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Shape optimization for the Helmholtz equation with complex Robin boundary conditions

Frédéric Magoulès*, Thi Phuong Kieu Nguyen*,
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July 6, 2017

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Abstract

In the aim to find the simplest and most efficient shape of a noise absorbing wall to dissipate the energy of a sound wave, we consider a frequency model (the Helmholtz equation) with a damping on the boundary. The damping on the boundary is firstly related with the damping in the volume, knowing the macroscopic parameters of a fixed porous medium. Once the well-posedness results are proved for the time-dependent and the frequency models in the class of bounded n -sets (for instance, locally uniform domains with a d -set boundary, containing self-similar fractals or Lipschitz domains as examples), the shape optimization problem of minimizing the acoustical energy for a fixed frequency is considered. To obtain an efficient wall shape for a large range of frequencies, we define the notion of ε -optimal shapes and prove their existence in a class of multiscale Lipschitz boundaries when we consider energy dissipation on a finite range of frequencies, and in a class of fractals for an infinite frequency range. The theory is illustrated by numerical results.

1 Introduction

The diffraction and absorption of waves by a system with both absorbing properties and irregular geometry is an open physical problem. From acoustics to optics, the wave absorption, related to the coupling between waves and matter, is a problem of interest. This has to be solved to understand why anechoic chambers (electromagnetic or acoustic) do work better with irregular absorbing walls. Therefore there is a question about the existence of an optimal shape of an absorbent wall (for a fixed absorbing material), optimal in the sense that it is as dissipative as possible for a large range of frequencies, and at the

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same time that such a wall could effectively be constructed. We consider the propagation of acoustic waves; the acoustic absorbent material of the wall is considered as a porous medium.

In the area of the optimization of acoustic performances of non absorbing walls, Duhamel [14, 15] studies sound propagation in a $2D$ vertical cut of a road wall and uses genetic algorithms to obtain optimal shapes (some of them are however not connected and thus could not be easily manufactured). The author also uses a branch and bound (combinatorial optimization) type linear programming in order to optimize the position sensors that allow an active noise control, firstly introduced by Lueg [19] in 1934. Abe et al. [1] consider a BE-based (boundary elements) shape optimization of a non absorbing two-dimensional wall in the framework of a two-dimensional sound scattering problem for a fixed frequency (for the Helmholtz equation) using a topological derivative with the principle that a new shape or topology is obtained by nucleating small scattering bodies. Also for the Helmholtz equation for a fixed frequency, using the shape derivative of a functional, representing the acoustical energy, Cao and Stanescu [10] consider a two-dimensional shape design problem for a non-absorbing part of the boundary to reduce the amount of noise radiated from aircraft turbofan engines. For the same problem, Farhadinia [17] developed a method based on measure theory, which does not require any information about gradients and the differentiability of the cost function.

On the other hand, for shape optimization problems there are a lot of theoretical results, reviewed in Refs. [3, 33], which rely on the topological derivatives of the cost functional to be minimized, with numerical application of the gradient method in both two and three dimensional cases (in the framework of solid mechanics). In this area, Achdou and Pironneau [2] considered the problem of optimization of a photocell using a complex-valued Helmholtz system with periodic boundary conditions with the aim to maximize the solar energy in a dissipative region.

For acoustic waves in the two-dimensional case, optimization of the shape of an absorbing inclusion placed in a lossless acoustic medium was considered in Refs. [34, 35]. The considered model is the linear damped wave equation [13, 4], which we also consider in our case to describe the wave propagation in a porous medium (see Section 2). Using the topology derivative approach, Münch et al. consider in [34, 35] the minimization of the acoustic energy of the solution of the damped wave equation at a given time $T > 0$ without any geometric restrictions and without the purpose of the design of an absorbent wall.

In this article, we study the two-dimensional optimization shape problem for a Helmholtz equation with a damping on the boundary, modeled by a complex-valued Robin boundary condition (see system (3) and Fig. 2). The shape of the damping boundary is to be found, in the aim to minimize the total acoustical energy of the system. The noise source can be imposed as a source term in the equation and/or by a Dirichlet boundary condition (on the boundary opposite to the absorbing wall) which models a noise coming from a road. As for acoustical cavities, the domain of computation is limited on its top and bottom by boundaries with Neumann boundary conditions.

In Section 2, we introduce the frequency model and its time-dependent analogue with a dissipation on the boundary and analyze its dissipative properties. We compare this model with dissipation by the boundary to the corresponding model with a dissipation

in the volume, described by a damped wave equation in which the values of the coefficients for a given porous medium are given as functions of its macroscopic parameters (as porosity, tortuosity and resistivity to the passage of air), as initially proposed by [21]. In particular, in Appendix A, we propose a possible way to find the complex parameter in the Robin boundary condition of the former model that best approximates the latter. All numerical calculations, in particular in Section 7, are performed for a porous material named ISOREL, frequently used in building isolation.

In Section 3 we establish the well-posedness results of the introduced models with a dissipation on the boundary in a large class of bounded domains ((ε, δ) -domains with a d -set boundary), containing Lipschitz domains and von Koch fractals as two particular cases. We need these well-posedness properties of the models to be able to consider in this class of domains the question of existence of an optimal shape, optimal in minimizing the acoustical energy (see Eqs. (17) and (19)).

In Section 4, for the case of a regular boundary in the classical framework of shape optimization, for any fixed frequency we have the existence of an optimal shape. To obtain an efficient wall shape for a large range of frequencies, we define ε -optimal shapes. Knowing empirically that for the efficient energy dissipation of an acoustic wave, its wavelength λ must be related with a geometric scale of the wall, we confirm this fact numerically by calculating the impact of the different geometric scales on the energy dissipation in time (see Fig. 5, which confirms a guess that the wall length scale must be of the order of $\lambda/2$). The developed optimization algorithm confirms that the optimal shape has its largest scale length of the order of $\lambda/2$ (see Subsection 7.1 and Fig. 7).

Moreover, using the fact that a wave with a wavelength λ_0 does not fit into a shape that has a characteristic scale much smaller than its wavelength (smaller than $\lambda_0/2$), we prove in Section 4 the existence of an ε -optimal shape which is a multiscale Lipschitz boundary when we consider energy dissipation on a finite range of frequencies, and which is a fractal for an infinite range. Thus, we prove that it is not possible to obtain the most efficient in the acoustical energy dissipation shape (an ε -optimal shape with a minimal $\varepsilon > 0$) for all frequencies without different geometric scales.

For the case of a regular boundary we provide in Section 5 the shape derivative of an objective functional chosen to describe the acoustical energy. Using the gradient descent method for the shape derivative, combined with the finite volume and level set methods introduced in Section 6, we find numerically the optimal shapes for a fixed frequency. In Subsection 7.2, we show the stability of the numerical algorithm and the non-uniqueness of the optimal shape, which can be explained by the non-uniqueness of the geometry providing the same spectral properties. In Subsection 7.3 we obtain an ε -optimal shape for a large range of frequencies. This shape is multiscale, and we show that if we keep only the largest scale, the new shape has the same good dissipation properties as the optimal one in the low frequencies corresponding to the chosen scale length, but is no more efficient in higher frequencies, for which the deleted geometry scales were important.

2 The model : Motivation and known properties

To describe the acoustic wave absorption by a porous medium, there are two possibilities. The first is to consider wave propagation in two media, typically the air and the wall, which corresponds to a damping in the volume. The most common mathematical model for it is the damped wave equation [4, 13]. The second possibility is to consider only one lossless medium, the air, and model energy dissipation by a damping condition on the boundary. In both cases, we need to ensure the same order of energy damping corresponding to the physical characteristics of the chosen porous medium as its porosity ϕ , tortuosity α_h and resistivity to the passage of air σ .

Thanks to Ref. [21], we can define the coefficients in the damped wave equation (the case of the absorption in volume) as functions of the above mentioned characteristics. More precisely, for a regular bounded domain $\Omega \subset \mathbb{R}^2$ (for instance $\partial\Omega \in C^1$) composed of two disjoint parts $\Omega = \Omega_0 \cup \Omega_1$ of two homogeneous media, air in Ω_0 and a porous material in Ω_1 , separated by an internal boundary Γ , we consider the following boundary value problem (for the pressure of the wave)

$$\begin{cases} \xi(x)\partial_t^2 u + a(x)\partial_t u - \nabla \cdot (\mu(x)\nabla u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n}|_{\mathbb{R}_t \times \partial\Omega} \equiv 0, \quad [u]_\Gamma = [\mu\nabla u \cdot n]_\Gamma = 0, \\ u|_{t=0} = u_0 \mathbf{1}_{\Omega_0}, \quad \partial_t u|_{t=0} = u_1 \mathbf{1}_{\Omega_0}, \end{cases} \quad (1)$$

with $\xi(x) = \frac{1}{c_0^2}$, $a(x) = 0$, $\mu(x) = 1$ in the air (*i.e.* in Ω_0) and $\xi(x) = \frac{\phi\gamma_p}{c_0^2}$, $a(x) = \sigma \frac{\phi^2\gamma_p}{c_0^2\rho_0\alpha_h}$, $\mu(x) = \frac{\phi}{\alpha_h}$ in the porous medium (*i.e.* in Ω_1). The external boundary $\partial\Omega$ is supposed to be perfectly rigid (Neumann boundary condition) and on the internal boundary Γ we have no-jump conditions on u and $\mu\nabla u \cdot n$, where n is the normal unit vector to Γ . Here, by c_0 and ρ_0 are denoted the sound velocity in and the density of the air respectively, and by $\gamma_p = 7/5$ the ratio of specific heats. The damped character of the wave propagation described by model (1) can be illustrated by the decreasing properties of the energy due to the damping term $a(x)u_t$ having its support in Ω_1 :

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} [\xi(\partial_t u)^2 + (\mu\nabla u) \cdot \nabla u] dx \right) = - \int_{\Omega_1} a(\partial_t u)^2 dx. \quad (2)$$

This model is numerically solved in Subsection 7.1.

But instead of the absorption in volume, especially for the sake of a simpler numerical treatment of the shape optimization, we consider the following frequency model of the damping by the boundary. Let Ω be a connected bounded domain of \mathbb{R}^2 with a Lipschitz boundary $\partial\Omega$. We suppose that the boundary $\partial\Omega$ is divided into three parts $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma$ (see Fig. 2 for an example of Ω , chosen for the numerical calculations) and consider

$$\begin{cases} \Delta u + \omega^2 u = -f(x) & x \in \Omega, \\ u = g(x) & \text{on } \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_N, \quad \frac{\partial u}{\partial n} + \alpha(x)u = 0 & \text{on } \Gamma, \end{cases} \quad (3)$$

where $\alpha(x)$ is a complex-valued regular function with a strictly positive real part ($\text{Re}(\alpha) > 0$) and a strictly negative imaginary part ($\text{Im}(\alpha) < 0$).

