



HAL
open science

Rigid cohomology of locally noetherian schemes Part 1 : Geometry

Bernard Le Stum

► **To cite this version:**

Bernard Le Stum. Rigid cohomology of locally noetherian schemes Part 1: Geometry. 2017. hal-01559040

HAL Id: hal-01559040

<https://hal.science/hal-01559040>

Preprint submitted on 10 Jul 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Rigid cohomology of locally noetherian schemes

Part 1 : Geometry

Bernard Le Stum

Version of July 10, 2017

Abstract

We set up the geometric background necessary to extend rigid cohomology from the case of algebraic varieties to the case of general locally noetherian formal schemes. In particular, we generalize Berthelot's strong fibration theorem to adic spaces: we show that if we are given a morphism of locally noetherian formal schemes which is partially proper and formally smooth around a formal subscheme, and we pull back along a morphism from an analytic space which is locally of noetherian type, then we obtain locally a fibration on strict neighborhoods.

Contents

Introduction	2
1 Adic formal schemes	7
2 Adic spaces	14
3 Adic spaces and formal schemes	30
4 Formal embeddings	40
5 Overconvergent spaces	44
6 Tubes	47
7 Strict neighborhoods	59

Introduction

State of the art

Pierre Berthelot developed rigid cohomology in the early 80's in [3] and [2] (see also [17]) as a p -adic cohomology theory for algebraic varieties X defined over a field k . It takes its values over a complete non archimedean field K of characteristic zero whose residue field is k . The main idea consists in showing that the geometry of X is reflected in the geometry of some subspace of a rigid analytic variety over K . More precisely, one embeds X into some formal scheme P and builds the tube $]X[$ of X in P as a subspace of the generic fibre of P . When P is proper and smooth around X , then the de Rham cohomology of a small neighborhood V of this tube is essentially independent of the choices. This is a consequence of the strong fibration theorem of Berthelot: if a morphism $Q \rightarrow P$ is proper and smooth around X , then it induces locally a fibration in the neighborhoods of the tubes.

There exists an alternative description of rigid cohomology that consists in putting together all the embeddings of X in P and all the neighborhoods V of $]X[$. As explained in [18], this gives rise to the overconvergent site whose cohomology is exactly rigid cohomology.

When the base field is not perfect, Christopher Lazda and Ambrus Pál showed in [16] that the theory can be refined. More precisely, in order to study algebraic varieties over $k((t))$, where k is a perfect field, they replace the Amice ring \mathcal{E} , which is the natural field of coefficients in Berthelot's theory, with the bounded Robba ring \mathcal{E}^\dagger . They also replace Tate's rigid analytic geometry with Huber theory of adic spaces: this is necessary because the bounded Robba ring lives on an adic space whose only closed point is a valuation of height 2.

In order to obtain constructible coefficients for rigid cohomology, Berthelot also developed in [4] a theory of arithmetic \mathcal{D} -modules. First of all, on any smooth formal scheme P , he builds a ring \mathcal{D}_P^\dagger of overconvergent differential operators on P . Then, when X is a subvariety of P , he considers a category of coherent $F - \mathcal{D}_{P\mathbb{Q}}^\dagger$ -modules with support on X . In the thesis of David Pigeon ([19]), and thereafter in the work of Daniel Caro and David Vauclair ([5]), the theory is extended to formal schemes that are only differentially p -smooth. More recently, Richard Crew showed in [7] that Berthelot's theory also extends to the case of formally smooth formal schemes that are not necessarily of finite type, but only "universally" noetherian (formally of finite type for example). Surprisingly, these two extensions of Berthelot's original theory are rather orthogonal: anticipating on forthcoming notations, we can say that when Berthelot considers $\mathbb{A} := \text{Spec}(k[[t]])$, then Caro and Vauclair will look at $\mathbb{A}^b := \text{Spec}(k[[[t]]])$ (which is not universally noetherian) and Crew will be interested

in $\mathbb{A}^- := \mathrm{Spf}(k[[t]])$ (which is not a p -adic formal scheme).

The main result

In this article, we set up the geometric background necessary to extend rigid cohomology from the case of an algebraic variety over a field to any morphism of locally noetherian formal schemes. More precisely, we extend Berthelot's strong fibration theorem to this setting. In order to state the theorem, it is convenient to extend the method and the vocabulary of [18]. The idea is to use formal schemes as a bridge between schemes and analytic spaces. We assume that all formal schemes are locally noetherian and all adic spaces are locally of noetherian type.

An overconvergent adic space is a pair made of a formal embedding $X \hookrightarrow P$ and a morphism of adic spaces $P^{\mathrm{ad}} \leftarrow V$. We will write $(X \hookrightarrow P \leftarrow V)$ for short. A formal morphism of overconvergent adic spaces is a pair of commutative diagrams

$$\begin{array}{ccc} Y \hookrightarrow Q & , & Q^{\mathrm{ad}} \longleftarrow W \\ \downarrow f & & \downarrow v^{\mathrm{ad}} \\ X \hookrightarrow P & & P^{\mathrm{ad}} \longleftarrow V \end{array}$$

The formal morphism is said to be right cartesian if W is a neighborhood of the inverse image of V along v^{ad} . It is called formally smooth (resp. partially proper) around Y if there exists an open subset U (resp. a closed subset Z) of Q that contains Y and such that the restriction of v to U (resp. to Z) is formally smooth (resp. partially proper).

Next, we introduce the notion of tube $]X[_V$ of X in V : this is derived from the closed embedding case where we take the inverse image of the adic space associated to the completion of P along X . We call a formal morphism a strict neighbourhood if f is an isomorphism, v is locally noetherian, u is an open immersion and the map induced on the tubes is surjective. The overconvergent adic site is obtained by turning strict neighborhoods into isomorphisms and using the topology coming from V . We will simply denote (X, V) the above overconvergent adic space viewed as an object of the overconvergent adic site (and forget about P in the notations).

The strong fibration theorem states that if we are given a right cartesian morphism which is formally smooth and partially proper around Y and such that f is an isomorphism, then it induces locally an isomorphism $(Y, W) \simeq (X, \mathbb{P}_V^r)$ in the overconvergent site.

Content

In the first section, we briefly review the theory of adic formal schemes. The main purpose is to set up the vocabulary and the notations and give elementary but fundamental examples. It will also be useful as a guideline for the theory of adic spaces that will be discussed just after. We also introduce the important notion of being partially proper for a morphism of formal schemes.

Section two is devoted to adic spaces. We mostly recall the main definitions and give some examples as in the previous section. We also introduce the notion of adic space associated to a usual scheme.

In section three, we recall how one may associate an adic space to a formal scheme and show that most classical properties of adic morphisms of formal schemes are reflected in the corresponding morphism of adic spaces. We also check that the same holds for formal smoothness. More interesting, we also introduce the notion of analytic property and show that if a morphism of formal schemes is partially proper then the corresponding morphism of adic spaces is analytically partially proper.

Section four is shorter: we essentially discuss the notion of formal embedding and prove the formal fibration theorem.

The purpose of section five is to extend the notion of overconvergent space (that was initially developed using Berkovich analytic spaces) to the adic world. This is essential for the development of a site-theoretic approach of rigid cohomology. Nevertheless, it should be noticed that one might also have followed Berthelot's original approach and completely avoided the notion of overconvergent space.

Section five is devoted to the notion of tube in the adic world which is sensibly different from the classical one which we call the naive tube. Anyway, we show that in the analytic situation, the tube of closed (resp. open) subset is equal to the closure (resp. interior) of the naive tube and we also introduce the notion of tube of finite radius.

In section seven, we define the overconvergent site as the localization of the category of overconvergent adic spaces (and formal morphisms) with respect to strict neighborhoods. We show that, even if the topology comes from the analytic side, the category is, in some sense, also local on the (formal) scheme side.

In the last section, we use all the material obtained so far in order to prove the strong fibration theorem.

Many thanks

Several parts of this paper were influenced by the conversations that I had with many mathematicians and I want to thank in particular Ahmed Abbes, Richard Crew, Kazuhiro Fujiwara, Michel Gros, Fumiharu Kato, Christopher Lazda, Vincent Mineo-Kleiner, Matthieu Romagny and Alberto Vezzani.

Notations/conventions

1. A *monoid* G is a set endowed with an action on itself which is associative with unit. Unless otherwise specified, all monoids and groups are assumed to be *commutative*, but we may insist for emphasis and call them *abelian* when the law is additive. A *ring* A is an abelian group endowed with a linear action on itself which is associative with unit. Unless otherwise specified, all rings and

fields are assumed to be *commutative*, but again, we may insist for emphasis.

2. If X is a G -set, $E \subset G$ and $F \subset X$, we will write

$$EF := \{gx : g \in E, x \in F\}.$$

By induction, if $E \subset G$, we will have

$$E^n = \{g_1 \cdots g_n : g_i \in E\}.$$

If M is an A -module, $E \subset A$ and $F \subset M$, we will denote by

$$E \cdot F := \left\{ \sum f_i x_i : f_i \in E, x_i \in F \right\}$$

the additive subgroup generated by EF . And if $E \subset A$, we will denote by

$$E^n := \left\{ \sum f_{i_1} \cdots f_{i_n} : f_{i_j} \in E \right\}$$

the additive subgroup generated by E^n . We shall make an exception and sometimes use the standard notation I^n instead of E^n when I is an ideal. We will also denote by $I^{(n)}$ the ideal generated by $\{f^n, f \in I\}$.

3. A *(topologically) ringed space* is a topological space X endowed with a sheaf of (topological) rings \mathcal{O}_X . A morphism is made of a continuous map $f : Y \rightarrow X$ and a (continuous) morphism of (topological) rings $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$. There exists an obvious forgetful functor from topologically ringed spaces to ringed spaces that commutes with all limits and colimits. Conversely, any ringed space may be seen as a topologically ringed space by using the fully faithful adjoint to the forgetful functor (discrete topology and sheaffifying). Note that a (topologically) ringed space is a sheaf on the big site of (topologically) ringed spaces: the site is subcanonical.
4. A *(topologically) locally ringed space* is a (topologically) ringed space X whose stalks are all local rings. A morphism of (topologically) locally ringed spaces is a morphism of (topologically) ringed spaces that induces a local homomorphism on the stalks. The above forgetful functor and its adjoint are compatible with these new conditions. And again, the sites are subcanonical. If $x \in X$, we will denote the residue field of the local ring $\mathcal{O}_{X,x}$ by $\kappa(x)$. If f is a section of \mathcal{O}_X in a neighborhood of x , we will denote by $f(x)$ the image of f in $\kappa(x)$.
5. A *(topologically) valued ringed space* is a (topologically) locally ringed space X whose stalks $\mathcal{O}_{X,x}$ are endowed with a valuation (up to equivalence) v_x whose kernel is exactly $\mathfrak{m}_{X,x}$. This is actually equivalent to choosing a valuation ring $\mathcal{V}(x)$ of $\kappa(x)$. By definition, the *height* of x is the height of v_x and x is called *trivial* when v_x is trivial (height = 0). A morphism of (topologically) valued ringed spaces is a morphism of (topologically) locally ringed spaces such that the induced morphism on the stalks is compatible (up to equivalence) with the valuations. This is equivalent to require that it induces a morphism between

the valuation rings. There exists an obvious functor that forgets the valuation. It possesses a fully faithful adjoint which is simply obtained by endowing $\mathcal{O}_{X,x}$ with the valuation induced by the trivial valuation on $\kappa(x)$. In particular, any (topologically) locally ringed space may be seen as a (topologically) valued ringed space.

6. A *doubly (topologically) (locally) ringed space* is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ where both (X, \mathcal{O}_X) and (X, \mathcal{O}_X^+) are (topologically) (locally) ringed spaces and $\mathcal{O}_X^+ \subset \mathcal{O}_X$. Morphisms are defined in the obvious way. We will still denote by X the (topologically) (locally) ringed space (X, \mathcal{O}_X) and then write $X^+ := (X, \mathcal{O}_X^+)$. This provides an adjoint and a coadjoint to the obvious functor $(X, \mathcal{O}_X) \mapsto (X, \mathcal{O}_X, \mathcal{O}_X)$.
7. If X is a (topologically) valued ringed space, then we shall denote by \mathcal{O}_X^+ the subsheaf locally defined by the conditions $v_x(f) \geq 0$. This defines a functor from (topologically) valued ringed spaces to doubly (topologically) locally ringed spaces which happens to be fully faithful.
8. Let X be a topological space and $x, y \in X$. We write $y \rightsquigarrow x$, and we say that x is a *specialization* of y or that y is a *generization* of x , if any neighborhood of x is also a neighborhood of y . Alternatively, it means that $x \in \overline{\{y\}}$ (the topological closure). This defines a partial order on the points of X .
9. A topological space is said to be *coherent* if it is quasi-compact, quasi-separated and admits a basis of quasi-compact open subsets. It is said to be *sober* if any irreducible subset has a unique generic point. It is called *spectral* if it is coherent and sober.
10. A subset of a topological space X is *(globally) constructible* (resp. *ind-constructible*, resp. *pro-constructible*) if it is a (resp. a union of, resp. an intersection of) boolean combination(s) of retrocompact open subspaces. If X is coherent, a locally constructible (resp. locally ind-constructible, resp. locally pro-constructible) subset is automatically (globally) constructible (resp. ind-constructible, resp. pro-constructible). The *constructible topology* on a topological space X is the coarsest topology for which the locally constructible subsets are open.
11. If X is a locally spectral space, then a subset is open (resp. closed, resp. open and closed) in the constructible topology if and only if it is locally ind-constructible (resp. locally pro-constructible, resp. locally constructible). A subset of a locally spectral space is closed (resp. open) if and only if it is locally pro-constructible (resp. locally ind-constructible) and stable under specialization (resp. generization). Any quasi-compact quasi-separated continuous map of locally spectral spaces is always continuous and closed for the constructible topology.

1 Adic formal schemes

In this section, we briefly review the basics of adic formal schemes, give some examples and generalize some notions to the case of non-adic morphisms. We send the reader to chapter 2 of [1] and chapter 1 of [8] for a recent presentation of the subject (see also chapter 10 of [10]). Note that the terminology may vary from one reference to another and we tend to prefer a vocabulary which is compatible with the usual conventions for adic spaces.

At some point, we will impose noetherian conditions but we can make some general definitions first.

Definition 1.1. *A topological ring A is called adic if there exists a finitely generated ideal I whose powers I^n form a fundamental system of neighborhoods of 0. Such an ideal is called an ideal of definition. A homomorphism $A \rightarrow B$ of adic rings is called adic when IB is an ideal of definition of B .*

Following Huber, we do *not* assume the topology to be complete or Hausdorff in general and will denote by $\hat{A} := \varprojlim A/I^{n+1}$ the (Hausdorff) completion of A . On the other hand, we require I to be *finitely generated* (which is not standard). It follows that \hat{A} will be a complete adic ring with ideal of definition $I\hat{A}$ and that $A/I = \hat{A}/I\hat{A}$ (and we may in practice replace A with \hat{A}).

If we are ready to always assume completeness, one may consider the more general notion of *admissible* ring. Note also that we can always see a usual ring as a complete adic ring because the discrete topology is adic (and complete) with respect to the zero (or any nilpotent) ideal.

A morphism of adic rings $A \rightarrow B$ is simply a continuous homomorphism of rings and it will always send an ideal of definition into some ideal of definition. We will always make it clear when we assume that the morphism is actually adic (which is a much stronger condition).

Definition 1.2. *If A is a topological ring, then the formal spectrum of A is the set $P := \text{Spf}(A)$ of open prime ideals of A .*

If A is a topological ring, $f \in A$ and $\mathfrak{p} \in P = \text{Spf}(A)$, then we will denote by $f(\mathfrak{p})$ the image of f in $\kappa(\mathfrak{p}) := \text{Frac}(A/\mathfrak{p})$. The set P is endowed with the topology such that the subsets

$$D(f) = \{\mathfrak{p} \in P : f(\mathfrak{p}) \neq 0\}$$

form a basis of open subsets. This is a spectral space.

When A is an adic ring, there exists a unique sheaf of topological rings \mathcal{O}_P such that

$$\Gamma(D(f), \mathcal{O}_P) = \widehat{A[1/f]} \tag{1}$$

(this is the completion for the I -adic topology if I denotes an ideal of definition of A). This turns P into a topologically locally ringed space.

If A is an adic ring, then there exists a canonical isomorphism of topologically locally ringed spaces $\mathrm{Spf}(\widehat{A}) \simeq \mathrm{Spf}(A)$ and this is why we may usually only consider complete adic rings. If one is ready to work only with complete rings, one can define more generally the topologically ringed space associated to an admissible ring A .

Note that we always have $\Gamma(P, \mathcal{O}_P) = A$ but theorem A and B do not hold without further hypothesis on A . For example, we have:

Proposition 1.3. *If A is a noetherian adic ring and if $P = \mathrm{Spf}(A)$, then the functors*

$$M \mapsto \mathcal{O}_P \otimes_A^{\mathbb{L}} M \quad \text{and} \quad \mathcal{F} \mapsto \mathbb{R}\Gamma(P, \mathcal{F})$$

induce an equivalence between finite \widehat{A} -modules and coherent \mathcal{O}_V -modules.

Proof. This is shown in [10] for example. □

The same result holds if we require the topology of A to be discrete (in which case it reduces to the analogous result on usual spectra).

Definition 1.4. *An adic formal scheme is a topologically locally ringed space P which is locally isomorphic to $\mathrm{Spf}(A)$ where A is an adic ring. It is said to be affine if it is actually isomorphic to some $\mathrm{Spf}(A)$.*

An adic formal scheme is a locally spectral space. Note that one can define the more general notion of *formal scheme* by using *admissible* rings.

If X is a scheme, then it is a locally ringed space and the topologically locally ringed space associated to X is an adic formal scheme. This gives rise to a fully faithful functor. Note that we recover X by simply forgetting the topology on the sheaf of rings. In practice, we will *identify* X with the corresponding adic formal scheme. Note that, with this identification, if A is an adic ring with ideal of definition I , we have

$$\mathrm{Spf}(A) = \varinjlim \mathrm{Spec}(A/I^n).$$

The functor

$$A \mapsto \mathrm{Spf}(A)$$

is fully faithful on complete adic rings. Better, there exists an adjunction

$$\mathrm{Hom}(X, \mathrm{Spf}(A)) \simeq \mathrm{Hom}(A, \Gamma(X, \mathcal{O}_X))$$

(continuous homomorphisms on the right hand side). As a consequence, the category of adic formal schemes has finite limits and we always have

$$\mathrm{Spf}(A) \times_{\mathrm{Spf}(R)} \mathrm{Spf}(B) = \mathrm{Spf}(A \otimes_R B).$$

Example If S is any adic formal scheme, then we may consider:

1. the relative *affine space*

$$\mathbb{A}_S^n = \underbrace{\mathbb{A} \times \cdots \times \mathbb{A}}_{n \text{ times}} \times S$$

in which $\mathbb{A} := \text{Spec}(\mathbb{Z}[T])$ (seen as a formal scheme). This is the same thing as the usual formal affine space which is usually denoted with an extra hat. If A is an adic ring with ideal of definition I , $S = \text{Spec}(A)$ and $A[T_1, \dots, T_n]$ has the I -adic topology, we have

$$\mathbb{A}_S^n = \text{Spf}(A[T_1, \dots, T_n]).$$

and we may sometimes write $\widehat{\mathbb{A}}_A^n$. Note that we could as well use the I -adic completion $\widehat{A}\{T_1, \dots, T_n\}$ of $A[T_1, \dots, T_n]$. This gives the same adic formal scheme.

2. the relative *projective space*

$$\mathbb{P}_S^n := \mathbb{P}^n \times S$$

(which is also usually denoted with an extra hat) in which

$$\mathbb{P}^n = \text{Proj}(\mathbb{Z}[T_0, \dots, T_n]).$$

We will also write $\widehat{\mathbb{P}}_A^n$ when $S = \text{Spf}(A)$. We may drop the exponent n when $n = 1$ and simply denote by \mathbb{P} the projective line.

3. the relative *open affine space*

$$\mathbb{A}_S^{n,-} := \underbrace{\mathbb{A}^- \times \cdots \times \mathbb{A}^-}_{n \text{ times}} \times S$$

in which $\mathbb{A}^- := \text{Spf}(\mathbb{Z}[T])$ where $\mathbb{Z}[T]$ has the T -adic topology. If A is an adic ring with ideal of definition I , $S = \text{Spf}(A)$ and $A[T_1, \dots, T_n]$ has the $I[T_1, \dots, T_n] + (T_1, \dots, T_n)$ -adic topology, then

$$\mathbb{A}_S^{n,-} = \text{Spf}(A[T_1, \dots, T_n]) = \text{Spf}(\widehat{A}[[T_1, \dots, T_n]])$$

and we will also write $\widehat{\mathbb{A}}_A^{n,-}$.

4. the relative *bounded affine space*

$$\mathbb{A}_S^{n,b} = \text{Spf}(A[[T_1, \dots, T_n]]),$$

(also denoted by $\widehat{\mathbb{A}}_A^{n,b}$) in which $S = \text{Spf}(A)$ and $A[[T_1, \dots, T_n]]$ has the I -adic topology. Be careful that $B\widehat{\otimes}_A A[[T_1, \dots, T_n]] \neq B[[T_1, \dots, T_n]]$ if completion is meant with respect to the topology of B (unless B is finite over A or $n = 0$). It follows that the bounded affine space is really a global object.

There exists a sequence of “inclusions”

$$\mathbb{A}_S^{n,-} \hookrightarrow \mathbb{A}_S^{n,b} \hookrightarrow \mathbb{A}_S^n \hookrightarrow \mathbb{P}_S^n.$$

We may also notice that \mathbb{A} represents the sheaf of rings $P \mapsto \Gamma(P, \mathcal{O}_P)$ and that \mathbb{A}^- represents the ideal of topologically nilpotent elements.

A *morphism* of adic formal schemes $Q \rightarrow P$ is simply a morphism of topologically locally ringed spaces. It is said to be *adic* if it comes locally from an adic map $A \rightarrow B$. In the affine case, it will then come *globally* from an adic morphism of adic rings. A morphism of adic formal schemes $Q \rightarrow P$ is *affine* if it comes locally *on* P from a continuous map $A \rightarrow B$. When P is an affine adic formal scheme, then Q will also be affine. Note that an affine map is not necessarily adic.

A morphism of adic formal schemes $Q \rightarrow P$ is a *closed immersion* (resp. *finite*) if it comes locally *on* P from an *adic* surjective (resp. finite) map $A \rightarrow B$. Again, in the affine case, it will come globally from an adic surjective (resp. finite) map. A *locally closed immersion* is the composition of a closed immersion with an open immersion (in that order). This gives in particular rise to the notions of (*locally*) *closed formal subscheme*.

A morphism $Q \rightarrow P$ of adic formal schemes is said to be *locally of finite type* if Q is locally isomorphic to a closed formal subscheme of a \mathbb{A}_P^n . This is always an *adic* morphism. If moreover, it is quasi-compact, then it is said to be *of finite type*. A morphism of affine formal schemes is of finite type if and only if it is *globally* a closed subscheme of some affine space. There also exists a stronger notion of morphism (*locally*) *finitely presented* by requiring that the subscheme of \mathbb{A}_P^n is defined by a finitely generated ideal but this will not matter to us when we work in the noetherian world.

A morphism $Q \rightarrow P$ of adic formal schemes is *locally quasi-finite* if it is locally of finite type with discrete fibres (locally on P and Q , this is the composition of an open immersion and a finite map). It is said to be *quasi-finite* if moreover it is quasi-compact.

If $u : Q \rightarrow P$ is any morphism of adic formal schemes, then the diagonal map $\Delta : Q \rightarrow Q \times_P Q$ is a locally closed immersion. The morphism u is said to be *quasi-separated* (resp. *separated*) if Δ is quasi-compact (resp. closed). The morphism u is said to be *proper* if it is separated of finite type and universally closed.

A morphism $u : Q \rightarrow P$ is said to be *formally unramified* (resp. *formally smooth*, resp. *formally étale*) if any commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\quad} & P \\ \uparrow & \dashrightarrow & \uparrow \\ \text{Spec}(R/\mathfrak{a}) & \hookrightarrow & \text{Spec}(R) \end{array}$$

with \mathfrak{a} nilpotent may be completed by the diagonal arrow in at most (resp. at least, resp. exactly) one way. When $P = \text{Spf}(A)$ and $Q = \text{Spf}(B)$ with A and B complete, it means that the morphism of rings $A \rightarrow B$ is formally unramified (resp. smooth, resp. étale). Be careful that being formally smooth is *not* a local property in general. The morphism u is called *unramified* (resp. *smooth*, resp. *étale*) if it is formally unramified (resp. smooth, resp. étale) and locally finitely presented (which is equivalent to locally of finite type in the noetherian situation below). There exists an intermediate notion: a morphism $u : Q \rightarrow P$ of adic formal schemes is *differentially smooth* if there exists, locally on Q , a formally étale morphism $Q \rightarrow \mathbb{A}_P^n$.

If $Q \hookrightarrow P$ is an immersion defined (on some open subset of P) by an ideal \mathcal{I}_Q , we may consider the *first infinitesimal neighborhood* $Q^{(1)}$ of Q in P defined by \mathcal{I}_Q^2 (which is always an adic formal scheme) and the corresponding short exact sequence

$$0 \rightarrow \check{\mathcal{N}}_{Q/P} \rightarrow \mathcal{O}_{Q^{(1)}} \rightarrow \mathcal{O}_Q \rightarrow 0$$

where $\check{\mathcal{N}}_{Q/P}$ is by definition the *conormal sheaf*. In the case of the diagonal immersion $Q \hookrightarrow Q \times_P Q$ associated to a morphism $Q \rightarrow P$, we obtain the *sheaf of differential forms* $\Omega_{Q/P}^1$. When $P = \mathrm{Spf}(A)$ and $Q = \mathrm{Spf}(B)$ are affine, we have $\Gamma(Q, \Omega_{Q/P}^1) = \hat{\Omega}_{B/A}^1$.

