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ON THE CAHN-HILLIARD-BRINKMAN SYSTEM WITH SINGULAR POTENTIAL AND NONCONSTANT VISCOSITY

MONICA CONTI, ANDREA GIORGINI

Abstract. The Cahn–Hilliard–Brinkman system has been recently proposed as a diffuse interface model for the phase separation of incompressible binary fluids in porous media. It consists of a Brinkman–Darcy equation governing the fluid velocity, nonlinearly coupled with a convective Cahn–Hilliard equation for the relative difference of the fluid concentrations. We prove existence and uniqueness of finite energy solutions in two space dimensions for a class of physically relevant and singular free energy density, in the case of concentration-dependent viscosity. Then, we discuss their regularization properties in finite time and we establish the strict separation property from the pure states.

Keywords: Diffuse interface models, Cahn–Hilliard equation, Darcy's law, singular potential, strict separation property.

MSC 2010: 35B65, 35Q35, 76D27, 76S05.

1. Introduction

The Cahn–Hilliard–Brinkman (CHB) system is a diffuse interface model describing the phase separation of binary immiscible and incompressible fluids in porous media (see [25, 28]). The model couples a modified Darcy's equation introduced by Brinkman in [8], which governes the volume-averaged fluid velocity $\boldsymbol{u}$, with a convective Cahn–Hilliard equation for the phase-field φ. This is the relative difference of the fluid concentrations, and takes value between -1 and 1, the extremals $\varphi = \pm 1$ representing the pure phases. Assuming that the binary fluid has constant mass density and occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the system reads as follows

$$\begin{cases} -\nabla \cdot (\nu D(\boldsymbol{u})) + \eta \boldsymbol{u} + \nabla p = \gamma \mu \nabla \varphi, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \varphi_t + \nabla \cdot (\varphi \boldsymbol{u}) = \nabla \cdot (m \nabla \mu), \\ \mu = -\varepsilon \Delta \varphi + \frac{1}{\varepsilon} \Psi'(\varphi), \end{cases} \qquad \text{in } \Omega \times (0, T),$$

completed with a nonslip boundary condition for $\boldsymbol{u}$ and homogeneous Neumann boundary conditions for the chemical potential μ and for φ. The latter ones entail that there is no mass flux and the interface separating the two fluids is orthogonal to the boundary. In the model, $2D(\boldsymbol{u}) = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{tr}}$ is the symmetric gradient and p denotes the fluid pressure. Besides, $\nu \geq 0$ is the (effective) viscosity, $\eta \geq 0$ the permeability, $\gamma > 0$ a surface tension parameter, $m \geq 0$ stands for the mobility and $\varepsilon > 0$ is related to the thickness of the interface separating the two fluids. The chemical potential μ is the variational derivative

of the Ginzburg–Landau free energy

$$\mathcal{E}(\varphi) = \int_\Omega \Big(\frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \Psi(\varphi) \Big) \, \mathrm{d}x,$$

where, in order to describe the phase segregation of an immiscible material (see [4, 9]), the potential Ψ has a double-well form in $[-1, 1]$. The two global minima are the equilibrium configurations, while the non-convex interval, called spinodal region, generates anti-diffusion and prevents a complete mixing of the fluids. The physically relevant homogeneous free energy Ψ firstly introduced by Cahn and Hilliard in [9] is

$$\Psi(s) = \frac{\Theta}{2} \Big((1+s) \ln(1+s) + (1-s) \ln(1-s) \Big) - \frac{\Theta_0}{2} s^2. \tag{1.1}$$

In this context, Θ is the absolute temperature of the mixture and Θ_0 is the so-called critical temperature. Since the phase separation takes place when the mixture undergoes a rapid cooling below the critical temperature, the parameters satisfy the physical relation

$$0 < \Theta < \Theta_0,$$

which induces the double-well form of the potential in (1.1). Besides, the equilibrium configurations are $\varphi_A = -\varphi_B = \beta$, where β is the positive root of

$$\ln \Big(\frac{1+\beta}{1-\beta} \Big) = \frac{2\Theta_0}{\Theta} \beta.$$

In the classical literature the singular potential Ψ has been often approximated by a quartic order polynomial defined on $\mathbb{R}$ as

$$\Psi_0(s) = \frac{\kappa}{4} (s^2 - \beta^2)^2, \tag{1.2}$$

where $\kappa > 0$ is related to Θ and Θ_0. This approximation is broadly justified in the regime Θ close to Θ_0. Nonetheless, the analysis replacing Ψ with Ψ_0 has a major drawback being impossible to ensure that the phase-field φ takes value in the physical reasonable interval between -1 and 1.

As anticipated, the CHB system belongs to a class of diffuse interface models which are employed to describe the behavior of multi-phase fluids, see [16] for a recent review. In the same direction, the Cahn–Hilliard–Navier–Stokes (CHNS) system has been investigated in several papers (see, e.g., [1, 6, 12, 20, 22]) as well as the Cahn–Hilliard–Hele–Shaw (CHHS) system (see [13, 17, 21, 30, 31] and the references therein). The latter is based on the classical Darcy's law, while the Brinkman equation involved in the derivation of CHB constitutes a relaxation via the term $\nabla \cdot (\nu D(\boldsymbol{u}))$. Accordingly, letting the viscosity approach zero, we recover the CHHS system. So far, the CHB model has been always studied with a regular potential of the form Ψ_0. In particular, subject to the aforementioned boundary conditions, the well-posedness and the longtime behavior are discussed in [5], setting ν, η and m as positive constants. The finite dimensionality of the attractor has been proved in [19]. Similar results have been obtained in [32] taking into account a dynamic boundary condition for φ. Finally, the case of nonconstant ν, η and m has been analyzed only from the numerical viewpoint in [10].

In this work we extend the mathematical study of the CHB system considering the physically relevant singular free energy Ψ. The second purpose is to understand the

role of the effective viscosity ν depending on φ in the Brinkman's law. Indeed, recent investigations suggest that such a dependence on the phase-field may have important effects on the morphology of the phase separation (see, e.g. [15, 26]). The proposed form of the local viscosity in a binary mixture is a linear combination of the individual components

$$\nu(s) = \nu_A \frac{1+s}{2} + \nu_B \frac{1-s}{2}, \tag{1.3}$$

where ν_A and ν_B are the positive viscosities of the two fluids. Note that $\nu(s)$ is strictly positive and satisfies

$$\nu(s) \geq \min\{\nu_A, \nu_B\}.$$

Hence, we assume hereafter that η and m are constant, addressing the reader to Section 8 for further comments. Taking for simplicity $\eta = m = 1$ as well as the physical parameters $\gamma = \varepsilon = 1$, the adimensional CHB system under investigation is

$$\begin{cases} -\nabla \cdot (\nu(\varphi) D(\boldsymbol{u})) + \boldsymbol{u} + \nabla p = \mu \nabla \varphi, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \varphi_t + \nabla \cdot (\varphi \boldsymbol{u}) = \Delta \mu, \\ \mu = -\Delta \varphi + \Psi'(\varphi), \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{1.4}$$

subject to the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}, \ \partial_{\boldsymbol{n}} \varphi = \partial_{\boldsymbol{n}} \mu = 0, \qquad \text{on } \partial\Omega \times (0,T), \tag{1.5}$$

being $\boldsymbol{n}$ the exterior normal on $\partial\Omega$, and endowed with the initial condition

$$\varphi(0) = \varphi_0, \quad \text{in } \Omega. \tag{1.6}$$

We present here a complete mathematical analysis of well-posedness and regularity for the system (1.4)–(1.6) focusing on the space dimension $d = 2$. Indeed, the existence result deals with finite energy solutions satisfying the equality

$$\mathcal{E}(\varphi(t)) + \int_s^t \left(\|\nabla \mu(\tau)\|^2 + \|\sqrt{\nu(\varphi(\tau))} D\boldsymbol{u}(\tau)\|^2 + \|\boldsymbol{u}(\tau)\|^2 \right) \mathrm{d}\tau = \mathcal{E}(\varphi(s))$$

for all $0 \leq s < t < \infty$. On the other hand, due to the presence of $\nu(\varphi)$, the regularity of the velocity field $\boldsymbol{u}$ obtained by the energy method is not enough to guarantee the uniqueness. To overcome this obstacle, we successfully gain higher regularity properties for $\boldsymbol{u}$ by studying the Stokes problem with variable viscosity. In particular, in dimension two we find

$$\int_t^{t+1} \left(\|\boldsymbol{u}(\tau)\|^4_{W^{1,2}(\Omega)} + \|\boldsymbol{u}(\tau)\|^2_{W^{1,3}(\Omega)} \right) \mathrm{d}\tau \leq C, \quad \forall\, t \geq 0,$$

which is the key tool to obtain a continuous dependence estimate and the instantaneous regularization of φ. Next, our task is proving the so-called strict separation property, namely, if the initial datum is not a pure state (i.e. $\varphi_0 \neq \pm 1$), then φ remains uniformly away from the pure states over time. More precisely, we show that

$$\forall\, \sigma > 0, \quad \exists\, \delta > 0 : \quad \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall\, t \geq \sigma. \tag{1.7}$$

Besides its deep physical meaning, such a property has an important consequence from the mathematical point of view. Indeed, on any time interval $[\sigma, \infty)$, the singularities of

Ψ and its derivatives play no longer any role and the original system can be treated as a model with regular potential having smooth classical solutions. The property (1.7) has been proven in dimension two for the Cahn–Hilliard equation in [23] and then recently generalized for the Cahn–Hilliard–Oono equation in [14]. Indeed, our proof of the separation property relies on the technique introduced in [14] with the aim of showing, through several regularization steps in finite time, the uniform estimate

$$\|\Psi'(\varphi(t))\|_{L^\infty(\Omega)} \le C, \quad \forall\, t \ge \sigma.$$

In its generality, the instantaneous separation property (1.7) is still an open issue in dimension three even for the solely Cahn–Hilliard equation.

Plan of the paper. In Section 2 we introduce the notation and the general assumptions on the viscosity and on the potential. In Section 3 we discuss the existence result. Section 4 is devoted to prove additional regularity properties of the finite energy solutions, which are crucial in Section 5 in order to guarantee the uniqueness and a continuous dependence estimate. In Section 6 we show that the system regularizes in finite time, entailing that any finite energy solution is indeed strong on $(0,\infty)$. The main result on the strict separation property is stated and proved in Section 7. In the final Section 8 we collect some comments and possible future investigations.

2. The Mathematical Setting

Let Ω be a connected bounded domain in $\mathbb{R}^2$ with smooth boundary. For any positive integer r, let $W^{r,p}(\Omega)$ be the Sobolev space of functions in $L^p(\Omega)$ with distributional derivative of order less or equal to r in $L^p(\Omega)$ and denote by $\|\cdot\|_{W^{r,p}(\Omega)}$ its norm. In particular, $H^r(\Omega) = W^{r,2}(\Omega)$ is a Hilbert space with respect to the inner product $\langle u, v\rangle_r = \sum_{|\kappa|\le r} \int_\Omega D^\kappa u(x) D^\kappa v(x)\,\mathrm{d}x$ and the induced norm $\|u\|_r = \sqrt{\langle u, u\rangle_r}$.