Remark 1 *This particular choice of the signs of the real and the imaginary parts of α are provided by the well-posedness properties and the energy decay of the corresponding time-dependent problem. In addition, as the frequency $\omega > 0$ is supposed to be fixed, α can contain a dependence on ω , i.e., $\alpha \equiv \alpha(x, \omega)$.*

Problem (3) is a frequency version of the following time-dependent wave propagation problem, considered in Ref. [7]:

$$\partial_t^2 u - \Delta u = e^{-i\omega t} f(x), \quad (4)$$

$$u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \quad (5)$$

$$u|_{\Gamma_D} = 0, \quad \left. \frac{\partial u}{\partial n} \right|_{\Gamma_N} = 0, \quad (6)$$

$$\frac{\partial u}{\partial n} - \text{Im}(\alpha(x))\partial_t u + \text{Re}(\alpha(x))u|_{\Gamma} = 0. \quad (7)$$

To show the energy decay, we follow Ref. [7] and introduce the Hilbert space $X_0(\Omega)$, defined as the Cartesian product of the set of functions $u \in H^1(\Omega)$, which vanish on Γ_D and satisfy the Robin type boundary condition (7) on Γ , with the space $L_2(\Omega)$. The equivalent norm on $X_0(\Omega)$ is defined by

$$\|(u, v)\|_{X_0(\Omega)}^2 = \int_{\Omega} (|\nabla_x u|^2 + |v|^2) dx + \int_{\Gamma} \text{Re}(\alpha(x))|u|^2 d\sigma$$

with the corresponding inner product

$$\langle (u_1, u_2), (v_1, v_2) \rangle = \int_{\Omega} (\nabla_x u_1 \nabla_x v_1 + u_2 v_2) dx + \int_{\Gamma} \text{Re}(\alpha(x))u_1 v_1 d\sigma. \quad (8)$$

The advantage of this norm is that the energy balance of the homogeneous problem (4)–(7) has the form

$$\partial_t (\|(u, \partial_t u)\|_{X_0(\Omega)}^2) = 2 \int_{\Gamma} \text{Im}(\alpha(x))|\partial_t u|^2 d\sigma.$$

Therefore, for $\text{Im}(\alpha) < 0$ on Γ , the energy decays in time.

For the case of a smooth boundary $\partial\Omega$ (at least Lipschitz), we have the well-posedness of two models:

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a closed domain with a smooth (at least Lipschitz) boundary $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma$. Let in addition $\text{Re}(\alpha(x)) > 0$, $\text{Im}(\alpha(x)) < 0$ be smooth functions (at least continuous) on Γ . Then the following results hold:*

1. (C. Bardos, J. Rauch [7]) *For all $f \in L^2(\Omega)$, $(u_0, u_1) \in X_0(\Omega)$ there exists a unique solution $(u, u_t) \in C([0, \infty[, X_0(\Omega))$ of system (4)–(7).*
2. (Gander et al. [18], Evans [16], Allaire [3]) *For all $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_D)$ and $\omega > 0$ there exists a unique $u \in H^1(\Omega)$ solution of problem (3) continuously depending on the data: there exists $C > 0$ not depending on f and g , such that*

$$\|u\|_{H^1(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_D)} \right).$$

In addition, if, for $m \in \mathbb{N}^$, $\partial\Omega \in C^{m+2}$, $f \in H^m(\Omega)$ and $g \in H^{m+\frac{3}{2}}(\Gamma_D)$, then the weak solution u belongs to $H^{m+2}(\Omega)$.*

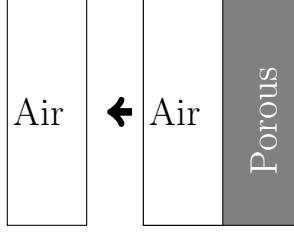


Figure 1: One medium for the absorption on the boundary (left) and two media for the absorption in the volume (right).

In order to relate the model with a damping on the boundary and the model with a damping in the volume (see Fig. 1), we propose in Appendix A a way to identify the parameter α in the Robin boundary condition that provides the best approximation of the latter model by the former, in the case of a flat boundary Γ .

3 Well-posedness of the boundary dissipation models for d -sets

We extend the well-posedness results of Theorem 1 to a more general class of boundaries, named Ahlfors d -regular sets or simply d -sets [26], using the functional analysis on (ε, δ) -domains [25, 26, 40], also called locally uniform domains [24]. Let us define the main notions [5].

Definition 1 (Ahlfors d -regular set or d -set [26, 40, 27]) Let F be a Borel subset of \mathbb{R}^n and m_d be the d -dimensional Hausdorff measure, $0 < d \leq n$, $d \in \mathbb{R}$. The set F is called a d -set, if there exist positive constants $c_1, c_2 > 0$,

$$c_1 r^d \leq m_d(F \cap B_r(x)) \leq c_2 r^d, \quad \text{for } \forall x \in F, 0 < r \leq 1,$$

where $B_r(x) \subset \mathbb{R}^n$ denotes the Euclidean ball centered at x and of radius r .

Definition 2 ((ε, δ) -domain [25, 26, 40]) An open connected subset Ω of \mathbb{R}^n is an (ε, δ) -domain, $\varepsilon > 0$, $0 < \delta \leq \infty$, if whenever $(x, y) \in \Omega^2$ and $|x - y| < \delta$, there is a rectifiable arc $\gamma \subset \Omega$ with length $\ell(\gamma)$ joining x to y and satisfying

1. $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$ and
2. $d(z, \partial\Omega) \geq \varepsilon|x-z|\frac{|y-z|}{|x-y|}$ for $z \in \gamma$.

It is known [40] that

- All (ε, δ) domains in \mathbb{R}^n are n -sets (d -set with $d = n$):

$$\exists c > 0 \quad \forall x \in \overline{\Omega}, \forall r \in]0, \delta[\cap]0, 1[\quad \mu(B_r(x) \cap \Omega) \geq C\mu(B_r(x)) = cr^n,$$

where $\mu(A)$ denotes the Lebesgue measure of a set A . This property is also called the measure density condition [20]. Let us notice that an n -set Ω cannot be “thin” close to its boundary $\partial\Omega$.

- If Ω is an (ε, δ) -domain and $\partial\Omega$ is a d -set ($d < n$) then $\overline{\Omega} = \Omega \cup \partial\Omega$ is a n -set.

In particular, a Lipschitz domain Ω is an (ε, δ) -domain and also an n -set [40]. But not every n -set is an (ε, δ) -domain: adding an in-going cusp to an (ε, δ) -domain we obtain an n -set which is not an (ε, δ) -domain anymore. Self-similar fractals (e.g., von Koch's snowflake domain) are examples of (ε, ∞) -domains with a d -set boundary [12, 40] for $d > n - 1$.

To extend the usual variational formulations introduced in Refs. [7, 18] to d -set type fractal boundaries, we use, as in Ref. [5], the existence of the d -dimensional Hausdorff measure m_d on $\partial\Omega$ (see Definition 1) and a generalization of the usual trace theorem and the Green formula in the sense of the Besov space $B_{\beta}^{2,2}(\partial\Omega)$ with $\beta = 1 - \frac{n-d}{2} > 0$ (for the definition of the Besov spaces on d -sets see Ref. [26] p.135 and Ref. [40] or Appendix in Ref. [6]). Note that for $d = n - 1$, one has $\beta = \frac{1}{2}$ and

$$B_{\frac{1}{2}}^{2,2}(\partial\Omega) = H^{\frac{1}{2}}(\partial\Omega).$$

Let us start with the generalization of the notion of the trace:

Definition 3 For an arbitrary open set Ω of \mathbb{R}^n , the trace operator Tr is defined [26, 8, 29] for $u \in L_{loc}^1(\Omega)$ by

$$\text{Tr} u(x) = \lim_{r \rightarrow 0} \frac{1}{m(\Omega \cap U_r(x))} \int_{\Omega \cap U_r(x)} u(y) dy,$$

where m denotes the Lebesgue measure. The trace operator Tr is considered for all $x \in \overline{\Omega}$ for which the limit exists.

Henceforth, the boundary $\partial\Omega$ is a d -set endowed with the d -dimensional Hausdorff measure, and $L_2(\partial\Omega)$ is defined with respect to this measure as well. Hence, the following Theorem (see Ref. [5] Section 2) generalizes the classical results [30, 31] for the Lipschitz boundaries $\partial\Omega$:

Theorem 2 Let Ω be an admissible in the sense of Ref. [5] domain in \mathbb{R}^n , i.e. Ω is an n -set, such that

- the norms $\|f\|_{H^1(\Omega)}$ and $\|f\|_{C_{\frac{1}{2}}^1(\Omega)} = \|f\|_{L_2(\Omega)} + \|f_{1,\Omega}^{\#}\|_{L_2(\Omega)}$ with

$$f_{1,\Omega}^{\#}(x) = \sup_{r>0} r^{-1} \inf_{c \in \mathbb{R}} \frac{1}{\mu(B_r(x))} \int_{B_r(x) \cap \Omega} |f(y) - c| dy$$

are equivalent on $H^1(\Omega)$;

- its boundary $\partial\Omega$ is a compact d -set, $n - 2 < d < n$.

Then,

1. $H^1(\Omega)$ is compactly embedded in $L_2^{loc}(\Omega)$ or in $L_2(\Omega)$ if Ω is bounded;
2. $\text{Tr}_{\Omega} : H^1(\mathbb{R}^n) \rightarrow H^1(\Omega)$ is a linear continuous and surjective operator with linear bounded inverse (the extension operator $E_{\Omega} : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$);

3. for $\beta = 1 - (n - d)/2 > 0$ the operators

$$\text{Tr} : H^1(\mathbb{R}^n) \rightarrow L_2(\partial\Omega), \quad \text{and} \quad \text{Tr}_{\partial\Omega} : H^1(\Omega) \rightarrow L_2(\partial\Omega)$$

are linear compact operators with dense image $\text{Im}(\text{Tr}) = \text{Im}(\text{Tr}_{\partial\Omega}) = B_\beta^{2,2}(\partial\Omega)$ and with linear bounded right inverse (the extension operators)

$$E : B_\beta^{2,2}(\partial\Omega) \rightarrow H^1(\mathbb{R}^n) \quad \text{and} \quad E_{\partial\Omega} : B_\beta^{2,2}(\partial\Omega) \rightarrow H^1(\Omega);$$

4. the Green formula holds (see also Refs. [29, 11] for the von Koch case in \mathbb{R}^2) for all u and v from $H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$:

$$\int_{\Omega} v \Delta u dx = \left\langle \frac{\partial u}{\partial \nu}, \text{Tr} v \right\rangle_{(B_\beta^{2,2}(\partial\Omega))', B_\beta^{2,2}(\partial\Omega)} - \int_{\Omega} \nabla v \nabla u dx, \quad (9)$$

where the dual Besov space $(B_\beta^{2,2}(\partial\Omega))' = B_{-\beta}^{2,2}(\partial\Omega)$ is introduced in Ref. [27].

Remark 2 Theorem 2 is a particular case of the results proven in Ref. [5], thanks to Refs. [20, 25, 26, 27, 40].

We also notice that in the framework the Sobolev space H^1 and the Besov spaces $B_\beta^{2,2}$ with $\beta < 1$, as here, we does not need to impose the Markov's local inequality on $\partial\Omega$ (see Ref. [26] p.39), as it is trivially satisfied (see Ref. [28] p. 198). If we are working with more regular spaces, H^k with $k \geq 2$, $k \in \mathbb{N}^*$, we need to add the assumption that $\partial\Omega$ preserves the Markov's local inequality: for every fixed $k \in \mathbb{N}^*$, there exists a constant $c = c(V, n, k) > 0$, such that

$$\max_{\partial\Omega \cap \overline{B_r(x)}} |\nabla P| \leq \frac{c}{r} \max_{\partial\Omega \cap \overline{B_r(x)}} |P|$$

for all polynomials $P \in \mathcal{P}_k$ and all closed balls $\overline{B_r(x)}$, $x \in \partial\Omega$ and $0 < r \leq 1$.

For $n = 2$, if Ω is a bounded connected domain, then, thanks to [25] and [20] (see also Proposition 1 in Ref. [5]), Ω is an (ε, δ) -domain.

