$  system reads as

$$\begin{cases} -\operatorname{div}(\nu(\varphi)D\mathbf{u}) + \eta(\varphi)\mathbf{u} + \nabla\pi = \mu\nabla\varphi, \\ \operatorname{div}\mathbf{u} = 0, \\ \partial_t\varphi + \mathbf{u} \cdot \nabla\varphi = \Delta\mu, \\ \mu = -\Delta\varphi + \Psi'_\lambda(\varphi), \end{cases}$$

endowed with (1.2)-(1.3). We have the following result proven in [6], whose proof is carried out by a standard Galerkin method based on the regularity of  $F_\lambda$  and energy estimates.

**Theorem C.1.** *Let  $\varphi_0 \in V$  and  $T > 0$ . Then, for any  $\lambda > 0$  the CHB $_\lambda$  problem has a weak solution  $(\varphi, \mathbf{u})$  which satisfies (3.3)-(3.4) for almost every  $t \in (0, T)$ , where  $\mu = -\Delta\varphi + \Psi'_\lambda(\varphi) \in L^2(0, T; V)$ , and  $\partial_n\varphi = 0$  almost everywhere on  $\partial\Omega \times (0, T)$ . Besides, the weak solution is such that*

$$\varphi \in \mathcal{C}([0, T], V) \cap L^2(0, T; H^3(\Omega)) \cap H^1(0, T; V'), \quad \mathbf{u} \in L^2(0, T; \mathbf{V}_\sigma).$$

**Remark C.2.** The regularity  $\varphi \in L^2(0, T; H^3(\Omega))$  is obtained by testing the equation of  $\mu$  by  $-\Delta^2\varphi$  (see [6]) and exploiting the smoothness of  $F'_\lambda$ . Nonetheless, this argument does not apply in presence of the singular potential.

**Energy estimates.** We now consider an admissible initial condition

$$\varphi_0 \in V \cap L^\infty(\Omega) \text{ with } \|\varphi_0\|_{L^\infty(\Omega)} \leq 1 \text{ and } \bar{\varphi}_0 = m \in (-1, 1). \quad (\text{C.1})$$

Let  $T > 0$  be given. For any  $\lambda \in (0, \bar{\lambda}]$  with  $\bar{\lambda}$  as in (iii), we denote by  $(\varphi, \mathbf{u})$  a solution to CHB $_\lambda$  on  $[0, T]$  departing from  $\varphi_0$  (we omit the dependence on  $\lambda$  for simplicity). The associated energy is given by

$$\mathcal{E}_\lambda(\varphi) = \frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega \Psi_\lambda(\varphi) \, dx.$$

We proceed by proving some energy estimates that are uniform with respect to  $\lambda$ . Accordingly, in what follows, the generic positive constant  $C$  may depend on  $\mathcal{E}(\varphi_0)$ ,  $m$  and  $T$ , but it is independent of  $\lambda \in (0, 1)$ .

We claim that

$$\|\varphi\|_{L^\infty(0,T;V)} + \|\nabla\mu\|_{L^2(0,T;H)} + \|\nabla\mathbf{u}\|_{L^2(0,T;H)} \leq C. \quad (\text{C.2})$$

Indeed, taking  $\mathbf{v} = \mathbf{u}$  in (3.3) and  $v = \mu$  in (3.4), and summing up the resulting equalities, we have

$$\frac{d}{dt}\mathcal{E}_\lambda(\varphi) + \|\nabla\mu\|^2 + (\nu(\varphi)D\mathbf{u}, D\mathbf{u}) + (\eta(\varphi)\mathbf{u}, \mathbf{u}) = 0.$$

In light of (3.1) and the Korn inequality, we obtain

$$\frac{d}{dt}\mathcal{E}_\lambda(\varphi) + \|\nabla\mu\|^2 + \nu_*\|\nabla\mathbf{u}\|^2 \leq 0.$$

Due to property (iv), we observe that  $F_\lambda(s) \leq F(s) \leq C$ , for all  $s \in [-1, 1]$ , which, in turn, entails  $\mathcal{E}_\lambda(\varphi_0) \leq \mathcal{E}(\varphi_0)$ . Then, an integration in time of the above differential inequality yields

$$\sup_{0 \leq t \leq T} \mathcal{E}_\lambda(\varphi(t)) + \int_0^T \|\nabla\mu(\tau)\|^2 + \|\nabla\mathbf{u}(\tau)\|^2 d\tau \leq \mathcal{E}(\varphi_0).$$

Owing to (iii), we have

$$\frac{1}{2}\|\nabla\varphi\|^2 \leq \mathcal{E}_\lambda(\varphi) + C,$$

for every  $\lambda \in (0, \bar{\lambda}]$ . Therefore, by the mass conservation, we infer that

$$\|\varphi\|_{L^\infty(0,T;V)} \leq C,$$

which completes the proof of (C.2). Now, exploiting (C.2), we have

$$\|\partial_t\varphi\|_{V'} \leq C(\|\nabla\mathbf{u}\| + \|\nabla\mu\|),$$

which implies

$$\|\partial_t\varphi\|_{L^2(0,T;V')} \leq C.$$

We proceed by testing  $\mu$  by  $-\Delta\varphi$ . By integrating by parts, we get

$$\|\Delta\varphi\|^2 + (\Psi_\lambda''(\varphi)\nabla\varphi, \nabla\varphi) = (\nabla\mu, \nabla\varphi).$$