As customary, we let $H = L^2(\Omega)$ and we denote its inner product by $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. We also set $V = H^1(\Omega)$ equipped with the norm

$$\|u\|_V^2 = \|\nabla u\|^2 + \|u\|^2,$$

and we indicate by V' the dual space of V, by $\|\cdot\|_{V'}$ its norm and simply by $\langle\cdot,\cdot\rangle$ the duality product $\langle\cdot,\cdot\rangle_{V',V}$. Let us recall that by the Poincaré-Wirtinger inequality

$$\|u\|_V \le C\left(\|\nabla u\| + \left|\int_\Omega u\,\mathrm{d}x\right|\right), \quad \forall\, u \in V. \tag{2.1}$$

Besides, in dimension $d = 2$, we have the Ladyzhenskaya inequality

$$\|u\|_{L^4(\Omega)} \le C\|u\|_V^{1/2}\|u\|^{1/2}. \tag{2.2}$$

Denoting by $\overline{f}$ the average of f over Ω, i.e.

$$\overline{f} := \frac{1}{|\Omega|}\langle f, 1\rangle, \quad \forall\, f \in V',$$

we introduce the space of zero-mean functions and its dual space

$$V_0 = \{v \in V : \overline{v} = 0\}, \quad V_0' = \left\{f \in V' : \overline{f} = 0\right\}.$$

We then consider the operator $A \in \mathcal{L}(V, V')$ defined by

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \forall\, u, v \in V.$$

Since the restriction of A in V_0 is an isomorphism from V_0 onto V_0', we define the inverse map $\mathcal{N} : V_0' \to V_0$. It is well known that for all $f \in V_0'$, $\mathcal{N} f$ is the unique $u \in V_0$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \langle f, v\rangle, \quad \forall\, v \in V.$$

On account of the above definitions, the following properties hold

$$\begin{aligned}
&\langle Au, \mathcal{N} f\rangle = \langle f, u\rangle, \quad \forall\, u \in V,\ \forall\, f \in V_0',\\
&\langle f, \mathcal{N} g\rangle = \langle g, \mathcal{N} f\rangle = \int_\Omega \nabla(\mathcal{N} f) \cdot \nabla(\mathcal{N} g)\mathrm{d}x, \quad \forall\, f, g \in V_0'.
\end{aligned}$$

Moreover, it is straightforward to prove that

$$\|f\|_* := \|\nabla \mathcal{N} f\| = \langle f, \mathcal{N} f\rangle^{1/2}$$

is an equivalent norm in V_0', and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_*^2 = \langle u_t(t), \mathcal{N} u(t)\rangle, \quad \text{for a.e. } t \in (0, T),\ \forall\, u \in H^1(0, T; V_0').$$

Note that

$$\|f\|_{-1}^2 := \|f - \overline{f}\|_*^2 + |\overline{f}|^2$$

is an equivalent norm in V'.

Next, to handle the velocity field $\boldsymbol{u}$, we introduce the solenoidal Hilbert space

$$\boldsymbol{H} = \left\{\boldsymbol{u} \in [L^2(\Omega)]^2 : \operatorname{div} \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \boldsymbol{n}\,|_{\partial\Omega} = 0\right\}.$$

In the sequel, we denote by $\langle \cdot, \cdot\rangle$ and $\|\cdot\|$ also the norm and the inner product, respectively, in $\boldsymbol{H}$. Then, we define the Hilbert space

$$\boldsymbol{V} = \left\{\boldsymbol{u} \in [H^1(\Omega)]^2 : \operatorname{div} \boldsymbol{u} = 0, \boldsymbol{u}\,|_{\partial\Omega} = 0\right\}$$

with inner product and norm

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{V}} = \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v}\rangle, \quad \|\boldsymbol{u}\|_{\boldsymbol{V}} = \|\nabla \boldsymbol{u}\|.$$

Let us recall that the Korn inequality gives

$$\|\nabla \boldsymbol{u}\|^2 \le 2\|D(\boldsymbol{u})\|^2 \le 2\|\nabla \boldsymbol{u}\|^2, \quad \forall\, \boldsymbol{u} \in \boldsymbol{V}.$$

In turn, $\|D(\boldsymbol{u})\|$ is an equivalent norm in $\boldsymbol{V}$. For easier notation, we simply denote by $\|\boldsymbol{u}\|_{W^{r,p}(\Omega)}$ the norm of the vector field in $[W^{r,p}(\Omega)]^2$.

Finally, we state and prove a generalized Young inequality.

Lemma 2.1. *Let $L > 0$ be given. Then, there exists $N = N(L) > 0$ such that*

$$xy\mathrm{e}^{Ly} \le \mathrm{e}^{Nx-1} + \frac{1}{2}y^2\mathrm{e}^{Ly}, \quad \forall\, x, y \ge 0. \tag{2.3}$$

Proof. Let us first show that, for every $a, b \geq 0$,

$$ab \leq b \ln b + \mathrm{e}^{a-1}. \tag{2.4}$$

Indeed, the function $f(b) = b \ln b + \mathrm{e}^{a-1} - ab$ satisfies $f(0) = \mathrm{e}^{a-1} > 0$ and $\lim_{b\to\infty} f(b) = \infty$. Besides $f'(b) = \ln b + 1 - a$, hence $\bar{b} = \mathrm{e}^{a-1}$ is the absolute minimum of f:

$$f(b) \geq f(\bar{b}) = \mathrm{e}^{a-1} \ln \mathrm{e}^{a-1} + \mathrm{e}^{a-1} - a\mathrm{e}^{a-1} = 0$$

for every $b \geq 0$, proving (2.4). Letting $a = Nx$ and $b = \frac{y}{N}\mathrm{e}^{Ly}$ in (2.4) for any given $N > 1, L > 0$, we easily find

$$\begin{aligned} xy\mathrm{e}^{Ly} &\leq \mathrm{e}^{Nx-1} + \frac{y}{N}\mathrm{e}^{Ly}\left(\ln \frac{y}{N} + \ln \mathrm{e}^{Ly}\right) \\ &\leq \mathrm{e}^{Nx-1} + \frac{L+1}{N} y^2 \mathrm{e}^{Ly}, \end{aligned}$$

and the thesis follows with $N > 2(L+1)$. □

Throughout the paper, $C > 0$ will stand for a generic constant which may be estimated by the parameters of the system and whose value may change even within the same line of a given equation.

3. Existence of Finite Energy Solutions

Let us state the main assumptions of this work. We require that the viscosity $\nu = \nu(s)$ is a bounded function satisfying $\nu \in \mathcal{C}^2(\mathbb{R})$ and

$$\nu(s) \geq 2\nu_1 > 0, \quad \forall\, s \in \mathbb{R}. \tag{3.1}$$

Next, we assume that Ψ is a quadratic perturbation of a singular (strictly) convex function in $[-1, 1]$, namely

$$\Psi(s) = F(s) - \frac{\Theta_0}{2} s^2,$$

where $F \in \mathcal{C}([-1, 1]) \cap \mathcal{C}^2(-1, 1)$ fulfils

$$\lim_{s\to -1} F'(s) = -\infty, \quad \lim_{s\to 1} F'(s) = +\infty,$$

and there exists $\Theta > 0$ such that

$$F''(s) \geq \Theta \quad \forall\, s \in (-1, 1). \tag{3.2}$$

Here, we study the physical case of the double well (singular) potential, namely we assume

$$\alpha := \Theta_0 - \Theta > 0.$$

We also extend $F(s) = +\infty$ for any $s \notin [-1, 1]$. Note that the above assumptions imply that there exists $s_0 \in (-1, 1)$ such that $F'(s_0) = 0$. Without loss of generality, we assume that $s_0 = 0$ and that $F(s_0) = 0$ as well. In particular, this entails that $F(s) \geq 0$ for all $s \in [-1, 1]$.

Remark 3.1. The assumptions are satisfied and motivated, in particular, by the logarithmic potential mentioned in the Introduction

$$F(s) = \frac{\Theta}{2}\Big((1+s)\log(1+s) + (1-s)\log(1-s)\Big).$$

In addition, the model viscosity (1.3) complies with (3.1) on the interval $[-1, 1]$.

In the sequel, we will always assume that all the hypothesis stated above are in place.

Definition 3.2. Let $\varphi_0 \in V$ with $\Psi(\varphi_0) \in L^1(\Omega)$ and $T > 0$ be given. A pair $(\varphi, \boldsymbol{u})$ is a finite energy solution to the CHB system (1.4)–(1.6) on $[0, T]$ if

$$\begin{aligned}
&\varphi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^4(0,T;H^2(\Omega)),\\
&\varphi \in L^\infty(\Omega \times (0,T)) \quad \text{with} \quad |\varphi(x,t)| < 1 \quad \text{a.e. } (x,t) \in \Omega \times (0,T),\\
&\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}),
\end{aligned}$$

and

$$\langle \nu(\varphi(t)) D\boldsymbol{u}(t), D\boldsymbol{v}\rangle + \langle \boldsymbol{u}(t), \boldsymbol{v}\rangle = \langle \mu(t)\nabla\varphi(t), \boldsymbol{v}\rangle, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \text{ a.e. } t \in [0,T], \tag{3.3}$$

$$\langle \varphi_t(t), w\rangle + \langle \nabla\cdot(\varphi(t)\boldsymbol{u}(t)), w\rangle + \langle \nabla\mu(t), \nabla w\rangle = 0, \quad \forall\, w \in V, \text{ a.e. } t \in [0,T], \tag{3.4}$$

with $\mu \in L^2(0,T;V)$ given by

$$\mu = -\Delta\varphi + \Psi'(\varphi). \tag{3.5}$$

Moreover, $\partial_{\boldsymbol{n}}\varphi = 0$ a.e. on $\partial\Omega \times (0,T)$ and $\varphi|_{t=0} = \varphi_0$ a.e. in Ω.

Remark 3.3. It is straightforward to observe that any solution satisfies the mass conservation property, namely,

$$\overline{\varphi}(t) = \overline{\varphi}_0, \quad \forall\, t \geq 0.$$

Moreover, its energy

$$\mathcal{E}(\varphi(t)) = \frac{1}{2}\|\nabla\varphi(t)\|^2 + \int_\Omega \Psi(\varphi(t))\,\mathrm{d}x$$

is finite for almost every $t \geq 0$.

Remark 3.4. Note that equation (3.3) is equivalent to

$$\langle \nu(\varphi) D\boldsymbol{u}, D\boldsymbol{v}\rangle + \langle \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \nabla\varphi \otimes \nabla\varphi, \nabla\boldsymbol{v}\rangle, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},$$

in light of the equality

$$\mu\nabla\varphi = \nabla\Big(\frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi)\Big) - \operatorname{div}(\nabla\varphi \otimes \nabla\varphi).$$

Remark 3.5. As customary, the pressure term is dropped in the weak formulation of the Brinkman's law. Indeed, the pressure can be recovered (up to a constant) thanks to the classical de Rham's theorem (see, for instance, [7]). In particular, since

$$S = \nabla\cdot(\nu(\varphi)D\boldsymbol{u}) - \boldsymbol{u} - \mu\nabla\varphi$$

is orthogonal (in the dual sense) to any element of $\boldsymbol{V}$, then there exists a function $p \in L^2(0,T;H)$ satisfying $\nabla p = S$.

We state our existence result.

Theorem 3.6. *Let $\varphi_0 \in V$ with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, there exists a global finite energy solution $(\varphi, \boldsymbol{u})$ to the CHB problem such that*

$$\varphi \in \mathcal{C}([0,\infty), V),$$

and the energy identity

$$\mathcal{E}(\varphi(t)) + \int_s^t \Big(\|\nabla \mu(\tau)\|^2 + \|\sqrt{\nu(\varphi(\tau))} D\boldsymbol{u}(\tau)\|^2 + \|\boldsymbol{u}(\tau)\|^2 \Big) \, \mathrm{d}\tau = \mathcal{E}(\varphi(s))$$

is satisfied for all $0 \le s < t < \infty$. Furthermore, we have the dissipative estimates

$$\mathcal{E}(\varphi(t)) + \int_t^{t+1} \Big(\|\nabla \mu(\tau)\|^2 + \|\nabla \boldsymbol{u}(\tau)\|^2 \Big) \, \mathrm{d}\tau \le C\mathcal{E}(\varphi_0)\mathrm{e}^{-\omega t} + C, \tag{3.6}$$

and

$$\int_t^{t+1} \Big(\|\varphi(\tau)\|^4_{H^2(\Omega)} + |\overline{\mu}(\tau)|^2 \Big) \, \mathrm{d}\tau \le C\mathcal{E}(\varphi_0)^2 \mathrm{e}^{-2\omega t} + C \tag{3.7}$$

for every $t \ge 0$, where ω and C are positive constants independent of the initial datum.

The rest of the section is devoted to the proof of Theorem 3.6, which is obtained via an approximation procedure and energy estimates.

3.1. **Approximation of the singular potential.** Let us recall some results in [11], concerning the existence of a sequence of regular functions F_λ which approximate the singular potential F. Precisely, for every $\lambda > 0$ there exists

$$F_\lambda : \mathbb{R} \to \mathbb{R}$$

such that

(i) F_λ is convex and $F_\lambda(s) \nearrow F(s)$, for all $s \in \mathbb{R}$, as $\lambda \to 0$.