In this framework we prove the following theorem

Theorem 3 Let Ω be a bounded domain in \mathbb{R}^n satisfying the conditions of Theorem 2 (for instance an (ε, δ) -domain) with a closed d -set boundary $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma$, $n - 2 < d < n$. By m_d is denoted the d -dimensional Hausdorff measure on $\partial\Omega$ (see Definition 1). Let in addition $\text{Re}(\alpha(x)) > 0$, $\text{Im}(\alpha(x)) < 0$ be continuous functions on Γ . Then the following results hold:

1. For all $f \in L^2(\Omega)$, $(u_0, u_1) \in X_0(\Omega)$, there exists an unique solution $U = (u, u_t) \in C([0, \infty[, X_0(\Omega))$ of system (4)–(7) satisfying the following energy identity

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} (|u_t|^2 + |\nabla_x u|^2) dx + \int_{\Gamma} \text{Re}(\alpha) |u|^2 dm_d \right) \\ - \int_{\Omega} \text{Im}(\alpha) |u_t|^2 dm_d = e^{-i\omega t} \int_{\Omega} f \overline{u_t} dx. \quad (10) \end{aligned}$$

2. For all $f \in L_2(\Omega)$ and $g \in B_\beta^{2,2}(\Gamma_D)$ (with $\beta = 1 - \frac{n-d}{2} > 0$) and $\omega > 0$ there exists a unique $u \in H^1(\Omega)$ solution of problem (3) in the following weak sense: for all $v \in H^1(\Omega)$

$$\int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - \omega^2 \int_{\Omega} u \bar{v} dx + \int_{\Gamma} \alpha u \bar{v} dm_d = \int_{\Omega} f \bar{v} dx + \int_{\Gamma_D} g \bar{v} dm_d. \quad (11)$$

The weak solution u continuously depends on the data: there exists $C > 0$, not depending on f , g and the values of α , such that

$$\|u\|_{H^1(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{B_\beta^{2,2}(\Gamma_D)} \right). \quad (12)$$

Proof. Let us focus on the proof of the second point in Theorem 3, *i.e.* on the well-posedness of the Helmholtz system (3). We introduce the space

$$V(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_D\} \quad (13)$$

with the norm

$$\|u\|_{V(\Omega)}^2 = \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} \operatorname{Re}(\alpha) |u|^2 dm_d.$$

Thanks to the continuity of the trace operator $\operatorname{Tr} : H^1(\Omega) \rightarrow L_2(\Gamma_D)$, the space $V(\Omega)$ is a Hilbert space with the inner product of $H^1(\Omega)$. As Γ is a part of a compact boundary $\partial\Omega$, the norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{V(\Omega)}$ are equivalent on $H^1(\Omega)$ (by Proposition 3 from Ref. [5]). Thus, the space $V(\Omega)$ is also a Hilbert space with the inner product

$$(u, v)_{V(\Omega)} = \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx + \int_{\Gamma} \operatorname{Re}(\alpha) u \bar{v} dm_d.$$

First, we consider the boundary value problem for the Laplacian ($\omega = 0$). We define a sesquilinear form on $V(\Omega)$ by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx + \int_{\Gamma} \alpha u \bar{v} dm_d$$

and an antilinear form

$$l(v) = \int_{\Omega} f \bar{v} dx.$$

We are looking for the weak solution $u \in V(\Omega)$ of the following variational problem

$$\forall v \in V(\Omega), \quad a(u, v) = l(v).$$

We apply the complex version of the Lax-Milgram theorem. Indeed, the coercivity and the continuity of the form $a(\cdot, \cdot)$ are immediate:

$$|a(u, u)| = \left| \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} \operatorname{Re}(\alpha) |u|^2 dm_d + i \int_{\Gamma} \operatorname{Im}(\alpha) |u|^2 dm_d \right| \geq \|u\|_{V(\Omega)}^2,$$

and

$$\begin{aligned}
|a(u, v)| &\leq \int_{\Gamma} |\alpha| |u\bar{v}| dm_d + \int_{\Omega} |\nabla u \nabla \bar{v}| dx \\
&\leq \int_{\Gamma} \operatorname{Re}(\alpha) \sqrt{1 + \left(\frac{\operatorname{Im}(\alpha)}{\operatorname{Re}(\alpha)}\right)^2} |u\bar{v}| dm_d + \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
&\leq C \left(\int_{\Gamma} \operatorname{Re}(\alpha) |u|^2 dm_d \right)^{\frac{1}{2}} \left(\int_{\Gamma} \operatorname{Re}(\alpha) |v|^2 dm_d \right)^{\frac{1}{2}} + \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
&\leq C \|u\|_{V(\Omega)} \|v\|_{V(\Omega)}.
\end{aligned}$$

We notice that, thanks to the boudness of Ω , the Poincaré inequality holds on $V(\Omega)$ and allows us to show the continuity of l :

$$|l(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq C(\Omega) \|f\|_{L^2(\Omega)} \|v\|_{V(\Omega)}.$$

Using now Theorem 2, which ensures that the extension operator $H^1(\Omega)$ into $H^1(\mathbb{R}^n)$ is continuous and that the embedding $H^1(\Omega)$ to $L^2(\Omega)$ is still compact, we conclude that the operator $-\Delta$ with the boundary conditions imposed in the weak sense of Besov spaces (see Eq. (9))

$$\forall \phi \in V(\Omega) \quad (u, \phi)_{V(\Omega)} + i \int_{\Gamma} \operatorname{Im}(\alpha) u \bar{\phi} dm_d = \lambda \int_{\Omega} u \bar{\phi} dx, \quad (14)$$

has a discrete spectrum and a compact resolvent as in the usual case of a regular boundary. Thus, as a corollary of the Fredholm theorem, we also have estimation (12).

Now let us prove that a real number ω^2 is not an eigenvalue of $-\Delta$. Suppose the converse: there exists an eigenfunction $u \in V(\Omega)$, such that (14) holds for $\lambda = \omega^2 \in \mathbb{R}^+$. Therefore, it also holds for $\phi = u$ and implies that $\operatorname{Im}(\alpha) \equiv 0$ on Γ , which contradicts our assumption $\operatorname{Im}(\alpha) < 0$. Consequently, the Helmholtz system (3) is well-posed for all $\omega \in \mathbb{R}$ in the weak sense of (11).

The well-posedness of system (4)–(7) is direct from results given in Ref. [7], which keep true in the general d -set case, thanks to the trace extension theorems (see Theorem 2 and Ref. [5] Section 2) and the above results for the Laplacian. Indeed, if we rewrite the system in the matrix form

$$\frac{d}{dt} U - BU = F,$$

with $U = (u, v) \in X_0(\Omega)$, $B = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 \\ e^{-i\omega t} f(x) \end{pmatrix}$, as for the regular case, the operator B is a closed anti-self-adjoint operator on $X_0(\Omega)$ with respect to the inner product defined in Eq. (8). In addition, B generates a strongly continuous contraction semigroup, has compact resolvent and $i\omega$ for $\omega \in \mathbb{R}$ does not in the spectrum of B , which concludes the proof. \square

We finish this section by the direct corollary of Theorems 1 and 3

Theorem 4 *Let Ω be a bounded domain in \mathbb{R}^n satisfying the conditions of Theorem 2 with a closed d -set boundary $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma$ ($n - 2 < d < n$). By m_d is denoted the d -dimensional Hausdorff measure on $\partial\Omega$ (see Definition 1). Let in addition $\operatorname{Re}(\alpha(x)) > 0$, $\operatorname{Im}(\alpha(x)) < 0$ be continuous functions on Γ . Then the following problem*

$$\begin{cases} \Delta u + \omega^2 u = -f(x) & x \in \Omega, \\ u = 0 & \text{on } \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_N, \quad \frac{\partial u}{\partial n} + \alpha(x) \operatorname{Tr} u = \operatorname{Tr} h(x) & \text{on } \Gamma, \end{cases} \quad (15)$$

has a unique weak solution $u \in V(\Omega)$ for all $f \in L^2(\Omega)$ and $h \in V(\Omega)$ in the following sense: for all $v \in V(\Omega)$

$$\int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - \omega^2 \int_{\Omega} u \bar{v} dx + \int_{\Gamma} \alpha \operatorname{Tr} u \operatorname{Tr} \bar{v} dm_d = \int_{\Omega} f \bar{v} dx + \int_{\Gamma} \operatorname{Tr} h \operatorname{Tr} \bar{v} dm_d.$$

The weak solution u continuously depends on the data: there exists $C > 0$, independent of f , h and the values of α , such that

$$\|u\|_{V(\Omega)} \leq C (\|f\|_{L^2(\Omega)} + \|h\|_{V(\Omega)}).$$

In addition, if, for $m \in \mathbb{N}^*$, $\partial\Omega \in C^{m+2}$, $f \in H^m(\Omega)$ and $h \in H^{m+1}(\Omega) \cap V(\Omega)$, then the weak solution u belongs to $H^{m+2}(\Omega) \cap V(\Omega)$.

4 Shape design problem

We consider the two dimensional shape design problem, which consists in optimizing the shape of Γ with the Robin dissipative condition in order to minimize the acoustic energy of the system (3). The boundaries with the Neumann and Dirichlet conditions Γ_D and Γ_N are supposed to be fixed. We denote by Ω_0 and Γ_0 the domain and the boundary respectively of the initial shape before optimization. The optimization step modifies the initial shape of Γ_0 to $\Gamma = (Id + \theta)\Gamma_0$, according to the map $x \in \Gamma_0 \mapsto (x + \theta(x)) \in \Gamma$ and following the vector field $\theta \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$. Here Id is the identity map $x \in \mathbb{R}^2 \mapsto x \in \mathbb{R}^2$, $W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ is the space of Lipschitz functions ϕ from \mathbb{R}^2 to \mathbb{R}^2 , such that ϕ and $\nabla\phi$ are uniformly bounded in \mathbb{R}^2 . Using the notations $|\cdot|_{\mathbb{R}^2}$ for the Euclidean norm in \mathbb{R}^2 and $|\cdot|_{\mathbb{R}^2 \times \mathbb{R}^2}$ for the matrices Euclidean norm on \mathbb{R}^2 , we define the norm on $W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ by

$$\|\phi\|_{W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)} = \sup_{x \in \mathbb{R}^2} (|\phi(x)|_{\mathbb{R}^2} + |\nabla\phi(x)|_{\mathbb{R}^2 \times \mathbb{R}^2}).$$

Hence $(W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2), \|\cdot\|_{W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)})$ is a Banach space. Following Ref. [3], p. 127, we also define for a fixed open D with a Lipschitz boundary the space

$$\mathcal{C}(\Omega_0) = \{\Omega \subset D \subset \mathbb{R}^2 \mid \exists \theta \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2), \|\theta\|_{W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)} < 1 \text{ such that } \Omega = (Id + \theta)\Omega_0\}. \quad (16)$$

Actually, as only a part of the boundary, Γ , changes its shape, we can impose that for a fixed open G with a Lipschitz boundary $\Gamma \subset \overline{G}$ (see the example of Fig. 2).

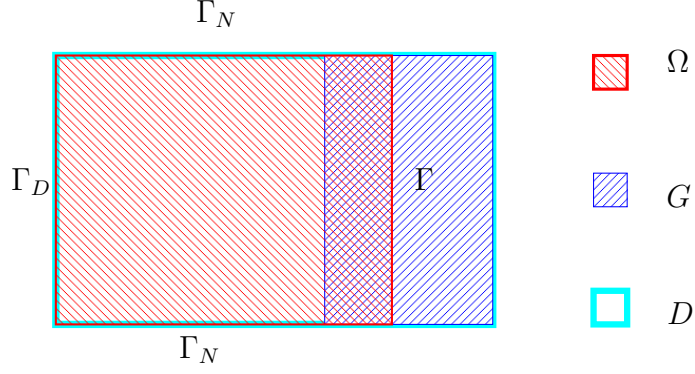


Figure 2: Example of a domain Ω in \mathbb{R}^2 with three types of boundaries: Γ_D and Γ_N are fixed and Γ can be changed in the restricted area \overline{G} . Here $\Omega \cup G = D$ and obviously $\Omega \subset D$.