Definition 1.5. *An adic formal scheme P is said to be locally noetherian if it is locally isomorphic to $\mathrm{Spf}(A)$ where A is a noetherian adic ring. It is said to be noetherian if, moreover, it is quasi-compact.*

When P is locally noetherian, there exists a unique reduced closed subscheme $P_{\mathrm{red}} \subset P$ having the same underlying space as P (in other words, there exists a biggest ideal of definition). This provides an adjoint to the embedding of the category of locally noetherian schemes into the category of locally noetherian formal schemes.

Recall that the product of two locally noetherian formal schemes is not necessarily locally noetherian (see example below). In order to get around this difficulty, the following notion was introduced by Richard Crew in [7]:

Definition 1.6. *A morphism of adic formal schemes $u: P' \rightarrow P$ is said to be (universally) locally noetherian if, whenever $Q \rightarrow P$ is a morphism of adic formal schemes with Q locally noetherian, then $u^{-1}(Q) := P' \times_P Q$ is locally noetherian. In the case u is quasi-compact, the morphism is said to be (universally) noetherian.*

From now on and unless otherwise specified, **all adic formal schemes will be assumed to be locally noetherian** and simply called *formal schemes*.

It is shown in [7] that the notion of locally noetherian morphism is stable under product and composition. Moreover, if $u: Q \rightarrow P$ is a locally noetherian morphism, then $\Omega_{Q/P}^1$ is a coherent sheaf and we have $\Omega_{Q/P}^1 = 0$ if and only if u is unramified. This is also the correct condition in order to have a Jacobian criterion:

Theorem 1.7 (Crew). *Assume that $P \rightarrow S$ and $Q \rightarrow S$ are two locally noetherian morphisms of formal schemes.*

1. *If $u: Q \rightarrow P$ is an S -morphism, then there exists a right exact sequence*

$$u^* \Omega_{P/S}^1 \rightarrow \Omega_{Q/S}^1 \rightarrow \Omega_{Q/P}^1 \rightarrow 0.$$

Assume moreover that Q is formally smooth over S . Then u is formally smooth (resp. étale) if and only if $u^ \Omega_{P/S}^1$ is locally a direct summand in (resp. is isomorphic to) $\Omega_{Q/S}^1$.*

2. If $i : Q \hookrightarrow P$ is an S -immersion, then there exists a right exact sequence

$$\check{\mathcal{N}}_{Q/P} \rightarrow i^* \Omega_{P/S}^1 \rightarrow \Omega_{Q/S}^1 \rightarrow 0.$$

Assume moreover that P is formally smooth over S . Then Q is formally smooth (resp. étale) over S if and only if $\check{\mathcal{N}}_{Q/S}$ is locally a direct summand in (resp. is isomorphic to) $i^* \Omega_{P/S}^1$.

Proof. This is proved in [7]. □

It is also shown that a locally noetherian morphism u is formally smooth if and only if it is differentially smooth.

Example We let S be a formal scheme (that we assume to be affine when we consider a bounded affine space over S).

1. The formal scheme $\mathbb{A}_S^b \times_S \mathbb{A}_S^b$ is *not* locally noetherian in general: the ring $A[[T]] \widehat{\otimes}_A A[[T]]$, where completion is meant with respect to the topology of A , is not noetherian in general (e.g. a big field with the discrete topology).
2. Let $S = \mathrm{Spf}(\mathcal{V})$ where \mathcal{V} is a discrete valuation ring with perfect residue field k of characteristic $p > 0$. Then the formal scheme \mathbb{A}_S^b is *not* smooth over S but it is differentially smooth. This holds more generally in the case $S = \mathrm{Spf}(A)$ when A has the p -adic topology and the absolute Frobenius of A/pA is finite.
3. The formal scheme $\mathbb{A}_{\bar{S}}$ is formally smooth. In particular, it is differentially smooth. Note however that this is not a smooth formal scheme over S because it is not even adic over S .
4. Of course, the formal scheme \mathbb{A}_S is smooth over S .

When we consider a morphism of formal schemes $u : Q \rightarrow P$, we may wonder how some properties of the morphism of schemes $u_{\mathrm{red}} : Q_{\mathrm{red}} \rightarrow P_{\mathrm{red}}$ might impact the morphism u . For example, one knows that u is *separated* or *affine* if and only if u_{red} is separated or affine. Also, u is universally locally noetherian if and only if u_{red} is (and this can be tested with usual locally noetherian schemes).

Definition 1.8. A morphism of formal schemes $u : Q \rightarrow P$ is said to be

1. formally (locally) of finite type (resp. formally (locally) quasi-finite) if u_{red} is (locally) of finite type (resp. (locally) quasi-finite).
2. partially proper (resp. partially finite) if u is formally locally of finite type and induces a proper (resp. finite) morphism of schemes $Z_{\mathrm{red}} \rightarrow T_{\mathrm{red}}$ whenever Z and T are irreducible components of Q and P such that $u(Z) \subset T$.

One may also call u a *formal thickening* when u_{red} is an isomorphism, in which case all the above properties are satisfied.

When u is an *adic* morphism, u being formally (locally) of finite type or quasi-finite, is equivalent to u being (locally) of finite type or quasi-finite, but this is not the case in general (see examples below). When u is quasi-compact, u being partially proper or finite is equivalent to u_{red} being proper or finite. If moreover, u is adic, this is equivalent to u being proper or finite.

It is important to notice that a morphism which is locally formally of finite type is automatically locally noetherian.

Example 1. If S is any formal scheme, then the structural map $\mathbb{A}_S^- \rightarrow S$ is a formal thickening. This provides an example of a morphism which is partially finite, and therefore partially proper and formally quasi-finite, and in particular formally of finite type. However, this morphism is *not* locally of finite type (and not locally quasi-finite, not proper, not finite either) because it is not adic.

2. Classically, the formal model of \mathbb{G}_m whose reduction is an infinite union of projective lines, each meeting another one in exactly one point, is a partially proper formal scheme. However, it is not quasi-compact.

When we write $\mathbb{A}^{\pm, N}$, we mean a product of N copies of \mathbb{A} or \mathbb{A}^- . Of course, they may be reordered so that

$$\mathbb{A}^{\pm, N} = \mathbb{A}^n \times \mathbb{A}^{-, m}$$

with $N = n + m$. In other words, if $S = \text{Spf}(A)$, we have

$$\mathbb{A}_S^{\pm, N} = \text{Spf}(A[T_1, \dots, T_n, S_1, \dots, S_m])$$

with the $I[T_1, \dots, T_n, S_1, \dots, S_m] + (S_1, \dots, S_m)$ -topology. Alternatively, we can write

$$\begin{aligned} \mathbb{A}_S^{\pm, N} &= \text{Spf}(A\{T_1, \dots, T_n\}[[S_1, \dots, S_m]]) \\ &= \text{Spf}(A[[S_1, \dots, S_m]]\{T_1, \dots, T_n\}). \end{aligned}$$

Lemma 1.9. *A morphism of formal schemes $u: Q \rightarrow P$ is formally locally of finite type if and only if it factors, locally on P and Q , through a closed immersion $Q \hookrightarrow \mathbb{A}_P^{\pm, N}$.*

Proof. The condition is clearly sufficient and, in order to prove that it is necessary, we may assume that u comes from a continuous morphism of noetherian adic rings $A \rightarrow B$ with B complete. If J is the biggest ideal of definition of B , then our hypothesis tell us that the composite map $A \rightarrow B \rightarrow B/J$ is of finite type and extends therefore to a surjective map $A' := A[T_1, \dots, T_n] \twoheadrightarrow B/J$. Assume that J is generated by g_1, \dots, g_m . Then, since B is complete, there exists a unique continuous map $A'' := A'[[S_1, \dots, S_m]] \rightarrow B$ that sends S_i to g_i for all $i = 1, \dots, m$. After completion, this map is surjective because it is adic and surjective onto $B/I''B$ where I'' is the biggest ideal of definition of A'' . \square

As a particular case, we see that a formal scheme is formally locally of finite type over a discrete valuation ring if and only if it is *special* in the sense of Berkovich.

Definition 1.10. *A morphism of formal schemes $u: Q \rightarrow P$ is said to satisfy the valuative criterion for properness if, given a field F and a valuation ring R of F , then any commutative diagram*

$$\begin{array}{ccc} Q & \xrightarrow{\quad} & P \\ \uparrow & \swarrow \text{---} & \uparrow \\ \text{Spec}(F) & \xrightarrow{\quad} & \text{Spec}(R) \end{array}$$

may be uniquely completed by the dotted arrow.

When $Q = \text{Spf}(B)$ is affine, the lifting property simply means that the image of $B \rightarrow F$ is contained in R .

Proposition 1.11. *If u is a partially proper morphism of formal schemes, then it satisfies the valuative criterion for properness.*

Proof. This follows directly from the valuative criterion for properness for usual schemes because properness is not sensitive to nilpotent immersions (and F and R have the discrete topology). \square

One can show that, for a morphism of finite type, the criterion is actually equivalent to properness.

2 Adic spaces

In this section, we give a brief description of the theory of adic spaces and some standard examples. We mostly follow the notes of Wedhorn in [23] but the vocabulary is also influenced by Scholze's lectures in Berkeley ([21]) and Conrad's lectures in Stanford ([6]) (see also [24]). Anyway, almost everything can be found in the original papers [14] and [12]. We keep the additive notation from Huber's thesis.

Definition 2.1. *If A is any ring and G is a totally ordered additive group, a valuation*

$$v: A \rightarrow G \cup \{+\infty\}$$

is a map that satisfies

1. $v(1) = 0, \quad v(0) = +\infty,$
2. $\forall f, g \in A, \quad v(fg) = v(f) + v(g),$
3. $\forall f, g \in A, \quad v(f + g) \geq \min(v(f), v(g)).$

The *height* of v is the height of the subgroup G_v generated by the image of v . We call v *trivial* if $G_v = \{0\}$ and *discrete* if $G_v \simeq \mathbb{Z}$.

Two valuations v and v' on the same ring A are said to be *equivalent* if they define the same preorder on A :

$$\forall f, g \in A, \quad v(f) \leq v(g) \Leftrightarrow v'(f) \leq v'(g)$$

(or equivalently, if they both factor through a common valuation).

Example 1. Up to equivalence, a trivial valuation corresponds to a unique prime $\mathfrak{p} \subset A$:

$$v_{\mathfrak{p}}(f) = 0 \Leftrightarrow f(\mathfrak{p}) \neq 0 \quad \text{and} \quad v_{\mathfrak{p}}(f) = +\infty \Leftrightarrow f(\mathfrak{p}) = 0.$$

Recall from our conventions that $f(\mathfrak{p})$ denotes the image of f in $\kappa(\mathfrak{p}) := \text{Frac}(A/\mathfrak{p})$ so that the condition also reads $f \notin \mathfrak{p}$ or $f \in \mathfrak{p}$.

2. If v is a valuation on a ring A , then the *Gauss valuation* on $A[T]$ is defined for $F := \sum_{i=0}^d f_i T^i \in A[T]$ by

$$v(F) = \min_{i=0}^d v(f_i) \in G \cup \{+\infty\}.$$

It has the same height as the original valuation (geometrically, it corresponds to the generic point of the unit disc).

3. With the same notations, one may always define valuations of higher height by

$$v^-(F) = \left(v(F), \min_{v(F)=v(f_i)} i \right) \in (G \times \mathbb{Z}) \cup \{+\infty\}$$

or

$$v^+(F) = \left(v(F), -\max_{v(F)=v(f_i)} i \right) \in (G \times \mathbb{Z}) \cup \{+\infty\},$$

where $G \times \mathbb{Z}$ has the lexicographical order (geometrically, they corresponds to the specializations of the generic point inside and outside the unit disc). When we start from a trivial valuation on an integral domain, then $v^- = \text{val}$ (standard valuation on polynomials) and $v^+ = -\text{deg}$ (where deg denotes the degree map on polynomials).

The *support* of a valuation v on A is the prime ideal

$$\text{supp}(v) := \{f \in A : v(f) = +\infty\}.$$

The residue field of v is $\kappa(v) := \kappa(\text{supp } v)$ and the image of $f \in A$ in $\kappa(v)$ will be denoted by $f(v) = f(\text{supp } v)$.

Some authors prefer the multiplicative notation for valuations. We will do that only for fields. In general, one can move back and forth between the additive and multiplicative notation by writing formally

$$\Gamma := \exp(-G) = \{\exp(-g), g \in G\} \quad \text{and} \quad G := -\ln(\Gamma) = \{-\ln \gamma, \gamma \in \Gamma\}.$$

Definition 2.2. If K is a field and Γ is a totally ordered (commutative) multiplicative group (of any height), then an (ultrametric) absolute value

$$|\cdot|: K \rightarrow \{0\} \cup \Gamma,$$

is a map that satisfies

1. $\forall a \in K, \quad |a| = 0 \Leftrightarrow a = 0,$
2. $\forall a, b \in K, \quad |ab| = |a||b|,$
3. $\forall a, b \in K, \quad |a + b| \leq \max(|a|, |b|).$

Besides the multiplicative notation and reversing order, this is nothing but a valuation on K and we will apply systematically the vocabulary of valuations to ultrametric absolute values. Note that we do *not* require an absolute value to have height at most one (this is not standard).

When K is endowed with an ultrametric absolute value (of any height), we call it a *valued field* and call

$$K^+ := \{a \in K : |a| \leq 1\}$$

its *valuation ring*. Up to equivalence, $|\cdot|$ is uniquely determined by K^+ , the value group $\Gamma_{|\cdot|}$ is isomorphic to $K^\times/K^{+\times}$ and the height of $|\cdot|$ is the same thing as the dimension of K^+ . The ultrametric absolute value induces a topology on K with basis of open subsets given by the open “discs”

$$D(c, \gamma) = \{a \in K, \quad |a - c| < \gamma\}$$

(this is the coarsest topology making the absolute value continuous - see below).

Any valuation v on a ring A will induce an ultrametric absolute value of the same height $|\cdot|$ on $\kappa(v)$ (and conversely). The formulas are simply given by

$$|f(v)| = \exp(-v(f)) \quad \text{and} \quad v(f) = -\ln(|f(v)|).$$

Definition 2.3. A topological ring A is called a *Huber ring* (an *f-adic ring* in the sense of Huber) if there exists an open adic subring $A_0 \subset A$. We then call A_0 a ring of definition of A and any ideal of definition I_0 of A_0 will be also called an ideal of definition of A . A ring homomorphism $A \rightarrow B$ between two Huber rings is called *adic* if it induces an adic map between some rings of definition.

When A is a Huber ring, then the completion \widehat{A} of A as an A_0 -module is also a Huber ring and it does not depend on the choice of A_0 . Moreover \widehat{A}_0 is a definition ring for \widehat{A} and we have $\widehat{A} = A \otimes_{A_0} \widehat{A}_0$.

By definition, a morphism of Huber rings $A \rightarrow B$ is a continuous homomorphism and it will always send a ring (resp. an ideal) of definition into some ring (resp. some ideal) of definition. We will make it clear when we assume that the morphism is actually adic (which is again a much stronger condition).

Definition 2.4. *Let A be a Huber ring. Then,*

1. *A is said to be of noetherian type if A is finitely generated over some noetherian ring of definition.*
2. *A is called a Tate ring if there exists a topologically nilpotent unit $\pi \in A$.*

Note that if A is of noetherian type or if A is a Tate ring, so is \widehat{A} .

- Example**
1. An adic ring A is always a Huber ring which is its own definition ring. It is of noetherian type if and only if A is noetherian. It cannot be a Tate ring unless A is the zero ring. This applies in particular to usual rings (endowed with the discrete topology).
 2. If A is an adic ring and π is a topologically nilpotent element which is not a zero-divisor, then $A[1/\pi]$ is a Tate ring with definition ring A . Any Tate ring arises in this way. The Tate ring $A[1/\pi]$ is of noetherian type if (but not only if) A is noetherian.
 3. Let $(K, |\cdot|)$ be a valued field (of some height). We endow K with the topology induced by $|\cdot|$. Then K is a Huber ring if and only if its topology *may be* defined by some absolute value of height at most one. When this is the case, we call $(K, |\cdot|)$ a *Huber valued field*. Actually, either $|\cdot|$ is trivial or the topology may be defined by an absolute value of height exactly one. When this is the case, $(K, |\cdot|)$ is called a *Tate valued field* and K is called a *non archimedean field*. A non archimedean field is of noetherian type if and only if the topology may be defined by a *discrete* valuation.
 4. Let A be a Huber ring with ring of definition A_0 and ideal of definition I_0 . If (f_1, \dots, f_r) is an *open* ideal in A , then the polynomial ring $A[T]$ is a Huber ring for the topology defined by the subring $A_0[f_1T^k, \dots, f_rT^k]$ endowed with the ideal of definition $I_0[f_1T^k, \dots, f_rT^k]$. Openness condition is necessary for $A[T]$ to be a topological ring (for this topology). These constructions preserve the noetherian type and the Tate conditions.

If A is a Huber ring, we will denote by A° the subring of power bounded elements of A (union of all rings of definition) and by $A^{\circ\circ}$ the ideal of topologically nilpotent elements in A° (union of all ideals of definition).

Definition 2.5. *A Huber pair (an affinoid ring in the sense of Huber) is a pair (A, A^+) where A is a Huber ring and A^+ is any subset A° . The integral closure $\overline{A^+}$ of the subring generated by A^+ and $A^{\circ\circ}$ is called the ring of integral elements.*

As we will see below, there is no harm in replacing A^+ with $\overline{A^+}$ (as Huber systematically does). There is however more flexibility in not making such an assumption. We will call the Huber pair (A, A^+) *complete* if A is complete and A^+ is the full ring of integral elements ($A^+ = \overline{A^+}$).

A Huber pair (A, A^+) is said to be of *noetherian type* or is called a *Tate pair* when A is of noetherian type or a Tate ring, respectively. Note that A^+ plays no role in this definition.

- Example**
1. If A is any Huber ring, then both (A, A°) and (A, \emptyset) are Huber pairs. The ring of integral elements is A° in the first case and this is the integral closure of the subring $\mathbb{Z} \cdot 1_A + A^\circ \subset A$ in the second case.
 2. If A is an adic ring, and this includes usual rings (with the discrete topology), then (A, A) is a Huber pair. If $\pi \in A$ is a non zero topologically nilpotent element which is not a zero divisor, then $(A[1/\pi], A)$ is a Tate pair with ring of integral elements equal to A . This cannot happen if A is a usual ring (with the discrete topology).
 3. Let $(K, |\cdot|)$ be a non trivially valued field with valuation ring K^+ . Then (K, K^+) is a Huber pair if and only if K^+ has a prime ideal of height one ($|\cdot|$ is *microbial* in Huber's words), and then, K^+ is exactly the ring of integral elements (but $K^+ \neq K^\circ$ in general). The ring K^+ is not noetherian unless the absolute value $|\cdot|$ is discrete, but (K, K^+) will be of noetherian type whenever the topology of K may be defined by a discrete valuation.
 4. In the case $A = \mathbb{Z}[1/p]$ (with the discrete topology), there exists essentially two Huber pairs, with rings of integral elements $\mathbb{Z}[1/p]$ and \mathbb{Z} . They will both play a role in the theory.
 5. Assume that (A, A^+) is a Huber pair, that $A[T]$ has the topology induced by $A_0[f_1T, \dots, f_rT]$ for some f_1, \dots, f_r generating an open ideal in A and ring of definition A_0 . Then, we will usually choose $A[T]^+ = A^+ \cup \{f_1T, \dots, f_rT\}$.

If G is a totally ordered additive group, then $G \cup \{+\infty\}$ is endowed with the topology whose open subsets U are defined by the conditions

$$+\infty \notin U \quad \text{or} \quad \exists M \in G,]M, +\infty[\subset U.$$

In particular, the topology of a valued field K is the coarsest topology making the absolute value continuous. As a consequence, we see that a valuation v on a topological ring A is continuous if and only if the quotient map $A \rightarrow \kappa(v)$ is continuous.

Definition 2.6. *The adic spectrum of a Huber pair (A, A^+) is the set $V := \text{Spa}(A, A^+)$ of equivalence classes of continuous valuations on A that are non-negative on A^+ .*

We always have

$$\overline{A}^+ = \{f \in A, \forall v \in V, v(f) \geq 0\} \quad \text{and} \quad V = \text{Spa}(A, \overline{A}^+)$$

This is why we may usually replace A^+ with \overline{A}^+ and assume that A^+ is the full ring of integral elements.

If A is a Huber ring, then there exists a smallest and a biggest adic spectrum associated to A which are given by

$$\mathrm{Spa}(A) := \mathrm{Spa}(A, A^\circ) \quad \text{and} \quad \mathrm{Cont}(A) := \mathrm{Spa}(A, \emptyset)$$

This notation will also be used when A has the *discrete* topology in which case $\mathrm{Spa}(A)$ denotes the set of all non-negative valuations on A , but we would then rather write $\mathrm{Spv}(A)$ instead of $\mathrm{Cont}(A)$ for the set of all valuations.

The adic spectrum $V := \mathrm{Spa}(A, A^+)$ is endowed with the topology for which a basis of open subsets is given by the *rational* subsets

$$R(f_1/f_0, \dots, f_r/f_0) = \{v : \forall i = 1, \dots, r, v(f_i) \geq v(f_0) \neq +\infty\}$$

in which (f_0, \dots, f_r) is an open ideal. Note that the condition (f_0, \dots, f_r) *open* is not necessary in order to obtain an open subset of V (although it would not be called *rational* anymore). As a consequence, the topology of V only depends on the ring A (and not the topology of A or the choice of A^+).

Example 1. If K is any field (endowed with the discrete topology) and k is any subring of K , then $\mathrm{Spa}(K, k)$ is the Riemann-Zariski space of K over k . As a standard example, note that $\mathrm{Spa}(\mathbb{C}(X), \mathbb{C}) \simeq \mathbb{P}_{\mathbb{C}}$ (more about this later).

2. For $n \in \mathbb{Z}$, set

$$0(n) = \begin{cases} 0 & \text{if } n \neq 0 \\ +\infty & \text{if } n = 0 \end{cases};$$

then, if p is prime, denote by v_p the usual p -adic valuation, and set

$$p(n) = \begin{cases} 0 & \text{if } p \nmid n \\ +\infty & \text{if } p \mid n \end{cases}.$$

Then, we have

$$\mathrm{Spv}(\mathbb{Z}) = \{0, v_p \text{ for } p \text{ prime}, p \text{ for } p \text{ prime}\}.$$

Any proper closed subset is a finite union of subsets of the form $\{p\}$ or $\{v_p, p\}$.

An important theorem of Huber states that V is a spectral space (coherent and sober). Let us say a few words about specialization/generization on adic spectra.

Lemma 2.7. *Let (A, A^+) be a Huber pair and $v, w \in V = \mathrm{Spa}(A, A^+)$. Then $w \rightsquigarrow v$ if and only if the valuations induce a morphism of ordered groups $G_v \rightarrow G_w$ such that*

$$\forall f, g \in A, \quad v(g) \leq v(f) = +\infty \Rightarrow w(g) \leq w(f). \quad (2)$$

Proof. Recall that, by definition, $w \rightsquigarrow v$ if and only if any neighborhood of v is also a neighborhood of w . It means that

$$\forall f, g \in A, \quad v(f) \geq v(g) \neq +\infty \Rightarrow w(f) \geq w(g) \neq +\infty. \quad (3)$$

Thus, we see that if $v(f) = v(g) \neq +\infty$, then necessarily $w(f) = w(g) \neq +\infty$ and there exists therefore a well defined map

$$G_v \cap v(A) \rightarrow G_w \cap w(A), \quad v(f) \mapsto w(f).$$

The condition also shows that this map preserves the order. Moreover, since we always have $v(fg) = v(f) + v(g)$ and $w(fg) = w(f) + w(g)$, it extends uniquely to a group homomorphism which is automatically order preserving because the group laws are compatible with the orders. Conversely, if the valuations induce a morphism of ordered groups, then condition (3) is automatically satisfied whenever $v(f) \neq +\infty$ and only condition (2) had to be checked. \square

Note that the support map is compatible with specialization: we have

$$w \rightsquigarrow v \Rightarrow \text{supp } w \rightsquigarrow \text{supp } v$$

(which means that $\text{supp } w \subset \text{supp } v$). There are two special kinds of specializations:

Definition 2.8. *Let $V = \text{Spa}(A, A^+)$ be an adic spectrum and $v, w \in V$. Then,*

1. *v is a horizontal (or primary) specialization of w if the valuations induce an injective morphism of ordered groups $G_v \hookrightarrow G_w$ and*

$$\forall f, g \in A, \quad v(g) < v(f) = +\infty \Rightarrow w(g) < w(f).$$

2. *v is a vertical (or secondary) specialization of w if the valuations induce a surjective morphism of ordered groups $G_v \twoheadrightarrow G_w$ and*

$$\forall f \in A, \quad v(f) = +\infty \Rightarrow w(f) = +\infty.$$

Note that a specialization $w \rightsquigarrow v$ is *vertical* if and only if $\text{supp } w = \text{supp } v$ (it lives in a fiber of the support map). On the other hand, given an inclusion $\text{supp } w \hookrightarrow \mathfrak{p}$ where \mathfrak{p} is some prime ideal in A , there will exist at most one horizontal specialization $w \rightsquigarrow v$ with $\text{supp } v = \mathfrak{p}$. A basic theorem of the theory states that any specialization is the composition of a vertical one and a horizontal one (in that order).