(ii) F'_λ is Lipschitz on $\mathbb{R}$ with constant $\frac{1}{\lambda}$ and $F''_\lambda(s)$ is nonnegative for all $s \in \mathbb{R}$.

(iii) $|F'_\lambda(s)| \nearrow |F'(s)|$ for $s \in (-1,1)$ and F'_λ converges uniformly to F' on any interval $[a,b] \subset (-1,1)$.

(iv) $F_\lambda(0) = F'_\lambda(0) = 0$, for all $\lambda > 0$.

Furthermore, we have the following uniform property with respect to λ.

Lemma 3.7. *For any $0 < \overline{\lambda} \le 1$, there exists $C > 0$ such that*

$$F_\lambda(s) \ge \frac{1}{4\overline{\lambda}} s^2 - C, \quad \forall\, s \in \mathbb{R},\ \forall\, \lambda \in (0, \overline{\lambda}].$$

3.2. **The approximating problems.** For any $\lambda \in (0,1)$ fixed, we introduce the quadratic perturbation of F_λ by

$$\Psi_\lambda(s) = F_\lambda(s) - \frac{\Theta_0}{2}s^2.$$

The corresponding regular CHB_λ problem reads as

$$\begin{cases} -\nabla\cdot(\nu(\varphi)D\boldsymbol{u}) + \boldsymbol{u} + \nabla p = \mu\nabla\varphi, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \varphi_t + \nabla\cdot(\varphi\boldsymbol{u}) = \Delta\mu, \\ \mu = -\Delta\varphi + \Psi_\lambda'(\varphi), \end{cases}$$

endowed with (1.5)-(1.6). Analogously to the singular case, given any $\varphi_0 \in V$, a pair $(\varphi, \boldsymbol{u})$ is a solution of CHB_λ on $[0,T]$ if

$$\begin{aligned} &\varphi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \\ &\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}), \end{aligned}$$

and

$$\langle \nu(\varphi(t))D\boldsymbol{u}(t), D\boldsymbol{v}\rangle + \langle \boldsymbol{u}(t), \boldsymbol{v}\rangle = \langle \mu(t)\nabla\varphi(t), \boldsymbol{v}\rangle, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \text{ a.e. } t \in [0,T], \tag{3.8}$$

$$\langle \varphi_t(t), w\rangle + \langle \nabla\cdot(\varphi(t)\boldsymbol{u}(t)), w\rangle + \langle \nabla\mu(t), \nabla w\rangle = 0, \quad \forall\, w \in V, \text{ a.e. } t \in [0,T], \tag{3.9}$$

with

$$\mu = -\Delta\varphi + \Psi_\lambda'(\varphi) \in L^2(0,T;V).$$

The Cahn–Hilliard–Brinkman system in presence of a regular potential with polynomial growth, satisfying suitable dissipation assumptions, has been studied in [5]. Indeed, we have

Theorem 3.8. *Let $\varphi_0 \in V$. Then, the CHB_λ problem has a solution $(\varphi, \boldsymbol{u})$ such that*

$$\varphi \in \mathcal{C}([0,T],V) \cap L^2(0,T;H^3(\Omega)).$$

The proof of Theorem 3.8 is carried out by a standard Galerkin method, based on the Lipschitz regularity of F_λ and on energy estimates, much more immediate than the uniform ones detailed below.

3.3. **Energy estimates.** Let $\lambda \in (0,1)$ be fixed. We denote the energy of a solution to CHB_λ by

$$\mathcal{E}_\lambda(\varphi) = \frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega \Psi_\lambda(\varphi)\,\mathrm{d}x.$$

In what follows, the generic positive constant C is independent of λ and of the initial datum.

Lemma 3.9. *There exist $\overline{\lambda} > 0$ and $\omega > 0$ such that, for any $\varphi_0 \in V$ and any $0 < \lambda \le \overline{\lambda}$, we have the dissipative estimates*

$$\mathcal{E}_\lambda(\varphi(t)) + \|\varphi(t)\|_V^2 \le C\mathcal{E}_\lambda(\varphi_0)\mathrm{e}^{-\omega t} + C(F_\lambda(\overline{\varphi}_0) + 1), \tag{3.10}$$

and

$$\int_t^{t+1} \Big(\|\nabla \mu(\tau)\|^2 + \|\nabla \boldsymbol{u}(\tau)\|^2 \Big) \, \mathrm{d}\tau \leq C\mathcal{E}_\lambda(\varphi_0)\mathrm{e}^{-\omega t} + C(F_\lambda(\overline{\varphi}_0) + 1), \tag{3.11}$$

for every $t \geq 0$.

Proof. We take $\boldsymbol{v} = \boldsymbol{u}$ in (3.8) and $w = \mu$ in (3.9). Summing up the resulting equalities, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\varphi) + \|\nabla \mu\|^2 + \langle \nu(\varphi) D\boldsymbol{u}, D\boldsymbol{u} \rangle + \|\boldsymbol{u}\|^2 = 0. \tag{3.12}$$

To reconstruct the energy in (3.12), we test μ by $\varphi - \overline{\varphi}$, getting

$$\langle F_\lambda'(\varphi), \varphi - \overline{\varphi} \rangle + \|\nabla \varphi\|^2 = \Theta_0 \langle \varphi, \varphi - \overline{\varphi} \rangle + \langle \mu - \overline{\mu}, \varphi - \overline{\varphi} \rangle. \tag{3.13}$$

By the convexity of F_λ, we know that

$$\int_\Omega F_\lambda(\varphi) \, \mathrm{d}x \leq \int_\Omega F_\lambda'(\varphi)(\varphi - \overline{\varphi}) \, \mathrm{d}x + \int_\Omega F_\lambda(\overline{\varphi}) \, \mathrm{d}x,$$

while

$$\Theta_0 \langle \varphi, \varphi - \overline{\varphi} \rangle + \langle \mu - \overline{\mu}, \varphi - \overline{\varphi} \rangle \leq \frac{1}{2}\|\nabla \varphi\|^2 + C\|\nabla \mu\|^2 + C\Theta_0^2 \|\varphi\|^2.$$

Then, we arrive at

$$\int_\Omega F_\lambda(\varphi) \, \mathrm{d}x + \frac{1}{2}\|\nabla \varphi\|^2 \leq C\|\nabla \mu\|^2 + CF_\lambda(\overline{\varphi}) + C\Theta_0^2 \|\varphi\|^2.$$

Now, exploiting Lemma 3.7 with a small $\overline{\lambda} = \overline{\lambda}(\Theta_0)$, we find

$$\frac{1}{2}\mathcal{E}_\lambda(\varphi) \leq C\|\nabla \mu\|^2 + CF_\lambda(\overline{\varphi}) + C.$$

Multiplying the above inequality by 2ω, where $\omega = 1/4C$, and, summing up with (3.12), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\varphi) + \omega \mathcal{E}_\lambda(\varphi) + \frac{1}{2}\|\nabla \mu\|^2 + \nu_1 \|\nabla \boldsymbol{u}\|^2 \leq CF_\lambda(\overline{\varphi}) + C. \tag{3.14}$$

Here, we have also used (3.1) and the Korn inequality. An application of the Gronwall lemma, together with the mass conservation, yields

$$\mathcal{E}_\lambda(\varphi(t)) \leq \mathcal{E}_\lambda(\varphi_0)\mathrm{e}^{-\omega t} + C(F_\lambda(\overline{\varphi}_0) + 1),$$

for some $\omega, C > 0$ that are independent of λ. In addition, owing to Lemma 3.7, for a possibly smaller $\overline{\lambda}$, there exists C such that

$$\mathcal{E}_\lambda(\varphi) \geq \frac{1}{2}\|\varphi\|_V^2 - C,$$

for every $\lambda \in (0, \overline{\lambda}]$. Therefore, we infer that

$$\|\varphi(t)\|_V^2 \leq C\mathcal{E}_\lambda(\varphi_0)\mathrm{e}^{-\omega t} + C(F_\lambda(\overline{\varphi}_0) + 1), \quad \forall\, t \geq 0.$$

A final integration of (3.14) on $[t, t+1]$ completes the proof. □

We prove two consequences of the dissipative nature of the system, referring hereafter to $\overline{\lambda}$ and ω as the parameters defined in Lemma 3.9. Accordingly, $\lambda \in (0, \overline{\lambda}]$.

Lemma 3.10. *We have*

$$\int_t^{t+1} \|\varphi_t(\tau)\|_{V'}^2 \,\mathrm{d}\tau \le C\Big(\mathcal{E}_\lambda(\varphi_0)\mathrm{e}^{-\omega t} + F_\lambda(\overline{\varphi}_0) + 1\Big)^2, \quad \forall\, t \ge 0.$$

Proof. We first observe that

$$\langle \nabla\cdot(\varphi\boldsymbol{u}), w\rangle \le \|\boldsymbol{u}\|_{L^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\|\nabla w\| \le C\|\nabla\boldsymbol{u}\|\|\varphi\|_V\|\nabla w\|, \quad w \in V.$$

Then, exploiting (3.10), we have

$$\int_t^{t+1} \|\nabla\cdot(\varphi(\tau)\boldsymbol{u}(\tau))\|_{V'}^2 \,\mathrm{d}\tau \le C\Big(\mathcal{E}_\lambda(\varphi_0)\mathrm{e}^{-\omega t} + F_\lambda(\overline{\varphi}_0) + 1\Big)\int_t^{t+1} \|\nabla\boldsymbol{u}(\tau)\|^2 \,\mathrm{d}\tau.$$

Therefore, by comparison

$$\int_t^{t+1} \|\varphi_t(\tau)\|_{V'}^2 \,\mathrm{d}\tau \le C\Big(\mathcal{E}_\lambda(\varphi_0)\mathrm{e}^{-\omega t} + F_\lambda(\overline{\varphi}_0) + 1\Big)\int_t^{t+1} \Big(\|\nabla\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\Big) \,\mathrm{d}\tau,$$

implying the desired conclusion. □

Lemma 3.11. *Let $\varphi_0 \in V$ with $\overline{\varphi}_0 = \kappa \in (-1,1)$. We have*

$$\int_t^{t+1} \Big(\|\Delta\varphi(\tau)\|^4 + \|F'(\varphi(\tau))\|_{L^1(\Omega)}^2 + |\overline{\mu}(\tau)|^2\Big) \,\mathrm{d}\tau \le C\Big(\mathcal{E}_\lambda(\varphi_0)\mathrm{e}^{-\omega t} + F_\lambda(\overline{\varphi}_0) + 1\Big)^2,$$

for every $t \ge 0$.

Proof. Testing μ by $-\Delta\varphi$ and integrating by parts, we get

$$\langle \nabla\mu, \nabla\varphi\rangle = \|\Delta\varphi\|^2 + \langle \Psi_\lambda'(\varphi), -\Delta\varphi\rangle.$$

An additional integration by parts, together with (3.2), yields

$$\langle \Psi_\lambda'(\varphi), -\Delta\varphi\rangle = \langle \Psi_\lambda''(\varphi)\nabla\varphi, \nabla\varphi\rangle \ge -\Theta_0\|\nabla\varphi\|^2.$$

Hence, we find

$$\|\Delta\varphi\|^2 \le C\|\nabla\varphi\|^2 + \|\nabla\mu\|\|\nabla\varphi\|.$$

Besides, since

$$\overline{\mu} = \langle \Psi_\lambda'(\varphi), 1\rangle,$$

we have

$$|\overline{\mu}| \le C|\overline{\varphi}| + \int_\Omega |F_\lambda'(\varphi)| \,\mathrm{d}x.$$

In order to control the right-hand side, we recall that there exists $K > 0$, independent of $\lambda \in (0, \overline{\lambda}]$, such that

$$\int_\Omega |F_\lambda'(\varphi)| \,\mathrm{d}x \le K\left|\int_\Omega F_\lambda'(\varphi)(\varphi - \overline{\varphi}) \,\mathrm{d}x\right| + K,$$

where K diverges to $+\infty$ as $|\kappa| = |\overline{\varphi}| \to 1$ (see [11] for the proof). Moreover, by virtue of (3.13), we know that

$$\begin{aligned}\langle F_\lambda'(\varphi), \varphi - \overline{\varphi}\rangle &\le \Theta_0\langle \varphi, \varphi - \overline{\varphi}\rangle + \langle \mu - \overline{\mu}, \varphi - \overline{\varphi}\rangle \\ &\le C\|\nabla\varphi\|^2 + C\|\nabla\mu\|\|\nabla\varphi\|.\end{aligned}$$

Combining the above controls, we arrive at

$$\|F'_\lambda(\varphi)\|_{L^1(\Omega)} + |\overline{\mu}| \leq C\|\nabla\varphi\|^2 + C\|\nabla\mu\|\|\nabla\varphi\| + C.$$

Finally, we deduce that

$$\|\Delta\varphi\|^4 + \|F'_\lambda(\varphi)\|^2_{L^1(\Omega)} + |\overline{\mu}|^2 \leq C(\|\nabla\varphi\|^2 + \|\nabla\mu\|\|\nabla\varphi\| + 1)^2.$$

In light of (3.10)-(3.11), the claim follows integrating on $[t, t+1]$. □

Remark 3.12. Note that $\varphi \in L^2(0,T;H^3(\Omega))$ is obtained by testing the equation of μ by $-\Delta^2\varphi$ (see [5]) and exploiting the Lipschitz regularity of F'_λ. In turn, $\varphi \in \mathcal{C}([0,T],V)$ immediately follows. Nonetheless, this argument does not work in presence of the singular potential.