To introduce the class of the admissible domains, on which we minimize the acoustical energy of the system (3), we define [3, 36] the quasi-distance $d(\Omega, \Omega_0)$ on $\mathcal{C}(\Omega_0)$

$$d(\Omega, \Omega_0) = \inf_{T \in \mathcal{T} | T(\Omega_0) = \Omega} (\|T - Id\|_{W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)} + \|T^{-1} - Id\|_{W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)})$$

with the following space of diffeomorphisms on \mathbb{R}^2 :

$$\mathcal{T} = \{T | (T - Id) \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2), T^{-1} - Id \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)\}.$$

Typically $T = Id + \theta$ with $\|\theta\|_{W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)} < 1$. If $d_H(\Omega_0, \Omega)$ is the Hausdorff distance between Ω_0 and Ω , we know [36] that $d_H(\Omega_0, \Omega) \leq d(\Omega_0, \Omega)$. Hence, in the following, our purpose is to minimize the acoustic energy in Ω on all admissible shapes of Γ keeping constant the volume of the initial domain Ω_0 , *i.e.*, to minimize

$$J(\Omega, u) = A \int_{\Omega} |u|^2 dx + B \int_{\Omega} |\nabla u|^2 dx + C \int_{\Gamma} |u|^2 d\sigma \quad (17)$$

for the domains $\Omega \in U_{ad}(\Omega_0)$ from the admissible class of domains

$$U_{ad}(\Omega_0) = \{\Omega \in \mathcal{C}(\Omega_0) | d(\Omega, \Omega_0) \leq \frac{1}{8}, \Gamma_D \cup \Gamma_N \subset \partial\Omega, \int_{\Omega} dx = \text{Vol}(\Omega_0)\} \quad (18)$$

with $\text{Vol}(\Omega_0) = |\Omega_0| = \int_{\Omega_0} dx$, $A \geq 0$, $B \geq 0$, $C \geq 0$ positive constants for all fixed $\omega > 0$. In what follows we also suppose that A , B and C are regular functions of ω :

Example 1 *If J is the acoustic energy of the Helmholtz problem (3), we typically have $A = 1$, $B = C = 0$ or equivalently, thanks to the variational form, $A = 0$, $B = \frac{1}{\omega^2}$, $C = \frac{\text{Re}(\alpha(\omega))}{\omega^2}$.*

In the definition of U_{ad} we take $d(\Omega, \Omega_0) \leq \frac{1}{8}$ according to Lemma 2.4 in Ref. [36] in the case $n = 2$ and $k = 1$. The restriction that all admissible domains $\Omega \in U_{ad}$ have the

fixed parts Γ_D and Γ_N in their boundaries is taken into account in the parametrization of $\Omega = (Id + \theta)\Omega_0$ by the vector field θ . It is sufficient to impose $\theta = 0$ on $\Gamma_D \cup \Gamma_N$, meaning that only the part Γ of the boundary may vary. In order to keep the volume constraint, instead of Eq. (17) we can also consider the objective function

$$J_1(\Omega, u) = A \int_{\Omega} |u|^2 dx + B \int_{\Omega} |\nabla u|^2 dx + C \int_{\Gamma} |u|^2 d\sigma + \mu(\text{Vol}(\Omega) - \text{Vol}(\Omega_0))^2, \quad (19)$$

where μ is some positive constant of the penalization.

Following the approach of F. Murat and J. Simon [36], also explained in Ref. [22], we have, using the continuity of u and J as functions of Ω [22], the existence of an optimal shape:

Theorem 5 *Let $\Omega_0 \subset D$ be a domain with a Lipschitz boundary $\partial\Omega_0$ such that $\Gamma_D \cup \Gamma_N \subset \partial\Omega_0$, U_{ad} be defined by (18) and $\omega > 0$ be fixed. For the objective function $J(\Omega)$, defined in (17), the shape optimization problem $\inf_{\Omega \in U_{ad}(\Omega_0)} J(\Omega)$ has at least one minimum point (there exists at least one optimal shape of Γ).*

See also Ref. [9] for a free discontinuity approach to a class of shape optimization problems involving Robin condition on the free boundary.

Let us notice that a bounded domain with a Lipschitz boundary is a particular case of the (ε, δ) -domains and hence the optimal shape domain is also an (ε, δ) -domain. For practical reasons, it is more realistic to find “the simplest” optimal shape, thus the general fractal or (ε, δ) -domains case is not really of interest. In addition, in most practical situations we need to find “the simplest” optimal shape not only for a fixed frequency, but for a large frequency interval $[\omega_0, \omega_1]$, such that walls with such a shape could actually be manufactured. Moreover, the functional J can be considered with frequency depending coefficients (regular functions of ω at least from C^1): $J(\Omega)(\omega)$. Hence, in the aim to find the simplest shape, efficient for the energy dissipation in a range of frequencies, we introduce the definition of an ε -optimal shape:

Definition 4 *The domain $\Omega^* \in U_{ad}$ is called an ε -optimal domain for the range of frequencies $[\omega_0, \omega_1]$, if for all $\omega \in [\omega_0, \omega_1]$ it holds*

$$| \min_{\Omega \in U_{ad}} J(\Omega)(\omega) - J(\Omega^*)(\omega) | < \varepsilon,$$

where by $J(\Omega^*)(\omega)$ is denoted the value of the functional J , calculated for the domain Ω^* at the frequency ω .

Remark 3 *To validate the notion of ε -optimal domain, let us verify that for a fixed $\varepsilon > 0$, if Ω^* is optimal for ω^* , there exists an interval $[\omega_0, \omega_1]$, such that $\omega^* \in [\omega_0, \omega_1]$, for which Ω^* is ε -optimal. Actually, we notice that*

- u depends continuously on ω and Ω ;
- for a fixed frequency, J is continuous as a function of Ω ;

- as the functional J is continuous on $\omega \in \mathbb{R}^+$, it is equicontinuous on the compact $[\omega_0, \omega_1] \subset \mathbb{R}^+$:

$$\forall \eta > 0 \quad \exists \delta_1(\eta) > 0 : |\omega^* - \omega| < \delta_1 \Rightarrow |J(\Omega^*)(\omega^*) - J(\Omega^*)(\omega)| < \eta;$$

- $J_{\min}(\omega) := \min_{\Omega \in U_{ad}} J(\Omega)(\omega)$ is a uniquely defined continuous function of ω :

$$\forall \hat{\eta} > 0 \quad \exists \delta_2(\hat{\eta}) > 0 : |\omega^* - \omega| < \delta_2 \Rightarrow \left| \min_{\Omega \subset U_{ad}} J(\Omega)(\omega) - J(\Omega^*)(\omega^*) \right| < \hat{\eta}.$$

Therefore, for a fixed $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that, if $|\omega^* - \omega| < \delta$, we have

$$\begin{aligned} & \left| \min_{\Omega \subset U_{ad}} J(\Omega)(\omega) - J(\Omega^*)(\omega) \right| \\ & \leq \left| \min_{\Omega \subset U_{ad}} J(\Omega)(\omega) - J(\Omega^*)(\omega^*) \right| + \left| J(\Omega^*)(\omega^*) - J(\Omega^*)(\omega) \right| \leq \eta + \hat{\eta} = \varepsilon. \end{aligned}$$

Remark 4 From Ref. [36] (Theorem 5.1 p.205 and Theorem 2.4 ii) p.59), we also know that, since $\Omega^* \in U_{ad}$ is optimal on ω^* , then there exists a sequence $(\Omega_n) \subset U_{ad}$, such that

$$\hat{d}_H(\Omega_n, \Omega^*) := d_H(\Omega_n, \Omega^*) + d_H(D \setminus \Omega_n, D \setminus \Omega^*) \rightarrow 0.$$

Consequently, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ Ω_n is ε -optimal around ω^* .

Conversely, if $\hat{\Omega}_n \in U_{ad}$ is ε_n -optimal on ω^* , i.e. $|J(\hat{\Omega}_n)(\omega^*) - J_{\min}(\omega^*)| < \varepsilon_n$, then for $\varepsilon_n \rightarrow 0$ there exists $\Omega^* \in U_{ad}$ such that $\hat{d}_H(\hat{\Omega}_n, \Omega^*) = \hat{\varepsilon}(\varepsilon_n) \rightarrow 0$ as $n \rightarrow +\infty$.

Let us also formulate the physical principle:

Assumption 1 (Physical principle) A wave with a wavelength λ_0 does not fit into a shape of characteristic scale much smaller than λ_0 . If the wave interaction with the dissipative media increases, then the energy of the wave decreases.

Taking into account the physical principle and numerical results from Section 7.1, we introduce the following definition of a “much smaller” wavelength and of “a much higher” frequency:

Definition 5 Let $\omega_1 > 0$ be a fixed frequency. A frequency ω_2 is called a much higher frequency for ω_1 , $\omega_2 \gg \omega_1$, if $\omega_2 \geq 2\omega_1$. Consequently, for the wavelengths: $\lambda_2 \ll \lambda_1$ (λ_2 is much smaller than λ_1), if $\lambda_2 \leq \frac{\lambda_1}{2}$. Conversely, the wavelength λ is called comparable to λ_0 , if $\lambda \in]\frac{\lambda_0}{2}, 2\lambda_0[$.

Since wall performances depend on the sizes of its components to compare to the wavelengths of the source, let us define the notion of the characteristic geometry size of a domain Ω (or Γ , since only the boundary Γ with the Robin boundary condition can change its shape). If Γ starts at a point $x_B \in \partial\Omega$ and ends at a point $x_E \in \partial\Omega$, we consider Γ_0 , which joins the points x_B and x_E by a straight line. For $x = (x^1, x^2) \in \mathbb{R}^2$, we suppose that the first coordinate axis x^1 follows Γ_0 and the second coordinate axis x^2 follows its normal direction. Since, the boundary Γ belongs to a fixed area \overline{G} , chosen in the beginning, the largest geometrical size of Γ parallel to x^2 depends on the chosen G . Thus, we does not consider the geometries with parts having a length along the axis x^1 much smaller the length along x^2 , and we especially interested in the shape sizes projected on x^1 .

Definition 6 (Shape lengths of Ω) Let the boundary Γ be a C^1 boundary of Ω starting from the point $x_B = (x_B^1, x_B^2)$ and ending in $x_E = (x_E^1, x_E^2)$ and Γ_0 be the straight line joining these two points (see Fig. 3 for an example). Let f_0 be the linear function defining Γ_0 by $x^2 = f_0(x^1)$. We suppose that Γ can be locally defined by the graph of a C^1 function (each time denoted by f , $f(x) = 0$ for $x \in \Gamma$). By x_i ($i = 1, \dots, N$) are denoted the intersection points of Γ with Γ_0 , ordered by the first coordinate from left to right

$$x_0^1 = x_B^1 < x_1^1 < x_2^1 < \dots < x_N^1 < x_E^1 = x_{N+1}^1,$$

for which in any neighborhood V of x_i there exists $x \in \Gamma$ such that

$$\frac{\partial}{\partial n} f(x) = \nabla_{x^1, x^2} f \cdot n \neq 0.$$

Here n is the unit normal vector to Γ_0 . These points define the deviation parts of Γ to compare to the straight line Γ_0 (see the filled regions on Figs. 3 and 4). Thus, we define the “horizontal” lengths

$$h_i = |x_i - x_{i-1}|_{\mathbb{R}^2} \quad \text{for } i = 1, \dots, N + 1.$$

Let, for $x \in \Gamma_0$, $m(x)$ be the number of intersections with Γ of the straight line, passing by the point x and following the unit normal vector to Γ_0 , denoted by $L(x)$.