Example 1. In $\text{Spv}(\mathbb{Z})$ we have the following vertical and horizontal specializations (for any prime p)

$$\begin{array}{c} 0 \\ \wr \\ \downarrow \\ v_p \rightsquigarrow p. \end{array}$$

2. If A is a Tate ring, all specializations are vertical and any valuation v has a unique (vertical) generization w of height 1.

For further use, we need to show some properties of specialization:

Lemma 2.9. *Let $V = \text{Spa}(A, A^+)$ be an adic spectrum, $v, w \in V$ and $f \in A$. Assume that $w \rightsquigarrow v$ in V . Then, we have*

1. $w(f) = +\infty \Rightarrow v(f) = +\infty$, and conversely if the specialization is vertical,
2. $w(f^n) \rightarrow +\infty \Rightarrow v(f^n) \rightarrow +\infty$,
3. $w(f) > 0 \Rightarrow v(f) > 0$ and conversely if the specialization is horizontal.

Proof. The first assertion follows from the existence of the map $G_v \rightarrow G_w$ and the very definition of a vertical specialization. For the second one, let us assume that the set $\{v(f^n)\}_{n \in \mathbb{N}}$ is not bounded. It means that there exists $\gamma_v \in G_v$ such that for all $n \in \mathbb{N}$ we have $v(f^n) \leq \gamma_v$. Then, if we denote by $\gamma_w \in G_w$ the image of γ_v , we will have for all $n \in \mathbb{N}$, $w(f^n) \leq \gamma_w$ which means that the set $\{w(f^n)\}_{n \in \mathbb{N}}$ is not bounded either. The last implication follows from the fact that the map $G_v \rightarrow G_w$ always preserves the order and that it even strictly preserves the order when it is injective. \square

We may also notice that we always have $v(f) > 0 \Leftrightarrow v(f^n) \rightarrow +\infty$ when v has height as most one but this is not true anymore in higher height.

We say that the Huber pair (A, A^+) or that $V := \text{Spa}(A, A^+)$ is *sheaffy* if there exists a (necessarily unique) structural sheaf \mathcal{O}_V for which

$$\Gamma(R(f_1/f_0, \dots, f_r/f_0), \mathcal{O}_V) = \widehat{A[1/f_0]}, \quad (4)$$

where $A[1/f_0]$ is the Huber ring whose ring of definition ring is $A_0[f_1/f_0, \dots, f_r/f_0]$ (and completion is meant relatively to this structure) for some ring of definition A_0 of A . When this is the case, $V := \text{Spa}(A, A^+)$ is a topologically valued ringed space.

If $V = \text{Spa}(A, A^+)$ is an adic spectrum, then the inclusion of valued fields

$$\text{Frac}(A/\text{supp } v) \hookrightarrow \mathcal{O}_{V,v}/\mathfrak{m}_{V,v}$$

(which is not an equality in general) induces an isomorphism on the *completions* and there should therefore be no problem in practice if we write $\kappa(v)$ for both fields. Following Berkovich, we will denote by $\mathcal{H}(v)$ their common completion.

Example 1. If $V = \text{Spa}(\mathbb{C}(X), \mathbb{C})$, then we have an isomorphism of locally ringed spaces $V^+ \simeq \mathbb{P}_{\mathbb{C}}$.

2. There exists isomorphisms

$$\text{Spv}(\mathbb{Z}) \setminus \{p\} \simeq \text{Spv}(\mathbb{Z}[1/p]) \quad \text{and} \quad \text{Spv}(\mathbb{Z}) \setminus \{v_p, p\} \simeq \text{Spa}(\mathbb{Z}[1/p]).$$

It is important for us to notice that, when A is of noetherian type, then V is automatically sheaffy. There are actually many various other conditions that would insure that V is sheaffy. This is the case for example also if A has the discrete topology. When A is of noetherian type, one can also prove Hilbert's theorems A and B as in the case of formal schemes:

Theorem 2.10 (Huber). *If A is of noetherian type and $V = \mathrm{Spa}(A, A^+)$, then the functors*

$$M \mapsto \mathcal{O}_V \otimes_{\hat{A}}^{\mathbb{L}} M \quad \text{and} \quad \mathcal{F} \mapsto \mathbb{R}\Gamma(V, \mathcal{F})$$

induce an equivalence between finite \hat{A} -modules and coherent \mathcal{O}_V -modules.

Proof. This is shown in [12]. □

Again, the same result also holds when A has the discrete topology.

Definition 2.11. *An adic space (or Huber space) is a topologically valued ringed space V , which is locally isomorphic to some (sheaffy) adic spectrum $\mathrm{Spa}(A, A^+)$. It is said to be affinoid if it is actually isomorphic to some $\mathrm{Spa}(A, A^+)$.*

Note that V is a locally spectral space. Alternatively, we could define an adic space as a doubly topologically locally ringed space $(V, \mathcal{O}_V, \mathcal{O}_V^+)$ locally isomorphic to an adic spectrum.

The functor

$$(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$$

is fully faithful on complete sheaffy pairs. Better, there exists an adjunction

$$\mathrm{Hom}(X, \mathrm{Spa}(A, A^+)) \simeq \mathrm{Hom}((A, A^+), (\Gamma(X, \mathcal{O}_X), \Gamma(X, \mathcal{O}_X^+)))$$

(compatible pairs of continuous homomorphisms on the right hand side). It is important to notice however that there is no fibered product in general in the category of adic spaces and that a (fibered) product of affinoid spaces is not necessarily affinoid ($\mathrm{Spv}(\mathbb{Z}[T]) \times \mathrm{Spa}(\mathbb{Q}_p)$ for example). Nevertheless, all the products used in the examples below are representable.

Example If O is any adic space, we may consider:

1. the *closed unit polydisc*

$$\mathbb{D}_O^n = \underbrace{\mathbb{D} \times \cdots \times \mathbb{D}}_{n \text{ times}} \times O$$

in which $\mathbb{D} := \mathrm{Spa}(\mathbb{Z}[T])$ (with the discrete topology). In the case $O = \mathrm{Spa}(A, A^+)$, we have

$$\mathbb{D}_O^n = \mathrm{Spa}(A[T_1, \dots, T_n], A^+[T_1, \dots, T_n])$$

(with the topology coming from A).

2. the *affine space*

$$\mathbb{A}_O^n = \underbrace{\mathbb{A}^{\mathrm{val}} \times \cdots \times \mathbb{A}^{\mathrm{val}}}_{n \text{ times}} \times O$$

in which $\mathbb{A}^{\mathrm{val}} := \mathrm{Spv}(\mathbb{Z}[T])$. Note that O being affinoid will *not* imply that \mathbb{A}_O^n is affinoid.

3. the *closed polydisc of radii* $\{f_{i,j}^{-\frac{1}{k}}\}$

$$\mathbb{D}_O^n(0, \{f_{i,j}^{-\frac{1}{k}}\}) = \mathbb{D}_O(0, \{f_{1,j}^{-\frac{1}{k}}\}) \times_O \cdots \times_O \mathbb{D}_O(0, \{f_{n,j}^{-\frac{1}{k}}\})$$

where $\mathbb{D}_O(0, \{f_j^{-\frac{1}{k}}\})$ is the disc of radius $\{f_j^{-\frac{1}{k}}\}$ defined as

$$\mathbb{D}_O(0, f_1^{-\frac{1}{k}}, \dots, f_r^{-\frac{1}{k}}) = \{v \in \mathbb{A}_O, v(f_1 T^k), \dots, v(f_r T^k) \geq 0\}.$$

In this definition, we assume that O lives over some affinoid space $\text{Spa}(A, A^+)$ and that $f_1, \dots, f_r \in A$ generate an open ideal of A . In the case $O = \text{Spa}(A, A^+)$, then a closed polydisc of some radii is always affinoid and

$$\mathbb{D}_O(0, f_1^{-\frac{1}{k}}, \dots, f_r^{-\frac{1}{k}}) = \text{Spa}(A[T], A^+[f_1 T^k, \dots, f_r T^k])$$

(with the topology induced by $A_0[f_1 T^k, \dots, f_r T^k]$ if A_0 denotes a ring of definition of A).

4. the *open unit polydisc*

$$\mathbb{D}_O^{n,-} := \underbrace{\mathbb{D}^- \times \cdots \times \mathbb{D}^-}_{n \text{ times}} \times O$$

in which $\mathbb{D}^- := \text{Spa}(\mathbb{Z}[T]) = \text{Spa}(\mathbb{Z}[[T]])$ with the T -adic topology. Again, the fact that O is affinoid will *not* imply that the open unit polydisc is affinoid.

The affine space is the union of all closed polydiscs of the same dimension. Actually, if $O = \text{Spa}(A, A^+)$ and f_1, \dots, f_n are topologically nilpotent elements of A that generate an open ideal in A , then it is easily checked that

$$\mathbb{A}_O := \bigcup_k \mathbb{D}_O(0, \{f_{\underline{i}}^{-1}\}_{|\underline{i}|=k}).$$

Here, we use the usual multiindex notation $f_{\underline{i}} := f_1^{i_1} \cdots f_n^{i_n}$ and $|\underline{i}| := i_1 + \cdots + i_n$ if $\underline{i} = (i_1, \dots, i_n) \in \mathbb{N}^n$.

Also, if O is an *analytic* space (see below), then the open unit polydisc is a union of closed polydiscs. Actually, if A is a Tate ring with topologically nilpotent unit π and $O = \text{Spa}(A, A^+)$, then we have

$$\mathbb{D}_O^- := \bigcup_k \mathbb{D}_O(0, \pi^{\frac{1}{k}}).$$

Be careful however that \mathbb{D}_O^- is usually strictly smaller than the subset $\{v \in \mathbb{D}_O, v(T) > 0\}$ which therefore is *not* a disc.

A morphism of adic spaces $W \rightarrow V$ is simply a morphism of topologically valued ring spaces. It is said to be adic if it comes locally from an adic morphism of Huber rings $A \rightarrow B$ (equivalently, it sends any analytic point to an analytic point - see below).

A *closed immersion* of adic spaces $W \hookrightarrow V$ is a morphism that comes locally on V from a surjective *adic* morphism $A \twoheadrightarrow B$ sending A^+ onto B^+ . One may then define the notions of *locally closed immersion* and (*locally*) *closed subspace*.

Definition 2.12. *An adic space V is locally of finite type over O if it is locally isomorphic to a locally closed subspace of \mathbb{A}_O^n . It is said to be of finite type if, moreover, it is quasi-compact.*

Note that one cannot replace the affine space by the *unit* polydisc in this definition but one may replace it with a polydisc of some radii which is then affinoid. It is also possible to define the notion of morphism locally finitely presented but this would be of no use to us when we enter the noetherian world. Huber proved that it is always possible to pull back along a morphism which is locally of finite type (although fibered products do not exist in general).

We used above the notation \mathbb{A}^{val} . Actually, to any locally ringed space X , one can associate a (topologically) valued ringed space X^{val} as follows. One sets

$$X^{\text{val}} := \{(x, v), x \in X, v \in \text{Spv}(\kappa(x))\}$$

(which is also sometimes written $\text{Spv}(X)$). It is made into a topological space by choosing as basis of open subsets, the subsets

$$\{(x, v) : x \in U, v(f(x)) \geq v(g(x)) \neq 0\} \subset X^{\text{val}},$$

in which U is an open subset of X and $f, g \in \Gamma(U, \mathcal{O}_X)$. There exists an obvious continuous map

$$\text{supp} : X^{\text{val}} \rightarrow X, \quad (x, v) \mapsto x,$$

and X^{val} is endowed with the sheaf of (topological) rings $\mathcal{O}_{X^{\text{val}}} := \text{supp}^{-1}(\mathcal{O}_X)$. In particular, we have $\mathcal{O}_{X^{\text{val}},(x,v)} = \mathcal{O}_{X,x}$ for all $(x, v) \in X^{\text{val}}$ and this local ring is endowed with the valuation induced by v .

Proposition 2.13. *If X is a scheme, then X^{val} is an adic space and the functor $X \mapsto X^{\text{val}}$ is fully faithful. Moreover, for any adic space V , there exists a natural bijection*

$$\text{Hom}(V, X) \simeq \text{Hom}(V, X^{\text{val}})$$

(morphisms of locally ringed spaces on one side and morphisms of adic spaces on the other). Finally, if $X = \text{Spec}(A)$, then there exists a natural isomorphism $X^{\text{val}} \simeq \text{Spv}(A)$.

Proof. We start with the last assertion: if A is any ring (endowed with the discrete topology), then there exists an obvious map

$$\text{Spv}(A) \rightarrow \text{Spec}(A)^{\text{val}}, \quad v \mapsto (\text{supp } v, \bar{v})$$

in which \bar{v} denotes the valuation induced by v on $\kappa(\text{supp } v)$. This is easily seen to be an isomorphism of (topologically) valued ringed spaces. Actually, by definition, we will always have

$$\Gamma(R(f_1/f_0, \dots, f_r/f_0), \mathcal{O}_V) = A[1/f_0]$$

(with the discrete topology). It follows that if X is a scheme, then X^{val} is an adic space. Now, both other questions are local. More precisely, for the fake adjunction,

we may assume that $X = \text{Spec}(A)$ and $V = \text{Spa}(B, B^+)$ with B complete. Then the result follows from the equality

$$\text{Hom}(A, B) = \text{Hom}((A, \emptyset), (B, B^+))$$

((automatically continuous) ring morphisms on one side and compatible pairs of (automatically continuous) ring morphisms on the other). Full faithfulness then results from the fact that $\Gamma(X^{\text{val}}, \mathcal{O}_{X^{\text{val}}}) = A$ when $X = \text{Spec}(A)$. \square

Corollary 2.14. *The functor $X \mapsto X^{\text{val}}$ commutes with finite limits of schemes.*

Proof. This is simply because finite limits of schemes are also finite limits in the whole category of locally ringed spaces. \square

In general, X^{val} is way too big and it is more convenient to rely on a relative version:

Corollary 2.15 (Huber). *Let $X \rightarrow S$ be a morphism of schemes which is locally of finite type, V any adic space and $V \rightarrow S$ a morphism of locally ringed spaces. Then, the functor*

$$W \mapsto \text{Hom}(W, X) \times_{\text{Hom}(W, S)} \text{Hom}(W, V)$$

is representable by the adic space

$$X_V := X \times_S V \simeq X^{\text{val}} \times_{S^{\text{val}}} V.$$

Proof. The only thing to check is that the fibered product on the right is representable and this follows from the fact that the morphism $X^{\text{val}} \rightarrow S^{\text{val}}$ is locally of finite type (because $X \rightarrow S$ is). \square

Example If V is an adic space, then we may consider

$$\mathbb{A}_V^n := \mathbb{A}^{n, \text{val}} \times V \quad \text{and} \quad \mathbb{P}_V^n := \mathbb{P}^{n, \text{val}} \times V$$

as we already did.

Definition 2.16. *An adic space V is said to be analytic if it is locally isomorphic to some $\text{Spa}(A, A^+)$ where A is a Tate ring.*

Be careful that an affinoid space which is analytic is not necessarily the adic spectrum of a Tate ring.

More generally, we will call a point v of an adic space V *analytic* if it has a neighborhood of the form $\text{Spa}(A, A^+)$ where A is a Tate ring. It is not difficult to see that a point is non-analytic if and only if it has a trivial (vertical) generization. We will denote by V^{an} (instead of Huber's V_a) the open subset of analytic points of V . More generally, if $T \subset V$, we should simply write T^{an} instead of $T \cap V^{\text{an}}$ for the set of analytic points of T .

Definition 2.17. *Let $(K, |\cdot|)$ be a Huber valued field with valuation ring K^+ . Then, $\text{Spa}(K^{\text{disc}}, K^+)$ is called a non analytic Huber point. If $|\cdot|$ is not trivial, then $\text{Spa}(K, K^+)$ is called an analytic Huber point.*

The topological space of a Huber point is totally ordered by generization with one maximal point of height 0 or 1 and one minimal point which is $|-|$.

Let V be an adic space and $v \in V$ a non-analytic point (resp. an analytic point). Then, there exists a canonical morphism

$$\mathrm{Spa}(\kappa(v)^{\mathrm{disc}}, \kappa(v)^+) \hookrightarrow V \quad (\text{resp. } \mathrm{Spa}(\kappa(v), \kappa(v)^+) \hookrightarrow V)$$

that identifies the Huber point with the set of vertical generizations of v .

When V is an *analytic* space, then the maximal points for generization in V are exactly the points of height 1 (the Berkovich points of V). We shall denote their set by V_1 (or V^{Berk}). There exists a retraction map $\mathrm{sep}: V \rightarrow V_1$ sending any point v to its maximal generization v_1 and V_1 is endowed with the *quotient* topology (and *not* the induced topology). Note that the canonical map $\kappa(v_1) \hookrightarrow \kappa(v)$ on the residue fields is dense and induces therefore an isomorphism $\overline{\mathcal{H}(v_1)} \simeq \overline{\mathcal{H}(v)}$ on the completed residue fields. Also, since we always have $\mathrm{sep}^{-1}(v) = \{v\}$, we see that the maps sep and sep^{-1} induce a bijection between the (locally closed, open, closed) subsets of V_1 and the (locally closed, open, closed) subsets of V that are stable under both specialization and generization. Moreover, V_1 is a *Fréchet* topological space (i.e. T_1) and the map sep is the adjunction map for a fake adjunction

$$\mathrm{Hom}(V_1, T) \simeq \mathrm{Hom}(V, T)$$

whenever T is a Fréchet topological space. Finally, when V is quasi-compact and quasi-separated, then the space V_1 is compact (Hausdorff) and the map sep is proper. It may be called the *Berkovich quotient* of V .

We will always assume that separated, proper or partially proper morphisms of adic spaces are locally of finite type as in [11]. More precisely, we make the following definitions:

Definition 2.18. *A morphism of adic spaces $u: W \rightarrow V$ is said to be*

1. *separated if u is locally of finite type and the diagonal map $\Delta: W \hookrightarrow W \times_V W$ is closed.*
2. *partially proper if u is separated and universally specializing with respect to adic pull back.*
3. *proper if u is partially proper and quasi-compact.*

You may visit section 1.3 of [14] for the details (even if some of his finiteness conditions are slightly weaker than ours). For example, when u is quasi-compact, (universally) specializing is equivalent to (universally) closed.

Example 1. If V is any adic space, then \mathbb{P}_V^r is proper over V .

2. It is not difficult to see that the open disc \mathbb{D}_V^- is partially proper over V when V is analytic. Since \mathbb{P}_V is proper over V , it is sufficient to show that the

open embedding $\mathbb{D}_V^- \hookrightarrow \mathbb{P}_V$ is specializing. And we may also assume that $V = \text{Spa}(B, B^+)$ where B is a Tate ring with topologically nilpotent unit π . But we have

$$\mathbb{D}_V^- = \bigcup_{n \in \mathbb{N}} \mathbb{D}_V(0, \pi^{\frac{1}{n}}) \quad \text{with} \quad \overline{\mathbb{D}_V(0, \pi^{\frac{1}{n}})} \subset \mathbb{D}_V(0, \pi^{\frac{1}{n+1}}).$$

Definition 2.19. *A morphism $W \rightarrow V$ of adic spaces satisfies the analytic valuative criterion for properness if, given a non-archimedean field F and valuation rings $R \subset F^+ \subsetneq F$, then any commutative diagram*

$$\begin{array}{ccc} W & \xrightarrow{\quad} & V \\ \uparrow & \swarrow \text{---} & \uparrow \\ \text{Spa}(F, F^+) & \longrightarrow & \text{Spa}(F, R) \end{array}$$

may be uniquely completed by the dotted arrow.

When $W = \text{Spa}(B, B^+)$ is affine, the lifting property means that the image of the map $B^+ \rightarrow \widehat{F}^+$ is contained in \widehat{R} .

Proposition 2.20. *Let $u : W \rightarrow V$ be a morphism of analytic spaces which is locally of finite type and quasi-separated. Then u satisfies the analytic valuative criterion for properness if and only if u is partially proper.*

Proof. This is exactly corollary 1.3.9 of [14]. □

There exists also a more subtle valuative criterion when V is *not* assumed to be analytic (proposition 3.12.2 of [11]).

One may also define a *finite* morphism of adic spaces as a morphism $u : W \rightarrow V$ that comes locally on V from a finite *adic* morphism $A \rightarrow B$ sending A^+ onto B^+ (or equivalently such that $\overline{A^+} \rightarrow \overline{B^+}$ is integral). And a *locally quasi-finite* morphism is a morphism which is locally of finite type with discrete fibres (*quasi-finite* when it is quasi-compact).

A morphism $u : W \rightarrow V$ is said to be *formally unramified* (resp. *formally smooth*, resp. *formally étale*) if any commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\quad} & V \\ \uparrow & \swarrow \text{---} & \uparrow \\ \text{Spa}(R/\mathfrak{a}) & \hookrightarrow & \text{Spa}(R) \end{array}$$

with \mathfrak{a} nilpotent may be completed by the diagonal arrow in at most (resp. at least, resp. exactly) one way. The morphism u is called *unramified* (resp. *smooth*, resp. *étale*) if it is formally unramified (resp. smooth, resp. étale) and locally finitely presented.

If $W \hookrightarrow V$ is an immersion defined (on some open subset of V) by an ideal \mathcal{I}_W , then we may consider the *first infinitesimal neighborhood* $W^{(1)}$ of W in V defined by \mathcal{I}_W^2 (which is always an adic space) and the corresponding short exact sequence

$$0 \rightarrow \check{\mathcal{N}}_{W/V} \rightarrow \mathcal{O}_{W^{(1)}} \rightarrow \mathcal{O}_W \rightarrow 0$$

where $\check{\mathcal{N}}_{W/V}$ is by definition the *conormal sheaf*. In the case of the diagonal immersion $W \hookrightarrow W \times_V W$ associated to a morphism $W \rightarrow V$, we obtain the *sheaf of differential forms* $\Omega_{W/V}^1$. One can show that a morphism of finite type $u : W \rightarrow V$ is unramified if and only if $\Omega_{W/V}^1 = 0$.

At some point, the adic spaces will only serve as a bridge between formal schemes and analytic spaces. The following notion will therefore be quite useful:

Definition 2.21. *Let \mathcal{P} be a property of morphisms of analytic spaces which is local on the base, stable under pull back and stable under composition. A morphism $u : W \rightarrow V$ of adic spaces is said to be analytically \mathcal{P} if for any analytic space V' over V , the morphism $u^{-1}(V') \rightarrow V'$ is \mathcal{P} .*

We may choose for \mathcal{P} the property of being (locally) of finite type (resp. separated, resp. (partially) proper, resp. unramified, resp. smooth, resp. étale, resp. an open or a (locally) closed immersion).

Note that the fact that $u^{-1}(V')$ is representable is part of the definition (and not automatic). We also insist on the fact that representability condition is also implicit when we require \mathcal{P} to be stable under pull back.

Actually, it is sufficient to check the condition when $V' = \mathrm{Spa}(A)$ and A is a Tate ring because the condition is local. Note also that, when V is analytic, the definition is equivalent to u itself satisfying \mathcal{P} because the property is stable under pull-back.

Be careful however that the image of V' into V is *not* necessarily contained inside the analytic locus V^{an} of V , simply because the analytic locus is not functorial (it is the non-analytic locus which is functorial). In particular, it is not sufficient to consider the case $V' = V^{\mathrm{an}}$ in the above definition.

Example The absolute open unit disc (we use the T -adic topology here)

$$\mathbb{D}^- := \mathrm{Spa}(\mathbb{Z}[[T]])$$

is analytically partially proper and analytically smooth over $\mathrm{Spv}(\mathbb{Z})$. Note however that \mathbb{D}^- is *not* proper and *not* smooth over $\mathrm{Spv}(\mathbb{Z})$ because the structural map is not even an adic morphism.

In order to be able to manipulate coherent sheaves, it is convenient to put some noetherian conditions on adic spaces:

Definition 2.22. *An adic space V is said to be locally of noetherian type if it is locally isomorphic to some $\mathrm{Spa}(A, A^+)$ with A of noetherian type.*

For adic spaces of noetherian type, we may usually work locally and make use of theorem A and Theorem B.

From now on and unless otherwise specified, **adic spaces will always be assumed to be locally of noetherian type** and simply called *adic spaces*.