3.4. **Existence of an energy solution to the CHB system.** We fix

$$\varphi_0 \in V \text{ with } \Psi(\varphi_0) \in L^1(\Omega) \text{ and } \overline{\varphi}_0 = \kappa \in (-1,1).$$

In light of Theorem 3.9, let $(\varphi_\lambda, \boldsymbol{u}_\lambda)$, $\lambda \in (0, \overline{\lambda}]$, be a family of solutions to CHB_λ departing from φ_0. Due to property (i), we observe that

$$F_\lambda(s) \leq F(s) \leq C, \quad \forall\, s \in [-1,1].$$

This, in turn, gives $\mathcal{E}_\lambda(\varphi_0) \leq \mathcal{E}(\varphi_0)$. Thus, from Lemmas 3.9, 3.10 and 3.11, we infer the uniform estimates

$$\|\varphi_\lambda(t)\|^2_V \leq C,$$

and

$$\int_t^{t+1} \Big(\|\varphi_\lambda(\tau)\|^4_{H^2(\Omega)} + \|\varphi_{\lambda,t}(\tau)\|^2_{V'} + \|\mu_\lambda(\tau)\|^2_V + \|\nabla \boldsymbol{u}_\lambda(\tau)\|^2 \Big)\, \mathrm{d}\tau \leq C,$$

for every $t \geq 0$, where the right-hand sides do no longer depend on λ.
Now, in the limit $\lambda \to 0$, we have the following convergences (up to subsequences)

$$\begin{aligned}
&\varphi_\lambda \to \varphi \text{ weakly star in } L^\infty(0,T;V),\\
&\varphi_\lambda \to \varphi \text{ weakly in } L^4(0,T;H^2(\Omega)),\\
&\varphi_{\lambda,t} \to \varphi_t \text{ weakly in } L^2(0,T;V'),\\
&\mu_\lambda \to \mu \text{ weakly in } L^2(0,T;V),\\
&\boldsymbol{u}_\lambda \to \boldsymbol{u} \text{ weakly in } L^2(0,T;\boldsymbol{V}).
\end{aligned}$$

By the classical Aubin-Lions Theorem, we also deduce that

$$\varphi_\lambda \to \varphi \text{ strongly in } L^2(0,T;V) \cap \mathcal{C}([0,T],H),$$

and

$$\varphi_\lambda(x,t) \to \varphi(x,t) \quad \text{a.e. } (x,t) \text{ in } \Omega \times (0,T).$$

We claim that the limit pair $(\varphi, \boldsymbol{u})$ is a finite energy solution according to Definition 3.2. Indeed, the required regularity of $(\varphi, \boldsymbol{u})$ immediately follows by the above convergences. Next, we show that φ fulfils

$$|\varphi(x,t)| < 1 \quad \text{a.e. } (x,t) \text{ in } \Omega \times (0,T).$$

To this aim, note that

$$\int_0^T \|F'_\lambda(\varphi_\lambda(t))\|_{L^1(\Omega)}\,\mathrm{d}t \le C,$$

for some $C > 0$ depending on T and on the initial datum. Following a standard argument, for any fixed $\eta \in (0, 1/2)$ we introduce the set

$$E^\lambda_\eta = \{(x,t) \in \Omega \times [0,T] \,:\, |\varphi_\lambda(x,t)| > 1-\eta\}\,.$$

It is easy to see that

$$|E^\lambda_\eta| \le \frac{C}{\min\{F'_\lambda(1-\eta), |F'_\lambda(-1+\eta)|\}}.$$

Hence, passing to the limit as $\lambda \to 0$ and then letting $\eta \to 0$, we conclude

$$|\,\{(x,t) \in \Omega \times (0,T) : |\varphi(x,t)| \ge 1\}\,| = 0.$$

Regarding the nonlinear potential, using the pointwise convergence of φ_λ and the uniform convergence of F'_λ to F' on any compact set in $(-1,1)$, we infer that

$$F'_\lambda(\varphi_\lambda) \to F'(\varphi) \quad \text{a.e. } (x,t) \in \Omega \times (0,T).$$

Moreover, using the definition of μ_λ, we get

$$F'_\lambda(\varphi_\lambda) \text{ is uniformly bounded w.r.t. } \lambda \text{ in } L^2(0,T;H).$$

Then, a well-known result implies that $F'_\lambda(\varphi_\lambda) \to F'(\varphi)$ weakly in $L^2(0,T;H)$, which allows us to identify

$$\mu = -\Delta\varphi + \Psi'(\varphi) \in L^2(0,T;V).$$

Finally, in a standard way, we pass to the limit in the weak formulation of CHB_λ proving the validity of (3.3)-(3.4).

3.5. **Energy equality and dissipativity.** Let us define the functional $\mathcal{J} : H \to H$ given by

$$\mathcal{J}(\varphi) = \frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega F(\varphi)\,\mathrm{d}x.$$

It is clear that $\mathcal{J}$ is proper, convex and lower-semicontinuous. Hence, appealing to [27, Lemma 4.1], we infer that $t \mapsto \mathcal{J}(\varphi)$ is absolutely continuous on $[0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(\varphi) = \langle \varphi_t, \mu - \Theta_0\varphi\rangle, \qquad \text{a.e. } t \in [0,T].$$

In particular, as a byproduct of the boundedness of F and $\varphi \in \mathcal{C}([0,T],H)$, it follows from the Lebesgue Theorem that

$$\int_\Omega F(\varphi(\cdot))\,\mathrm{d}x \in \mathcal{C}([0,T]),$$

which in turn gives $\varphi \in \mathcal{C}([0,T],V)$. Now, taking $w = \mu$ in (3.4) and exploiting the standard chain rule, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varphi) + \|\nabla\mu\|^2 + (\boldsymbol{u}\nabla\varphi, \mu) = 0, \qquad \text{a.e. } t \in [0,T]. \tag{3.15}$$

At this time, taking $\boldsymbol{v} = \boldsymbol{u}$ in (3.3) and summing up to (3.15), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varphi) + \|\nabla\mu\|^2 + \|\sqrt{\nu(\varphi)}D\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2 = 0, \qquad \text{a.e. } t \in [0,T],$$

proving the energy equality claimed in Theorem 3.6. We are left to establish the dissipative estimate (3.6). As a matter of fact passing to the limit as $\lambda \to 0$ in (3.10)-(3.11), we deduce that

$$\mathcal{E}(\varphi(t)) + \int_t^{t+1} \Big(\|\nabla\mu(\tau)\|^2 + \|\nabla\boldsymbol{u}(\tau)\|^2\Big)\,\mathrm{d}\tau \le C\mathcal{E}(\varphi_0)\mathrm{e}^{-\omega t} + C,$$

for almost every $t \ge 0$. The continuity $\varphi \in \mathcal{C}([0,T],V)$ allows us to conclude that the inequality holds true for all $t \ge 0$. Finally, the control (3.7) follows by Lemma 3.11. This finishes the proof of Theorem 3.6.

4. Further Regularity of the Finite Energy Solutions

We establish some further regularity properties of the energy solutions $(\varphi, \boldsymbol{u})$ given by Theorem 3.6. In the sequel, the generic constant $C > 0$ may depend on $\mathcal{E}(\varphi_0)$ and $\overline{\varphi}_0$.

Lemma 4.1. *For any $p \ge 2$, there exists $C = C(p)$ such that*

$$\int_t^{t+1} \Big(\|\varphi(\tau)\|^2_{W^{2,p}(\Omega)} + \|F'(\varphi(\tau))\|^2_{L^p(\Omega)}\Big)\,\mathrm{d}\tau \le C, \quad \forall\, t \ge 0.$$

Proof. We argue as in [1, Lemma 2] (see also [14, Corollary 4.3]). For $k \in \mathbb{N}$, let

$$h_k(s) = \begin{cases} -1 + \frac{1}{k}, & s \in (-1, -1+\frac{1}{k}), \\ s, & s \in [-1+\frac{1}{k}, 1-\frac{1}{k}], \\ 1 - \frac{1}{k}, & s \in (1-\frac{1}{k}, 1), \end{cases}$$

and let us consider $\varphi_k(x) = h_k(\varphi(x))$. Note that

$$\nabla\varphi_k = \nabla\varphi \cdot \chi_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}.$$

Given $p \ge 2$, we test the elliptic equation

$$-\Delta\varphi + F'(\varphi) = \mu^*, \quad \mu^* := \mu + \Theta_0\varphi, \tag{4.1}$$

by $|F'(\varphi_k)|^{p-2}F'(\varphi_k)$. Since F'' is well-defined and positive, we learn that

$$\langle -\Delta\varphi, |F'(\varphi_k)|^{p-2}F'(\varphi_k)\rangle = (p-1)\langle |F'(\varphi_k)|^{p-2}F''(\varphi_k)\nabla\varphi \cdot \chi_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}, \nabla\varphi\rangle \ge 0,$$

which in turn gives

$$\langle F'(\varphi), |F'(\varphi_k)|^{p-2}F'(\varphi_k)\rangle \le \langle \mu^*, |F'(\varphi_k)|^{p-2}F'(\varphi_k)\rangle.$$

Recalling that F' is increasing and $F'(s)s \ge 0$, we are lead to

$$\|F'(\varphi_k)\|^p_{L^p(\Omega)} \le \langle F'(\varphi), |F'(\varphi_k)|^{p-2}F'(\varphi_k)\rangle.$$

By the Hölder inequality

$$\langle \mu^*, |F'(\varphi_k)|^{p-2}F'(\varphi_k)\rangle \le \|F'(\varphi_k)\|^{p-1}_{L^p(\Omega)}\|\mu^*\|_{L^p(\Omega)}.$$

In light of the embedding $V \subset L^p(\Omega)$, we thus obtain

$$\|F'(\varphi_k)\|_{L^p(\Omega)} \le \|\mu^*\|_{L^p(\Omega)} \le C\big(1 + \|\mu\|_V\big).$$

Having in mind (3.6) and (3.7), an integration in time on $[t, t+1]$ yields

$$\int_t^{t+1} \|F'(\varphi_k(\tau))\|^2_{L^p(\Omega)} \, \mathrm{d}\tau \leq C.$$

Finally, passing to the limit as $k \to \infty$, we get the above estimate replacing φ_k with φ. The desired control for φ in $W^{2,p}(\Omega)$ follows by the regularity theory for elliptic equations applied to (4.1). □

Lemma 4.2. *We have*

$$\int_t^{t+1} \|\nabla \boldsymbol{u}(\tau)\|^4 \, \mathrm{d}\tau \leq C, \quad \forall\, t \geq 0.$$

Proof. We take $\boldsymbol{v} = \boldsymbol{u}$ in (3.3) (cfr. Remark 3.4)

$$\langle \nu(\varphi) D\boldsymbol{u}, D\boldsymbol{u} \rangle + \|\boldsymbol{u}\|^2 = \langle \nabla\varphi \otimes \nabla\varphi, \nabla \boldsymbol{u} \rangle.$$