If a part of Γ , limited by two lines $L(x_{i-1})$ and $L(x_i)$, can be described by a bijection f , i.e.

$$\text{for all } x \in \Gamma_0 \text{ with } x^1 \in]x_{i-1}^1, x_i^1[, \quad m(x) = 1,$$

then we find the function (f or f_0), which is firstly intersected by the line $L(x)$ on $[x_{i-1}, x_i]$, and define the low part f_l as its graph. The other function in the couple (f and f_0) defines the upper part f_u and corresponds to the second intersection of $L(x)$ (see Fig. 4). Thus, we find the “vertical” lengths for each of such geometrical parts, by setting

$$v_i = \|f_u - f_l\|_{C([x_{i-1}^1, x_i^1])},$$

and form the couples (h_i, v_i) .

For the intervals with $m(x) > 1$, we find firstly the maximum interval $[x_{i,b}^*, x_{i,e}^*] \subset \Gamma_0$, such that for all $x \in [x_{i,b}^*, x_{i,e}^*]$ $m(x) > 1$ (i.e. if $x = x_{i,b}^* - \varepsilon$ or $x = x_{i,e}^* + \varepsilon$ for all sufficiently small $\varepsilon > 0$, then $m(x) = 1$) and define

$$h_i^* = |x_{i,e}^* - x_{i,b}^*|_{\mathbb{R}^2}, \quad v_i^* = d_H(f_h, f_b)|_{[x_{i,b}^*, x_{i,e}^*]},$$

i.e. the vertical length v_i^* is given by the Hausdorff distance between f_h and f_b restricted on the area between $L(x_{i,b}^*)$ and $L(x_{i,e}^*)$. We fix, as previous, the couples (h_i^*, v_i^*) .

Going in smaller geometrical details, on each bijection interval $[x_i, x_{i-1}]$ with $m(x) = 1$ we consider all points \tilde{x}_j for $j = 1, \dots, K$ for which

$$\partial_\tau f(\tilde{x}_j) = \nabla_{x^1, x^2} f \cdot \tau = 0 \quad \text{and} \quad P(\tilde{x}_j) \neq P(\tilde{x}_{j+1}),$$

where the tangential derivative is taken for the unit vector τ parallel to Γ_0 and $P(y)$ is the orthogonal projection on Γ_0 . Let us set $\tilde{x}_0 = x_{i-1}$ and $\tilde{x}_{K+1} = x_i$. If $K \geq 3$ is an

odd number (i.e. the derivative changes its sign more than twice between x_{i-1} and x_i), then we set (as K is odd, then $K + 1$ is even)

$$\tilde{h}_j = |P(\tilde{x}_{2j}) - P(\tilde{x}_{2(j-1)})|_{\mathbb{R}^2}, \quad \tilde{v}_j = \|f_u - f_l\|_{C([P(\tilde{x}_{2(j-1)}), P(\tilde{x}_{2j})])} \text{ for } j = 1, \dots, \frac{K+1}{2}.$$

To avoid difficult notations, the obtained sequence

$$(h_i, v_i)_{i=1 \dots N} \cup (h_i^*, v_i^*)_{i=1 \dots N^*} \cup (\tilde{h}_j, \tilde{v}_j)_{j=1, \dots, \frac{K+1}{2}}$$

is still denoted by $(h_i, v_i)_{i=1 \dots N}$.

If there exists $x \in \Gamma_0$, such that $x^1 \in]x_{i^*}^1, x_{i^*+1}^1[$ for a fixed i^* and $m(x) > 1$, then the corresponding part of Γ , given by f , is no more bijective between $L(x_{i^*})$ and $L(x_{i^*+1})$. Let us consider the union of the joint intervals $]x_{i^*}^1, x_{i^*+k}^1[$ ($k \geq 1$) on which f is not bijective. Then, going from the left to the right on Γ inside of the area delimited by $L(x_{i^*})$ and $L(x_{i^*+k})$, we find points $y_j \in \Gamma$ ($j \in \mathbb{N}$), such that

$$\partial_n f(y_j) = \nabla_{x^1, x^2} f \cdot n = 0 \quad \text{and} \quad P(y_j) \neq P(y_{j+1}),$$

where the normal derivative is taken for the normal unit vector n of Γ_0 . Hence, we define the horizontal lengths

$$h_{N+j} = |P(y_{j+1}) - P(y_j)|_{\mathbb{R}^2}.$$

To define the vertical lengths we say that a curve in the found bijection area is the lower geometrical part f_l , if the number of intersections of $L(x)$ with it is odd ($m(x)$ is odd) and is the upper part f_u , if the number of intersections with it is even (see Fig. 4). Therefore, we define h_{N+j} as $\|f_u - f_l\|_{C(I)}$ in the bijective compact segment I , constructed using the points x_i and the projections on Γ_0 of points y_j .

For each bijective interval inside of $]x_{i^*}^1, x_{i^*+1}^1[$, actually between all couples of points (y_{j-1}, y_j) , we also find, as for the bijective case, all points \tilde{y}_k for $k = 1, \dots, K$, for which $\partial_\tau f(\tilde{x}_k) = \nabla_{x^1, x^2} f \cdot \tau = 0$, and, in the case of $K \geq 3$, we add $\tilde{h}_k = |P(\tilde{y}_{2k}) - P(y_{2(k-1)})|_{\mathbb{R}^2}$ and $\tilde{v}_k = \|f_u - f_l\|_{C([P(\tilde{x}_{2(k-1)}), P(\tilde{x}_{2k})])}$ for $k = 1, \dots, \frac{K+1}{2}$ to the sequence $(h_i, v_i)_{i=1 \dots N}$.

Repeating the above procedure for all intervals (or more generally, for all unions of joint intervals) on Γ_0 , where at least in one point $m(x) > 1$, we construct the sequence $(h_i, v_i)_{i \in \mathbb{N}}$, finite or not. Now, for each i we compare h_i and v_i :

1. if $v_i \ll h_i$, then $d_i := h_i$ (the fragment is a quasi-plane),
2. if v_i and h_i are comparable, then $d_i := \frac{h_i + v_i}{2}$.

The case $h_i \ll v_i$ is forbidden by the assumption and by the choice of the open set G . The lengths $(d_i)_{i \in \mathbb{N}}$ are characteristic lengths of each element of Γ to compare to Γ_0 and are called shape's length of Ω .

Definition 7 (Characteristic geometric size of Ω) Let $(d_i)_{i \in \mathbb{N}}$ be the sequence of shape lengths of Ω . Independently of its finiteness, there exists the length $d = \max_i d_i$, which is called the largest geometric size of Ω . If the length of Γ is finite, then the

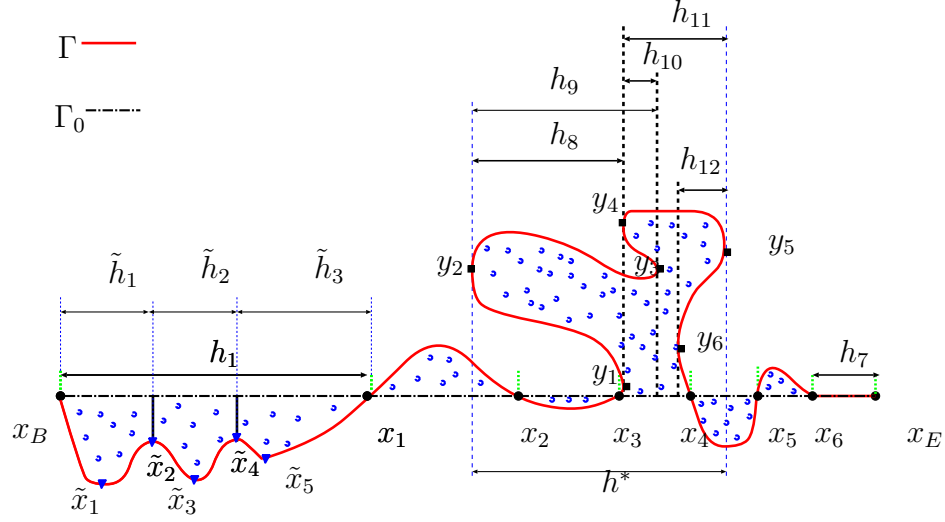


Figure 3: Example of the definition of horizontal shape lengths of Γ by the construction introduced in Definition 6. Here we have 6 intersections of Γ with Γ_0 and 6 additional points y_i in the region $]x_1, x_5[$, where there are points $x \in \Gamma_0$ for which $m(x) > 1$. The segment $[x_B^1, x_1^1]$ contains the projections of 5 points \tilde{x}_i , in which the tangential derivative of f parallel to Γ_0 is equal to zero. The maximum interval $[P(y_2), P(y_5)]$, where $m(x) > 1$, gives the length h^* . Therefore, the sequence of the horizontal shape lengths is given by h_i for $i = 1, \dots, 11$, h^* and \tilde{h}_j for $j = 1, \dots, 3$. The intervals, on which Γ can be described by a bijection for $x^1 \in [P(y_2), P(y_5)]$, are $[x_2, x_3]$, $[x_3, P(y_1)]$, $[P(y_2), P(y_1)]$, $[P(y_1), P(y_3)]$, $[P(y_4), P(y_3)]$, $[P(y_3), P(y_6)]$, $[P(y_6), P(y_5)]$, $[P(y_6), x_4]$.

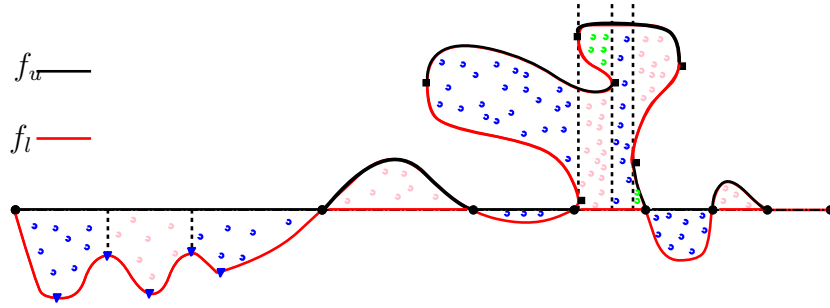


Figure 4: The choice of f_u and f_l for each interval, where Γ given on Fig. 3 is defined by a bijection.

number of its shape lengths N is finite too, and there exists the minimal geometric size $d_{\min} = \min_{i=1, \dots, N} d_i$. Hence, let $(d_i)_{i \in \mathbb{N}}$ be ordered decreasing by

$$d = d_0 \geq d_1 \geq d_2 \geq \dots, \quad d_i \rightarrow 0 \text{ if } i \rightarrow +\infty \text{ or } d_N = d_{\min} \text{ if } N < \infty.$$

The sequence (d_i) defines the distribution of geometrical sizes of Γ . Let $\text{Vol}(\Gamma_0)$ be the length of Γ_0 : $\text{Vol}(\Gamma_0) = |x_E - x_B|_{\mathbb{R}^2}$. A positive number $\ell_k(\Omega)$ ($k \in \mathbb{N}$) is a characteristic

geometric size of Ω number k if there exists $K(k) \geq 2^k \left\lceil \frac{\text{Vol}(\Gamma_0)}{2^d} \right\rceil$ shape lengths of Ω

$$d_{i_m} \in \left] \frac{d}{2^{k+1}}, \frac{d}{2^{k-1}} \right[\cap [d_{\min}, d] \quad (m = 0, \dots, K(k)) \text{ such that } \ell_k(\Omega) = \frac{1}{K(k)} \sum_{m=1}^{K(k)} d_{i_m}.$$

If $k = 0$, $\ell_0(\Omega)$, denoted in what follows by $\ell(\Omega)$, is the largest characteristic geometry size of Ω . If $k \in \mathbb{N}$ is such that $\frac{d}{2^{k+1}} \leq d_{\min} < \frac{d}{2^{k-1}}$, then the corresponding $\ell_k(\Omega)$, denoted in what follows by $\ell_{\min}(\Omega)$, is the smallest characteristic geometric size of Ω .