If $u : W \rightarrow V$ is a morphism locally of finite type, then $\Omega_{W/V}^1$ is a coherent sheaf. Moreover, there exists a Jacobian criterion:

Theorem 2.23 (Huber). *1. Let $u : W \rightarrow V$ be a morphism between adic spaces that are locally of finite type over O . Then, there exists a right exact sequence*

$$u^*\Omega_{V/O}^1 \rightarrow \Omega_{W/O}^1 \rightarrow \Omega_{W/V}^1 \rightarrow 0.$$

Assume moreover that W is smooth over O . Then u is smooth (resp. étale) if and only if $u^\Omega_{V/O}^1$ is locally a direct summand in (resp. is isomorphic to) $\Omega_{W/O}^1$*

2. Let $i : W \hookrightarrow V$ be an immersion of adic spaces locally of finite type over O . Then, there exists a right exact sequence

$$\check{N}_{W/V} \rightarrow i^*\Omega_{V/O}^1 \rightarrow \Omega_{W/O}^1 \rightarrow 0.$$

Assume moreover that V is smooth over O . Then W is smooth (resp. étale) over O if and only if $\check{N}_{W/O}$ is locally a direct summand in (resp. is isomorphic to) $i^\Omega_{V/O}^1$.*

Proof. The existence of the right exact sequences follow from proposition 1.6.3 in [14] and the other assertions are proved in proposition 1.6.9 of [14]. \square

If one has to deal with formal schemes that are *not* locally noetherian, then it seems necessary to replace Huber adic spaces with the *generalized adic spaces* of Scholze and Weinstein (section 2.1 of [22] - see also Peter Scholze's lectures [21]). The point is that $\mathrm{Spa}(A, A^+)$ need not be sheafy. In Scholze/Weinstein theory, the *affinoid space* $\mathrm{Spa}(A, A^+)$ is redefined to be the sheaf associated to the presheaf of sets

$$(B, B^+) \mapsto \mathrm{Hom}((A, A^+), (B, B^+))$$

on the category \mathcal{C} opposite to the category of complete Huber pairs. Here, we endow \mathcal{C} with the topology generated by rational coverings (be careful that this is not a pretopology in the usual sense: the inverse image of a rational embedding is only a filtered direct limit of rational embeddings). A *generalized adic space* is a sheaf V on \mathcal{C} which is locally ind-representable by rational embeddings.

There exists a functor $V \mapsto |V|$ from generalized adic spaces to topological spaces which may be defined by glueing from the affinoid case. If $V = \mathrm{Spa}(A, A^+)$, then

$$|V| := \{\text{continuous valuations on } A \text{ non - negative on } A^+\} / \sim$$

is simply the underlying space of the former $\mathrm{Spa}(A, A^+)$ (in practice, one still writes V instead of $|V|$). The sheaf \mathcal{O} (resp. \mathcal{O}^+) on \mathcal{C} is defined as the sheaf associated to the presheaf

$$(A, A^+) \mapsto A \quad (\text{resp. } A^+).$$

By restriction, they both induce a sheaf of topological rings on $|V|$ that we may denote by \mathcal{O}_V and \mathcal{O}_V^+ respectively (be careful however that we may have $\Gamma(V, \mathcal{O}_V) \neq \widehat{A}$ when $V = \mathrm{Spa}(A, A^+)$ is not *sheaffy*).

We can identify the category of Huber adic spaces (Scholze and Weinstein call them *honest* adic spaces) with the full subcategory of generalized adic spaces V that satisfy for all Huber pairs (A, A^+) ,

$$\mathrm{Hom}(V, \mathrm{Spa}(A, A^+)) \simeq \mathrm{Hom}((A, A^+), (\Gamma(V, \mathcal{O}_V), \Gamma(V, \mathcal{O}_V^+))).$$

3 Adic spaces and formal schemes

In this section, we recall how a formal scheme may be seen as an adic space in such a way that most geometric properties translate directly into the new world. Unless otherwise specified, all formal schemes (resp. adic spaces) are locally noetherian (resp. locally of noetherian type).

If I_0 is a definition ideal of a Huber ring A , then A' will denote the ring A endowed with the I_0A -adic topology (and \widehat{A}' will denote the completion of A for the I_0A -adic topology). The topology of A' is coarser than the topology of A but an ideal \mathfrak{a} of A is open for one topology if and only if it is open for the other. Concerning completions, we have

$$\widehat{A}' = \varprojlim A/(I_0A)^n \neq \widehat{A} = \varprojlim A/I_0^n$$

in general.

Example 1. If A is an adic ring, then $A' = A$ as topological rings.

2. If A is a Tate ring, then A' has the coarse topology and $\widehat{A}' = \{0\}$.

Recall that a point of an adic space is said to be trivial if the corresponding valuation is trivial (has height zero).

Proposition 3.1. *Let V be an adic space, V_0 the subset of trivial points and $i = V_0 \hookrightarrow V$ the inclusion map. Then, we have $i^{-1}\mathcal{O}_V^+ = i^{-1}\mathcal{O}_V$ and $(V_0, i^{-1}\mathcal{O}_V)$ is a formal scheme. If $V = \mathrm{Spa}(A, A^+)$, then there exists a natural isomorphism $V_0 \simeq \mathrm{Spf}(A')$.*

Proof. Clearly, $\mathcal{O}_{V,v}^+ = \mathcal{O}_{V,v}$ when v is trivial. Now, the second statement is local and it is therefore sufficient to prove the last assertion. We first notice that the support induces a bijection

$$\mathrm{supp} : V_0 \simeq \mathrm{Spf}(A'). \tag{5}$$

More precisely, the inverse map sends an open prime \mathfrak{p} of A to

$$v_{\mathfrak{p}}(f) = \begin{cases} 0 & \text{if } f(\mathfrak{p}) \neq 0 \\ +\infty & \text{otherwise} \end{cases}.$$

Now, let (f_0, \dots, f_r) be an open ideal in A . Since any $v \in V_0$ will only take the values 0 and $+\infty$, we see that

$$v(f_1), \dots, v(f_r) \geq v(f_0) \neq +\infty \Leftrightarrow f_0(v) \neq 0.$$

In other words, we have

$$R(f_1/f_0, \dots, f_r/f_0) \cap V_0 = D(f_0) \subset V_0$$

and it immediately follows that the support map (5) is a homeomorphism.

The sheaf $i^{-1}\mathcal{O}_V$ is the sheaf associated to

$$D(f_0) \mapsto \varinjlim \Gamma(R(f_1/f_0, \dots, f_r/f_0), \mathcal{O}_V).$$

In order to see that the support map (5) is an isomorphism, we will compute this limit. After localizing, we may assume that $f_0 = 1$ and then, after adding some functions, that $A = A_0[f_0, \dots, f_r]$ where A_0 is some ring of definition. Then by definition, we have

$$\Gamma(R(f_1/f_0, \dots, f_r/f_0), \mathcal{O}_V) = \widehat{A}. \quad \square$$

Note that we used the general assumption that A is finitely generated over A_0 in order to finish this proof.

One may also remark that $V_0 \cap V^{\text{an}} = \emptyset$. More precisely, the closure $\overline{V_0}$ of V_0 in V is exactly the set of non-analytic points.

The proof of the following lemma is straightforward:

Lemma 3.2. *1. The functor $A \mapsto A'$ is (left) adjoint to the forgetful functor from adic rings to Huber rings.*

2. The functor $(A, A^+) \mapsto A'$ is (left) adjoint to the functor $A \mapsto (A, A)$ from adic rings to Huber pairs. \square

Recall from [13], section 4 (or [23], chapter 9.2) that one can associate functorially an adic space to any (locally noetherian) formal scheme P . We will denote it by P^{ad} (and not $t(P)$ as Huber does). More precisely, we have the following:

Proposition 3.3. *The functor $V \mapsto V_0$ has an adjoint $P \mapsto P^{\text{ad}}$ which is fully faithful. When $P = \text{Spf}(A)$, we have $P^{\text{ad}} = \text{Spa}(A)$.*

Proof. We have to show that if P a formal scheme, then there exists a unique adic space P^{ad} such that if V is an adic space, then there exists a natural bijection

$$\text{Hom}(P^{\text{ad}}, V) \simeq \text{Hom}(P, V_0).$$

The question is local on P and V and we may therefore assume that $P = \mathrm{Spf}(A)$ is affine and $V := \mathrm{Spa}(B, B^+)$ is affinoid. We may also assume that A is complete. Then, it follows from lemma 3.2 that if we set $P^{\mathrm{ad}} = \mathrm{Spa}(A)$, we have

$$\mathrm{Hom}(P^{\mathrm{ad}}, V) \simeq \mathrm{Hom}(P, V_0).$$

It only remains to show that the functor is fully faithful, or equivalently, that the adjunction map is an isomorphism $P_0^{\mathrm{ad}} \simeq P$. Again this is a local question. But if A is an adic ring, we have $A^{\wedge} = A$. \square

As a consequence of the proposition, the functor $P \mapsto P^{\mathrm{ad}}$ commutes with all colimits, the functor $V \mapsto V_0$ commutes with all limits and there exists a natural inclusion $P \hookrightarrow P^{\mathrm{ad}}$ (identification of the points of P with the trivial points in P^{ad}). We may also notice that

$$(V_0)^{\mathrm{ad}} = \{v \in V : \forall f \in \mathcal{O}_{V,v}, v(f) \geq 0\} \subset V$$

is the set of all points having a trivial horizontal specialization.

We will denote by P^{an} the analytic part of P^{ad} (instead of $P^{\mathrm{ad}^{\mathrm{an}}}$). Note that $X^{\mathrm{an}} = \emptyset$ if X is a usual scheme (and not merely a formal scheme).

Proposition 3.4. *If P is a formal scheme, then restriction along the inclusion map $P \hookrightarrow P^{\mathrm{ad}}$ induces an equivalence between coherent modules on both sides.*

Proof. We already know that the restriction of $\mathcal{O}_{P^{\mathrm{ad}}}$ to P is exactly \mathcal{O}_P . Moreover, the question is local and we may therefore assume that $P = \mathrm{Spf}(A)$ with A complete. In this case, we have at our disposal Theorem A and B on both sides and restriction is simply given by $M \otimes_A \mathcal{O}_{P^{\mathrm{ad}}} \mapsto M \otimes_A \mathcal{O}_P$ (for some finite A -module M). \square

We have the following fake adjunction on the other side (recall that we denote by V^+ the topologically locally ringed space V, \mathcal{O}_V^+):

Proposition 3.5 (Huber). *If V is an adic space and P is a formal scheme, then there exists a natural bijection*

$$\mathrm{Hom}(V^+, P) \simeq \mathrm{Hom}(V, P^{\mathrm{ad}})$$

(morphisms of topologically ringed spaces on one side and morphisms of adic spaces on the other).

Proof. This is shown by Huber in Proposition 4.1 of [13]. \square

Note that when $P = \mathrm{Spf}(A)$ with A complete, we have

$$\mathrm{Hom}(V, P^{\mathrm{ad}}) \simeq \mathrm{Hom}(A, \Gamma(V, \mathcal{O}_V^+))$$

(continuous maps on the right hand side). As a consequence of this proposition, we have the following:

Corollary 3.6. *The functor $P \mapsto P^{\text{ad}}$ commutes with all finite limits.*

Proof. This is not completely formal because we do not have a true adjunction. Since $\text{Spf}(\mathbb{Z})^{\text{ad}} = \text{Spa}(\mathbb{Z})$, we only have to show that, given two morphisms of formal schemes $P_i \rightarrow P$ for $i = 1, 2$ and a commutative diagram

$$\begin{array}{ccc}
 V & & \\
 \swarrow & \searrow & \\
 & (P_1 \times_P P_2)^{\text{ad}} & \longrightarrow P_1^{\text{ad}} \\
 \downarrow & \downarrow & \downarrow \\
 & P_2^{\text{ad}} & \longrightarrow P^{\text{ad}},
 \end{array}$$

there exists a unique dotted arrow that preserves the commutativity of the diagram. This is a local question and we may therefore assume that P, P_1, P_2 are affine with rings of functions A, A_1 and A_2 respectively. But then, using the remark following the proposition, we can build a canonical map $A_1 \otimes_A A_2 \rightarrow \Gamma(V, \mathcal{O}_V^+)$ and we are done. \square

Example Let S be a formal scheme.

1. We have

$$\mathbb{A}_S^{n,\text{ad}} = \mathbb{D}_{S^{\text{ad}}}^n, \quad \mathbb{P}_S^{n,\text{ad}} = \mathbb{P}_{S^{\text{ad}}}^n \quad \text{and} \quad \mathbb{A}_S^{n,-,\text{ad}} = \mathbb{D}_{S^{\text{ad}}}^{n,-}.$$

2. When $S = \text{Spf}(A)$ and $V = S^{\text{ad}}$, we will also consider the *relative bounded disc*

$$\mathbb{D}_V^{\text{b},n} = \mathbb{A}_S^{\text{b},n,\text{ad}}.$$

There is a sequence of (strict) inclusions

$$\mathbb{D}^- \subset \mathbb{D}^{\text{b}} \subset \mathbb{D}^+ \subset \mathbb{A}^{\text{val}} \subset \mathbb{P}^{\text{val}}.$$

Definition 3.7. *If P is a formal scheme, then the composition of the obvious morphism of locally topologically ringed spaces and the adjunction morphism*

$$\text{sp}: P^{\text{ad}} \rightarrow P^{\text{ad},+} \rightarrow P$$

is the specialization map.

Recall that, by construction, if $P = \text{Spf}(A)$, then we have

$$\text{sp}(v) = \{f \in A : v(f) > 0\}.$$

Alternatively, any $v \in P^{\text{ad}}$ has a unique trivial *horizontal* specialization v_0 and we have $\text{sp}(v) = \text{supp } v_0$. This is a good reason for calling this map the “specialization” map.

Example In the case $X = \text{Spec}(\mathbb{Z})$, we have $X^{\text{ad}} = X^{\text{val}} = \text{Spv}(\mathbb{Z})$ and $\text{Spec}(\mathbb{Z})$ embeds naturally in $\text{Spv}(\mathbb{Z})$ by sending the ideal (p) to the trivial valuation modulo p (works also for $p = 0$). The only other valuations on \mathbb{Z} are the p -adic valuations v_p for p prime. There exists two *different* sections for this embedding because $\text{supp}(v_p) = (0)$ but $\text{sp}(v_p) = (p)$.

Proposition 3.8. *Let P be a formal scheme. Then the specialization map $P^{\text{ad}} \rightarrow P$ (resp. the adjunction map $P^{\text{ad},+} \rightarrow P$) is a flat retraction for the canonical embedding $P \hookrightarrow P^{\text{ad}}$ (resp. $P \hookrightarrow P^{\text{ad},+}$).*

Proof. One first checks that the composite morphism

$$P \hookrightarrow P^{\text{ad}} \rightarrow P^{\text{ad},+} \rightarrow P$$

is the identity. This is a local question which becomes trivial in the affine case. It only remains to show that the specialization and adjunction maps are flat. This follows from the fact that, if $v \in P^{\text{ad}}$, then the localization maps

$$\mathcal{O}_{P^{\text{ad}},v}^+ \rightarrow \mathcal{O}_{P^{\text{ad},v_0}}^+ \quad \text{and} \quad \mathcal{O}_{P^{\text{ad}},v} \rightarrow \mathcal{O}_{P^{\text{ad},v_0}}$$

are flat (noetherian hypothesis used here) and we have by definition

$$\mathcal{O}_{P^{\text{ad},v_0}}^+ = \mathcal{O}_{P^{\text{ad},v_0}} = \mathcal{O}_{P,\text{sp}(v)}. \quad \square$$

Corollary 3.9. *If P is a formal scheme, then Rsp_* induces an equivalence of categories between coherent sheaves on P^{ad} and coherent sheaves on P .*

Proof. Follows from proposition 3.4. □

Actually, such an equivalence may be shown directly: it follows from the construction of the fake adjunction of proposition 3.5 that, if $Q \subset P$ is an open formal subscheme, then $\text{sp}^{-1}(Q) = Q^{\text{ad}}$. In particular, the inverse image of an affine open subset is always an affinoid open subset.

We insist on the fact that the functor $P \mapsto P^{\text{ad}}$ is fully faithful and commutes with all finite limits and colimits. We can also list the following properties:

- Proposition 3.10.**
1. *A morphism of formal schemes $Q \rightarrow P$ is a locally closed (resp. an open, resp. a closed) immersion if and only if $Q^{\text{ad}} \rightarrow P^{\text{ad}}$ is a locally closed (resp. an open, resp. a closed) immersion. Actually, there exists a bijection between closed formal subschemes of P and closed adic subspaces of P^{ad} .*
 2. *A family $\{P_i \rightarrow P\}_{i \in I}$ of morphisms of formal schemes is an open covering if and only if $\{P_i^{\text{ad}} \rightarrow P^{\text{ad}}\}_{i \in I}$ is an open covering.*
 3. *A morphism of formal schemes $Q \rightarrow P$ is affine (resp. quasi-compact, resp. quasi-separated) if and only if $Q^{\text{ad}} \rightarrow P^{\text{ad}}$ is affinoid (resp. quasi-compact, resp. quasi-separated).*

4. A morphism of formal schemes $Q \rightarrow P$ is adic (resp. (locally) of finite type, resp. (locally) quasi-finite, resp. locally of finite type and separated, resp. proper, resp. finite) if and only if $Q^{\text{ad}} \rightarrow P^{\text{ad}}$ is so.
5. A morphism of formal schemes $Q \rightarrow P$ is unramified (resp. smooth, resp. étale) if and only if $Q^{\text{ad}} \rightarrow P^{\text{ad}}$ is so.

Proof. By definition, if $Q \subset P$ is an open subset, then Q^{ad} is open in P^{ad} . And since the functor $V \mapsto V_0$ has the same property, we see that the converse is also true. The question about closed immersions is local on P and we may therefore assume that $P = \text{Spf}(A)$. In this case, closed immersions on both sides correspond bijectively to ideals in A . Here again, noetherianness is crucial. This finishes the proof of assertion 1).

Assertion 2) results from the fact that both functors $P \mapsto P^{\text{ad}}$ and $V \mapsto V_0$ preserve open coverings.

We know that when P is affine, then P^{ad} is affinoid. And conversely, if V is affinoid, then V_0 is affine. Assertion 3) follows easily.

In assertion 4), the adic and (locally) of finite type cases follow easily from the definitions (see also proposition 4.2 of [13]). For the finite case, we use the fact that there exists a bijection between coherent sheaves on both sides. Since a locally quasi-finite morphism of formal schemes is locally the composition of an open immersion and a finite map, the direct implication is clear in the (locally) quasi-finite case. Conversely, if $Q^{\text{ad}} \rightarrow P^{\text{ad}}$ has discrete fibres, the same is true for the induced map $Q \rightarrow P$. The separated case is an automatic consequence of the fact that the functor $P \mapsto P^{\text{ad}}$ preserves fibered products and closed immersions. For properness, it seems more reasonable to rely on the valuative criterion (see [11], proposition 3.12.5 for a proof).

Assertion 5) will follow from lemma 3.11 below (formal case) because we already know that u is locally of finite type if and only if u^{ad} is. \square

There exists also a beautiful theorem of Huber ([11], proposition 3.12.7) that states that any proper adic space over P^{ad} comes from a (unique and proper) formal scheme over P . The finite case should be clear.

Be careful that a morphism $Q \rightarrow P$ might be surjective although $Q^{\text{ad}} \rightarrow P^{\text{ad}}$ is not: if we endow \mathbb{Z}_p with the p -adic topology, then the closed immersion $\text{Spec}(\mathbb{F}_p) \hookrightarrow \text{Spf}(\mathbb{Z}_p)$ is actually a homeomorphism (of one point spaces) but the closed immersion $\text{Spa}(\mathbb{F}_p) \hookrightarrow \text{Spa}(\mathbb{Z}_p)$ identifies the unique point of the first space with the unique *closed* point of the second (which has also an *open* point). The same example shows that assertion 2) is not true for closed or locally closed coverings.

Lemma 3.11. *A locally noetherian morphism of formal schemes $u: Q \rightarrow P$ is formally unramified (resp. formally smooth, resp. formally étale) if and only if $u^{\text{ad}}: Q^{\text{ad}} \rightarrow P^{\text{ad}}$ is.*

Proof. The question is local on P and Q (thanks to the locally noetherian hypothesis in the smooth case) that we may assume to be affine, say $P = \mathrm{Spf}(A)$ and $Q = \mathrm{Spf}(B)$. We have to check that the lifting property on the formal side is equivalent to the corresponding lifting property on the adic side (complete rings and complete pairs) :

$$\begin{array}{ccc} B & \longleftarrow & A \\ \downarrow & \dashrightarrow & \downarrow \\ R/\mathfrak{a} & \longleftarrow & R \end{array} \quad \text{and} \quad \begin{array}{ccc} (B, B) & \longleftarrow & (A, A) \\ \downarrow & \dashrightarrow & \downarrow \\ (S/\mathfrak{b}, (S/\mathfrak{b})^+) & \longleftarrow & (S, S^+) \end{array} .$$

This should be clear because $(S/\mathfrak{b})^+ = S^+ / (\mathfrak{b} \cap S^+)$ when \mathfrak{b} is nilpotent. \square

Recall that, when P is a formal scheme, there exists an equivalence of categories $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{ad}}$ between coherent sheaves on P and P^{ad} . As a consequence, one easily sees that, if $u: Q \rightarrow P$ is a locally noetherian morphism, then $(\Omega_{Q/P}^1)^{\mathrm{ad}} = \Omega_{Q^{\mathrm{ad}}/P^{\mathrm{ad}}}^1$. More generally, if $Q \hookrightarrow P$ is any immersion of formal schemes, then $\check{\mathcal{N}}_{Q/P}^{\mathrm{ad}} = \check{\mathcal{N}}_{Q^{\mathrm{ad}}/P^{\mathrm{ad}}}$.

If X is a usual scheme, then there exists a natural inclusion $X^{\mathrm{ad}} \subset X^{\mathrm{val}}$ which is locally given by $\mathrm{Spa}(A) \subset \mathrm{Spv}(A)$. Moreover, if Z is a closed subset of X , then $Z^{\mathrm{ad}} = Z^{\mathrm{val}} \cap X^{\mathrm{ad}}$.

The above example of the projective space generalizes as follows:

Proposition 3.12. *A morphism of schemes $Y \rightarrow X$ satisfies the valuative criterion for properness if and only if*

$$Y^{\mathrm{ad}} \simeq Y_{X^{\mathrm{ad}}}.$$

Proof. The condition means that any commutative diagram

$$\begin{array}{ccc} V & & \\ \downarrow & \dashrightarrow & \downarrow \\ Y^{\mathrm{ad}} & \longrightarrow & X^{\mathrm{ad}} \\ \downarrow & & \downarrow \\ Y^{\mathrm{val}} & \longrightarrow & X^{\mathrm{val}}, \end{array}$$

may be uniquely completed. Using the universal properties of our functors, we see that this is equivalent to the same assertion about the commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & V^+ \\ \downarrow & \dashrightarrow & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

This can be checked pointwise and we may therefore assume that $V = \mathrm{Spa}(K, K^+)$ is a Huber point. We are then reduced to the same assertion relative to the

commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(K^+) \\ \downarrow & \dashrightarrow & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

This is exactly the valuative criterion for properness. \square

For further use, let us prove the following:

Proposition 3.13. *If $u: P' \rightarrow P$ is the blowing up of a usual subscheme Z in a formal scheme P and $Z' = u^{-1}(Z) := P' \times_P Z$, then u induces an isomorphism $P'^{\mathrm{ad}} \setminus Z'^{\mathrm{ad}} \simeq P^{\mathrm{ad}} \setminus Z^{\mathrm{ad}}$. In particular, it induces an isomorphism $P'^{\mathrm{an}} \simeq P^{\mathrm{an}}$.*

Proof. Assume first that $u: X' \rightarrow X$ is the blowing up of a subscheme Z of a scheme X . Then, it induces an isomorphism $X' \setminus Z' \simeq X \setminus Z$ and therefore also an isomorphism $X'^{\mathrm{val}} \setminus Z'^{\mathrm{val}} \simeq X^{\mathrm{val}} \setminus Z^{\mathrm{val}}$. By restriction, we deduce an isomorphism $X'^{\mathrm{ad}} \setminus Z'^{\mathrm{ad}} \simeq X^{\mathrm{ad}} \setminus Z^{\mathrm{ad}}$. In general, the question is local and we may assume that $P = X/Y$ where X is a scheme and Y a closed subscheme of X containing Z . By restriction again, we obtain an isomorphism $P'^{\mathrm{ad}} \setminus Z'^{\mathrm{ad}} \simeq P^{\mathrm{ad}} \setminus Z^{\mathrm{ad}}$. The last assertion follows from the fact that $Z^{\mathrm{an}} = \emptyset$ and $Z'^{\mathrm{an}} = \emptyset$ (note that Z' is a *usual* subscheme because a blowing up is an *adic* map). \square

Alternatively, in the previous proof, one may assume from the beginning that P is the completion of an affine scheme and do the computations locally. Anyway, it should be noticed that the inclusion $(P \setminus Z)^{\mathrm{ad}} \subset P^{\mathrm{ad}} \setminus Z^{\mathrm{ad}}$ is *strict* in general and this is why the above proof needs some care.

If one is only interested in the *analytic* part P^{an} of P^{ad} , then one may also use Raynaud's method as in [1] or [8]. This is a consequence of the following theorem:

Theorem 3.14 (Raynaud). *If P is a quasi-compact formal scheme, then specialization induces an isomorphism of locally topologically ringed spaces*

$$P^{\mathrm{an},+} \simeq \varprojlim_{P' \rightarrow P} P'$$

where $P' \rightarrow P$ runs through all blowing ups of usual subschemes.