Hence, exploiting (3.1) and the Korn inequality, we have

$$\nu_1 \|\nabla \boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2 \leq \langle \nabla\varphi \otimes \nabla\varphi, \nabla \boldsymbol{u} \rangle.$$

By (2.2), we deduce

$$\langle \nabla\varphi \otimes \nabla\varphi, \nabla \boldsymbol{u} \rangle \leq \|\nabla \boldsymbol{u}\| \|\nabla\varphi\|^2_{L^4(\Omega)} \leq C \|\nabla \boldsymbol{u}\| \|\varphi\|_V \|\varphi\|_{H^2(\Omega)} \leq \frac{\nu_1}{2} \|\nabla \boldsymbol{u}\|^2 + C \|\varphi\|^2_{H^2(\Omega)},$$

so we end up with the control

$$\|\nabla \boldsymbol{u}\| \leq C \|\varphi\|_{H^2(\Omega)}.$$

The thesis follows by (3.7). □

Remark 4.3. It is worth noticing that the above regularity for $\boldsymbol{u}$ holds true also for the Galerkin approximations. Unfortunately, this is not enough to ensure uniqueness of the energy solution, and we need to gain some extra regularity properties for $\boldsymbol{u}$. To this aim, let us observe that (3.3) is also equivalent to

$$\langle \nu(\varphi(t)) D\boldsymbol{u}(t), D\boldsymbol{v} \rangle + \langle \boldsymbol{u}(t), \boldsymbol{v} \rangle = -\langle \varphi(t) \nabla\mu(t), \boldsymbol{v} \rangle, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \quad \text{a.e. } t \in [0, T]$$

where

$$\varphi \nabla\mu \in L^2(0, T; H),$$

according to the boundedness of φ stated in the Definition 3.2. Hence, having in mind the classical regularity result for the Stokes problem, one might expect that $\boldsymbol{u}(t) \in H^2(\Omega)$ for almost every t. Indeed, this is the case when ν is constant. On the contrary, when ν is variable we are not able to reach such a regularity.

We prove the following weaker, but still crucial result.

Lemma 4.4. *We have*

$$\int_t^{t+1} \|\boldsymbol{u}(\tau)\|^{8/5}_{W^{2,4/3}(\Omega)} \mathrm{d}\tau \leq C, \tag{4.2}$$

and

$$\int_t^{t+1} \|\boldsymbol{u}(\tau)\|^2_{W^{1,3}(\Omega)} \mathrm{d}\tau \leq C, \tag{4.3}$$

for every $t \geq 0$.

Proof. Let us first observe that the velocity equivalently solves

$$\langle \nu(\varphi) D\boldsymbol{u}, D\boldsymbol{v}\rangle = \langle f, \boldsymbol{v}\rangle, \quad \boldsymbol{v} \in \boldsymbol{V},$$

where

$$f = -\varphi\nabla\mu - \boldsymbol{u}.$$

Now, in order to prove the first estimate, we argue as in [1, Lemma 4]. Accordingly, we rewrite the equation above as a standard Stokes problem, namely

$$\langle D\boldsymbol{u}, \nabla\boldsymbol{v}\rangle = \langle g, \boldsymbol{v}\rangle, \quad \boldsymbol{v} \in [\mathcal{C}_0^\infty(\Omega)]^2 \text{ with } \nabla\cdot\boldsymbol{v} = 0,$$

where g satisfies

$$\|g\|_{L^{4/3}(\Omega)} \le C(\|f\| + \|\nabla\boldsymbol{u}\|)(1 + \|\nabla\varphi\|_{L^4(\Omega)}).$$

By (2.2) and the Young inequality, we have

$$\|g\|^{8/5}_{L^{4/3}(\Omega)} \le C(\|f\|^{8/5} + \|\nabla\boldsymbol{u}\|^{8/5})(1 + \|\varphi\|^{4/5}_{H^2(\Omega)}) \le C(1 + \|f\|^2 + \|\nabla\boldsymbol{u}\|^2 + \|\varphi\|^4_{H^2(\Omega)}).$$

Recalling that

$$\int_t^{t+1} \Big(\|f(\tau)\|^2 + \|\nabla\boldsymbol{u}(\tau)\|^2 + \|\varphi(\tau)\|^4_{H^2(\Omega)}\Big)\,\mathrm{d}\tau \le C,$$

we conclude

$$\int_t^{t+1} \|g(\tau)\|^{8/5}_{L^{4/3}(\Omega)}\mathrm{d}\tau \le C.$$

By the well-known regularity theory for the Stokes problem (see e.g. [7]),

$$\|\boldsymbol{u}\|_{W^{2,4/3}(\Omega)} \le C\|g\|_{L^{4/3}(\Omega)},$$

and the proof of (4.2) is done. In order to prove (4.4), we recall the following Gagliardo-Nirenberg inequality

$$\|\boldsymbol{u}\|_{W^{1,3}(\Omega)} \le \|\boldsymbol{u}\|^{2/3}_{W^{2,4/3}(\Omega)}\|\boldsymbol{u}\|^{1/3}_{W^{1,2}(\Omega)}.$$

Hence, we deduce

$$\|\boldsymbol{u}\|^2_{W^{1,3}(\Omega)} \le C\|\boldsymbol{u}\|^{4/3}_{W^{2,4/3}(\Omega)}\|\nabla\boldsymbol{u}\|^{2/3} \le \|\boldsymbol{u}\|^{8/5}_{W^{2,4/3}(\Omega)} + C\|\nabla\boldsymbol{u}\|^4,$$

and the conclusion follows by collecting (4.2) with Lemma 4.2. □

5. Continuous Dependence and Uniqueness

We are now in the position to prove the uniqueness of the finite energy solution.

Theorem 5.1. *Let φ_{01}, φ_{02} be such that $\varphi_{0i} \in V$, $\Psi(\varphi_{0i}) \in L^1(\Omega)$ and $|\varphi_{0i}| < 1$, $i = 1, 2$. Then, any couple of energy solutions $(\varphi_1, \boldsymbol{u}_1)$ and $(\varphi_2, \boldsymbol{u}_2)$ of the CHB problem on $[0, T]$ with initial data φ_{01} and φ_{02}, respectively, satisfies*

$$\|\varphi_1(t) - \varphi_2(t)\|_{V'} \le C\|\varphi_1(0) - \varphi_2(0)\|_{V'} + C|\overline{\varphi}_1(0) - \overline{\varphi}_2(0)|^{1/2},$$

for any $t \in [0, T]$. In particular, the energy solution to CHB is unique.

Proof. Let us consider $(\varphi_1, \boldsymbol{u}_1)$ and $(\varphi_2, \boldsymbol{u}_2)$ two finite energy solutions to the CHB system with total mass $\overline{\varphi}_1(0)$ and $\overline{\varphi}_2(0)$. Their difference $\varphi = \varphi_1 - \varphi_2$, $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ solves

$$\begin{aligned}\langle \nu(\varphi_1) D\boldsymbol{u}, D\boldsymbol{v}\rangle + \langle \boldsymbol{u}, \boldsymbol{v}\rangle + \langle \nu(\varphi_1) - \nu(\varphi_2) D\boldsymbol{u}_2, D\boldsymbol{v}\rangle \\ = \langle \nabla\varphi \otimes \nabla\varphi_1, \nabla\boldsymbol{v}\rangle + \langle \nabla\varphi \otimes \nabla\varphi_2, \nabla\boldsymbol{v}\rangle, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\end{aligned} \tag{5.1}$$

and

$$\langle \varphi_t, w\rangle + \langle \nabla\cdot(\varphi\boldsymbol{u}_1), w\rangle + \langle \nabla\cdot(\varphi_2\boldsymbol{u}), w\rangle + \langle \nabla\mu, \nabla w\rangle = 0, \quad \forall\, w \in V, \tag{5.2}$$

where

$$\mu = -\Delta\varphi + \Psi'(\varphi_1) - \Psi'(\varphi_2).$$

We note that $\overline{\varphi}(t) = \overline{\varphi}_1(0) - \overline{\varphi}_2(0)$ for all $t \geq 0$.

Taking $w = \mathcal{N}(\varphi - \overline{\varphi})$ in (5.2), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \overline{\varphi}\|_*^2 + \langle \mu, \varphi - \overline{\varphi}\rangle = I_1 + I_2,$$

having set

$$I_1 = \langle \varphi\boldsymbol{u}_1, \nabla\mathcal{N}(\varphi - \overline{\varphi})\rangle, \quad I_2 = \langle \varphi_2\boldsymbol{u}, \nabla\mathcal{N}(\varphi - \overline{\varphi})\rangle.$$

By the assumptions on Ψ, we deduce

$$\begin{aligned}\langle \mu, \varphi - \overline{\varphi}\rangle &= \|\nabla\varphi\|^2 + \langle \Psi'(\varphi_1) - \Psi'(\varphi_2), \varphi_1 - \varphi_2\rangle + \langle \Psi'(\varphi_1) - \Psi'(\varphi_2), \overline{\varphi}\rangle \\ &\geq \|\nabla\varphi\|^2 - \alpha\|\varphi\|^2 - |\langle \Psi'(\varphi_1) - \Psi'(\varphi_2), \overline{\varphi}\rangle| \\ &\geq \|\varphi\|_V^2 - (\alpha+1)\|\varphi\|^2 - (\|\Psi'(\varphi_1)\|_{L^1(\Omega)} + \|\Psi'(\varphi_2)\|_{L^1(\Omega)})|\overline{\varphi}|.\end{aligned}$$

Besides, recalling that $|\overline{\varphi}| \leq 2$

$$\begin{aligned}(\alpha+1)\|\varphi\|^2 &\leq C\|\varphi - \overline{\varphi}\|^2 + C\|\overline{\varphi}\|^2 \\ &\leq C\|\nabla\varphi\|\|\varphi - \overline{\varphi}\|_* + C|\overline{\varphi}|^2 \\ &\leq \frac{1}{2}\|\varphi\|_V^2 + C\|\varphi\|_{-1}^2.\end{aligned}$$

Owing to the mass conservation, and setting

$$\Upsilon(t) = C(\|\Psi'(\varphi_1(t))\|_{L^1(\Omega)} + \|\Psi'(\varphi_2(t))\|_{L^1(\Omega)}),$$

we thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{-1}^2 + \frac{1}{2}\|\varphi\|_V^2 \leq C\|\varphi\|_{-1}^2 + \Upsilon|\overline{\varphi}| + I_1 + I_2. \tag{5.3}$$

We proceed by estimating I_1 and I_2. First,

$$|I_1| \leq \|\boldsymbol{u}_1\|_{L^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\|\varphi - \overline{\varphi}\|_* \leq \frac{1}{4}\|\varphi\|_V^2 + C\|\boldsymbol{u}_1\|_{L^3(\Omega)}^2\|\varphi - \overline{\varphi}\|_*^2.$$

Next, since by definition of solution $\|\varphi_2\|_{L^\infty(\Omega)} \leq 1$,

$$|I_2| \leq \|\boldsymbol{u}\|\|\varphi_2\|_{L^\infty(\Omega)}\|\varphi - \overline{\varphi}\|_* \leq \|\boldsymbol{u}\|\|\varphi - \overline{\varphi}\|_*.$$

Now, in order to find a control for $\|\boldsymbol{u}\|$, we take $\boldsymbol{v} = \boldsymbol{u}$ in (5.1) yielding

$$\langle \nu(\varphi_1) D\boldsymbol{u}, D\boldsymbol{u}\rangle + \|\boldsymbol{u}\|^2 + J = \langle \nabla\varphi(t) \otimes \nabla\varphi_1, \nabla\boldsymbol{u}\rangle + \langle \nabla\varphi(t) \otimes \nabla\varphi_2(t), \nabla\boldsymbol{u}\rangle, \tag{5.4}$$

where

$$J = \langle \nu(\varphi_1) - \nu(\varphi_2) D\boldsymbol{u}_2, D\boldsymbol{u} \rangle.$$