Remark 5 Let $\omega_0 = \frac{2\pi}{\lambda_0}$ and Ω_{λ_0} be a domain with the unique characteristic geometric size $\ell(\Omega_{\lambda_0}) = \ell_{\min}(\Omega_{\lambda_0}) = \frac{\lambda_0}{2}$. In addition, let Ω be a domain with $\ell(\Omega) = \ell(\Omega_{\lambda_0})$, $\ell_{\min}(\Omega) \ll \ell_{\min}(\Omega_{\lambda_0})$ and $d_H(\Omega, \Omega_{\lambda_0}) \leq \ell_{\min}(\Omega)$. Then the domain Ω has two different scales: $\ell(\Omega)$ and $\ell_{\min}(\Omega)$. Actually, we suppose that the boundary Γ of the domain Ω is obtained by adding to Γ_{λ_0} smaller scales. Adding a scale to the boundary of Ω_{λ_0} , we increase the wave interaction with the dissipative part of the boundary. Therefore, the energy $J(\Omega)(\omega_0)$ can only decrease, as compared to $J(\Omega_{\lambda_0})(\omega_0)$, or stay equal (in the case of no fitting of the wave inside the smallest parts of the boundary). Thus, if $\Omega_{\lambda_0}^o$ is an optimal domain for the frequency ω_0 , then we have

$$J(\Omega_{\lambda_0}^o)(\omega_0) \leq J(\Omega)(\omega_0) \leq J(\Omega_{\lambda_0})(\omega_0).$$

We also use the following hypothesis, coming from the empirical physics and confirmed by the numerical results in Section 7.1:

Assumption 2 Let Ω^* be an optimal domain for $\omega^* > 0$ with n ($n \in \mathbb{N}$) characteristic geometric scales $(\ell_j(\Omega^*))_{j=0, \dots, n-1}$. Then there exists j_0 ($0 \leq j_0 \leq n-1$), such that $\ell_{j_0}(\Omega^*) = \frac{\lambda^*}{2}$.

In the framework of ε -optimal shapes, the physical principle with Definition 5 directly ensures

Corollary 1 1. Let Ω^* be an optimal domain for $\omega^* > 0$ with n ($n \in \mathbb{N}$) characteristic geometric scales, such that $\ell_{j_0}(\Omega^*) = \frac{\lambda^*}{2}$ with $0 \leq j_0 \leq n-1$, as in Assumption 2. Then any $\Omega \in U_{ad}(\Omega_0)$ with the same characteristic geometric scales as Ω^* up to number j_0 :

$$\forall j = 0, \dots, j_0 \quad \ell_j(\Omega) = \ell_j(\Omega^*),$$

and $d_H(\Omega, \Omega^*) < \frac{\lambda^*}{4}$, is also optimal on ω^* :

$$J(\Omega)(\omega^*) = J_{\min}(\omega^*) = J(\Omega^*)(\omega^*).$$

2. Let $\varepsilon > 0$ be a fixed real number and $\Omega \in U_{ad}(\Omega_0)$ (see Eq. (18) for the definition of $U_{ad}(\Omega_0)$) be an ε -optimal domain on a range of frequencies $[\omega_0, \omega_1]$ with the smallest characteristic geometric size:

$$\ell_{\min}(\Omega) = \frac{\lambda}{2} \quad \text{for a } \lambda \in [\lambda_1, \lambda_0].$$

If $\Omega^\circ \in U_{ad}(\Omega_0)$ is a domain, such that

$$\exists j \in \mathbb{N} : \quad \ell_j(\Omega^\circ) = \ell_{\min}(\Omega) \quad \text{and} \quad d_H(\Omega^\circ, \Omega) \leq \frac{\lambda}{4},$$

then Ω° is also ε -optimal on $[\omega_0, \omega_1]$ (with the same ε).

Proof. Let us prove the first point. Without loss of generality, let us suppose Ω^* is such that $j_0 = n - 1$, i.e. $\ell_{\min}(\Omega^*) = \frac{\lambda^*}{2}$. Thus, if Ω has additional characteristic geometric sizes $\ell_j \ll \frac{\lambda^*}{2}$ ($j \geq n$) with $d_H(\Omega, \Omega^*) < \frac{\lambda^*}{4}$, then by Assumption 1 and Remark 5, the greater interaction of the wave with the dissipative boundary implies the non increasing of the energy $J(\Omega)(\omega^*) \leq J(\Omega^*)(\omega^*)$. But, since Ω^* is optimal on ω^* , we also have $J(\Omega^*)(\omega^*) = J_{\min}(\omega^*) \leq J(\Omega)(\omega^*)$. Consequently, $J(\Omega)(\omega^*) = J_{\min}(\omega^*)$, which ensures that Ω is optimal on ω^* . Therefore, to be optimal on a fixed frequency ω^* , it is sufficient to be optimal for the geometric sizes with $\ell = \frac{\lambda^*}{2}$, all sizes much smaller than $\frac{\lambda^*}{2}$ (in the sense of $d_H(\Omega^*, \Omega) \leq \frac{\lambda^*}{4}$), do not change the optimal property at one frequency point.

Let us prove the second point. By the assumption, the scaling $\lambda/2$ is sufficient to be ε -optimal on $[\omega_0, \omega_1]$:

$$\forall \omega \in [\omega_0, \omega_1] \quad |J(\Omega)(\omega) - J_{\min}(\omega)| < \varepsilon.$$

If we keep this scale and add characteristic scales much smaller than $\lambda/2$, such that $d_H(\Omega^\circ, \Omega) \leq \frac{\lambda}{4}$, then it holds $J_{\min}(\omega) \leq J(\Omega^\circ)(\omega) \leq J(\Omega)(\omega)$. Since the adding of small characteristic scales does not change the properties to be optimal on ω , it also does not change the property to be not farther than ε from the optimal domain. Therefore, the properties of ε -optimality still hold for Ω° with the same ε . \square

Remark 6 Thanks to the first point of Corollary 1, in what follows, for a solution of $\min_{\Omega \in U_{ad}} J(\Omega)(\omega)$ we always take Ω with $\ell_{\min}(\Omega) = \frac{\lambda}{2}$ (for $\lambda = 2\pi/\omega$) (i.e. we consider the “simplest” such Ω in terms of characteristic scales).

Definition 8 (*n -times wavelength preserving optimal domain*)

Let Ω_{λ_0} be a fixed initial domain with at least Lipschitz boundary, $\lambda_0 = \frac{2\pi}{\omega_0} \leq \frac{1}{2}$ for a fixed $\omega_0 \geq 4\pi$ and $n \geq 1$, $n \in \mathbb{N}$. The domain $\Omega^{opt} \in U_{ad}(\Omega_{\lambda_0})$ is called an n -times wavelength preserving optimal domain based on Ω_{λ_0} for the frequency ω_0 , if there exists

$$\lambda_n < \dots < \lambda_k < \dots < \lambda_1 < \lambda_0, \quad \text{such that } \forall k = 1, \dots, n \quad \lambda_k \leq \frac{\lambda_0}{2^k},$$

and $\Omega^{opt} = \Omega_{\lambda_n}$ is a solution of the following minimization problem (see Eq. (18) for the definition of the admissible domains)

$$J(\Omega_{\lambda_n})(\omega_n) = \inf_{\Omega \subset U_{ad}(\Omega_{\lambda_{n-1}}), d_H(\Omega, \Omega_{\lambda_{n-1}}) \leq \frac{\lambda_n}{4}} J(\Omega)(\omega_n),$$

where for $k = 1, \dots, n - 1$ the domains Ω_{λ_k} are solutions of the minimization problems

$$J(\Omega_{\lambda_k})(\omega_k) = \inf_{\Omega \subset U_{ad}(\Omega_{\lambda_{k-1}}), d_H(\Omega, \Omega_{\lambda_{k-1}}) \leq \frac{\lambda_k}{4}} J(\Omega)(\omega_k).$$

Remark 7 We know that $d_H(\Omega_{\lambda_0}, \Omega) \leq d(\Omega_{\lambda_0}, \Omega)$. In Definition 8 we have supposed that the largest wavelength $\lambda_0 \leq \frac{1}{2}$ (or the smallest frequency $\omega_0 \geq 4\pi$) in the aim to ensure $d_H(\Omega_{\lambda_0}, \Omega) \leq \frac{\lambda_0}{4} \leq \frac{1}{8}$, where $\frac{1}{8}$ comes from the condition from Lemma 2.4 in Ref. [36]. The additional restriction $d_H(\Omega_k, \Omega_{\lambda_{k-1}}) \leq \frac{\lambda_k}{4}$ keeps the new set of the admissible domains closed by the Hausdorff convergence and, thus, we still have the existence of an optimal shape in this class [22, 36].

Proposition 1 Let Ω^{opt} be an n -times wavelength preserving optimal domain based on a domain Ω_{λ_0} with $\Gamma_{\lambda_0} \in C^3$ for the frequency $\omega_0 \geq 4\pi$ (thus $2\lambda_0 \leq 1$). Then

$$\Omega^{opt} \in U_{ad}(\Omega_{\lambda_0}) \quad \text{with} \quad d_H(\Omega_{\lambda_0}, \Omega^{opt}) \leq \frac{\lambda_0}{4}$$

and Ω^{opt} has at least n multiscale characteristic shape lengths

$$\ell_k(\Omega^{opt}) = \frac{\lambda_k}{2} \leq \frac{\lambda_0}{2^{k+1}} \quad (k = 1, \dots, n).$$

Moreover,

1. for $n \geq 2$, every Ω_{λ_k} , $k = 1, \dots, n-1$ is an $(n-k)$ -times wavelength preserving optimal domain based on the domain Ω_{λ_0} for the frequency ω_0 ,
2. for all $k = 1, \dots, n$, Ω_{λ_k} has the characteristic geometric sizes of $\Omega_{\lambda_{k-1}}$ and $\ell_k = \ell_{\min}(\Omega_{\lambda_k}) = \frac{\lambda_k}{2}$,
3. if Ω_{λ_0} is an ε -optimal domain on $[\omega_a, \omega_b]$ ($\omega_a > 0$, $\omega_0 \in [\omega_a, \omega_b]$) with $\ell_{\min}(\Omega_{\lambda_0}) = \frac{\lambda_0}{2}$ and $\omega_b \leq 2\omega_0$, then for $n \geq 1$ all domains Ω_{λ_k} ($k = 1, \dots, n$) are ε -optimal on $[\omega_a, \omega_b]$ (with the same ε).