Proof. (sketch) For the details, we refer to part II, appendix A of [8]. First of all, the existence of the map follows from proposition 3.13. In order to show that this is an isomorphism, after blowing up an ideal of definition and localizing, we may assume that $P = \mathrm{Spf}(A)$ for some adic ring A and that there exists a principal ideal of definition (π) . In this case, we have $P^{\mathrm{an}} = \mathrm{Spa}(A[\frac{1}{\pi}])$. Now, if we denote by V the right hand side of our isomorphism, and pick up some $v \in V$, then one first show that the completion $\hat{\mathcal{O}}_{V,v}$ of the local ring is a valuation ring. By composition, this provides a valuation on $A[\frac{1}{\pi}]$. This way, we obtain an inverse for our map. It is not hard to finish the proof because blowing up an open ideal in A provides a rational covering on the other side (and conversely). \square

It should be remarked that the map $P \mapsto P^{\text{an}}$ is only functorial in *adic* maps. However, non adic maps will play an important role in our constructions and this is why we cannot simply rely on this more restrictive notion. Before stating the next lemma, recall that the (resp. the analytic) valuative criterion for properness requires the existence of a lifting in the following situation (R a valuation ring)

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \uparrow & \dashrightarrow & \uparrow \\ \text{Spec}(F) & \longrightarrow & \text{Spec}(R) \end{array} \quad (\text{resp.} \quad \begin{array}{ccc} W & \longrightarrow & V \\ \uparrow & \dashrightarrow & \uparrow \\ \text{Spa}(F, F^+) & \longrightarrow & \text{Spa}(F, R). \end{array})$$

Lemma 3.15. *If $u: Q \rightarrow P$ is a morphism of formal schemes that satisfies the valuative criterion for properness, then $u^{\text{ad}}: Q^{\text{ad}} \rightarrow P^{\text{ad}}$ satisfies the analytic valuative criterion for properness.*

Proof. We give ourselves a non-archimedean field F and valuation rings $R \subset F^+ \subsetneq F$. We start with a morphism $\text{Spa}(F, F^+) \rightarrow Q^{\text{ad}}$ and we denote by w the image of the closed point. Without loss of generality, we may assume that F is complete. Then, by definition, there exists a natural morphism $\mathcal{O}_w \rightarrow F$ which is local and compatible with the valuations (when F is endowed with the valuation associated to F^+). It follows that $\kappa(w) \subset F$ and

$$F^+ \cap \kappa(w) = \kappa(w)^+ = \{\alpha \in \kappa(w) / w(\alpha) \geq 0\}$$

is the valuation ring of $\kappa(w)$. In particular, we may as well assume from now on that $F = \kappa(w)$ and $F^+ = \kappa(w)^+$ (and we replace R with $R \cap \kappa(w) \subset \kappa(w)^+$).

We consider now the composition of the local morphisms of local rings

$$\mathcal{O}_{\text{sp}(w)} \rightarrow \mathcal{O}_w^+ \twoheadrightarrow \kappa(w)^+.$$

We denote by $S \subset \mathcal{O}_{\text{sp}(w)}$ the inverse image of $R \subset \kappa(w)^+$ (see diagram below) and by $\overline{S} \subset \kappa(\text{sp}(w))$ the image of S which is a valuation ring of $\kappa(\text{sp}(w))$. In order to make sure that \overline{S} is a valuation ring, one may consider the image \overline{R} of R in the residue field of $\kappa(w)^+$, which is a valuation ring, and notice that $\overline{S} = \overline{R} \cap \kappa(\text{sp}(w))$.

We suppose now that the composite map $\text{Spa}(\kappa(w), \kappa(w)^+) \rightarrow Q^{\text{ad}} \rightarrow P^{\text{ad}}$ factors through some morphism $\text{Spa}(\kappa(w), R) \rightarrow P^{\text{ad}}$ and we denote by v the image of the closed point under this last map. Then, as before, we have $\kappa(v) \subset \kappa(w)$ and $\kappa(v)^+ = R \cap \kappa(v)$. Thus, if we consider the commutative diagram of local morphisms of local rings

$$\begin{array}{ccccc} \mathcal{O}_{\text{sp}(v)} & \longrightarrow & \mathcal{O}_v^+ & \twoheadrightarrow & \kappa(v)^+ \equiv R \cap \kappa(v) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{\text{sp}(w)} & \longrightarrow & \mathcal{O}_w^+ & \twoheadrightarrow & \kappa(w)^+ \\ \uparrow & & \uparrow & & \uparrow \\ S & \longrightarrow & R, & & \end{array}$$

we see that the image of $\mathcal{O}_{\text{sp}(v)}$ in $\mathcal{O}_{\text{sp}(w)}$ is contained in S . It follows that its image in $\kappa(\text{sp}(w))$ is contained in \overline{S} . From this, we deduce the existence of a morphism

$\mathrm{Spec}(\overline{S}) \rightarrow P$ that we may uniquely lift as $\mathrm{Spec}(\overline{S}) \rightarrow Q$ because we assume that u satisfies the valuative criterion for properness. Let us denote by \mathfrak{q} the image of the closed point (so that, by construction, $\mathrm{sp}(v) = u(\mathfrak{q})$).

After replacing Q with an affine neighborhood of \mathfrak{q} , we may assume that $Q = \mathrm{Spf}(B)$. The lifting property tells us that the image of the natural map $B \rightarrow \kappa(\mathrm{sp}(w))$ is contained in \overline{S} . A quick look at the above diagram and we see that the image of the natural map $B \rightarrow \kappa(w)^+$ is contained in R and we are done. \square

Theorem 3.16. *If $u: Q \rightarrow P$ is a morphism of formal schemes which is formally locally of finite type (resp. formally locally of finite type and separated, resp. partially proper) then $u^{\mathrm{ad}}: Q^{\mathrm{ad}} \rightarrow P^{\mathrm{ad}}$ is analytically locally of finite type, resp. analytically separated, resp. analytically partially proper).*

Proof. We consider the first assertion. This is a local question and we may therefore assume that u splits into a closed immersion followed with the projection $\mathbb{A}^n \times \mathbb{A}^{-,m} \times P \rightarrow P$. We are therefore reduced to the case where u is either a closed immersion, the projection of the affine line \mathbb{A} onto $\mathrm{Spec}(\mathbb{Z})$, or the projection of the formal affine line \mathbb{A}^- onto $\mathrm{Spec}(\mathbb{Z})$. In the first two cases, we already know that u^{ad} is of finite type. It is therefore sufficient to recall that $\mathbb{D}_{\overline{V}}$ is locally of finite type over V when V is analytic. More precisely, we may assume that $V = \mathrm{Spa}(B, B^+)$ where B is a Tate ring with topologically nilpotent unit π in which case

$$\mathbb{D}_{\overline{V}} = \bigcup_{n \in \mathbb{N}} \mathbb{D}_V(0, |\pi|^{\frac{1}{n}})$$

is a (increasing) union of affinoid open subsets of finite type.

Assume now that u is also separated which means that the diagonal embedding $Q \hookrightarrow P \times_Q P$ is a closed immersion. First of all, since u is locally formally of finite type, this is a locally noetherian morphism, and the fibered product is representable by a locally noetherian formal scheme. It follows that the map $Q^{\mathrm{ad}} \hookrightarrow P^{\mathrm{ad}} \times_{Q^{\mathrm{ad}}} P^{\mathrm{ad}}$ is also a closed immersion (but we may not call u^{ad} separated because our definition requires a finiteness condition). Pulling back preserves products and closed immersions. It follows that u^{ad} is analytically separated.

We assume now that u is partially proper and we consider the pull back $(u^{\mathrm{ad}})^{-1}(V) \rightarrow V$ of the morphism $u^{\mathrm{ad}}: Q^{\mathrm{ad}} \rightarrow P^{\mathrm{ad}}$ along some morphism $V \rightarrow P^{\mathrm{ad}}$ with V analytic. We already know that our map is locally of finite type and separated and we apply the *analytic valuative criterion*. It is actually sufficient to show that the map u^{ad} itself satisfies the analytic valuative criterion (using the universal property of fibered products) and this was proved in lemma 3.15. \square

The above functors $P \mapsto P^{\mathrm{ad}}$, $V \mapsto V_0$ and $V \mapsto V^+$ (or even $X \mapsto X^{\mathrm{val}}$) extend naturally to the context of generalized adic spaces (without any noetherian hypothesis) but the adjunctions of proposition 3.3 and 3.5 (as well as proposition 2.13) are not valid anymore in this full generality. And there are many other issues as well.

Example If \mathcal{V} is a (non discrete) valuation ring with fraction field K and P is a formal scheme which is formally of finite type over \mathcal{V} , then P^{ad} is only a generalized adic space but $P_K^{\text{ad}} := P^{\text{ad}} \times_{\text{Spa}(\mathcal{V})} \text{Spa}(K)$ is an analytic Huber (honest) space. In other words, in the non noetherian case, we would mostly use generalized adic spaces as a bridge between formal schemes and analytic Huber spaces. Note that one could also use directly Raynaud's generic fiber and completely avoid adic spaces but we would then have to stick to adic morphisms and this is not our philosophy here.

4 Formal embeddings

In this section, we study the notion of formal embedding and prove the formal fibration theorem. Recall that all formal schemes (resp. adic spaces) are supposed to be locally noetherian (resp. locally of noetherian type).

Definition 4.1. A formal embedding is a locally closed embedding of formal schemes $X \hookrightarrow P$.

A formal embedding is automatically universally noetherian. Formal embeddings form a category with compatible pairs of morphisms

$$\begin{array}{ccc} Y & \hookrightarrow & Q \\ \downarrow f & & \downarrow u \\ X & \hookrightarrow & P. \end{array}$$

Of course, f is uniquely determined by u when it exists. The forgetful functor $(X \hookrightarrow P) \mapsto P$ commutes with all limits and all colimits because it has obvious adjoint $P \mapsto (P \hookrightarrow P)$ and coadjoint $P \mapsto (\emptyset \hookrightarrow P)$. And the forgetful functor $(X \hookrightarrow P) \mapsto X$ commutes with all limits because it has an obvious adjoint $X \mapsto (X \hookrightarrow X)$.

For later use, we endow the category of formal embeddings $X \hookrightarrow P$ with the coarsest topology making cocontinuous the forgetful functor $(X \hookrightarrow P) \mapsto P$. This is the topology generated by the pretopology made of families $\{(X_i \hookrightarrow P_i) \mapsto (X \hookrightarrow P)\}_{i \in I}$ where $X = \bigcup_{i \in I} X_i$ and $P = \bigcup_{i \in I} P_i$ are both open coverings. If we wish, we may always assume that for all $i \in I$, we have $X_i = X \cap P_i$. One easily sees that this topology is subcanonical. Moreover, both functors $(X \hookrightarrow P) \mapsto P$ and $(X \hookrightarrow P) \mapsto X$ are left exact continuous and cocontinuous.

We implicitly endowed the category of formal schemes with the Zariski topology but we may have chosen another topology such as the coarse or the étale topology for example. The corresponding topology on the category of formal embeddings $X \hookrightarrow P$ will again be defined as the coarsest topology making cocontinuous the forgetful functor $(X \hookrightarrow P) \mapsto P$.

When $X \hookrightarrow P$ is a formal embedding, we will often identify X with its image in P and consider it as a locally closed subspace of P . Conversely, if X is a locally closed subspace of a formal scheme P , then there always exists a structure of formal

subscheme on X and we will often consider X as endowed with such a structure. Actually, there always exists a unique structure of reduced subscheme on X but it might be more convenient sometimes to use another one. This applies in particular to the closure \overline{X} of a formal subscheme X in P . Note that the closure map is functorial in the sense that any morphism of formal embeddings

$$\begin{array}{ccc} Y \hookrightarrow & Q \\ \downarrow f & \downarrow u \\ X \hookrightarrow & P. \end{array}$$

will induce a morphism $\overline{f} : \overline{Y} \rightarrow \overline{X}$ (for the reduced structures for example).

For future reference, we state and prove the following result:

Lemma 4.2. *Assume that we are given a morphism of formal embeddings*

$$\begin{array}{ccc} X' \hookrightarrow & P' \\ \downarrow f & \downarrow u \\ X \hookrightarrow & P \end{array}$$

with f formally étale and u differentially smooth in the neighborhood of X' . Then, locally on P and P' , there exists a morphism $v : P' \rightarrow \mathbb{A}_P^n$ which is formally étale in the neighborhood of X and extends both f and \overline{f} when X and \overline{X} are embedded into \mathbb{A}_P^n through the zero section.

Proof. We follow the proof of theorem 1.3.7 of [3]. Since f is formally étale, there exists an isomorphism

$$\check{\mathcal{N}}_{X'/u^{-1}(X)} \simeq (\Omega_{u^{-1}(X)/X}^1)_{|X'}$$

between the conormal sheaf on one side and the restriction of the sheaf of differential forms on the other. Now, since the question is local and u is differentially smooth in the neighborhood of X' , we may assume that there exists a basis of the conormal sheaf which is induced by global sections f_1, \dots, f_n of the ideal $\mathcal{I}_{X'}$ that define X' in $u^{-1}(X)$. After multiplication by a common factor, we may assume that the sections f_1, \dots, f_n also induce global sections of the ideal $\mathcal{I}_{\overline{X}'}$ defining \overline{X}' in $u^{-1}(\overline{X})$. Lifting these sections to P' provides a morphism $v : P' \rightarrow \mathbb{A}_P^n$ which extends f and \overline{f} (when X and \overline{X} are seen as formal subschemes of \mathbb{A}_P^n via the zero section). By construction, the morphism $v^* \Omega_{\mathbb{A}_P^n/P}^1 \rightarrow \Omega_{P'/P}^1$ is an isomorphism in the neighborhood of X' . This implies that the map v is formally étale in the neighborhood of X' . \square

With a projective version of the same argument, we can prove the following:

Lemma 4.3. *Assume that we are given a commutative diagram*

$$\begin{array}{ccc} Y \hookrightarrow & Z & \\ \downarrow f & \searrow g & \\ X \hookrightarrow & & P \end{array}$$

where f is étale, g is locally quasi-projective and j is a dense immersion. It will extend locally on Y and P to a morphism of formal embeddings

$$\begin{array}{ccccc} Y & \xrightarrow{j} & Z & \longrightarrow & Q \\ \downarrow f & & & \searrow g & \downarrow u \\ X & \longrightarrow & & & P \end{array}$$

where u is a projective morphism which is étale in the neighborhood of Y .

Proof. This is taken from the proof of theorem 2.3.5 of [3]. The question being local on P , we may assume that Z is a formal subscheme of the projective space \mathbb{P}_P^N . We may also assume that P is affine. As in the proof of lemma 4.2, there exists an isomorphism

$$\check{\mathcal{N}}_{Y/\mathbb{P}_X^N} \simeq \left(\Omega_{\mathbb{P}_X^N/X}^1 \right) |_Y.$$

Since the question is local on Y (but not on Z), we may assume that there exists a basis of the conormal sheaf which is defined on some open subset U of \mathbb{P}_X^N . We may actually assume that $U = V \cap \mathbb{P}_X^N$ where $V := D^+(s)$ for some $s \in \Gamma(\mathbb{P}_P^N, \mathcal{O}(m))$ and that our basis comes from some sequence $f_1, \dots, f_N \in \Gamma(\mathbb{P}_P^N, \mathcal{O}(n))$. We set $Q := V(f_1, \dots, f_N) \subset \mathbb{P}_P^N$. The Jacobian criterion shows that Q is étale in the neighborhood of Y . \square

Formal completion is usually only defined for usual schemes. We extend it to formal schemes as follows:

Definition 4.4. *Let X be a closed formal subscheme of a formal scheme P . If \mathcal{I}_X denotes the ideal of X in P and \mathcal{I} is some ideal of definition of P , we let X_n denote the closed subscheme of P defined by $(\mathcal{I}_X + \mathcal{I})^{n+1}$. Then the completion $P^{/X}$ of P along X is $P^{/X} := \varinjlim X_n$.*

It is not difficult to check that this definition is independent of the choice of the ideal of definition. Note that if we did not have noetherian hypothesis, it would be necessary to assume that the closed immersion is locally (radically) finitely presented in order to obtain an adic formal scheme. Note also that completion is usually written with an extra hat as $\hat{P}^{/X}$ but we will rather not do that. Finally, it is not hard to see that $P^{/X}$ is (representable as) a formal scheme: when $P = \mathrm{Spf}(A)$ and $X = \mathrm{Spf}(A/\mathfrak{a})$, we will have $P^{/X} = \mathrm{Spf}(A^{\mathfrak{a}})$ where $A^{\mathfrak{a}}$ is the ring A endowed with the $\mathfrak{a} + I$ -adic topology where I is some ideal of definition for A .

It is possible to extend the notion of completion to the locally closed situation but we want to avoid this because this would only create confusion later once we introduce the notion of tube.

There exists a natural map $P^{/X} \rightarrow P$ and the inclusion $X \hookrightarrow P$ factors through a closed embedding $X \hookrightarrow P^{/X}$ which is actually a formal thickening (and in particular a *homeomorphism*). The map $P^{/X} \rightarrow P$ is formally étale and universally noetherian.

The functor $(X \hookrightarrow P) \mapsto P^/X$ commutes with all limits because it has an adjoint $P \mapsto (P_{\text{red}} \hookrightarrow P)$. Since this last functor is fully faithful, the map $P^/X \rightarrow P$ is a monomorphism (although not an immersion in general). Finally, one easily checks that the functor $(X \hookrightarrow P) \mapsto P^/X$ is left exact continuous and cocontinuous.

Let us state some other basic properties of completion:

Proposition 4.5. *Let X be a closed formal subscheme of a formal scheme P .*

1. *The formal scheme $P^/X$ only depends on the underlying subspace of X (and not on the structure of formal scheme).*
2. *If $P = \bigcup_{i \in I} P_i$ is an open covering and, for all $i \in I$, $X_i := P_i \cap X$, then we have an open covering*

$$P^/X = \bigcup_{i \in I} P_i^/X_i.$$

3. *If $u: Q \rightarrow P$ is a morphism of formal schemes, then*

$$u^{-1}(P^/X) = Q^{/u^{-1}(X)}.$$

Proof. All questions are local and easily checked. □

As a particular case of the last assertion, we see that if Y is a closed formal subscheme of P that contains X and $Q := P^/Y$, then

$$Q^/X = P^/X.$$

When X is only a closed *subspace* of a formal scheme P , we will denote by $P^/X$ the completion of P with respect to any structure of formal subscheme on X . For example, if X is a locally closed subscheme and \overline{X} denotes its closure in P , it has a meaning to consider $P^/\overline{X}$.

We will now use lemma 4.2 in the case $X = \overline{X}$, $X' = \overline{X}'$ and f is an isomorphism:

Proposition 4.6 (Formal fibration theorem). *Assume that we are given two closed formal embeddings $X \hookrightarrow P$ and $X' \hookrightarrow P'$. If a differentially smooth morphism $u: P' \rightarrow P$ induces an isomorphism $X' \simeq X$, then it induces, locally on P and P' , an isomorphism*

$$P'^{/X'} \simeq \mathbb{A}^{n,-} \times P^/X.$$

Proof. Using lemma 4.2, we may assume that there exists a morphism $v: P' \rightarrow \mathbb{A}_P^n$ which is formally étale and induces an isomorphism between X' and the image of X via the zero section. We conclude with the forthcoming lemma 4.7. □

Lemma 4.7. *In the situation of the proposition,*

1. *if u is formally étale, then it induces an isomorphism $P'^{/X'} \simeq P^/X$,*
2. *if $P' = \mathbb{A}_P^n$ and X' is contained in the zero section, then $P'^{/X'} \simeq \mathbb{A}^{n,-} \times P^/X$.*

Proof. The first assertion is an immediate consequence of the definition of formal étaleness. The second one follows from left exactness of completion of formal schemes. \square

We want now to make precise the notion of a morphism having some property “around” a formal subscheme. Actually, this splits into two cases: properties that are *open* in nature (such as formally smooth for example) and properties that are *closed* in nature (such as partially proper for example). Note that some properties are neither open or closed in nature (such as smooth which is at the same time formally smooth (open) and locally of finite type (closed)).

Definition 4.8. *A morphism of formal schemes $u: Q \rightarrow P$ is said to be flat (resp. formally smooth, resp. formally unramified, resp. formally étale) around a formal subscheme Y of Q if there exists a neighborhood Q' of Y in Q such that the induced map $Q' \rightarrow P$ has this property.*

This is a standard notion and we turn now to the other side:

Definition 4.9. *A morphism $u: Q \rightarrow P$ of formal schemes is said to be separated (resp. affine, resp. (locally) of finite type, (locally) quasi-finite, resp. partially proper, resp. partially finite) around a formal subscheme Y of Q if there exists a closed subspace Z of Q containing Y such that $Q^Z \rightarrow P$ is separated (resp. affine, resp. formally (locally) of finite type, resp. formally (locally) quasi-finite, resp. partially proper, resp. partially finite).*

We will often consider a morphism of formal embeddings

$$\begin{array}{ccc} Y & \hookrightarrow & Q \\ \downarrow f & & \downarrow u \\ X & \hookrightarrow & P \end{array}$$

in which case the condition is equivalent to $\bar{f}: \bar{Y}_{\text{red}} \rightarrow \bar{X}_{\text{red}}$ being separated (resp. affine, resp. (locally) of finite type, resp. (locally) quasi-finite, resp. locally of finite type and proper on irreducible components, resp. locally of finite type and finite on irreducible components). We may replace P and Q by their completions along \bar{X} and \bar{Y} without changing the conditions. Note that we may then also replace X and Y with the open formal subschemes having the same underlying space if we wish.

5 Overconvergent spaces

In this section, we extend the notion of overconvergent space from [18]. We recall that all formal schemes are supposed to be locally noetherian and all adic spaces are assumed to be locally of noetherian type.

Definition 5.1. *An adic overconvergent space is a pair*

$$(X \hookrightarrow P, P^{\text{ad}} \xleftarrow{\lambda} V)$$

where $X \hookrightarrow P$ is a locally closed embedding of formal schemes and $\lambda: V \rightarrow P^{\text{ad}}$ is a morphism of adic spaces. We will say analytic overconvergent space when V is analytic.

Sometimes, we will write

$$X \hookrightarrow P \rightleftarrows P^{\text{ad}} \xleftarrow{\lambda} V$$

but we will usually make it shorter as $(X \hookrightarrow P \leftarrow V)$. Recall from proposition 3.5, that giving the morphism $\lambda: V \rightarrow P^{\text{ad}}$ is equivalent to giving a morphism of locally topologically ringed spaces $V^+ \rightarrow P$ and we will call *specialization* the composition

$$\text{sp}_V: V \rightarrow V^+ \rightarrow P$$

with the obvious morphism of locally topologically ringed spaces (as before, we write $V^+ := (V, \mathcal{O}_V^+)$). Alternatively, sp_V may be defined as the composition of the morphism $\lambda: V \rightarrow P^{\text{ad}}$ (seen as a morphism of locally topologically ringed spaces) with usual specialization $\text{sp}: P^{\text{ad}} \rightarrow P$

Note that, in the definition of an analytic overconvergent space, we only require that V is analytic and not at all that λ factors through P^{an} . It may even happen that the analytic locus of P is empty.

Example Let K be a complete discretely valued field of mixed characteristic p . We denote by \mathcal{V} the valuation ring of K and by k its residue field.

1. We endow \mathcal{V} with the p -adic topology. We embed $\text{Spec}(k)$ into $\text{Spf}(\mathcal{V})$ and let $V = \text{Spa}(K)$ (λ is the inclusion). Then,

$$(\text{Spec}(k) \hookrightarrow \text{Spf}(\mathcal{V}) \leftarrow \text{Spa}(K))$$

is the usual basis for rigid cohomology.

2. Now, we endow $\mathcal{V}[[t]]$ with the p -adic topology (and *not* the (p, t) -adic topology). We embed $\eta_k := \text{Spec}(k((t)))$ into $\mathbb{A}_{\mathcal{V}}^{\text{b}} := \text{Spf}(\mathcal{V}[[t]])$ and let

$$V := \mathbb{D}_K^{\text{b}} = \text{Spa}(K \otimes_{\mathcal{V}} \mathcal{V}[[t]])$$

be the bounded unit disc (λ is the inclusion again). Then

$$(\eta_k \hookrightarrow \mathbb{A}_{\mathcal{V}}^{\text{b}} \leftarrow \mathbb{D}_K^{\text{b}})$$

is the refined basis for rigid cohomology over a Laurent series field (see [16]).

Definition 5.2. A formal morphism of adic overconvergent spaces

$$(Y \hookrightarrow Q, Q^{\text{ad}} \xleftarrow{\mu} W) \rightarrow (X \hookrightarrow P, P^{\text{ad}} \xleftarrow{\lambda} V)$$

is a triple of morphisms $f: Y \rightarrow X, v: Q \rightarrow P, u: W \rightarrow V$ making commutative the diagrams

$$\begin{array}{ccc} Y \hookrightarrow Q & , & Q^{\text{ad}} \xleftarrow{\mu} W \\ \downarrow f & & \downarrow v^{\text{ad}} \\ X \hookrightarrow P & & P^{\text{ad}} \xleftarrow{\lambda} V \end{array} \quad \begin{array}{ccc} & & \downarrow u \\ & & V \end{array}$$

We might sometimes draw a full diagram

$$\begin{array}{ccccccc}
Y & \hookrightarrow & Q & \xrightleftharpoons{\quad} & Q^{\text{ad}} & \xleftarrow{\mu} & W \\
\downarrow f & & \downarrow v & & \downarrow v^{\text{ad}} & & \downarrow u \\
X & \hookrightarrow & P & \xrightleftharpoons{\quad} & P^{\text{ad}} & \xleftarrow{\lambda} & V
\end{array}$$

or simply write

$$(f, u, v) : (Y \hookrightarrow Q \leftarrow W) \rightarrow (X \hookrightarrow P \leftarrow V).$$

Note also that the condition is equivalent to requiring the commutativity of the diagram

$$\begin{array}{ccc}
Y \hookrightarrow Q & \xleftarrow{\text{sp}_W} & W \\
\downarrow f & & \downarrow v \\
X \hookrightarrow P & \xleftarrow{\text{sp}_V} & V.
\end{array}$$

Example Let K be a complete discretely valued field of mixed characteristic p with valuation ring \mathcal{V} and residue field k . Then, the *Amice ring* $\mathcal{A} := \mathcal{V}[\widehat{[[t]]}][\frac{1}{t}]$ is a complete discrete valuation ring with residue field $k((t))$ and its fraction field is the *Amice field* $\mathcal{E} = \mathcal{A}[\frac{1}{p}]$. We may then consider the formal morphism of adic overconvergent spaces

$$\begin{array}{ccccc}
\text{Spec}(k((t))) & \hookrightarrow & \text{Spf}(\mathcal{A}) & \longleftarrow & \text{Spa}(\mathcal{E}) \\
\parallel & & \downarrow & & \downarrow \\
\eta & \hookrightarrow & \mathbb{A}_{\mathcal{V}}^{\text{b}} & \longleftarrow & \mathbb{D}_K^{\text{b}}.
\end{array}$$

This is the morphism that refines the usual basis for rigid cohomology over $k((t))$. The map induced on the tubes (see below) will be given by the inclusion of the bounded Robba ring \mathcal{E}^\dagger (also denoted by \mathcal{R}^{b}) into the Amice field \mathcal{E} .