We have

$$\langle \nabla\varphi(t) \otimes \nabla\varphi_1, \nabla\boldsymbol{u} \rangle \leq \|\nabla\varphi\| \|\nabla\varphi_1\|_{L^\infty(\Omega)} \|\nabla\boldsymbol{u}\| \leq \frac{\nu_1}{4}\|\nabla\boldsymbol{u}\|^2 + C\|\varphi_1\|^2_{W^{2,6}(\Omega)}\|\nabla\varphi\|^2.$$

Dealing analogously with the last term, on account of (3.1) and of the Korn inequality, we arrive at

$$\frac{\nu_1}{2}\|\nabla\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2 + J \leq C(\|\varphi_1\|^2_{W^{2,6}(\Omega)} + \|\varphi_2\|^2_{W^{2,6}(\Omega)})\|\varphi\|^2_V. \tag{5.5}$$

Regarding J, by $\nu \in \mathcal{C}^1(\mathbb{R})$, we find the control

$$\begin{aligned} |J| &\leq C\|\varphi\|_{L^6(\Omega)}\|\nabla\boldsymbol{u}_2\|_{L^3(\Omega)}\|\nabla\boldsymbol{u}\| \\ &\leq \frac{\nu_1}{2}\|\nabla\boldsymbol{u}\|^2 + C\|\varphi\|^2_V\|\nabla\boldsymbol{u}_2\|^2_{L^3(\Omega)}. \end{aligned}$$

Thus, we learn by (5.5) that

$$\|\boldsymbol{u}\| \leq C\big(\|\varphi_1\|_{W^{2,6}(\Omega)} + \|\varphi_2\|_{W^{2,6}(\Omega)} + \|\boldsymbol{u}_2\|_{W^{1,3}(\Omega)}\big)\|\varphi\|_V,$$

and, exploiting this in I_2, we find

$$|I_2| \leq \frac{1}{4}\|\varphi\|^2_V + C\big(\|\varphi_1\|^2_{W^{2,6}(\Omega)} + \|\varphi_2\|^2_{W^{2,6}(\Omega)} + \|\boldsymbol{u}_2\|^2_{W^{1,3}(\Omega)}\big)\|\varphi - \overline{\varphi}\|^2_*.$$

Collecting the above estimates for I_1 and I_2 in (5.3), we obtain the final differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2_{-1} \leq \Gamma\|\varphi\|^2_{-1} + \Upsilon|\overline{\varphi}|,$$

having set

$$\Gamma(t) = C\big(1 + \|\varphi_1(t)\|^2_{W^{2,6}(\Omega)} + \|\varphi_2(t)\|^2_{W^{2,6}(\Omega)} + \|\boldsymbol{u}_2(t)\|^2_{W^{1,3}(\Omega)} + \|\nabla\boldsymbol{u}_1(t)\|^2\big),$$

which is summable in light of Lemma 4.1 and Lemma 4.4. An application of the Gronwall lemma gives

$$\|\varphi(t)\|^2_{-1} \leq \|\varphi(0)\|^2_{-1}\mathrm{e}^C + |\overline{\varphi}(0)|\|\Upsilon\|_{L^1(0,T)}\mathrm{e}^C, \quad \forall\, t \in [0,T],$$

where $C = \|\Gamma\|_{L^1(0,T)}$. In particular, if $\varphi_1(0) = \varphi_2(0)$, then $\varphi_1 \equiv \varphi_2$ and by (5.4), $\boldsymbol{u}_1 \equiv \boldsymbol{u}_2$ as well, thus uniqueness follows. □

6. Regularization in Finite Time

Let $R > 0$ and $\kappa \in (-1,1)$ be given. In the sequel, we consider bundles of trajectories $(\varphi, \boldsymbol{u})$ departing from φ_0 such that

$$\mathcal{E}(\varphi_0) \leq R \quad \text{and} \quad \overline{\varphi}_0 = \kappa.$$

The aim is proving higher order regularity estimates for the trajectories which depend on R and κ but are independent of the specific choice of the initial datum. Accordingly, the generic constant $C > 0$ depends on R and κ.

Theorem 6.1. *For every $\sigma > 0$, there exists $C = C(\sigma)$ such that*

$$\|\mu\|_{L^\infty(\sigma,\infty;V)} + \|\boldsymbol{u}\|_{L^\infty(\sigma,\infty;\boldsymbol{V})} \leq C,$$

and

$$\int_t^{t+1} \Big(\|\boldsymbol{u}(\tau)\|^2_{H^2(\Omega)} + \|\varphi_t(\tau)\|^2_V \Big) \, \mathrm{d}\tau \leq C, \quad \forall\, t \geq \sigma. \tag{6.1}$$

Moreover, for every $p \geq 2$, there exists $C = C(\sigma, p)$ such that

$$\|\varphi\|_{L^\infty(\sigma,\infty;W^{2,p}(\Omega))} + \|F'(\varphi)\|_{L^\infty(\sigma,\infty;L^p(\Omega))} \leq C. \tag{6.2}$$

Remark 6.2. At this point, by the classical Aubin-Simon embedding we learn that

$$\varphi \in \mathcal{C}([\sigma,\infty), W^{1,p}(\Omega)),$$

for every $p \geq 2$ and $\sigma > 0$. In particular,

$$\varphi \in \mathcal{C}(\overline{\Omega} \times [\sigma,\infty)).$$

Proof. Let us first recall that the dissipative inequalities (3.6) and (3.7) yield

$$\mathcal{E}(\varphi(t)) + \int_t^{t+1} \Big(\|\varphi(\tau)\|^4_{H^2(\Omega)} + \|\nabla\mu(\tau)\|^2 + \|\nabla\boldsymbol{u}(\tau)\|^4 \Big) \, \mathrm{d}\tau \leq C, \quad \forall\, t \geq 0. \tag{6.3}$$

Besides, arguing by comparison, we deduce

$$\|\varphi_t\|_{V'} \leq C(\|\nabla\mu\| + \|\nabla\boldsymbol{u}\|), \tag{6.4}$$

and, reasoning as in the proof of Lemma 3.11, we learn that

$$\|\mu\|_V \leq C(1 + \|\nabla\mu\|). \tag{6.5}$$

We take $w = \mu_t$ in (3.4) getting

$$\langle \varphi_t, \mu_t \rangle + \langle \nabla \cdot (\varphi\boldsymbol{u}), \mu_t \rangle + \langle \nabla\mu, \nabla\mu_t \rangle = 0.$$

Since

$$\langle \varphi_t, \mu_t \rangle = \langle -\Delta\varphi_t, \varphi_t \rangle + \langle \Psi''(\varphi)\varphi_t, \varphi_t \rangle \geq \frac{1}{2}\|\nabla\varphi_t\|^2 - C\|\varphi_t\|^2_{V'},$$

we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mu\|^2 + \frac{1}{2}\|\nabla\varphi_t\|^2 + \langle \boldsymbol{u} \cdot \nabla\varphi, \mu_t \rangle \leq C\|\varphi_t\|^2_{V'}.$$

A differentiation in time of (3.3) entails

$$\langle \nu(\varphi) D\boldsymbol{u}_t, D\boldsymbol{v} \rangle + \langle \nu'(\varphi)\varphi_t D\boldsymbol{u}, D\boldsymbol{v} \rangle + \langle \boldsymbol{u}_t, \boldsymbol{v} \rangle = \langle \mu_t \nabla\varphi, \boldsymbol{v} \rangle + \langle \mu\nabla\varphi_t, \boldsymbol{v} \rangle, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

For $\boldsymbol{v} = \boldsymbol{u}$, this gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\langle \nu(\varphi) D\boldsymbol{u}, D\boldsymbol{u} \rangle + \|\boldsymbol{u}\|^2 \big) = \langle \mu_t \nabla\varphi, \boldsymbol{u} \rangle + \langle \mu\nabla\varphi_t, \boldsymbol{u} \rangle - \frac{1}{2}\langle \nu'(\varphi)\varphi_t D\boldsymbol{u}, D\boldsymbol{u} \rangle.$$

Hence, setting

$$\boldsymbol{\Lambda} = \langle \nu(\varphi) D\boldsymbol{u}, D\boldsymbol{u} \rangle + \|\boldsymbol{u}\|^2 + \|\nabla\mu\|^2,$$

we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda} + \frac{1}{2}\|\nabla\varphi_t\|^2 \leq C\|\varphi_t\|^2_{V'} + \langle \mu\nabla\varphi_t, \boldsymbol{u} \rangle - \frac{1}{2}\langle \nu'(\varphi)\varphi_t D\boldsymbol{u}, D\boldsymbol{u} \rangle.$$

Now, taking $\boldsymbol{v} = -\Delta \boldsymbol{u}$ in (3.3), we have

$$\langle \nu(\varphi)\Delta \boldsymbol{u}, \Delta \boldsymbol{u}\rangle + \|\nabla \boldsymbol{u}\|^2 = -\langle \mu\nabla\varphi, \Delta \boldsymbol{u}\rangle - \langle \nu'(\varphi)\nabla\varphi D\boldsymbol{u}, \Delta \boldsymbol{u}\rangle.$$

Recalling (3.1), we deduce

$$\langle \nu(\varphi)\Delta \boldsymbol{u}, \Delta \boldsymbol{u}\rangle + \|\nabla \boldsymbol{u}\|^2 \geq \nu_1\|\Delta \boldsymbol{u}\|^2 + \|\nabla \boldsymbol{u}\|^2 \geq \nu_*\|\boldsymbol{u}\|^2_{H^2(\Omega)},$$

for some constant $\nu_* > 0$. Summing up with the above differential inequality for $\boldsymbol{\Lambda}$, we are lead to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda} + \frac{1}{2}\|\nabla\varphi_t\|^2 + \nu_*\|\boldsymbol{u}\|^2_{H^2(\Omega)} &\leq C\|\varphi_t\|^2_{V'} + |\langle \mu\nabla\varphi_t, \boldsymbol{u}\rangle| \\ &\quad + C|\langle \nu'(\varphi)\varphi_t D\boldsymbol{u}, D\boldsymbol{u}\rangle| + |\langle \mu\nabla\varphi, \Delta\boldsymbol{u}\rangle| + |\langle \nu'(\varphi)\nabla\varphi D\boldsymbol{u}, \Delta\boldsymbol{u}\rangle|.\end{aligned}$$

First, we estimate

$$|\langle \mu\nabla\varphi_t, \boldsymbol{u}\rangle| \leq \frac{1}{8}\|\nabla\varphi_t\|^2 + C\|\boldsymbol{u}\|^2_{L^3(\Omega)}\|\mu\|^2_V.$$

Next, by the Ladyzhenskaya inequality (2.2), we control

$$\begin{aligned}C|\langle \nu'(\varphi)\varphi_t D\boldsymbol{u}, D\boldsymbol{u}\rangle| &\leq C\|\varphi_t\|\|D\boldsymbol{u}\|^2_{L^4(\Omega)} \\ &\leq C\|\varphi_t\|^{1/2}_{V'}\|\nabla\varphi_t\|^{1/2}\|\nabla\boldsymbol{u}\|\|\boldsymbol{u}\|_{H^2(\Omega)} \\ &\leq \frac{\nu_*}{4}\|\boldsymbol{u}\|^2_{H^2(\Omega)} + \frac{1}{8}\|\nabla\varphi_t\|^2 + C\|\nabla\boldsymbol{u}\|^4\|\varphi_t\|^2_{V'},\end{aligned}$$

and

$$\begin{aligned}|\langle \nu'(\varphi)\nabla\varphi D\boldsymbol{u}, \Delta\boldsymbol{u}\rangle| &\leq C\|\nabla\varphi\|_{L^4(\Omega)}\|D\boldsymbol{u}\|_{L^4(\Omega)}\|\Delta\boldsymbol{u}\| \\ &\leq C\|\nabla\varphi\|^{1/2}\|\varphi\|^{1/2}_{H^2(\Omega)}\|\nabla\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|^{3/2}_{H^2(\Omega)} \\ &\leq \frac{\nu_*}{4}\|\boldsymbol{u}\|^2_{H^2(\Omega)} + C\|\varphi\|^2_{H^2(\Omega)}\|\nabla\boldsymbol{u}\|^2.\end{aligned}$$

In addition,

$$|\langle \mu\nabla\varphi, \Delta\boldsymbol{u}\rangle| \leq \frac{\nu_*}{4}\|\boldsymbol{u}\|^2_{H^2(\Omega)} + C\|\mu\|^2_V\|\varphi\|^2_{H^2(\Omega)}.$$