Observe that  $\Psi_\lambda''(s) \geq -\theta_0$ , for all  $s \in \mathbb{R}$  (cf. (i)). Using (C.2), we thus find

$$\|\Delta\varphi\|^2 \leq C(1 + \|\nabla\mu\|),$$

which, in turn, gives us

$$\|\Delta\varphi\|_{L^4(0,T;H)} \leq C.$$

The next goal is to control the total norm of  $\mu$  in  $V$ . Notice that  $\bar{\mu} = \overline{F'_\lambda(\varphi)} - \theta_0\bar{\varphi}$ . We recall that there exists  $C = C(m) > 0$ , independent of  $\lambda \in (0, \bar{\lambda}]$ , such that

$$\int_\Omega |F'_\lambda(\varphi)| dx \leq C \left| \int_\Omega F'_\lambda(\varphi)(\varphi - \bar{\varphi}) dx \right| + C,$$

where  $C$  diverges to  $+\infty$  as  $|m| \rightarrow 1$  (see [15] for the proof). In order to estimate the right-hand side, we multiply  $\mu$  by  $\varphi - \bar{\varphi}$  and integrate by parts. Using (C.2), we find

$$\begin{aligned} \|\nabla\varphi\|^2 + (F'_\lambda(\varphi), \varphi - \bar{\varphi}) &= \theta_0(\varphi, \varphi - \bar{\varphi}) + (\mu - \bar{\mu}, \varphi - \bar{\varphi}) \\ &\leq C(1 + \|\nabla\mu\|). \end{aligned} \quad (\text{C.3})$$

Combining the above inequalities, we arrive at

$$\|\mu\|_{L^2(0,T;V)} \leq C.$$

Finally, we infer by comparison from the definition of  $\mu$  that

$$\|F'_\lambda(\varphi)\|_{L^2(0,T;H)} \leq C. \quad (\text{C.4})$$

**Existence of a weak solution to the CHB system.** We now consider  $(\varphi_\lambda, \mathbf{u}_\lambda)$ ,  $\lambda \in (0, \bar{\lambda}]$ , a family of solutions to  $\text{CHB}_\lambda$  departing from  $\varphi_0$  as in (C.1). In light of the above uniform estimates we can pass to the limit  $\lambda \rightarrow 0$  with the following convergences (up to subsequences)

$$\begin{aligned} \varphi_\lambda &\rightarrow \varphi \text{ weakly star in } L^\infty(0, T; V), \\ \varphi_\lambda &\rightarrow \varphi \text{ weakly in } L^4(0, T; H^2(\Omega)), \\ \partial_t \varphi_\lambda &\rightarrow \partial_t \varphi \text{ weakly in } L^2(0, T; V'), \\ \mu_\lambda &\rightarrow \mu \text{ weakly in } L^2(0, T; V), \\ \mathbf{u}_\lambda &\rightarrow \mathbf{u} \text{ weakly in } L^2(0, T; \mathbf{V}_\sigma). \end{aligned}$$

By the classical Aubin-Lions Theorem (cf. [9, Theorem II.5.16]) , we also deduce that

$$\varphi_\lambda \rightarrow \varphi \text{ strongly in } L^2(0, T; V) \cap \mathcal{C}([0, T], H),$$

and

$$\varphi_\lambda(x, t) \rightarrow \varphi(x, t) \quad \text{a.e. } (x, t) \text{ in } \Omega \times (0, T).$$

We claim that the limit pair  $(\varphi, \mathbf{u})$  is a weak solution according to Definition 3.2. Indeed, the required regularity of  $(\varphi, \mathbf{u})$  immediately follows by the above convergences. Next, following a standard argument, for any fixed  $\eta \in (0, 1/2)$  we introduce the set

$$E_\eta^\lambda = \{(x, t) \in \Omega \times [0, T] : |\varphi_\lambda(x, t)| > 1 - \eta\}.$$

It is easy to see from (C.4) that

$$|E_\eta^\lambda| \leq \frac{C}{\min\{F'_\lambda(1 - \eta), |F'_\lambda(-1 + \eta)|\}}.$$

Hence, passing to the limit as  $\lambda \rightarrow 0$  and then letting  $\eta \rightarrow 0$ , we conclude

$$|\{(x, t) \in \Omega \times (0, T) : |\varphi(x, t)| \geq 1\}| = 0,$$

meaning that  $\varphi \in L^\infty(\Omega \times (0, T))$  with  $|\varphi(x, t)| < 1$  for almost every  $(x, t) \in \Omega \times (0, T)$ . Regarding the nonlinear potential, using the pointwise convergence of  $\varphi_\lambda$  and the uniform convergence of  $F'_\lambda$  to  $F'$  on any compact set in  $(-1, 1)$ , we infer that  $F'_\lambda(\varphi_\lambda) \rightarrow F'(\varphi)$ , for almost every  $(x, t) \in \Omega \times (0, T)$ . Then, in light of (C.4), a weak form of the Lebesgue convergence theorem implies that  $F'_\lambda(\varphi_\lambda) \rightarrow F'(\varphi)$  weakly in  $L^2(0, T; H)$ , which allows us to identify  $\mu = -\Delta\varphi + \Psi'(\varphi) \in L^2(0, T; V)$ . In a standard way, we pass to the limit in the weak formulation of  $\text{CHB}_\lambda$  proving the validity of (3.3)-(3.4). Finally, it is easily verified that  $\partial_n \varphi = 0$  for almost every  $(x, t) \in \partial\Omega \times (0, T)$ .

**Remark C.3.** We observe that, letting  $\nu \rightarrow 0$ , it is easily deduced the existence of a weak solution to the Cahn–Hilliard–Hele–Shaw system with unmatched viscosities (in this context  $\eta$  plays the role of the viscosity). Nonetheless, the velocity field  $\mathbf{u}$  belongs only to  $L^2(0, T; H)$ .

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