The following observation will allow us to split some proofs in two separate cases:

Proposition 5.3. *Any formal morphism*

$$\begin{array}{ccccccc}
Y & \hookrightarrow & Q & \xleftarrow{\mu} & W \\
\downarrow f & & \downarrow v & & \downarrow u \\
X & \hookrightarrow & P & \xleftarrow{\lambda} & V.
\end{array}$$

is the composition of a formal morphism with $u = \text{Id}_V$ and another one with both $f = \text{Id}_X$ and $v = \text{Id}_P$.

Proof. It is sufficient to split our morphism as follows:

$$\begin{array}{ccccc}
Y & \hookrightarrow & Q & \xleftarrow{\mu} & W \\
\downarrow f & & \downarrow v & & \parallel \\
X & \hookrightarrow & P & \xleftarrow{v^{\text{ad}} \circ \mu} & W & \square \\
\parallel & & \parallel & & \downarrow u \\
X & \hookrightarrow & P & \xleftarrow{\lambda} & V.
\end{array}$$

Endowed with formal morphisms, the adic overconvergent spaces form a category (that we will need to refine later). The functor $(X \hookrightarrow P \leftarrow V) \mapsto (X \hookrightarrow P)$ has an adjoint $(X \hookrightarrow P) \mapsto (X \hookrightarrow P \leftarrow \emptyset)$ and a coadjoint $(X \hookrightarrow P) \mapsto (X \hookrightarrow P \leftarrow P^{\text{ad}+})$. In particular, it commutes with all limits and all colimits. On the other hand, the functor $(X \hookrightarrow P \leftarrow V) \mapsto V$ has a coadjoint $V \mapsto (\text{Spec}(\mathbb{Z}) \hookrightarrow \text{Spec}(\mathbb{Z}) \leftarrow V)$ and commutes therefore with all colimits.

One can also check directly from the definition that the functor $(X \hookrightarrow P \leftarrow V) \mapsto V$ actually commutes with all limits. More precisely, if we are given a diagram $\{(X_i \hookrightarrow P_i \leftarrow V_i)\}_{i \in I}$, and we assume that $\varprojlim X_i$, $\varprojlim P_i$ and $\varprojlim V_i$ exist, then our diagram has a limit which is $(\varprojlim X_i \hookrightarrow \varprojlim P_i \leftarrow \varprojlim V_i)$. We should mostly apply this to fibered products.

We endow the category of adic overconvergent spaces and formal morphisms with the *adic* topology: this is the coarsest topology making cocontinuous the forgetful functor $(X \hookrightarrow P \leftarrow V) \mapsto V$. It is generated by the pretopology made of families

$$\{(X \hookrightarrow P \leftarrow V_i) \rightarrow (X \hookrightarrow P \leftarrow V)\}_{i \in I}$$

where $V = \bigcup_{j \in J} V_j$ is an open covering. This topology is subcanonical. Moreover, the functor $(X \hookrightarrow P \leftarrow V) \mapsto V$ is left exact continuous and cocontinuous, giving rise to two morphisms of topos.

We implicitly endowed the category of adic spaces with the adic topology. If we use another topology such as the étale topology for example, then we can consider the corresponding topology on the category of adic overconvergent spaces and formal morphisms which is defined exactly as before. On the other hand, we may also endow our category with the image topology of the functor

$$(X \hookrightarrow P) \mapsto (X \hookrightarrow P \leftarrow P^{\text{ad}})$$

where the first category is endowed for example with the Zariski topology (or some other topology if we wish). We obtain the topology generated by the pretopology made of families

$$\{(X_i \hookrightarrow P_i \leftarrow V_i) \rightarrow (X \hookrightarrow P \xleftarrow{\lambda} V)\}_{i \in I}$$

where $P = \bigcup_{j \in J} P_j$ is an open covering, and for each $i \in I$, $X_i = X \cap P_i$ and $V_i = \lambda^{-1}(P_i^{\text{ad}})$. Finally, note that it is also possible to endow the category of adic overconvergent spaces and formal morphisms with a topology coming both from the adic and the formal side (coarsest topology finer than both). This would give rise for example to the Zariski-adic topology.

6 Tubes

In this section, we introduce the notion of tube that may be seen as an adic version of completion. We recall that all formal schemes (resp adic spaces) are assumed to be locally noetherian (resp. of noetherian type).

We start with the following observation:

Lemma 6.1. *If $X \hookrightarrow P$ is a closed formal embedding, then the canonical map $P^{/X} \rightarrow P$ induces a homeomorphism between $P^{/X,\text{ad}}$ and its image inside P^{ad} .*

Proof. This is a local question and we may therefore assume that $P = \text{Spf}(A)$ is affine and that X is defined by an ideal $\mathfrak{a} \subset A$. It is then sufficient to recall that the topology on $\text{Spa}(A)$ and $\text{Spa}(A/\mathfrak{a})$ only depend on the ring A (and not on its topology) and are actually both induced by the topology of $\text{Spv}(A)$. \square

Example 1. In the simplest non trivial case $X := \text{Spec}(\mathbb{F}_p)$ and $P := \text{Spec}(\mathbb{Z})$, we have $P^{/X} = \text{Spf}(\mathbb{Z}_p)$ (with the p -adic topology). It follows that $P^{\text{ad}} = \text{Spv}(\mathbb{Z})$ and $P^{/X,\text{ad}} = \text{Spa}(\mathbb{Z}_p)$. The homeomorphic image of $\text{Spa}(\mathbb{Z}_p)$ into $\text{Spv}(\mathbb{Z})$ is the closed subset consisting in the the point v_p and its (horizontal) specialization p .

2. If we consider the inclusion of the origin $0 \hookrightarrow \mathbb{A}$, then we have $\mathbb{A}^{/0} = \mathbb{A}^-$ and therefore

$$\mathbb{A}^{/0,\text{ad}} = \mathbb{D}^- \subset \mathbb{D} = \mathbb{A}^{\text{ad}}.$$

Recall that this inclusion is not open or closed (although it is analytically an open immersion).

We will usually identify the topological space of $P^{/X,\text{ad}}$ with its homeomorphic image inside P^{ad} .

Definition 6.2. *Let $(X \hookrightarrow P \xleftarrow{\lambda} V)$ be an adic overconvergent space.*

1. *When X is closed in P , then the tube of X in V is the subset*

$$]X[_V := \lambda^{-1}(P^{/X,\text{ad}}) \subset V.$$

2. *In general, if \overline{X} denotes the closure of X in P and $\infty_X := \overline{X} \setminus X$ its locus at infinity, then the tube of X in V is the subset*

$$]X[_V :=]\overline{X}[_V \setminus]\infty_X[_V \subset V.$$

Be careful that even if P does not usually appear in the notation, the tube $]X[_V$ also depends on P , and it might sometimes be necessary to write $]X[_{P,V}$ in order to remove any ambiguity. On the contrary, when $V = P^{\text{ad}}$ (and $\lambda = \text{Id}_V$), we will write $]X[_P$ and call it the *tube of X in P* . In particular, when X is closed in P , then $]X[_P$ is the homeomorphic image of $P^{/X,\text{ad}}$ in P^{ad} discussed above.

The tube $]X[_V$ is endowed with the subspace topology coming from the topology of V . When we consider it as a topologically locally ringed space, we use the restriction sheaf $i_X^{-1}\mathcal{O}_V$ where $i_X:]X[_V \hookrightarrow V$ denotes the inclusion map. When X is closed in P , the homeomorphism $P^{/X,\text{ad}} \simeq]X[_P$ is *not* an isomorphism of locally topologically ringed spaces in general.

Note that we could also consider the *naive tube* $\text{sp}_V^{-1}(X)$ which is different from the true tube just introduced (more about this later).

Example 1. Recall that $] \text{Spec}(\mathbb{F}_p)[_{\text{Spec}(\mathbb{Z})}$ is a *closed* subset of $\text{Spv}(\mathbb{Z})$ which is homeomorphic, but *not* isomorphic, to $\text{Spa}(\mathbb{Z}_p)$ (where \mathbb{Z}_p has the p -adic topology). More precisely, a function on the tube is an element of the local ring $\mathbb{Z}_{(p)}$ which is smaller than \mathbb{Z}_p .

2. With the adic overconvergent space

$$(\text{Spec}(\mathbb{F}_p) \hookrightarrow \text{Spec}(\mathbb{Z}) \leftarrow \text{Spa}(\mathbb{Q}_p)),$$

in which \mathbb{Q}_p has the “ p -adic” topology, then we have

$$] \text{Spec}(\mathbb{F}_p)[_{\text{Spa}(\mathbb{Q}_p)} = \text{Spa}(\mathbb{Q}_p).$$

3. Let us consider now an adic overconvergent space $(X \hookrightarrow P \leftarrow O)$ where $O := \text{Spa}(K, \mathcal{O}_v)$ is a *Huber point*. Then, we have:

(a) If O is non-analytic, then $]X[_O$ can be any “interval” in O (totally ordered by specialization).

(b) If O is analytic, then $]X[_O = \emptyset$ or O .

4. If we consider the adic overconvergent space $(\{0\} \hookrightarrow \mathbb{A} \leftarrow \mathbb{D}_{\mathbb{Q}_p})$, we have

$$]0[_{\mathbb{D}_{\mathbb{Q}_p}} = \mathbb{D}_{\mathbb{Q}_p}^- \subset \mathbb{D}_{\mathbb{Q}_p},$$

which is an *open* subset.

5. (Lazda and Pàl setting) Let K be a discretely valued field of mixed characteristic p with valuation ring \mathcal{V} and residue field k . We embed the point $X := \eta_k$ into $P := \mathbb{A}_{\mathcal{V}}^b$, and we let $V := \mathbb{D}_K^b$. The tube of the closed point of P is the open unit disc \mathbb{D}_K^- and it follows that

$$]X[_V = \mathbb{D}_K^b \setminus \mathbb{D}_K^- = \overline{\{v\}} = \{v, v^-\},$$

where v is the *Gauss point*. A basis of neighborhoods of $]X[_V$ is given by the $V_n := \mathbb{D}_K^b \setminus \mathbb{D}_K(0, p^{\frac{1}{n}})$. It follows that

$$\Gamma(]X[_V, i^{-1}\mathcal{O}_V) = \mathcal{E}^\dagger$$

is the bounded Robba ring of K (see [16]).

6. (Monsky-Washnitzer setting) Let R be a noetherian ring and X an affine scheme of finite type over R . From a presentation of X over R , we obtain a sequence of inclusions $X \subset \mathbb{A}_R^n \subset \mathbb{P}_R^n$ and we denote by \overline{X} the projective closure of X . We fix a morphism of formal scheme $S \rightarrow \text{Spec}(R)$ and a morphism of adic spaces $O \rightarrow S^{\text{ad}}$ and we consider the adic overconvergent space

$$X_S \hookrightarrow \overline{X}_S \leftarrow \overline{X}_O.$$

Then we have

$$]X_S[_{\overline{X}_O} =]\mathbb{A}_S^n[_{\mathbb{P}_O^n} \cap X_O \subset \mathbb{A}_O^n.$$

In order to lighten the notations, when we are given a fixed morphism of adic spaces $\lambda : V \rightarrow W$, some $v \in V$ and some function f defined in a neighborhood of $\lambda(v)$ in W , we will simply write $v(f)$ instead of $v(\lambda^{-1}(f))$.

Lemma 6.3. *Let $P := \mathrm{Spf}(A)$ be an affine formal scheme, $X \subset P$ the closed formal subscheme defined by an ideal $\mathfrak{a} \subset A$ and $\lambda : V \rightarrow P^{\mathrm{ad}}$ any morphism of adic spaces. Then, we have*

$$\mathrm{sp}_V^{-1}(X) = \{v \in V : \forall f \in \mathfrak{a}, v(f) > 0\}$$

and

$$]X[_V = \{v \in V : \forall f \in \mathfrak{a}, v(f^n) \rightarrow +\infty\}.$$

Proof. The first assertion immediately follows from the definition of specialization. Now, we have

$$P^{/X, \mathrm{ad}} = \mathrm{Spa}(A^{\mathfrak{a}}) \subset \mathrm{Spa}(A) = P^{\mathrm{ad}}.$$

By definition, a point $v \in V$ belongs to $]X[_V$ if and only if $\lambda(v)$, which is a point in P^{ad} , falls into $P^{/X, \mathrm{ad}}$. It means that $\lambda(v)$ is not only continuous for the topology of A but also for the \mathfrak{a} -adic topology. It exactly means that for any $f \in \mathfrak{a}$, we will have $v(f^n) \rightarrow +\infty$. \square

It will sometimes be convenient to use the multiplicative notation, or even work directly inside the residue field, and we want to recall that

$$v(f^n) \rightarrow +\infty \Leftrightarrow |f^n(v)| \rightarrow 0 \Leftrightarrow f^n(v) \rightarrow 0.$$

In other words, $\{v(f^n)\}_{n \in \mathbb{N}}$ is unbounded in G_v if and only if $f(v)$ is topologically nilpotent in $\kappa(v)$.

The lemma shows that $]X[_V \subset \mathrm{sp}_V^{-1}(X)$ when X is *closed*. Unlike in Tate or Berkovich theory, this inclusion is usually *strict* (as we shall see). As a consequence, there is *no* inclusion in general when X is only assumed to be locally closed. In particular, some points of $]X[_P$ might specialize *outside* X (but not too far as we shall also see).

Example We let $S = \mathrm{Spf}(A)$ be an affine formal scheme and we consider the zero-section $S \hookrightarrow \mathbb{A}_S$. If v is any valuation on A and v^- denotes the corresponding valuation on $A[T]$ that specializes inside the unit disc, we have $v^-(T) = (0, 1) > (0, 0)$ but, if the valuation v is not trivial, we have $v^-(T^n) = (0, n) \dashrightarrow +\infty$ because $(0, n) < (g, 0)$ whenever $g > 0$ (we use the lexicographical order). In this case, we see that $]S[_P$ is strictly smaller than $\mathrm{sp}^{-1}(S)$.

Proposition 6.4. *Let $(X \hookrightarrow P \xleftarrow{\lambda} V)$ be an adic overconvergent space. Then, we have the following:*

1. *if $Y \subset P$ is another locally closed subscheme and $X \subset Y$, then $]X[_V \subset]Y[_V$,*
2. *if $X = X_1 \cap X_2$ where X_1, X_2 are two other locally closed subschemes, then $]X[_V =]X_1[_V \cap]X_2[_V$,*

3. if $X = X_1 \cup X_2$ where X_1, X_2 are two other locally closed subschemes, then $]X[_V =]X_1[_V \cup]X_2[_V$.

Proof. Clearly, it is sufficient to consider the case $V = P^{\text{ad}}$. Let us first check that these assertions hold for *closed* subspaces. This is a local question on P and we may therefore rely on lemma 6.14. Assertion 1) should then be clear. For the other two, we may assume that $P = \text{Spf}(A)$ and X, X_1, X_2 are defined by $\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2$, respectively. In assertion 2) (resp. 3)), we suppose that $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ (resp. $\mathfrak{a} = \mathfrak{a}_1 \mathfrak{a}_2$). It is then sufficient to notice that if $f, g \in A$, we have

$$v((f+g)^n) \rightarrow +\infty \Leftrightarrow (v(f^n) \rightarrow +\infty \text{ and } v(g^n) \rightarrow +\infty)$$

and

$$v((fg)^n) \rightarrow +\infty \Leftrightarrow (v(f^n) \rightarrow +\infty \text{ or } v(g^n) \rightarrow +\infty).$$

Note that, for the second equivalence, it is necessary to use the fact that our valuations are *non-negative* on A .

We derive now from the closed case that if U and Z are open and closed complements in the topology of P , then $]U[_P$ and $]Z[_P$ are complements in P^{ad} (this is not obvious from the definition). Equivalently, it means that $] \overline{U}[_P \cap]Z[_P =]\infty_U[_P$ and this equality follows from assertion 2) in the closed case since $\overline{U} \cap Z = \infty_U$. Note that this implies that the proposition also holds in the open case.

As an intermediate step, we prove now that if Z is closed, U open and $X \subset U \cap Z$, then we have $]X[_P \subset]U[_P \cap]Z[_P$. From the closed case, we have $] \overline{X}[_P \subset]Z[_P$ and therefore also $]X[_P \subset]Z[_P$. In order to prove that $]X[_P \subset]U[_P$, we introduce a closed complement F for U and use the previous remark. We may then rewrite our inclusion on the form $] \overline{X}[_P \cap]F[_P \subset]\infty_X[_P$, and this follows from the closed case again since $\overline{X} \cap F \subset \infty_X$.

Assume that we actually have an equality $X = U \cap Z$. Then one shows that we also have an equality $]X[_P =]U[_P \cap]Z[_P$. We only have to prove the reverse inclusion $]U[_P \cap]Z[_P \subset]X[_P$. Equivalently, we have to show that $]Z[_P \subset]X[_P \cup]F[_P$ (where F is a closed complement for U as before), and this follows again from the closed case since $Z \subset X \cup F$.

It is now easy to finish the proof of assertions 1) and 2). For assertion 1), we may write $Y = U \cap Z$ with U open and Z closed. We assume that $X \subset Y = U \cap Z$ and it follows that $]X[_P \subset]U[_P \cap]Z[_P$ but, from what we just proved, we know that $]Y[_P =]U[_P \cap]Z[_P$. For assertion 2), we can write $X_i = U_i \cap Z_i$ with U_i open and Z_i closed, and use the stability by intersection for open or closed that we already know.

Assertion 3) also results from the closed case because closure commutes with union, and therefore, $\overline{X} = \overline{X}_1 \cup \overline{X}_2$. More precisely, thanks to assertion 1), only the direct inclusion needs a proof and this is equivalent to $] \overline{X}[_P \subset]X_1[_P \cup]X_2[_P \cup]\infty_X[_P$. Actually, since, for $i = 1, 2$, we have $\infty_{X_i} \subset \infty_X$, the same is true for the tubes, and we are reduced to check that $] \overline{X}[_P \subset] \overline{X}_1[_P \cup] \overline{X}_2[_P \cup]\infty_X[_P$. Thus, as expected, this again follows from the closed case. \square

Corollary 6.5. *Let $(X \hookrightarrow P \xleftarrow{\lambda} V)$ be an adic overconvergent space. If $P := \mathrm{Spf}(A)$ is affine and X is defined modulo an ideal of definition by*

$$\forall i \in \{1, \dots, r\}, f_i(x) = 0 \quad \text{and} \quad \exists j \in \{1, \dots, s\}, g_j(x) \neq 0,$$

then $\mathrm{sp}_V^{-1}(X)$ is defined in V by

$$\forall i \in \{1, \dots, r\}, v(f_i) > 0 \quad \text{and} \quad \exists j \in \{1, \dots, s\}, v(g_j) \leq 0.$$

and $]X[_V$ is defined by

$$\forall i \in \{1, \dots, r\}, v(f_i^n) \rightarrow +\infty \quad \text{and} \quad \exists j \in \{1, \dots, s\}, v(g_j^n) \nrightarrow +\infty.$$

Proof. When X is a closed subset defined by an open ideal \mathfrak{a} , our hypothesis means that $\mathfrak{a} = I + (f_1, \dots, f_r)$ for some ideal of definition I of A . Since we always have $v(f^n) \rightarrow +\infty$ for $f \in I$, the assertion follows from proposition 6.14. In general, X is the intersection of a closed subset and an open subset and we may use assertion 2) of proposition 6.4. \square

The standard properties of completion translate directly into the language of tubes:

Proposition 6.6. *Let $(X \hookrightarrow P \xleftarrow{\lambda} V)$ be an adic overconvergent space. Then,*

1. *The tube $]X[_V$ only depends on the underlying subspace of X (and not on the structure of formal scheme).*
2. *if $P = \bigcup_{i \in I} P_i$ is an open covering, $X_i := X \cap P_i$ and $V_i := \lambda^{-1}(P_i^{\mathrm{ad}})$, then we have an open covering*

$$]X[_{P,V} = \bigcup_{i \in I}]X_i[_{P_i, V_i},$$

3. *if $u: Q \rightarrow P$ is a morphism of formal schemes and λ factors through Q^{ad} , then*

$$]X[_{P,V} =]u^{-1}(X)[_{Q,V}.$$

Proof. Thanks to proposition 6.4, we may assume that X is closed in P in which case everything follows from proposition 4.5 and the standard properties of the functor $(-)^{\mathrm{ad}}$. \square

Be careful that, when Q is an *open* formal subscheme of P containing X , then $]X[_{Q \neq }]X[_P$ in general: we may not replace P with a neighborhood of X (this is a striking difference with the notion of tube in Tate or Berkovich theory). On the contrary, if Y is a closed formal subscheme of P that contains X and $Q := P^Y$, then we do have

$$]X[_{P=}]X[_Q.$$

As a consequence of assertion 3), we see that the tube is functorial in the sense that any morphism of adic overconvergent spaces

$$(Y \hookrightarrow Q \leftarrow W) \mapsto (X \hookrightarrow P \leftarrow V)$$

induces a morphism of locally topologically ringed spaces

$$]f[_v:]Y[_W \rightarrow]X[_V$$

(this is not built into the definition). The functor $(X \hookrightarrow P \leftarrow V) \mapsto]X[_V$ is clearly continuous and cocontinuous. Be careful however that the (composite) functor $(X \hookrightarrow P) \mapsto]X[_P$ is still continuous, but it is not cocontinuous because the adic topology is usually finer than the Zariski topology.

We can also prove that the tube is left exact in the following sense:

Proposition 6.7. *Assume that we are given two morphisms of formal embeddings $(X_i \hookrightarrow P_i) \rightarrow (X \hookrightarrow P)$ for $i = 1, 2$ and that $P_1 \times_P P_2$ is representable by a locally noetherian formal scheme. Then, if we are given $\lambda: V \rightarrow P_1^{\text{ad}} \times_{P^{\text{ad}}} P_2^{\text{ad}}$, we have*

$$]X_1 \times_X X_2[_V =]X_1[_V \cap]X_2[_V.$$

Proof. We may assume that $V = P_1^{\text{ad}} \times_{P^{\text{ad}}} P_2^{\text{ad}}$. In other words, if we denote by $p_i: P_1 \times_P P_2 \rightarrow P_i$ the projections, we have to show that

$$]X_1 \times_X X_2[_{P_1 \times_P P_2} = (p_1^{\text{ad}})^{-1}(]X_1[_{P_1}) \cap (p_2^{\text{ad}})^{-1}(]X_2[_{P_2}).$$

We may assume that $X = P = \text{Spf}(A)$, that $P_i = \text{Spf}(A_i)$ and that X_i is closed in P_i defined by \mathfrak{a}_i for $i = 1, 2$. It follows that $P_1 \times_P P_2 = \text{Spf}(A_1 \otimes_A A_2)$ and that $X_1 \times_X X_2$ is defined by $\mathfrak{a}_1 \otimes_A A_2 + A_1 \otimes_A \mathfrak{a}_2$. Therefore, when $v \in P_1 \times_P P_2$, we will have

$$\begin{aligned} v \in]X_1 \times_X X_2[_{P_1 \times_P P_2} &\Leftrightarrow \forall f \in \mathfrak{a}_i, v(p_i^{-1}(f)^n) \rightarrow +\infty \text{ for } i = 1, 2 \\ &\Leftrightarrow (p_i^{\text{ad}})(v) \in]X_i[_{P_i} \text{ for } i = 1, 2 \end{aligned}$$

since $v(p_i^{-1}(f)^n) = ((p_i^{\text{ad}})(v))(f^n)$ for $i = 1, 2$. □

Unfortunately, the following fibration theorem will not be very useful but it is worth mentioning however, as a formal consequence of the formal fibration theorem:

Proposition 6.8 (Weak fibration theorem). *Assume that we are given two formal embeddings $X \hookrightarrow P$ and $X' \hookrightarrow P'$. If a differentially smooth morphism $u: P' \rightarrow P$ induces an isomorphism $X' \simeq X$ and an isomorphism $\overline{X'} \simeq \overline{X}$, then, it induces locally on P , an isomorphism*

$$]X'[_{P' \simeq \mathbb{D}^{-,n} \times]X[_P.$$

Proof. We may assume that X and X' are closed in P and P' respectively, in which case our assertion follows directly from the formal fibration theorem 4.6. □

One would like to relax our hypothesis in the proposition and only assume that u induces an isomorphism $X' \simeq X$ (and not necessarily $\overline{X'} \simeq \overline{X}$). Unfortunately, the conclusion will not hold anymore as the case of an open immersion $P' \hookrightarrow P$ shows

(with $X = X' = P'$). The tube inside P will be strictly bigger than the tube inside P' in general.