Therefore, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda} + \frac{1}{4}\|\nabla\varphi_t\|^2 + \frac{\nu_*}{4}\|\boldsymbol{u}\|^2_{H^2(\Omega)} &\leq C\|\varphi_t\|^2_{V'} + C\|\boldsymbol{u}\|^2_{L^3(\Omega)}\|\mu\|^2_V \\ &\quad + C\|\nabla\boldsymbol{u}\|^4\|\varphi_t\|^2_{V'} + C\|\varphi\|^2_{H^2(\Omega)}\big(\|\nabla\boldsymbol{u}\|^2 + \|\mu\|^2_V\big).\end{aligned}$$

Keeping in mind (6.4) and (6.5), in light of the equivalence

$$\frac{1}{C}(\|\nabla\mu\|^2 + \|\nabla\boldsymbol{u}\|^2) \leq \boldsymbol{\Lambda} \leq C(\|\nabla\mu\|^2 + \|\nabla\boldsymbol{u}\|^2),$$

we finally obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda} + \frac{1}{8}\|\nabla\varphi_t\|^2 + \frac{\nu_*}{4}\|\boldsymbol{u}\|^2_{H^2(\Omega)} \leq \Upsilon\boldsymbol{\Lambda} + \Upsilon, \tag{6.6}$$

where

$$\Upsilon(t) = C\big(1 + \|\varphi(t)\|^2_{H^2(\Omega)} + \|\boldsymbol{u}(t)\|^2_{L^3(\Omega)} + \|\nabla\boldsymbol{u}(t)\|^4\big).$$

Owing to (6.3), an application of the uniform Gronwall lemma yields

$$\|\nabla\mu(t)\| + \|\nabla\boldsymbol{u}(t)\| \leq \frac{C}{\sqrt{\sigma}}, \quad \forall\, t \geq \sigma.$$

The integral inequality (6.1) follows by an integration of (6.6) on any interval $[t, t+1]$. Finally, reasoning as in the proof of Lemma 4.1, from $\mu \in L^\infty(\sigma, \infty; V)$ we easily deduce the desired control (6.2). □

Remark 6.3. The proof of Theorem 6.1 is obtained by formal computations. However, they can be rigorously justified through the Galerkin scheme mentioned in Section 3.

Remark 6.4. As a consequence of Theorem 6.1, we learn that $\mu \in L^2(t, t+1; H^2(\Omega))$ for every $t \geq \sigma$. Then, it is immediate to deduce that

$$\varphi_t + \nabla \cdot (\boldsymbol{u}\varphi) = \Delta\mu, \qquad \text{a.e. } (x,t) \in \Omega \times (\sigma, \infty)$$

and $\partial_{\boldsymbol{n}}\mu = 0$ a.e. in $\partial\Omega \times (\sigma, \infty)$. Accordingly, the energy solution is indeed a strong solution on $\Omega \times (\sigma, \infty)$.

7. Strict Separation Property

We prove the validity of the strict separation property for a class of singular potentials which includes, in particular, the physical relevant logarithmic free energy. As above, let us fix $R > 0$ and $\kappa \in (-1, 1)$, and let $(\varphi, \boldsymbol{u})$ be the solution to the CHB system departing from φ_0 satisfying

$$\mathcal{E}(\varphi_0) \leq R \quad \text{and} \quad \overline{\varphi}_0 = \kappa.$$

In what follows, the generic constant $C > 0$ depends on R and κ.

Theorem 7.1. *Let $\sigma > 0$. Assume that F'' is convex and*

$$F''(s) \leq \mathrm{e}^{K|F'(s)|+K}, \quad \forall\, s \in (-1, 1),$$

for some $K > 0$. Then, there exists $\delta = \delta(\sigma, R, \kappa) > 0$ such that

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall\, t \geq 2\sigma. \tag{7.1}$$

We need two preparatory lemmas. The first is based on a key idea introduced in [23, Lemma 7.1] (see also [14]).

Lemma 7.2. *Let the assumptions of Theorem 7.1 hold. For any $p \geq 2$, there exists $C = C(\sigma, p)$, such that*

$$\|F''(\varphi)\|_{L^p(t,t+1;L^p(\Omega))} \leq C, \quad \forall\, t \geq \sigma.$$

Proof. For $k \in \mathbb{N}$, let φ_k be the truncation of φ defined in Lemma 4.1. Given $L > 0$, arguing as in Lemma 4.1 and [14, Lemma 5.3], we test (4.1) by $F'(\varphi_k)\mathrm{e}^{L|F'(\varphi)|}$ getting

$$\int_\Omega |F'(\varphi_k)|^2 \mathrm{e}^{L|F'(\varphi_k)|}\,\mathrm{d}x \leq \int_\Omega |\mu^*||F'(\varphi_k)|\mathrm{e}^{L|F'(\varphi_k)|}\,\mathrm{d}x.$$

We now estimate the right-hand side by the generalized Young inequality (2.3) with the choice

$$x = |\mu^*|, \quad y\mathrm{e}^{Ly} = |F'(\varphi_k)|\mathrm{e}^{L|F'(\varphi_k)|}.$$

Accordingly, we find $N = N(L)$ such that

$$\int_\Omega |\mu^*||F'(\varphi_k)|\mathrm{e}^{L|F'(\varphi_k)|}\,\mathrm{d}x \le \int_\Omega \frac{1}{2}|F'(\varphi_k)|^2\mathrm{e}^{L|f(\varphi_k)|}\,\mathrm{d}x + \int_\Omega \mathrm{e}^{N|\mu^*|}\,\mathrm{d}x,$$

and we obtain

$$\frac{1}{2}\int_\Omega |F'(\varphi_k)|^2\mathrm{e}^{L|F'(\varphi_k)|}\,\mathrm{d}x \le \int_\Omega \mathrm{e}^{N|\mu^*|}\,\mathrm{d}x.$$

Due to the Trudinger-Moser inequality in dimension two (see [24])

$$\int_\Omega e^{|v|}\mathrm{d}x \le e^{C(\|v\|_V^2+1)}, \quad \forall\, v \in V,$$

and owing to Theorem 6.1, we have the following controls

$$\begin{aligned}\int_\Omega \mathrm{e}^{L|F'(\varphi_k)|}\,\mathrm{d}x &\le C\Big(1 + \int_\Omega |F'(\varphi_k)|^2\mathrm{e}^{L|F'(\varphi_k)|}\,\mathrm{d}x\Big)\\ &\le C\Big(1 + \int_\Omega \mathrm{e}^{N|\mu^*|}\,\mathrm{d}x\Big) \le C\Big(1 + \mathrm{e}^{CN^2\|\mu^*\|_V^2}\Big) \le C.\end{aligned}$$

Exploiting the growth condition on F'' and applying this inequality with $L = pK$, we thus infer

$$\int_\Omega |F''(\varphi_k)|^p\,\mathrm{d}x \le \int_\Omega \mathrm{e}^{pK|F'(\varphi_k)|+pK}\,\mathrm{d}x \le C,$$

where $C > 0$ depends on p also. An integration in time on $[t, t+1]$ with $t \ge \sigma$, and a final passage to the limit as $k \to \infty$ complete the proof. □

Lemma 7.3. *Let the assumptions of Theorem 7.1 hold. There exists $C = C(\sigma)$ such that*

$$\|\varphi_t\|_{L^\infty(\sigma,\infty;H)} \le C.$$

Proof. Given $h > 0$, let us introduce the incremental ratio

$$\partial_t^h u = \frac{1}{h}[u(t+h) - u(t)].$$

Owing to Remark 6.4, the solution solves

$$\big(\partial_t^h\varphi\big)_t + \partial_t^h\boldsymbol{u}\cdot\nabla\varphi(t+h) + \boldsymbol{u}\cdot\nabla\partial_t^h\varphi = \Delta\partial_t^h\mu.$$

Testing the above equation by $\partial_t^h\varphi$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2 = \langle\Delta\partial_t^h\mu, \partial_t^h\varphi\rangle + I_1 + I_2, \tag{7.2}$$

having set

$$I_1 = -\langle\partial_t^h\boldsymbol{u}\cdot\nabla\varphi(t+h), \partial_t^h\varphi\rangle, \quad I_2 = -\langle\boldsymbol{u}\cdot\nabla\partial_t^h\varphi, \partial_t^h\varphi\rangle.$$

Integrating by parts and making use of the boundary conditions, we get

$$\begin{aligned}\langle\Delta\partial_t^h\mu, \partial_t^h\varphi\rangle &= \langle\partial_t^h\mu, \Delta\partial_t^h\varphi\rangle\\ &= -\|\Delta\partial_t^h\varphi\|^2 + \Theta_0\|\nabla\partial_t^h\varphi\|^2 + \langle\frac{1}{h}[F'(\varphi(t+h)) - F'(\varphi(t))], \Delta\partial_t^h\varphi\rangle.\end{aligned}$$

Exploiting the convexity of F'', we estimate

$$\frac{1}{h}\Big|F'(\varphi(t+h))-F'(\varphi(t))\Big| \le \int_0^1 F''(\tau\varphi(t+h)+(1-\tau)\varphi(t))\,|\partial_t^h\varphi|\,\mathrm{d}\tau$$
$$\le \int_0^1 \Big(\tau F''(\varphi(t+h))+(1-\tau)F''(\varphi(t))\Big)\,|\partial_t^h\varphi|\,\mathrm{d}\tau \le \Big(F''(\varphi(t+h))+F''(\varphi(t))\Big)|\partial_t^h\varphi|,$$

and we deduce

$$\Big|\langle\frac{1}{h}[F'(\varphi(t+h))-F'(\varphi(t))],\Delta\partial_t^h\varphi\rangle\Big|$$
$$\le \frac{1}{2}\|\Delta\partial_t^h\varphi\|^2 + C\Big(\|F''(\varphi(t+h))\|^2_{L^3(\Omega)}+\|F''(\varphi(t))\|^2_{L^3(\Omega)}\Big)\|\partial_t^h\varphi\|^2_{L^6(\Omega)}.$$

By interpolation

$$\|\partial_t^h\varphi\|^2_{L^6(\Omega)} \le C\|\nabla\partial_t^h\varphi\|^2 \le C\|\partial_t^h\varphi\|\|\Delta\partial_t^h\varphi\|,$$

and we easily derive from (7.2) the differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2+\frac{1}{4}\|\Delta\partial_t^h\varphi\|^2 \le \Upsilon\|\partial_t^h\varphi\|^2+I_1+I_2, \tag{7.3}$$

where

$$\Upsilon(t)=C(1+\|F''(\varphi(t+h))\|^4_{L^3(\Omega)}+\|F''(\varphi(t))\|^4_{L^3(\Omega)}).$$

Let us now consider the equation for $\partial_t^h\boldsymbol{u}$ as in Remark 3.4. Testing by $\partial_t^h\boldsymbol{u}$, we find

$$\langle\nu(\varphi(t+h))D\partial_t^h\boldsymbol{u},D\partial_t^h\boldsymbol{u}\rangle+\|\partial_t^h\boldsymbol{u}\|^2=J_1+J_2, \tag{7.4}$$

having set

$$J_1=-\langle\frac{1}{h}[\nu(\varphi(t+h))-\nu(\varphi(t))]D\boldsymbol{u},D\partial_t^h\boldsymbol{u}\rangle,$$

and

$$J_2=\langle\nabla\partial_t^h\varphi\otimes\nabla\varphi(t+h),\nabla\partial_t^h\boldsymbol{u}\rangle+\langle\nabla\varphi(t)\otimes\nabla\partial_t^h\varphi,\nabla\partial_t^h\boldsymbol{u}\rangle.$$

Note that by (3.1) and the Korn inequality

$$\langle\nu(\varphi(t+h))D\partial_t^h\boldsymbol{u},D\partial_t^h\boldsymbol{u}\rangle \ge \nu_1\|\nabla\partial_t^h\boldsymbol{u}\|^2,$$

hence, summing up (7.3) and (7.4), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2+\frac{1}{4}\|\Delta\partial_t^h\varphi\|^2+\nu_1\|\nabla\partial_t^h\boldsymbol{u}\|^2 \le \Upsilon\|\partial_t^h\varphi\|^2+I_1+I_2+J_1+J_2.$$