Proof. Firstly, we notice that for all $n \geq 1$

$$\begin{aligned} d_H(\Omega_{\lambda_0}, \Omega^{opt}) &\leq \frac{1}{4}(\lambda_1 + \dots + \lambda_n) \leq \frac{\lambda_0}{8} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} \right) \\ &= \left(2 - \frac{1}{2^{n-1}} \right) \frac{\lambda_0}{8} = \left(1 - \frac{1}{2^n} \right) \frac{\lambda_0}{4} \leq \frac{\lambda_0}{4}. \end{aligned}$$

Application of Assumption 1 and its Corollary 1 finishes the proof. Actually, point (2) is also a direct corollary of Assumption 2. More precisely, for point (3), we have that for all $k \geq 1$

$$d_H(\Omega_{\lambda_k}, \Omega_{\lambda_{k-1}}) \leq \frac{\lambda_k}{4} \leq \frac{\lambda_0}{4 \cdot 2^k} \leq \frac{\lambda_b}{4},$$

with $\ell_j(\Omega_{\lambda_k}) = \ell_{\min}(\Omega_{\lambda_j})$ ($0 \leq j \leq k$) by point (2). Thus, if for all $k \geq 1$ $\frac{\lambda_0}{4 \cdot 2^k} \leq \frac{\lambda_b}{4}$ (i.e. for $\lambda_b \geq \frac{\lambda_0}{2}$), by Corollary 1, Ω_{λ_k} is ε -optimal on the interval of ε -optimality of $\Omega_{\lambda_{k-1}}$. \square

Definition 9 The interval $[\omega_0, \omega_1]$ is called the maximum interval of ε -optimality of a domain Ω , if for all $\omega \in]\omega_0, \omega_1[$ Ω is ε -optimal on ω , but no more on ω_0 nor on ω_1 :

$$\forall \omega \in]\omega_0, \omega_1[\quad |J(\Omega)(\omega) - J_{\min}(\omega)| < \varepsilon, \quad \text{but} \quad |J(\Omega)(\omega_i) - J_{\min}(\omega_i)| \geq \varepsilon \quad (i = 0, 1).$$

Now, we give the following theorem for the existence of an ε -optimal domain for a fixed range of frequency for the problem (15), *i.e.* with $g = 0$ on Γ_D :

Theorem 6 *Let $[\omega_0, \omega_{\max}]$ with $\omega_0 \geq 4\pi$ and $\omega_{\max} \leq +\infty$ be a fixed frequency interval, and Ω_0 be a fixed regular domain in \mathbb{R}^2 ($\Gamma \in C^3$) with $\ell(\Omega_0) = \ell_{\min}(\Omega_0) = \frac{\lambda_0}{2}$.*

Define $N = \left\lceil \log_2 \frac{\omega_{\max}}{\omega_0} \right\rceil$ for $\omega_{\max} < \infty$. For all $\omega \in [\omega_0, \infty[$ consider

$$J_{\min}(\omega) = \inf_{\Omega \in U_{ad}(\Omega_0)} J(\Omega)(\omega),$$

the minimum of the acoustical energy for the Helmholtz problem (15) with f , h and α smooth functions of ω (of the class C^1), such that for all fixed $\omega > 0$ they satisfy the assumptions of Theorem 4 and in the high frequency limit verify

$$\text{for } \omega \rightarrow +\infty \quad \frac{f}{\omega^2} \rightarrow 0 \text{ and } \frac{h}{\omega} \rightarrow 0 \text{ in } L_2(D), \quad \frac{\text{Re}\alpha}{\omega} \rightarrow 0, \quad \frac{\omega}{|\text{Im}\alpha|} \rightarrow c_0, \quad (20)$$

where $c_0 > 0$ is a real constant.

Then $J_{\min}(\omega) \rightarrow 0$ for $\omega \rightarrow +\infty$ and there exists $\delta_0 \geq 0$, depending on ω_0 , such that for all $\varepsilon > \delta_0$ there exists a domain $\hat{\Omega}_0 \in U_{ad}(\Omega_0)$ with

$$\frac{\lambda_0}{4} \leq \ell_{\min}(\hat{\Omega}_0) \leq \ell(\Omega_0),$$

which is ε -optimal on a maximal interval $[\omega_0, \omega_1]$ with $\omega_1 \geq 2\omega_0$ (see Definition 9).

Moreover, there exists $\varepsilon_0 \geq \delta_0$, depending on ω_0 and ω_{\max} , such that for all $\varepsilon > \varepsilon_0$ there exists an ε -optimal domain Ω^ on $[\omega_0, \omega_{\max}]$:*

$$\forall \omega \in [\omega_0, \omega_{\max}] \quad |J(\Omega^*)(\omega) - J_{\min}(\omega)| < \varepsilon, \quad (21)$$

such that $d_H(\hat{\Omega}_0, \Omega^) \leq \frac{\lambda_0}{4}$*

1. *for $\omega_{\max} < \infty$, $\Omega^* \in U_{ad}(\hat{\Omega}_0)$ with at least N characteristic scales $\ell_k(\Omega^*) \leq \frac{\lambda_0}{2^{k+1}}$ for $k = 1, \dots, N$, where $\ell_N(\Omega^*) = \ell_{\min}(\Omega^*)$;*
2. *for $\omega_{\max} = \infty$, Ω^* is a fractal domain (an (ε, ∞) -domain), obtained as a limit for $N \rightarrow +\infty$ of the finite case, with $\ell_k(\Omega^*) \leq \frac{\lambda_0}{2^{k+1}}$ for $k \in \mathbb{N}^*$.*

Remark 8 *From the theory of the transparent or absorbing boundary conditions, it is known that the increasing of $|\text{Im}\alpha|$ corresponds to the increasing of the wave absorption by the boundary Γ , while the coefficient $\text{Re}\alpha$ corresponds to the reflection by Γ . Intuitively, if $h = 0$, the condition $\frac{\partial u}{\partial |x|} + (\text{Re}\alpha - i|\text{Im}\alpha|)u = 0$ is satisfied by the wave $e^{-i\alpha|x|} = e^{-i\text{Re}\alpha|x|}e^{-|\text{Im}\alpha||x|}$, where the term $e^{-i\text{Re}\alpha|x|}$ gives the propagating wave and the second term give the exponential dissipation, growing with the modulus of $\text{Im}\alpha$. Hence, the condition $\frac{\text{Re}\alpha}{|\text{Im}\alpha|} \rightarrow 0$ as soon as $\omega \rightarrow +\infty$, imposed in Eq. (20) reads that the dissipation by Γ dominates the reflection especially for the high frequencies.*

Proof. Theorem 4 with condition (20) ensures that $J_{\min}(\omega) \rightarrow 0$ for $\omega \rightarrow +\infty$. Let us prove it for a fixed admissible domain Ω . Indeed, for any fix $\omega > 0$, by Theorem 4, the operator

$$B : L_2(\Omega) \times V(\Omega) \rightarrow V(\Omega), \text{ defining by } B(f, h) = u$$

with u , the solution of problem (15), is linear and continuous (see also Ref. [5] for the real Robin boundary condition). Thus, if $\frac{f}{\omega^2} \rightarrow 0$ in $L_2(\Omega)$, then $\frac{1}{\omega^2} \|f\|_{L_2(\Omega)}^2 \rightarrow 0$ and, by the analogy, $\frac{h}{\omega} \rightarrow 0$ in $L_2(\Gamma)$ implies $\frac{1}{\omega} \|h\|_{L_2(\Gamma)}^2 \rightarrow 0$. The speeds of the decays on ω come from the fundamental solution of the Helmholtz equation: a derivative of the order m of u is of the order ω^m . Consequently, by the continuity of B , of the trace operator, and of all functions on ω , with the help of the variational formulation we have that for $\omega \rightarrow +\infty$ the solution of problem (15) u tends in $V(\Omega)$ to 0, the unique solution (thanks to $c_0 \neq 0$ the real line belongs to the resolvent set) of the limit homogeneous problem

$$\begin{cases} \Delta u + u = 0 & x \in \Omega, \\ u = 0 & \text{on } \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_N, \quad \frac{\partial u}{\partial n} - ic_0^{-1} \text{Tr}u = 0 & \text{on } \Gamma. \end{cases}$$

In Theorem 6 we impose the decay conditions on the domain D , since it is fixed and contains all admissible Ω . Since for all admissible Ω (which are bounded!) we have the homogeneous Dirichlet condition on Γ_D , then for all Ω it holds the Poincaré inequality, ensuring the damping of the acoustical energy (see also Example 1, from where we directly have $J(\Omega)(\omega) = \frac{1}{\omega^2} \|u\|_{V(\Omega)}^2$)

$$J(\Omega)(\omega) = \|u\|_{L_2(\Omega)}^2 \leq C_\Omega \|u\|_{V(\Omega)}^2 \rightarrow 0 \text{ as } \omega \rightarrow +\infty.$$

Hence, the same is true for $J_{\min}(\omega)$.

Let us now prove the existence of an ε -optimal domain $\hat{\Omega}_0 \in U_{ad}(\Omega_0)$ on an maximal interval $[\omega_0, \omega_1]$ with $\omega_1 \geq 2\omega_0$ for a sufficiently large ε . It turns on the question how to approximate a continuous function on a compact, here $J_{\min}(\omega)$ on $[\omega_0, 2\omega_0]$, by its value in one point. Thanks to Remark 3, for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all ω satisfying $|\omega^* - \omega| < \delta$, a domain Ω^* , optimal for ω^* , is ε -optimal on $]\omega^* - \delta, \omega^* + \delta[$. Thus, taking $\omega^* = \frac{3\omega_0}{2}$, the question is in the possibility to have $\delta \geq \frac{\omega_0}{2}$.

As $J_{\min}(\omega)$ is a continuous function on the compact $[\omega_0, 2\omega_0]$, by the Mean-value Theorem, there exists a frequency $\omega_0^* \in [\omega_0, 2\omega_0]$, such that

$$J_{\min}(\omega_0^*) = \frac{1}{\omega_0} \int_{\omega_0}^{2\omega_0} J_{\min}(\omega) d\omega.$$

Therefore, if Ω^* is optimal on ω_0^* , *i.e.* $J(\Omega^*)(\omega_0^*) = J_{\min}(\omega_0^*)$, then for all

$$\varepsilon > \|J_{\min}(\omega) - J_{\min}(\omega_0^*)\|_{C([\omega_0, 2\omega_0])} + \|J(\Omega^*)(\omega) - J(\Omega^*)(\omega_0^*)\|_{C([\omega_0, 2\omega_0])} =: \delta_0$$

the domain Ω^* is ε -optimal on $[\omega_0, 2\omega_0]$ with the maximal interval of ε -optimality $[\omega_0, \omega_1] \supset [\omega_0, 2\omega_0]$.

Obviously, if we minimize the distance $\|J(\Omega^*)(\omega) - J(\Omega^*)(\omega_0^*)\|_{C([\omega_0, 2\omega_0])}$ on the set of optimal domains for ω_0^* , we have

$$\inf_{\Omega^*} \|J(\Omega^*)(\omega) - J(\Omega^*)(\omega_0^*)\|_{C([\omega_0, 2\omega_0])} \geq \|J_{\min}(\omega) - J_{\min}(\omega_0^*)\|_{C([\omega_0, 2\omega_0])} =: r,$$

and hence δ_0 cannot be less than $2r$, which gives a limit of the precision. For instance, for $r_0 < r$ there does not exist $\omega_0^* \in [\omega_0, 2\omega_0]$, such that

$$\|J_{\min}(\omega) - J_{\min}(\omega_0^*)\|_{C([\omega_0, 2\omega_0])} \leq r_0.$$

If $J_{\min}(\omega)$ is constant on $[\omega_0, 2\omega_0]$, then the limit precision r becomes equal to zero:

$$\forall \omega \in [\omega_0, 2\omega_0] \quad |J_{\min}(\omega) - \frac{1}{\omega_0} \int_{\omega_0}^{2\omega_0} J_{\min}(\omega) d\omega| = 0.$$

The converse is also true: if for all $\varepsilon > 0$ a fixed domain Ω^* is ε -optimal on a compact interval $[\omega_a, \omega_b]$, then $J_{\min}(\omega)$ is constant on $[\omega_a, \omega_b]$.