A tube has no reason to be locally closed (even for the constructible topology) as the example of the absolute open unit disc inside the absolute closed unit disc shows. The situation will be much nicer when we stick to analytic spaces (as we shall see later) or if we consider the naive tube:

Proposition 6.9. *Assume that X is a locally closed (resp. a closed, resp. an open) formal subscheme of a formal scheme P and let $\lambda : V \rightarrow P^{\text{ad}}$ be any morphism of adic spaces. Then, $\text{sp}_V^{-1}(X)$ is a locally closed (resp. a closed, resp. an open) locally constructible subset of V .*

Proof. Note that only constructibility really needs a proof because specialization is continuous. We may clearly assume that $P = \text{Spf}(A)$ and that X is the closed subset defined by an ideal \mathfrak{a} . If f_1, \dots, f_r are generators of \mathfrak{a} , then

$$\text{sp}_V^{-1}(X) = \left\{ v \in V : \min_{i=1}^r v(f_i) > 0 \right\}$$

is a closed constructible subset of V (finite intersection of complements of rational open subsets which are quasi-compact). \square

We can now describe how far our tube differs from the naive tube. In particular, we will see that the points of the tube of X cannot specialize too far from X itself (we will denote by $\overset{\circ}{W}$ and \overline{W} the interior and the closure of a subset W):

Proposition 6.10. *Let $(X \rightarrow P \leftarrow V)$ be an adic overconvergent space. If we let $W := \text{sp}_V^{-1}(X)$, then we have*

$$\overset{\circ}{W} \subset]X[_V \subset \overline{W}.$$

Moreover, if V is analytic, then $]X[_V = \overset{\circ}{W}$ (resp. $]X[_V = \overline{W}$) when X is closed (resp. open) in the topology of P .

Proof. First of all, we should recall that, if we write $W(X)$ instead of W , then W sends closed to closed and preserves standard set operations. And we know from proposition 6.4 that the tube also preserves these operations (although it does not send closed to closed). Assume for a while that we know that the conclusion holds in the closed case. If we write $X = Z \cap U$ with Z closed and U open in the topology of P , we will have

$$\overline{W(Z)} \subset]Z[_V \subset W(Z)$$

and, by considering a closed complement of U , we deduce that

$$W(U) \subset]U[_V \subset \overline{W(U)}.$$

Since $]Z \cap U[_V =]Z[_V \cap]U[_V$, this will imply that

$$W(\overline{Z \cap U}) = \overline{W(Z)} \cap W(U) \subset]Z \cap U[_V \subset W(Z) \cap \overline{W(U)} = \overline{W(Z \cap U)}$$

and we will be done.

Thus, we assume from now on that X is closed in P . Since the question is local on V , we may also assume that $V = \text{Spa}(B, B^+)$ with (B, B^+) complete, that P is affine and that X is defined by some ideal \mathfrak{a} . Then $]X[_V$ is defined as a subset of V by the conditions $v(f^n) \rightarrow +\infty$ and W is defined by the conditions $v(f) > 0$ for $f \in \mathfrak{a}$. We proved in lemma 2.9 that both conditions are stable under specialization. Now, any $v \in V$ has a generization $w \in V$ of minimal height ≤ 1 and it follows that

$$w(f) > 0 \Leftrightarrow w(f^n) \rightarrow +\infty \Rightarrow v(f^n) \rightarrow +\infty.$$

Now, since W is a (closed) constructible subset of V , one knows that $v \in \overset{\circ}{W}$ if and only if any generization of v belongs to W . This implies the first assertion and we may now assume that V is analytic. In this case, w has height exactly 1 and this is the maximum (vertical) generization of v . Moreover, there exists an isomorphism $\mathcal{H}(w) \simeq \mathcal{H}(v)$ on the completed residue fields, and it follows that

$$w(f) > 0 \Leftrightarrow w(f^n) \rightarrow +\infty \Leftrightarrow v(f^n) \rightarrow +\infty$$

(because this last condition is purely topological). Thus, we see that $w \in W$ when $v \in]X[_V$. Actually, the same holds for any other generization of v because it is necessarily a specialization of w . \square

Corollary 6.11. *The tube of a locally closed (resp. closed, resp. open) formal subscheme in an analytic space is a locally closed (resp. open, resp. closed) subset.*

In particular, the tube of a *closed* formal subscheme into an *analytic* overconvergent space has a natural structure of analytic space (as an open subset).

Assume that $X \hookrightarrow P$ is a locally closed embedding such that X is *open* in the topology of P . Then, there exists a unique *open* formal subscheme Q of P having the same underlying space as X and we have $\text{sp}^{-1}(X) = Q^{\text{ad}}$ which is an open subset of P^{ad} . It follows that $]X[_{P^{\text{an}}}$ is the same thing as the closure of Q^{an} in P^{an} . More generally, if we are given $\lambda: V \rightarrow P^{\text{ad}}$ with V analytic, then $]X[_V$ will be the closure of $\lambda^{-1}(Q^{\text{ad}})$ in V .

Recall that we denote by V_1 the set of points of height 1 in an analytic space V and by $\text{sep}: V \rightarrow V_1$ the canonical retraction (the projection on the Berkovich quotient).

Proposition 6.12. *If $(X \hookrightarrow P \leftarrow V)$ is an analytic overconvergent space and $v \in V$, then*

$$v \in]X[_V \Leftrightarrow \text{sp}_V(\text{sep}(v)) \in X.$$

Proof. We may assume that X is open in the topology of P - and use complement and intersection in order to deduce the general case. Then, we know from proposition 6.10 that $]X[_V = \overline{W}$ with $W := \text{sp}_V^{-1}(X)$. Since W is a constructible subset, we will have $v \in \overline{W}$ if and only if v has a generization v' in W . Since W is open, the maximum generization v_1 of v' will also be in W and we may therefore assume that $v' = v_1$, which is also the maximum generization of v . In other words, we have $v \in \overline{W} =]X[_V$ if and only if $\text{sep}(v) = v_1 \in W = \text{sp}_V^{-1}(X)$ as asserted. \square

As a consequence of this proposition, when V is analytic, we can recover the true tube from the naive one:

$$]X[_V = \text{sep}^{-1}(\text{sp}_V^{-1}(X) \cap V_1).$$

In particular, we see that, when V is analytic, the tube $]X[_V$ is stable both under specialization and generization.

We want now to describe more precisely the analytic points of the tube of a closed formal subscheme. Recall that if \mathfrak{a} is an ideal in a ring A , we denote by $\mathfrak{a}^{(n)}$ the ideal generated by all f^n with $f \in \mathfrak{a}$.

Definition 6.13. *Let $(X \hookrightarrow P \xleftarrow{\lambda} V)$ be an adic overconvergent space with X closed in P . Let $V \rightarrow \text{Spa}(B, B^+)$ be an adic morphism and J an ideal of definition for B . Then the tube of radius $n \in \mathbb{N}$ of X in V (relatively to J) is*

$$[X]_{V,n} := \left\{ v \in V : \inf_{f \in \mathcal{I}_{X, \text{sp}(v)}^{(n)}} v(f) \geq \inf_{g \in J} v(g) \neq +\infty \right\}.$$

We will give some examples below but we want to first prove the following:

Proposition 6.14. *Let $(X \hookrightarrow P \leftarrow V)$ be an adic overconvergent space with X closed in P , $V \rightarrow \text{Spa}(B, B^+)$ an adic morphism and J an ideal of definition for B . Then, for all $n \in \mathbb{N}$, $[X]_{V,n}$ is a constructible open subset of V such that*

$$\overline{[X]_{V,n}} \subset [X]_{V,n+1}$$

and we have

$$]X[_V^{\text{an}} = \bigcup_{n \in \mathbb{N}} [X]_{V,n}.$$

Proof. All questions are local on V and therefore also on P . We may therefore assume that $P := \text{Spf}(A)$, so that X is defined by an ideal $\mathfrak{a} \subset A$, and that $V = \text{Spa}(B, B^+)$. In this situation, if f_1, \dots, f_r are some generators for \mathfrak{a} modulo J and g_1, \dots, g_s some generators for J , we have

$$[X]_{V,n} := \left\{ x \in V : \min_{i=1}^r v(f_i^n) \geq \min_{j=1}^s v(g_j) \neq +\infty \right\}.$$

One may then notice that $[X]_{V,n}$ is the finite union of the rational subsets

$$\left\{ x \in V : \min_{i,j=1}^{r,s} \{v(f_i^n), v(g_j)\} \geq v(g_k) \neq +\infty \right\}$$

for $k = 1, \dots, s$. This is therefore a quasi-compact (which is equivalent to constructible since V is affinoid) open subset of V as asserted. Now that we know that $[X]_{V,n}$ is constructible, in order to prove the second assertion, it is sufficient to check that if $w \in [X]_{V,n}$ specializes to $v \in V$, then $v \in [X]_{V,n+1}$. At this point, it seems more natural to work in the residue fields and use the multiplicative

notation (even if the absolute value might have higher height). Let us assume that $|f^n(w)| \leq |g(w)| \neq 0$ with $f \in \mathfrak{a}$ and $g \in J$. Then necessarily $f(w)$ is topologically nilpotent (because $g(w)$ is) and $\left| \frac{f^n(w)}{g(w)} \right| \leq 1$. It follows that $\frac{f^{n+1}(w)}{g(w)}$ is also topologically nilpotent. Since $\mathcal{H}(v)$ is homeomorphic to $\mathcal{H}(w)$, we also have $g(v) \neq 0$ and $\frac{f^{n+1}(v)}{g(v)}$ topologically nilpotent. In particular, we must have $\left| \frac{f^{n+1}(v)}{g(v)} \right| \leq 1$ and therefore $|f^{n+1}(v)| \leq |g(v)| \neq 0$.

For the last assertion, it is sufficient to check that, for $f \in \mathfrak{a}$ and $v \in V$, we have

$$(v(f^n) \rightarrow +\infty \text{ and } v \in V^{\text{an}}) \Leftrightarrow (\exists n \in \mathbb{N}, \exists g \in J, v(f^n) \geq v(g) \neq +\infty).$$

First of all, we know that

$$v \in V^{\text{an}} \Leftrightarrow \exists g \in J, v(g) \neq +\infty,$$

because J is an ideal of definition. Moreover, once we know that $v(g) \neq +\infty$, then the equivalence

$$v(f^n) \rightarrow +\infty \Leftrightarrow \exists n \in \mathbb{N}, v(f^n) \geq v(g)$$

follows from the fact that $v(g^n) \rightarrow +\infty$ (because $g \in J$ and our valuations are continuous). \square

Example Assume that we are given a morphism of adic spaces $O \rightarrow \text{Spa}(B, B^+)$ where B is a Tate ring with topologically nilpotent unit π (in which case the morphism is automatically adic and O is automatically analytic).

1. We consider the analytic overconvergent space (zero section)

$$\text{Spec}(\mathbb{Z}) \hookrightarrow \mathbb{A} \longleftarrow \mathbb{D}_O.$$

Then, we have

$$]\text{Spec}(\mathbb{Z})[_{\mathbb{D}_O} = \mathbb{D}_O^-$$

(even as adic spaces, as we shall see below), and

$$[\text{Spec}(\mathbb{Z})]_{\mathbb{D}_O, n} = \mathbb{D}_O(0, \pi^{\frac{1}{n}}) := \{v \in \mathbb{A}_O : v(T^n) \geq v(\pi)\}$$

is a closed disk which is an affinoid space for each $n \in \mathbb{N}$. The formula of the proposition reads

$$\mathbb{D}_O^- = \bigcup_{n \in \mathbb{N}} \mathbb{D}_O(0, \pi^{\frac{1}{n}}).$$

Recall that the open unit disc does not contain the v^- -points.

2. We consider now the analytic overconvergent space

$$\mathbb{A} \hookrightarrow \mathbb{P} \longleftarrow \mathbb{P}_O.$$

Then, from the first example, one can easily deduce that

$$]\mathbb{A}[_{\mathbb{P}_O}^{\text{an}} = \bigcap_{n \in \mathbb{N}} \mathbb{D}_O(0, \pi^{-\frac{1}{n}})$$

where

$$\mathbb{D}_O(0, \pi^{-\frac{1}{n}}) := \{v \in \mathbb{A}_O : v(\pi T^n) \geq 0\}$$

is also an affinoid open subset for each $n \in \mathbb{N}$. This tube is slightly bigger than the closed unit disc \mathbb{D}_O : it contains the v^+ -points.

3. To make it more explicit, we may consider the case $O = \text{Spa}(\mathbb{Q}_p)$ (with the “ p -adic” topology). Then we have

$$\mathbb{P}_{\mathbb{Q}_p}^{\text{an}} = \underbrace{\mathbb{D}(0, 1^+) \sqcup \{v_p^+\}}_{]A[} \sqcup \underbrace{\mathbb{D}(\infty, 1^-)}_{] \infty[}.$$

4. This can be extended to the Monsky-Washnitzer setting. We let R be a noetherian ring and $S \rightarrow \text{Spec}(R)$ be any morphism of formal schemes. Then, we consider an R -algebra of finite type A and denote by \overline{X} some projective closure of $X := \text{Spec}(A)$. Finally, we give ourselves a morphism $R \rightarrow B_0$ where B is a Tate ring with topologically nilpotent unit π and let $O = \text{Spa}(B, B^+)$. Then we have

$$\Gamma(]X_S[_{\overline{X}_O}, i^{-1}\mathcal{O}_{\overline{X}_O}) = (B_0 \otimes_R A)^\dagger[1/\pi].$$

We can now prove the following adic variant of the famous proposition 0.2.7 of [3]:

Proposition 6.15. *If $(X \hookrightarrow P \xleftarrow{\lambda} V)$ is an analytic overconvergent space with X closed in P , then there exists an isomorphism of adic spaces:*

$$\lambda^{-1}(P/X, \text{ad}) := P/X, \text{ad} \times_{P, \text{ad}} V \simeq]X[_{V \subset V}.$$

Proof. The question is local. We may therefore assume that $P = \text{Spf}(A)$ is affine with X defined by an ideal $\mathfrak{a} = (f_1, \dots, f_r)$ and that $V = \text{Spa}(B, B^+)$ with B a complete Tate ring with topologically nilpotent unit π . Recall that, in this situation, $P/X = \text{Spa}(A/\mathfrak{a})$ where A/\mathfrak{a} is identical to A as a ring but its topology is defined by $\mathfrak{a} + I$ if I denotes an ideal of definition for A . Proposition 6.14 tells us that $]X[_{V} = \bigcup_{n \in \mathbb{N}}]X[_{V, n}$ with

$$]X[_{V, n} = \left\{ x \in V : \min_{i=1}^r \{v(f_i^n)\} \geq v(\pi) \neq +\infty \right\},$$

and we need to show that the natural map $]X[_{V, n} \rightarrow P^{\text{ad}}$ factors canonically through $P/X, \text{ad}$. After replacing \mathfrak{a} with $\mathfrak{a}^{(n)}$, we may assume that $n = 1$. If we write $]X[_{V, 1} = \text{Spf}(C, C^+)$ with (C, C^+) complete, we need to show that the composite map $A \rightarrow B \rightarrow C$ is continuous for the \mathfrak{a} -adic topology. Thus, we have to check that $f_i^n \rightarrow 0$ in C for all $i = 1, \dots, r$ (we may re-use the letter n). This follows from the fact that, by definition, f_i/π is power-bounded in C and π is topologically nilpotent. \square

As a consequence of this proposition, we see that, in an *analytic* overconvergent space $(X \hookrightarrow P \leftarrow V)$, we may replace P with its completion along \overline{X} and then replace X with the open formal subschemes Q of P having the same underlying

space as X , without modifying the tube. In practice, we are therefore often reduced to the case when $X = Q$ is an open formal subscheme of P as we shall see later.

Note that it is necessary to assume V analytic in the proposition in order to consider $]X[_V$ as an adic space (as an open subset of V). Anyway, the map $\lambda^{-1}(P/X, \text{ad}) \rightarrow]X[_V$ is not even an isomorphism of locally topologically ringed spaces in general.

In order to generalize the constructions of this section to the case where P is *not* locally noetherian, it is necessary to assume that X is *locally radically finitely presented*: it means that locally, X is defined modulo an ideal of definition by an ideal \mathfrak{a} such that $\sqrt{\mathfrak{a}} = \sqrt{A \cdot S}$ for some *finite* set $S \in A$. One may then proceed in the same way as above, using generalized adic spaces of Scholze and Weinstein instead of the honest adic spaces of Huber. We will not work out the details here.

Example If \mathcal{V} is a (non discrete) valuation ring with residue field k and $X \hookrightarrow P$ is a locally closed embedding of a k -variety into a formal scheme which is locally finitely presented over \mathcal{V} , then X is locally radically finitely presented. This follows from the fact that, if \mathfrak{m} denotes the maximal ideal of \mathcal{V} , then we have $\sqrt{\mathfrak{m}} = \mathfrak{m} = \sqrt{(\pi)}$ for any topologically nilpotent unit π .

7 Strict neighborhoods

In this section, we relax our category of overconvergent spaces in order to make the role of the ambient formal scheme secondary and to allow the replacement of the adic space by some neighborhood of the tube. As usual, formal schemes are always locally noetherian and adic spaces are locally of noetherian type.

Definition 7.1. A strict neighborhood is a formal morphism

$$\begin{array}{ccccc} X' & \hookrightarrow & P' & \longleftarrow & V' \\ \downarrow f & & \downarrow v & & \downarrow u \\ X & \hookrightarrow & P & \longleftarrow & V. \end{array}$$

where

1. f is an isomorphism,
2. v is locally noetherian,
3. u is an open immersion,
4. $]f[_u$ is surjective.

Note that a strict neighborhood automatically induces an isomorphism

$$]f[_u:]X'[_V \simeq]X[_V$$

of topologically locally ringed spaces (for the induced structures).

Up to isomorphism, we may always assume that $X' = X$, that $f = \text{Id}_X$ and that u is the inclusion of an open subset so that $]X[_{V'} =]X[_V$, and we would then write

$$\begin{array}{ccccc} X & \hookrightarrow & P' & \longleftarrow & V' \\ \parallel & & \downarrow v & & \downarrow \\ X & \hookrightarrow & P & \longleftarrow & V. \end{array}$$

Example Assume that we are given a formal scheme S and a morphism $O \rightarrow S^{\text{ad}}$ with O analytic. Then there exists a sequence of strict neighborhoods

$$(S \hookrightarrow \mathbb{A}_{\bar{S}} \longleftarrow \mathbb{D}_{\bar{O}}) \rightarrow (S \hookrightarrow \mathbb{A}_S \longleftarrow \mathbb{D}_O) \rightarrow (S \hookrightarrow \mathbb{P}_S \longleftarrow \mathbb{P}_O)$$

(this is not true however when O is not analytic).

A strict neighborhood always splits as follows:

Proposition 7.2. *Any strict neighborhood*

$$\begin{array}{ccccc} X' & \hookrightarrow & P' & \longleftarrow & V' \\ \downarrow f & & \downarrow v & & \downarrow u \\ X & \hookrightarrow & P & \longleftarrow & V. \end{array}$$

is the composition of a strict neighborhood with $u = \text{Id}_V$ and another strict neighborhood with both $f = \text{Id}_X$ and $v = \text{Id}_P$.

Proof. Follows from proposition 5.3. □

Strict neighborhoods will allow us to increase (or shrink) P as the next proposition shows.

Proposition 7.3. *If $(X \hookrightarrow P \longleftarrow V)$ is an adic overconvergent space and $P \hookrightarrow Q$ is a locally closed embedding, then we have a strict neighborhood:*

$$\begin{array}{ccccc} X & \hookrightarrow & P & \longleftarrow & V \\ \parallel & & \downarrow & & \parallel \\ X & \hookrightarrow & Q & \longleftarrow & V. \end{array}$$

Proof. Since the left hand square is cartesian, we have $]X[_{Q \cap P^{\text{ad}}} =]X[_P$. Everything follows because a locally closed embedding is locally noetherian. □

Be careful however that the morphism

$$\begin{array}{ccccc} X & \hookrightarrow & P & \xlongequal{\quad} & P^{\text{ad}} \\ \parallel & & \downarrow & & \downarrow \\ X & \hookrightarrow & Q & \xlongequal{\quad} & Q^{\text{ad}}. \end{array}$$

is *not* a strict neighborhood in general as the case $X = P = \mathbb{A}$ and $Q = \mathbb{P}$ shows: we have $]X[_P = \mathbb{D} \neq]\mathbb{A}[_{\mathbb{P}} =]X[_Q$. In the statement of the proposition, it is therefore important to keep the same V on the right.

Using the next property, it will also be possible to assume that the locus at infinity is a divisor:

Proposition 7.4. *Let $(X \hookrightarrow P \leftarrow V)$ be an adic overconvergent space. If $v: P' \rightarrow P$ is a blowing up of a usual subscheme centered outside X , then it extends to a strict neighborhood*

$$\begin{array}{ccccc} X \hookrightarrow & P' & \xleftarrow{\lambda'} & V' & \\ \parallel & \downarrow v & & \downarrow u & \\ X \hookrightarrow & P & \xleftarrow{\lambda} & V & \end{array}$$

Proof. We may clearly assume that $V = P^{\text{ad}}$. Now, the assertion results from proposition 3.13. More precisely, if we denote by Z the center of the blowing up and let $Z' = v^{-1}(Z)$ so that $X \cap Z = \emptyset$ and $X \cap Z' = \emptyset$, then v induces an isomorphism $P'^{\text{ad}} \setminus Z'^{\text{ad}} \simeq P^{\text{ad}} \setminus Z^{\text{ad}}$ between an open neighborhood of $]X[_{P'}$ and an open neighborhood of $]X[_P$. We may set $V' = P'^{\text{ad}} \setminus Z'^{\text{ad}}$. \square

As a consequence, we obtain Chow's lemma for overconvergent spaces:

Corollary 7.5. *Assume that we are given a formal morphism*

$$\begin{array}{ccccc} Y \hookrightarrow & Q & \longleftarrow & W & \\ \downarrow f & \downarrow v & & \downarrow u & \\ X \hookrightarrow & P & \longleftarrow & V & \end{array}$$

in which $v: Q \rightarrow P$ is separated of finite type around Y (see definition 4.9). If Y is reduced, f is quasi-projective and \overline{X} quasi-compact, then, there exists a strict neighborhood

$$\begin{array}{ccccc} Y' \hookrightarrow & Q' & \longleftarrow & W' & \\ \downarrow & \downarrow & & \downarrow & \\ Y \hookrightarrow & Q & \longleftarrow & W & \end{array}$$

such that the composite map $v': Q' \rightarrow P$ is quasi-projective in the neighborhood of Y' .

Proof. We may assume that X also is reduced. We are in the situation to apply the precise Chow's lemma (corollary 5.7.14 of [20]): there exists a blowing-up $Z \rightarrow \overline{Y}$ centered outside Y such that the composite map $Z \rightarrow \overline{X}$ is projective. This blowing up is always induced by some blowing up $Q' \rightarrow Q$ and we may apply proposition 3.13 (we have $Y' = Y$ and $\overline{Y'} = Z$). \square

When V is analytic, we also have the following:

Proposition 7.6. *If $(X \hookrightarrow P \leftarrow V)$ is an analytic overconvergent space, then there exists a strict neighborhood*

$$\begin{array}{ccccc} X & \hookrightarrow & P' & \xleftarrow{\lambda'} & V' \\ \parallel & & \downarrow v & & \downarrow u \\ X & \hookrightarrow & P & \xleftarrow{\lambda} & V. \end{array}$$

with X open and dense in the topology of P' and $]X[_{V'}$ closed in V' .

Proof. Simply choose $P' = P^{\overline{X}}$ and $V' =]\overline{X}[_{V}$. The result is then an immediate consequence of proposition 6.15. \square

The next two propositions will allow us to prove that strict neighborhoods form a right multiplicative system.

Lemma 7.7. *Strict neighborhoods are stable under pull back.*

Proof. Immediate consequence of proposition 6.7. \square

Lemma 7.8. *If a strict neighborhood*

$$\begin{array}{ccccc} X' & \hookrightarrow & P' & \longleftarrow & V' \\ \downarrow f & & \downarrow v & & \downarrow u \\ X & \hookrightarrow & P & \longleftarrow & V. \end{array}$$

has a section, then this section is also a strict neighborhood.