We estimate the right-hand side term by term as follows. By Theorem 6.1, we have

$$|I_1| \le \|\partial_t^h\boldsymbol{u}\|_{L^6(\Omega)}\|\nabla\varphi(t+h)\|_{L^3(\Omega)}\|\partial_t^h\varphi\| \le \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2+C\|\partial_t^h\varphi\|^2,$$

and

$$|I_2| \le \|\boldsymbol{u}\|_{L^6(\Omega)}\|\nabla\partial_t^h\varphi\|_{L^3(\Omega)}\|\partial_t^h\varphi\| \le \frac{1}{24}\|\Delta\partial_t^h\varphi\|^2+C\|\partial_t^h\varphi\|^2.$$

Besides, recalling that $\nu \in \mathcal{C}^1(\mathbb{R})$, and making use of (2.2), we find the control

$$\begin{aligned}|J_1| &\leq C\|\partial_t^h\varphi\|_{L^4(\Omega)}\|\nabla \boldsymbol{u}\|_{L^4(\Omega)}\|\nabla\partial_t^h\boldsymbol{u}\| \\ &\leq \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2 + C\|\nabla\boldsymbol{u}\|\|\boldsymbol{u}\|_{H^2(\Omega)}\|\partial_t^h\varphi\|\|\Delta\partial_t^h\varphi\| \\ &\leq \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2 + \frac{1}{24}\|\Delta\partial_t^h\varphi\|^2 + C\|\boldsymbol{u}\|^2_{H^2(\Omega)}\|\partial_t^h\varphi\|^2.\end{aligned}$$

Finally, the embedding $W^{1,3}(\Omega) \subset L^\infty(\Omega)$ together with (6.2) yields

$$\begin{aligned}|J_2| &\leq \|\nabla\partial_t^h\varphi\|\big(\|\nabla\varphi(t)\|_{L^\infty(\Omega)} + \|\nabla\varphi(t+h)\|_{L^\infty(\Omega)}\big)\|\nabla\partial_t^h\boldsymbol{u}\| \\ &\leq \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2 + \frac{1}{24}\|\Delta\partial_t^h\varphi\|^2 + C\|\partial_t^h\varphi\|^2.\end{aligned}$$

Collecting all the above estimates, we end up with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2 + \frac{1}{8}\|\Delta\partial_t^h\varphi\|^2 + \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2 \leq C(1 + \|\boldsymbol{u}\|^2_{H^2(\Omega)} + \Upsilon)\|\partial_t^h\varphi\|^2.$$

Note that

$$\int_t^{t+1}\Big(\|\partial_t^h\varphi(\tau)\|^2 + \|\boldsymbol{u}(\tau)\|^2_{H^2(\Omega)} + \Upsilon(\tau)\Big)\,\mathrm{d}\tau \leq C, \quad \forall\, t \geq \sigma,$$

in light of Theorem 6.1 and Lemma 7.2. An application of the uniform Gronwall lemma and a final passage to the limit as $h \to 0$ complete the proof. □

Proof of Theorem 7.1. Thanks to Lemma 7.3, arguing by comparison and owing to the regularity of $\boldsymbol{u}$ and φ, we infer that

$$\mu \in L^\infty(2\sigma,\infty; H^2(\Omega)) \subset L^\infty(2\sigma,\infty; L^\infty(\Omega)).$$

Therefore, following the proof of Lemma 4.1, we find

$$\|F'(\varphi)\|_{L^p(\Omega)} \leq C(1 + \|\mu\|_{L^\infty(\Omega)}), \quad \forall\, p \geq 2,$$

where the positive constant C is independent of p. This implies

$$\|F'(\varphi)\|_{L^\infty(\Omega\times(t,t+1))} \leq C, \quad \forall\, t \geq 2\sigma.$$

Since F' diverges at ± 1 and φ is continuous as established in Remark 6.2, we immediately deduce the existence of $\delta > 0$ such that

$$|\varphi(x,t)| \leq 1 - \delta, \quad \forall\,(x,t) \in \overline{\Omega} \times [2\sigma,\infty).$$

The proof is completed. □

8. Further Developments

In this final section we collect some remarks and natural developments of our work.

• The longtime behavior of the CHB system can be characterized by virtue of the regularity properties here established. More specifically, on account of Theorem 3.6 and Theorem 5.1, (1.4)–(1.6) generates a semigroup of operators via the rule

$$S(t)\varphi_0 = \varphi(t), \quad \forall\, t \geq 0,$$

being $(\varphi, \boldsymbol{u})$ the unique global energy solution to the CHB problem with initial condition

$$\varphi_0 \in \mathcal{H}_\kappa = \{\varphi \in V \,:\, F(\varphi) \in L^1(\Omega),\, \overline{\varphi} = \kappa\}, \qquad \kappa \in (-1,1).$$

The semigroup turns out to be strongly continuous, see [14] for the proof, and dissipative due to (3.6). Then, in light of Theorem 6.1, the existence of a unique (compact and connected) global attractor $\mathcal{A}_\kappa$ for $S(t)$ on $\mathcal{H}_\kappa$ follows by the classical semigroup theory (see, e.g., [29]). Furthermore, once the strict separation is reached, a further investigation of the asymptotic behavior is possible. In particular, the existence of exponential attractors $\mathcal{E}_\kappa$ (see [23]) and the convergence of each trajectory to a single stationary state (see [5]) could be proved.

• A relevant generalization of our work would be to consider the complete CHB model with permeability η and mobility m depending on the concentration in dimension two. First, the dependence of η on φ proposed in literature (see e.g. [13]) is similar to (1.3), namely

$$\eta(s) = \eta_A \frac{1+s}{2} + \eta_B \frac{1-s}{2},$$

where η_A, η_B are the positive fluid permeabilities. Being able to handle the higher nonlinear term $\nabla \cdot (\nu(\varphi) D(\boldsymbol{u}))$, we observe that the presence of $\eta(\varphi)\boldsymbol{u}$ in the Brinkman's law does not affect significantly the proofs of the present paper. On the contrary, concerning the variable mobility, the non-degenerate form proposed in [3] is

$$m(s) = 1 - \sigma s^2,$$

for some $\sigma \in (0,1)$. Its mathematical analysis is more delicate and deserves future investigations.

• A further interesting problem is the study of the CHB system with singular potential and constant viscosity ν in dimension three. In which case, the (weak) continuous dependence estimate and the regularization in finite time can be achieved by the same techniques exploited in this work. Nonetheless, the instantaneous separation property (1.7) seems to be an hard task, being an open issue even for the solely Cahn–Hilliard equation. On the other hand, borrowing the method of [2], a weaker version of (1.7) is expected. This is the existence of a certain time $t^* > 0$, depending on the initial datum and eventually large, such that the solution is bounded away from the pure phases when t is larger than t^*.

References

[1] H. Abels, *On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Ration. Mech. Anal., **194** (463–506), 2009.

[2] H. Abels, M. Wilke, *Convergence to equilibrium for the Cahn–Hilliard equation with a logarithmic free energy*, Nonlinear Anal., **67** (3176–3193), 2007.

[3] J.W. Barrett, J.F. Blowey, *Finite element approximation of the Cahn–Hilliard equation with concentration dependent mobility*, Math. Comp., **68** (487–517), 1999.

[4] J.F. Blowey, C.M. Elliott, *The Cahn–Hilliard gradient theory for phase separation with nonsmooth free energy. I. Mathematical analysis*, European J. Appl. Math., **2** (233–280), 1991.

[5] S. Bosia, M. Conti, M. Grasselli, *On the Cahn–Hilliard–Brinkman system*, Commun. Math. Sci., **13** (1541–1567), 2015.

[6] F. Boyer, *Mathematical study of multi-phase flow under shear through order parameter formulation*, Asymptot. Anal., **20** (175–212), 1999.

[7] F. Boyer, P. Fabrie, *Mathematical tools for the study of the incompressible Navier–Stokes equations and related models*, Springer, New York, 2013.

[8] H.C. Brinkman, *A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles*, Appl. Sci. Res., **A1** (27–36), 1947.

[9] J.W. Cahn, J.E. Hilliard, *Free energy of a nonuniform system. I. Interfacial energy*, J. Chem. Phys., **28** (258–267), 1958.

[10] C. Collins, J. Shen, S.M. Wise, *An efficient, energy stable scheme for the Cahn–Hilliard–Brinkman system*, Commun. Comput. Phys., **13** (929–957), 2013.

[11] C.G. Gal, A. Giorgini, M. Grasselli, *The nonlocal Cahn–Hilliard equation with singular potential: well-posedness, regularity and strict separation property*, J. Differential Equations, to appear.

[12] C.G. Gal, M. Grasselli, *Asymptotic behavior of a Cahn–Hilliard–Navier–Stokes system in 2D*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **27** (401–436), 2010.

[13] H. Garcke, K.F. Lam, *Global weak solutions and asymptotic limits of a Cahn–Hilliard–Darcy system modelling tumour growth*, AIMS Math., **1** (318–360), 2016.

[14] A. Giorgini, M. Grasselli, A. Miranville, *Cahn–Hilliard–Oono equations with singular potentials*, preprint.

[15] Y. Huo, H. Zhang, Y. Yang, *Effects of reversible chemical reaction on morphology and domain growth of phase separating binary mixtures with viscosity difference*, Macromol. Theory Simul., **13** (280–289), 2004.

[16] J.S. Kim, *Phase-field models for multi-component fluid flows*, Commun. Comput. Phys., **12** (613–661), 2012.

[17] H-G. Lee, J.S. Lowengrub, J. Goodman, *Modeling pinchoff and reconnection in a Hele–Shaw cell. I. The models and their calibration*, Phys. Fluids, **14** (492–513), 2002.

[18] H-G. Lee, J.S. Lowengrub, J. Goodman, *Modeling pinchoff and reconnection in a Hele–Shaw cell. II. Analysis and simulation in the nonlinear regime*, Phys. Fluids, **14** (514–545), 2002.

[19] F. Li, C. Zhong, B. You, *Finite-dimensional global attractor of the Cahn–Hilliard–Brinkman system*, J. Math. Anal. Appl., **434** (599–616), 2016.

[20] C. Liu, J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier spectral method*, Phys. D, **179** (211–228), 2003.

[21] J. Lowengrub, E. Titi, K. Zhao, *Analysis of a mixture model of tumor growth*, European J. Appl. Math., **24** (691–734), 2013.

[22] A. Miranville, R. Temam, *On the Cahn–Hilliard–Oono–Navier–Stokes equations with singular potentials*, Appl. Anal., **95** (2609–2624), 2016.

[23] A. Miranville and S. Zelik, *Robust exponential attractors for Cahn–Hilliard type equations with singular potentials*, Math. Methods Appl. Sci., **27** (545–582), 2004.

[24] T. Nagai, T. Senba, K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., **40** (411–433), 1997.

[25] W. Ngamsaad, J. Yojina, W. Triampo, *Theoretical studies of phase-separation kinetics in a Brinkman porous medium*, J. Phys. A, **43** (202001), 2010.

[26] F.J. Valdes-Parada, J.A. Ochoa-Tapia, J. Alvarez-Ramirez, *On the effective viscosity for the Darcy–Brinkman equation*, Phys. A, **385** (69–79), 2007.

[27] E. Rocca, G. Schimperna, *Universal attractor for some singular phase transition systems*, Phys. D, **192** (279–307), 2004.

[28] M. Schmuck, M. Pradas, G.A. Pavliotis, S. Kalliadasis, *Derivation of effective macroscopic Stokes–Cahn–Hilliard equations for periodic immiscible flows in porous media*, Nonlinearity, **26** (3259–3277), 2013.

[29] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1997.

[30] X. Wang, H. Wu, *Long-time behavior for the Hele–Shaw–Cahn–Hilliard system*, Asymptot. Anal., **78** (217–245), 2012.

[31] X. Wang, Z. Zhang, *Well-posedness of the Hele–Shaw–Cahn–Hilliard system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **30** (367–384), 2013.

[32] B. You, F. Li, *Well-posedness and global attractor of the Cahn–Hilliard–Brinkman system with dynamic boundary conditions*, Dyn. Partial Differ. Equ. **13** (75–90), 2016.

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