Proof. Condition 1) and 4) are clearly satisfied because f and $]f[_u$ are always isomorphisms and a section of an isomorphism is necessarily an isomorphism. Actually, the same holds for condition 3) because an open immersion with a section is also an isomorphism. Finally, condition 2) follows from the fact that a section of a locally noetherian map is locally noetherian. \square

Proposition 7.9. *Adic overconvergent spaces and formal morphisms admit right calculus of fractions with respect to strict neighborhoods.*

Proof. This is a formal consequence of lemmas 7.7 and 7.8 as lemma 7.10 below shows. \square

Lemma 7.10. *Let \mathcal{C} be a category and S a set of morphisms which contains identities, is stable by composition, is stable by pull-back and is stable by taking sections. Then, \mathcal{C} admits right calculus of fractions with respect to S .*

Proof. According to definition 2.3 of [9], there are four (sometimes called after Ore) conditions to check, the first two of which coincide with our first two. For the third condition, we have to show that any diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow t & & \downarrow s \\ Y & \xrightarrow{f} & X \end{array}$$

with $s \in S$ may be completed into a commutative diagram with $t \in S$. We may simply choose for t the pull back of s along f . For the fourth condition, we must show that any commutative diagram

$$Y' \xrightarrow{t} Y \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} X \xrightarrow{s} X'$$

with $s \in S$ may be completed into a commutative diagram with $t \in S$. First of all, if we pull back s along itself, we see that $X \times_{X'} X$ is representable and that the projection $p : X \times_{X'} X \rightarrow X$ belongs to S . It follows that the diagonal map $\delta : X \hookrightarrow X \times_{X'} X$ also belongs to S because this is a section of p . We may then pull back δ along the map $(f, g) : Y \rightarrow X \times_{X'} X$ in order to obtain $t : Y' \rightarrow Y$ (which is actually the kernel of f and g). \square

Definition 7.11. *The adic overconvergent site is the category of adic overconvergent spaces localized with respect to strict neighborhoods and endowed with the image topology.*

In other words, an object of the adic overconvergent site is an adic overconvergent space and a morphism is, up to equivalence, a diagram

$$\begin{array}{ccc} (Y' \hookrightarrow Q' \leftarrow W') & & \\ \downarrow & \searrow & \\ (Y \hookrightarrow Q \leftarrow W) & & (X \hookrightarrow P \leftarrow V) \end{array}$$

where the vertical map is a strict neighborhood. It means that we may always replace W with some neighborhood of the tube and modify Q almost as we wish.

Our category is endowed with the *image topology* which is the coarsest topology making continuous the localization map. This is the topology generated by the pretopology made of families of formal morphisms

$$\{(X \hookrightarrow P \leftarrow V_i) \rightarrow (X \hookrightarrow P \leftarrow V)\}_{i \in I}$$

where, for each $i \in I$, V_i is open in V , and $]X[_V = \bigcup_{i \in I}]X[_{V_i}$. Again, it is subcanonical.

If necessary, we will call this topology the *adic topology*. We may also endow our category with the *Zariski topology* (image of the Zariski topology) or even with the *Zariski-adic topology* which is the coarsest topology finer than the two others, for example. Many other choices are possible.

Since the formal scheme P plays a very loose role in the theory, we will usually denote by (X, V) an object of the adic overconvergent site. The localization functor

$$(X \hookrightarrow P \leftarrow V) \mapsto (X, V)$$

from the category of overconvergent adic spaces and formal morphism to the overconvergent adic site commutes with finite limits ([9] again) and is continuous (by

definition). Let us remark that, as a consequence, the functor $(X \hookrightarrow P) \mapsto (X, P^{\text{ad}})$ from the category of formal embeddings to the category of adic overconvergent sites also commutes with finite limits. It is continuous for the Zariski or Zariski-adic topology but not for the adic-topology (see proposition 7.16 however). Also, the functor $(X, V) \mapsto X$ commutes with all limits because it has an adjoint $X \mapsto (X, \emptyset)$. And it is also continuous. Finally, giving a sheaf (or a presheaf) on the adic overconvergent site is equivalent to giving a sheaf (or a presheaf) with respect to *formal* morphisms which becomes an isomorphism on strict neighborhoods.

We insist on the fact that we usually write $(X \hookrightarrow P \leftarrow V)$ for an adic overconvergent space seen as an object before localization (with formal morphisms) and (X, V) when we see it as an object of the adic overconvergent site (up to strict neighborhoods).

Proposition 7.12. *A formal morphism*

$$(Y \hookrightarrow Q \leftarrow W) \rightarrow (X \hookrightarrow P \leftarrow V)$$

induces an isomorphism $(Y, W) \simeq (X, V)$ if and only if there exists a commutative diagram

$$\begin{array}{ccc} & (Y' \hookrightarrow Q' \leftarrow W') & \\ \swarrow & & \searrow \\ (Y \hookrightarrow Q \leftarrow W) & \xrightarrow{\quad\quad\quad} & (X \hookrightarrow P \leftarrow V) \end{array}$$

where both diagonal arrows are strict neighborhoods.

We may always assume that $Y' = Y$, that the corresponding map is the identity and that W' is an open subset of W .

Proof. Only the direct implication needs a proof. According to proposition 7.1.20. (i) of [15] (be careful that they call left what we call right) or section 3.5 of chapter I in [9], there exists a commutative diagram of formal morphisms:

$$\begin{array}{ccc} (Y' \hookrightarrow Q' \leftarrow W') & \longrightarrow & (X' \hookrightarrow P' \leftarrow V') \\ \downarrow & \swarrow & \downarrow \\ (Y \hookrightarrow Q \leftarrow W) & \longrightarrow & (X \hookrightarrow P \leftarrow V) \end{array}$$

with strict neighborhoods as vertical maps. It is sufficient to prove that the upper map is a strict neighborhood. Since the composite map $Q' \rightarrow P' \rightarrow Q$ is locally noetherian, the first one $Q' \rightarrow P'$ is also necessarily locally noetherian (see [7]) and condition 2) holds. The proofs that the other three conditions hold are very similar to each other and we will only do condition 3) which is the one that requires some care. Let us denote by V'' the inverse image of W' in V' through the diagonal map, and identify W' and V'' with their images in W and V respectively. Then, there exists a commutative diagram

$$\begin{array}{ccc} W' & \longrightarrow & V'' \\ \parallel & \swarrow & \parallel \\ W' & \longrightarrow & V'' \end{array}$$

The diagonal map has both a section and a retraction and must be an isomorphism. It follows that the upper map is also an isomorphism and we are done. \square

Be careful that the map $Q \rightarrow P$ is not necessarily locally noetherian as the following example shows:

$$(\emptyset \subset \mathbb{A}^b \leftarrow \emptyset) \rightarrow (\emptyset \subset \text{Spec}(\mathbb{Z}) \leftarrow \emptyset.)$$

We will mostly be interested in the following consequence:

Corollary 7.13. *Assume that we are given two adic overconvergent spaces $(X_i \hookrightarrow P_i \leftarrow V_i)$ for $i = 1, 2$. Then, $(X_1, V_1) \simeq (X_2, V_2)$ if and only if there exists a common strict neighborhood*

$$\begin{array}{ccc} & (X \hookrightarrow P \leftarrow V) & \\ \swarrow & & \searrow \\ (X_1 \hookrightarrow P_1 \leftarrow V_1) & & (X_2 \hookrightarrow P_2 \leftarrow V_2). \quad \square \end{array}$$

Note that when $X_2 = X_1$, we may also assume that $X = X_1 = X_2$ and that the corresponding maps are the identity.

Most of the time, we will work over a given adic overconvergent space $(C \hookrightarrow S \leftarrow O)$. It is important to notice that, by construction, the diagonal maps in corollary 7.13 will be defined over $(C \hookrightarrow S \leftarrow O)$ as well.

It will be convenient to call a formal morphism $(X \hookrightarrow P \leftarrow V) \rightarrow (C \hookrightarrow S \leftarrow O)$ *locally noetherian* when the morphism $P \rightarrow S$ is locally noetherian.

Proposition 7.14. *Assume that we are given two locally noetherian adic overconvergent spaces $(X_i \hookrightarrow P_i \leftarrow V_i)$ over $(C \hookrightarrow S \leftarrow O)$ for $i = 1, 2$. Then, $(X_1, V_1) \simeq (X_2, V_2)$ if and only if the projections extend to strict neighborhoods*

$$\begin{array}{ccc} & (X \hookrightarrow P_1 \times_S P_2 \leftarrow V) & \\ \swarrow & & \searrow \\ (X_1 \hookrightarrow P_1 \leftarrow V_1) & & (X_2 \hookrightarrow P_2 \leftarrow V_2). \end{array}$$

Again, when $X_2 = X_1$, we may also assume that $X = X_1 = X_2$ and that the corresponding maps are the identity.

Proof. In the situation of corollary 7.13, we may always assume (as we indicate after the statement of the corollary) that the morphisms are compatible with the structural maps. Then the common strict neighborhood factors as

$$\begin{array}{ccc} & (X \hookrightarrow P \leftarrow V) & \\ & \downarrow & \\ & (X \hookrightarrow P_1 \times_S P_2 \leftarrow V) & \\ \swarrow & & \searrow \\ (X_1 \hookrightarrow P_1 \leftarrow V_1) & & (X_2 \hookrightarrow P_2 \leftarrow V_2). \end{array}$$

It should be clear that both diagonal arrows are strict neighborhoods. For example, the composite map

$$]X[_{P,V \rightarrow}]X[_{P_1 \times_S P_2, V \rightarrow}]X_1[_{P_1, V_1}$$

being surjective, the second map must also be surjective and condition 4) holds. \square

The following obvious consequence of proposition 7.14 will often make it possible to assume that X is a usual scheme when considering an adic overconvergent space (X, V) .

Corollary 7.15. *Assume that (X, V) and (X, V') are two locally noetherian adic overconvergent spaces over (C, O) . Then we have*

$$(X, V) \simeq (X, V') \Leftrightarrow (X_{\text{red}}, V) \simeq (X_{\text{red}}, V')$$

Note that this is the same X on both sides and we implicitly assume that the map induced at this level is the identity.

There is no chance for the forgetful functor $(X, V) \rightarrow X$ to be continuous (for the Zariski topology of X) because we choose to use the coarse topology on the algebraic side when we defined the topology on the adic overconvergent site. However, we have the following consequence of the proposition:

Corollary 7.16. *Let $(X \hookrightarrow P \xleftarrow{\lambda} V)$ and $(X \hookrightarrow P' \xleftarrow{\lambda'} V')$ be two locally noetherian adic overconvergent spaces over some $(C \hookrightarrow S \leftarrow O)$. Assume that there exists two open coverings $P = \bigcup_{i \in I} P_i$ and $P' = \bigcup_{i \in I} P'_i$ such that, for each $i \in I$, $X \cap P_i = X \cap P'_i$ and, locally, we have $(X \cap P_i, \lambda^{-1}(P_i^{\text{ad}})) \simeq (X \cap P'_i, \lambda'^{-1}(P'_i{}^{\text{ad}}))$. Then, locally, we have $(X, V) \simeq (X, V')$.*

Proof. Let us write $X_i := X \cap P_i = X \cap P'_i$, $V_i := \lambda^{-1}(P_i^{\text{ad}})$ and $V'_i := \lambda'^{-1}(P'_i{}^{\text{ad}})$. We assume that there exists, for each $i \in I$, a family of open subsets $\{V_{ij}\}_{j \in J_i}$ of V_i such that $]X_i[_{V_i} = \bigcup_{j \in J_i}]X_i[_{V_{ij}}$, a family of open subsets $\{V'_{ij}\}_{j \in J_i}$ of V'_i such that $]X'_i[_{V'_i} = \bigcup_{j \in J_i}]X'_i[_{V'_{ij}}$, and an isomorphism $(X_i, V_{ij}) \simeq (X_i, V'_{ij})$. Then, proposition 7.14 tells us that for each $i \in I, j \in J_i$, the projections extend to a common strict neighborhood

$$\begin{array}{ccc} & (X_i \hookrightarrow P_i \times_S P'_i \leftarrow W_{ij}) & \\ & \swarrow \qquad \searrow & \\ (X_i \hookrightarrow P_i \leftarrow V_{ij}) & & (X_i \hookrightarrow P'_i \leftarrow V'_{ij}). \end{array}$$

It follows that the projections

$$\begin{array}{ccc} & (X \hookrightarrow P \times_S P' \leftarrow W_{ij}) & \\ & \swarrow \qquad \searrow & \\ (X \hookrightarrow P \leftarrow V_{ij}) & & (X \hookrightarrow P' \leftarrow V'_{ij}). \end{array}$$

also provide a common strict neighborhood and (X, V) is locally isomorphic to (X, V') . \square

8 The strong fibration theorem

At last, we give the the strong fibration theorem, building on all the previous results. As usual, formal schemes are supposed to be locally noetherian and adic spaces are assumed to be locally of noetherian type.

Recall that adic overconvergent spaces were first made into a category by using formal morphisms and that we turned strict neighborhoods into isomorphisms in order to obtain the adic overconvergent site. In the process, the formal scheme which is used as a link between the algebraic and the analytic world almost disappears. We will see that, however, many properties of the formal morphism are somehow transferred to the (true) morphism.

We start by explaining how we can extend some notions coming from formal schemes (resp. adic spaces) to adic overconvergent spaces and formal morphisms (resp. to the adic overconvergent site):

Definition 8.1. 1. A formal morphism

$$\begin{array}{ccccc} Y & \hookrightarrow & Q & \longleftarrow & W \\ \downarrow f & & \downarrow v & & \downarrow u \\ X & \hookrightarrow & P & \longleftarrow & V. \end{array}$$

of adic overconvergent spaces is

- (a) right cartesian W is a neighborhood of Y in $(v^{\text{ad}})^{-1}(V)$,
 - (b) said to satisfy a property \mathcal{P} (of formal schemes) if v satisfies the property \mathcal{P} around Y .
2. A morphism of adic overconvergent spaces $(Y, W) \rightarrow (X, V)$ is said to satisfy a property \mathcal{P} (of adic spaces) if there exists a neighborhood V' of X in V and a neighborhood W' of Y in W such that the induced map $W' \rightarrow V'$ satisfies the property \mathcal{P} .

Recall from definitions 4.8 and 4.9 that the expression “around Y ” has two different meanings depending on the “open” or “closed” nature of the property.

Proposition 8.2. If a right cartesian formal morphism

$$\begin{array}{ccccc} Y & \hookrightarrow & Q & \longleftarrow & W \\ \downarrow f & & \downarrow v & & \downarrow u \\ X & \hookrightarrow & P & \longleftarrow & V. \end{array}$$

of analytic overconvergent spaces is formally locally of finite type and formally unramified (resp. and formally smooth, resp. and formally étale), then it induces an unramified (resp. a smooth, resp. an étale) morphism

$$(Y, W) \rightarrow (X, V).$$

Proof. We may assume, after completion, that X and Y are open in P and Q respectively.

If v is formally unramified around Y , then the coherent sheaf $\Omega_{Q/P}^1$ has a support Z which is a closed formal subscheme of Q that does not meet Y . The inclusion $Z \hookrightarrow P^Z$ induces an inclusion $Z^{\text{ad}} \hookrightarrow]Z[_P$ and it follows that $\mu^{-1}(Z^{\text{ad}}) \subset]Z[_V$. In particular, the support of $\Omega_{W/V}^1$, which is a closed subset contained in $\mu^{-1}(Z^{\text{ad}})$, does not meet $]Y[_V$. It follows that there exists a neighborhood W' of Y in W such that $\Omega_{W'/V}^1 = 0$. Moreover, since v is locally of finite type around Y , we may assume thanks to corollary 8.3 that $W' \rightarrow V$ is locally of finite type. It follows that u is unramified in the neighborhood of Y .

We proceed in an analogous way for smoothness using the jacobian criterion. Since the question is local and v is formally locally of finite type, we may assume that there exists a closed embedding $Q \hookrightarrow \mathbb{A}_P^{\pm, N}$ defined by an ideal (f_1, \dots, f_r) and that the minor $\det[\partial f_i / \partial T_j]_{i,j=1}^r$ is invertible outside a closed subset Z not meeting Y . Pulling back, we obtain a locally closed embedding $W' \hookrightarrow \mathbb{D}_V^{\pm, N}$ and the above minor stays invertible outside Z^{ad} . As above, $\mu^{-1}(Z^{\text{ad}})$ does not meet $]Y[_{W'}$ and we know that the locus where the minor is invertible is open. \square

Theorem 3.16 has the following consequence:

Corollary 8.3. *If a right cartesian formal morphism*

$$(Y \hookrightarrow Q \leftarrow W) \rightarrow (X \hookrightarrow P \leftarrow V)$$

of analytic overconvergent spaces is formally locally of finite type (resp. formally locally of finite type and separated, resp. partially proper), then the associated morphism

$$(Y, W) \rightarrow (X, V)$$

is locally of finite type (resp. separated, resp. partially proper).

Proof. Simply replace V and W with $V' :=]\overline{X}[_V$ and $W' :=]\overline{Y}[_W$. \square

In order to go further, it will be necessary to mix open (such as formally smooth) and closed (such as partially proper) conditions.

Theorem 8.4. *If a right cartesian formal morphism of analytic overconvergent spaces*

$$\begin{array}{ccccc} X' & \hookrightarrow & P' & \xleftarrow{\lambda'} & V' \\ \downarrow f & & \downarrow v & & \downarrow u \\ X & \hookrightarrow & P & \xleftarrow{\lambda} & V \end{array}$$

is partially proper and formally étale, and induces an isomorphism $f: X' \simeq X$, then it induces an isomorphism

$$(X', V') \simeq (X, V).$$

Proof. After completing along \overline{X} (resp. \overline{X}'), we may assume that X (resp. X') is open in the topology of P (resp. P'). If we denote for the moment by \mathcal{U} (resp. \mathcal{U}') the open subset of P (resp. P') having the same underlying space as X (resp. X'), then the induced morphism $\mathcal{U}' \rightarrow \mathcal{U}$ is formally étale and induces an isomorphism on the maximal reduced subschemes. This is necessarily an isomorphism. We may therefore replace X (resp. X') by \mathcal{U} (resp. \mathcal{U}') and assume that it is a formal open subscheme of P (resp. P'). In this situation, we have $\mathrm{sp}^{-1}(X) = X^{\mathrm{ad}}$ (resp. $\mathrm{sp}^{-1}(X') = X'^{\mathrm{ad}}$). Let us consider now the open subspaces $W = \mathrm{sp}_V^{-1}(X)$ and $W' = \mathrm{sp}_{V'}^{-1}(X')$ of V and V' respectively. Since the right hand square of our diagram is cartesian in the neighborhood of the tubes, we see that u induces an isomorphism

$$W' = \lambda'^{-1}(X'^{\mathrm{ad}}) \simeq \lambda^{-1}(X^{\mathrm{ad}}) = W.$$

Since v is partially proper around X' , we know from corollary 8.3 that u is partially proper in the neighborhoods of the tubes. In particular, it is specializing and the isomorphism $W' \simeq W$ extends therefore uniquely to a homeomorphism

$$]X'[_{V'} = \overline{W}' \simeq \overline{W} =]X[_V$$

(recall that W and W' are locally constructible). Now, any $v' \in]X'[_{V'}$ generalizes to some $w' \in W'$ and since V' is analytic, the canonical map $\mathcal{O}_{v'} \rightarrow \mathcal{O}_{w'}$ (is local and) induces an isomorphism $\mathcal{H}(v') \simeq \mathcal{H}(w')$. For the same reason, we have $\mathcal{H}(u(v')) \simeq \mathcal{H}(u(w'))$. On the other hand, the isomorphism $W' \simeq W$ provides an isomorphism $\mathcal{H}(u(w')) \simeq \mathcal{H}(w')$ and it follows that $\mathcal{H}(u(v')) \simeq \mathcal{H}(v)$. We may therefore apply proposition 2.3.7 of [14] (analytic variant) which tells us that u induces an isomorphism between the étale topos of $(V',]X'[_V)$ and the étale topos of $(V,]X[_V)$. Moreover, thanks to proposition 8.2, we may assume that the map $u: V' \rightarrow V$ is étale. Then, necessarily, u induces an isomorphism between a neighborhood of $]X'[_V$ in V' and a neighborhood of $]X[_V$ in V . \square

Note that the étale site of a pseudo-adic space (V, T) is not subcanonical because any open immersion $V' \hookrightarrow V$ with $T \subset V'$ will induce an isomorphism between the étale topos of (V', T) and (V, T) . This is why we need to shrink V and V' once more at the end of the proof. Alternatively, one could use the modified étale site obtained by formally inverting such an open immersion (which is then subcanonical).

Theorem 8.5 (Strong fibration theorem). *If a right cartesian formal morphism of analytic overconvergent spaces*

$$(X' \hookrightarrow P' \leftarrow V') \rightarrow (X \hookrightarrow P \leftarrow V)$$

is partially proper and formally smooth, and induces an isomorphism $X' \simeq X$, then there exists, locally in the adic overconvergent site, an isomorphism

$$(X', V') \simeq (X, \mathbb{P}_V^n).$$

Here, we denote by (X, \mathbb{P}_V^n) the adic overconvergent space associated to $(X \hookrightarrow \mathbb{P}_P^n \leftarrow \mathbb{P}_V^n)$.

Proof. The strategy is due to Berthelot and we follow essentially the proof of theorem 4.1.3 in [18]. First of all, thanks to corollary 7.15, we may assume that both X and X' are reduced. Also, using proposition 7.4, we may assume that X' is the complement of a divisor in \overline{X}' . Now, since the question is local for the adic topology of the overconvergent site, we may assume that V is affinoid. Moreover, thanks to corollary 7.16, the question is local on X and therefore also on P that we may both assume to be affine. Also thanks to proposition 7.8, we may assume that P' is quasi-compact. We are then in the situation of applying our version of Chow's lemma (corollary 7.5) and we may therefore assume that the map $\overline{X}' \rightarrow \overline{X}$ induced on the compactifications is projective. Next, proposition 4.3 grants us the existence of a projective morphism of formal embeddings $(X' \hookrightarrow Q) \rightarrow (X \hookrightarrow P)$ which is étale around X' and such that the closure of X' in Q is identical to the closure of X' in P' . The formal morphism $(X' \hookrightarrow Q \leftarrow Q \times_P V) \rightarrow (X \hookrightarrow P \leftarrow V)$ is right cartesian, partially proper and étale around X' and induces therefore, thanks to theorem 8.4, an isomorphism in the adic overconvergent site. After pulling back along this formal morphism, we may therefore assume that v actually induces an isomorphism $\overline{f} : \overline{X}' \simeq \overline{X}$. We may now use lemma 4.2 and apply theorem 8.4 again. \square

In order to state the last result, we recall that if \mathcal{B} is a site, then a morphism $T' \rightarrow T$ of presheaves on \mathcal{B} is called a *local epimorphism* (resp. *local isomorphism*)¹ if the morphism of associated sheaves $\tilde{T}' \rightarrow \tilde{T}$ is an epimorphism (resp. isomorphism). And a morphism $Y \rightarrow X$ in \mathcal{B} is a *local epimorphism* (resp. *local isomorphism*) if the corresponding morphism of presheaves $\hat{Y} \rightarrow \hat{X}$ is.

Corollary 8.6. *If a formal morphism of analytic overconvergent spaces*

$$(Y \hookrightarrow Q \leftarrow W) \rightarrow (X \hookrightarrow P \leftarrow V)$$

is partially proper and formally smooth, and induces an isomorphism $X' \simeq X$, then it induces a local epimorphism

$$(X', V') \rightarrow (X, V)$$

in the adic overconvergent site.

Proof. It follows from theorem 8.5 that, locally, the morphism $(X', V') \rightarrow (X, V)$ has a section. \square

References

- [1] Ahmed Abbes. *Éléments de géométrie rigide. Volume I*, volume 286 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2010. Construction et étude géométrique des espaces rigides. [Construction and geometric study of rigid spaces], With a preface by Michel Raynaud.

¹Also called *covering* (resp. *bicovering*) morphism

- [2] Pierre Berthelot. Cohomologie rigide et cohomologie à support propre, seconde partie. Incomplete, 1983.
- [3] Pierre Berthelot. Cohomologie rigide et cohomologie à support propre, première partie. *Prépublication de l'IRMAR*, 96(03):89, 1996.
- [4] Pierre Berthelot. Introduction à la théorie arithmétique des \mathcal{D} -modules. *Astérisque*, (279):1–80, 2002. Cohomologies p -adiques et applications arithmétiques, II.
- [5] Daniel Caro and David Vauclair. Logarithmic p -bases and arithmetical differential modules. *Preprint*, 2015.
- [6] Brian Conrad. Perfectoid spaces. *Number Theory Learning Seminar*, 2015.
- [7] Richard Crew. Arithmetic d -modules on adic formal schemes. *Preprint*, 2017.
- [8] Kazyhiro Fujiwara and Fumiharu Kato. Foundations of rigid geometry. 2014.
- [9] Pierre Gabriel and Michel Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.
- [10] A. Grothendieck and J. A. Dieudonné. *Éléments de géométrie algébrique. I*, volume 166 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1971.
- [11] Roland Huber. *Bewertungsspektrum und rigide Geometrie*, volume 23 of *Regensburger Mathematische Schriften [Regensburg Mathematical Publications]*. Universität Regensburg, Fachbereich Mathematik, Regensburg, 1993.
- [12] Roland Huber. Continuous valuations. *Math. Z.*, 212(3):455–477, 1993.
- [13] Roland Huber. A generalization of formal schemes and rigid analytic varieties. *Math. Z.*, 217(4):513–551, 1994.
- [14] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.
- [15] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
- [16] Christopher Lazda and Ambrus Pál. *Rigid cohomology over Laurent series fields*, volume 21 of *Algebra and Applications*. Springer, [Cham], 2016.
- [17] Bernard Le Stum. *Rigid cohomology*, volume 172 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.
- [18] Bernard Le Stum. The overconvergent site. *Mémoires de la SMF*, 127, 2011.

- [19] David Pigeon. *Les D -modules arithmétiques dans le cas des p -bases et un algorithme pour le calcul de fonctions zêta*. PhD thesis, Univeristé de Caen, 2014.
- [20] Michel Raynaud and Laurent Gruson. Critères de platitude et de projectivité. Techniques de “platification” d’un module. *Invent. Math.*, 13:1–89, 1971.
- [21] Peter Scholze. Peter scholze’s lectures on p -adic geometry. 2014.
- [22] Peter Scholze and Jared Weinstein. Moduli of p -divisible groups. *Camb. J. Math.*, 1(2):145–237, 2013.
- [23] Thorsten Wedhorn. Adic spaces. 2012.
- [24] Thorsten Wedhorn. Short introduction to adic spaces. 2015.