By Assumption 2 and Remark 6 $\ell_{\min}(\hat{\Omega}_0) = \frac{\pi}{\omega_0^*} = \frac{\lambda_0^*}{2}$, which is, by its definition, less or equal to $\frac{\lambda_0}{2}$ and bigger or equal to $\frac{\lambda_0}{4}$.

Now, let us consider on frequency intervals of the form $[2^k \omega_0, 2^{k+1} \omega_0]$ the corresponding limit precisions

$$r_k := \|J_{\min}(\omega) - \frac{1}{2^k \omega_0} \int_{2^k \omega_0}^{2^{k+1} \omega_0} J_{\min}(\omega) d\omega\|_{C([2^k \omega_0, 2^{k+1} \omega_0])}.$$

Since $J_{\min}(\omega) \rightarrow 0$ for $\omega \rightarrow +\infty$, *i.e.* J_{\min} converges towards a constant value, thus $r_k \rightarrow 0$ for $k \rightarrow +\infty$. Hence, it is easier to approximate J_{\min} (with more precision) for high than for low frequencies. Consequently, there exist a finite number $K \in \mathbb{N}$ of the frequency ranges of the form $[2^k \omega_0, 2^{k+1} \omega_0]$ and $k_0 \in \mathbb{N}$, such that if $r_{k_0} = \max_{i=1, \dots, K} r_i$ is the minimal precision on these K intervals, then for all $k \geq k_0$ the limit precisions in higher frequencies are better: $r_k \leq r_{k_0}$.

Therefore, taking ε_0 large enough, such that $\varepsilon_0 \geq 2r_{k_0} \geq 0$ and $\varepsilon_0 > \delta_0$, to prove the theorem it is sufficient to show that for all $\varepsilon > \varepsilon_0$ there exists a sequence $(\Omega_n)_{n=1, \dots, N} \subset U_{ad}(\hat{\Omega}_0)$, such that

1. for all $\omega \in [\omega_0, 2^n \omega_0]$ ($n \leq N$), the domain Ω_n is ε -optimal with characteristic scales $\ell_k(\Omega_n) \leq \frac{\lambda_0}{2^{k+1}}$ for $k = 1, \dots, n$, where $\ell_n(\Omega_n) = \ell_{\min}(\Omega_n)$;
2. there exists Ω^* , which is equal to $\Omega_N \in U_{ad}(\hat{\Omega}_0)$ if $N < +\infty$ and which is a fractal (an (ε, ∞) -domain) if $N = +\infty$ with $\ell_k(\Omega^*) \leq \frac{\lambda_0}{2^{k+1}}$ for $k \in \mathbb{N}^*$, such that $\Omega_n \xrightarrow{*} \Omega^*$ in the class of locally uniform domains (see Definition 2), *i.e.*:

$$\forall \omega \in [\omega_0, \omega_{\max}] \quad \forall \eta > 0 \quad \exists M(\omega, \eta) > 0 : \forall n \geq M(\omega, \eta) \quad |J(\Omega_n)(\omega) - J(\Omega^*)(\omega)| < \eta,$$

and finally conclude that this Ω^* is ε -optimal on $[\omega_0, \omega_{\max}]$ (by point 1):

$$\forall \omega \in [\omega_0, \omega_{\max}] \quad |J(\Omega^*)(\omega) - J_{\min}(\omega)| < \varepsilon.$$

Let us fix $\varepsilon > \varepsilon_0$. For instance, a sequence of n -times wavelength preserving optimal domains $\hat{\Omega}_n$ based on $\hat{\Omega}_0$ for the frequency ω_0 verifies the properties (1)-(2). Indeed, we have constructed a domain $\hat{\Omega}_0 \in U_{ad}(\Omega_0)$ with $\ell_{\min}(\hat{\Omega}_0) = \frac{\lambda_0^*}{2} \in [\frac{\lambda_0}{4}, \frac{\lambda_0}{2}]$, such that it is ε -optimal on a maximal interval $[\omega_0, \omega_0^e] \supset [\omega_0, 2\omega_0]$. Let now, $\hat{\Omega}_1 \in U_{ad}(\Omega_0)$ be optimal on $\omega_1^* \in [2\omega_0, 4\omega_0]$, such that

$$J_{\min}(\omega_1^*) = \frac{1}{2\omega_0} \int_{2\omega_0}^{4\omega_0} J_{\min}(\omega) d\omega \quad \text{and} \quad d_H(\hat{\Omega}_1, \hat{\Omega}_0) \leq \frac{\lambda_1^*}{4}.$$

Thus, $\hat{\Omega}_1$ is an 1-times wavelength preserving optimal domain based on $\hat{\Omega}_0$ for the frequency ω_0 (see Definition 8). Consequently, by point (3) of Proposition 1, the domain $\hat{\Omega}_1$ is ε -optimal on $[\omega_0, 2\omega_0]$ and by the optimality on the mean-value frequency ω_1^* and by the fact that ε is sufficiently large, $\hat{\Omega}_1$ is also ε -optimal on $[2\omega_0, 4\omega_0]$. Hence, $\hat{\Omega}_1$ is ε -optimal on $[\omega_0, 4\omega_0]$. Taking each time optimal domains on the mean-values frequencies ω_k^* with the restriction

$$d_H(\hat{\Omega}_k, \hat{\Omega}_{k-1}) \leq \frac{\lambda_k^*}{4},$$

we obtain a sequence of n -times wavelength preserving optimal domains $(\hat{\Omega}_n)_{n=1, \dots, N}$ based on $\hat{\Omega}_0$ for the frequency ω_0 . Proposition 1 ensures the point (1) and that $\Omega^* = \Omega_N$ if N is finite.

For $N = +\infty$ we have

$$\forall n \in \mathbb{N} \quad d_H(\hat{\Omega}_n, \hat{\Omega}_{n+1}) \leq \frac{1}{4} \frac{\lambda_0}{2^{n+1}},$$

and thus, independently on $n \in \mathbb{N}$, by Proposition 1,

$$d_H(\hat{\Omega}_0, \hat{\Omega}_n) \leq \frac{\lambda_0}{4}.$$

For all $n \geq 1$ the domain $\hat{\Omega}_n \in U_{ad}(\hat{\Omega}_0)$ is ε -optimal on $[\omega_0, 2^n \omega_0]$ with

$$\hat{d}(\hat{\Omega}_n, \hat{\Omega}_{n+1}) \rightarrow 0 \quad \text{for} \quad n \rightarrow +\infty.$$

Hence Point (2) holds by the construction and also by the compactness results of Ref. [36] (Theorem 5.1 p.205 and Theorem 2.4 ii) p.59). \square

Remark 9 *To be coherent with numerical results of Section 7, Theorem 6 is given in the two-dimensional case. But, thanks to the general properties of the wave propagation, the physical principle in Assumption 1 obviously holds for the three dimensional case too, which directly applies Theorem 6 also for three-dimensional domains: to be the most efficient to dissipate the acoustical energy in \mathbb{R}^2 or \mathbb{R}^3 for almost all frequencies, the boundary Γ must be fractal.*

5 Shape derivative

Following the ideas Ref. [3], we provide two types of the shape derivation of J_1 : the first method is a formal derivation of the Lagrangian, associated with the optimization problem, which allows in the simplest way to obtain formula (24), but does not allow to prove it rigorously. To have a rigorous proof, as it is explained in Ref. [3], we need to use a direct derivative approach, involving the Eulerian derivative over domain, which is much more complicated. Two methods give the same formula (24). Let us start by introducing the definition of the shape derivative of a function (see Ref. [3]). Without lost of generality, we consider all times the two dimensional case ($n = 2$).

Definition 10 (*Shape derivative*) *The shape derivative of a function $K(\Omega) : \mathcal{C}(\Omega_0) \rightarrow \mathbb{R}$ at Ω_0 is defined as the Fréchet derivative in $W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ at 0 of the function $\theta \mapsto K(Id + \theta)(\Omega_0)$, i.e.,*

$$K(Id + \theta)(\Omega_0) = K(\Omega_0) + K'(\Omega_0)(\theta) + o(\theta) \quad \text{with} \quad \lim_{\theta \rightarrow 0} \frac{\|o(\theta)\|_{L^\infty(\mathbb{R}^2)}}{\|\theta\|_{W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)}} = 0,$$

where $K'(\Omega_0)$ is a continuous linear form on $W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$.

As in Ref. [3], we introduce two types of derivative. The first is the Eulerian derivative (or shape derivative), denoted by U , and the second is the Lagrangian derivative (or material derivative), denoted by Y .

Definition 11 (*Eulerian derivative*) *Assume that x belongs both to the initial domain Ω_0 and to the deformed domain $\Omega = (Id + \theta)(\Omega_0)$. A continuous linear form of $\theta \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$, denoted by $U(\theta, x)$, is called the Eulerian derivative, if it is defined by the expression:*

$$u((Id + \theta)(\Omega_0), x) = u(\Omega_0, x) + U(\theta, x) + o(\theta), \quad \text{with} \quad \lim_{\theta \rightarrow 0} \frac{\|o(\theta)\|}{\|\theta\|} = 0, \quad (22)$$

i.e., U is the directional derivative of u in the direction θ .

Let us notice that Definition 11 is local and takes sense for all x in an open set Ω_0 , since for small enough by norm θ the points x also belong to Ω . However, it does not ensure that if $x \in \partial\Omega_0$, then necessarily $x \in \Omega = (Id + \theta)(\Omega_0)$. Hence, we also need to introduce the Lagrangian derivative:

Definition 12 (*Lagrangian derivative*) *Let $u(\Omega)$ be defined on the domain Ω and $\Omega = (Id + \theta)(\Omega_0)$. For all $x \in \Omega_0$, let us define the function*

$$\hat{u}(\theta, x) = u((Id + \theta)(\Omega_0)) \circ (Id + \theta) = u((Id + \theta)(\Omega_0), x + \theta(x)). \quad (23)$$

The Lagrangian derivative, denoted by $Y(\theta, x)$, is a continuous linear form of θ , defined by the expression (it is the directional derivative of $\hat{u}(\theta, x)$ in the θ -direction)

$$\hat{u}(\theta, x) = \hat{u}(0, x) + Y(\theta, x) + o(\theta), \quad \text{with} \quad \lim_{\theta \rightarrow 0} \frac{\|o(\theta)\|}{\|\theta\|} = 0,$$

where $\hat{u}(0, x) = u(\Omega_0, x)$.

We notice that this time, varying θ , all functions $\hat{u}(\theta, x)$ are defined on the same domain Ω_0 , and we have no problem for the boundary points $x \in \partial\Omega_0$.

By composition of derivatives, we have the following relation between U and Y :

$$Y(\theta, x) = U(\theta, x) + \theta(x) \cdot \nabla u(\Omega_0, x).$$

We recall two important results from Ref. [3], which will be used to compute the shape derivative of the objective functions J and J_1 .

Theorem 7 (G. Allaire [3] Proposition 6.28 p.137) *Let Ω_0 be open bounded smooth domain in \mathbb{R}^2 . Let $u(\Omega)$ be a function from $C(\Omega_0)$ to $L^1(\mathbb{R}^2)$. Let, in addition, $\hat{u}(\theta)$ be the function from $W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$, defined by Eq. (23), derivable at 0 with the derivative Y (the Lagrangian derivative of $u(\Omega)$). Then the function K_1 from $C(\Omega_0)$ to \mathbb{R} , defined by*

$$K_1(\Omega) = \int_{\Omega} u(\Omega) dx,$$

is differentiable at Ω_0 and for all $\theta \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$, we have

$$K'_1(\Omega_0)(\theta) = \int_{\Omega_0} (u(\Omega_0) \operatorname{div} \theta + Y(\theta)) dx.$$

Similarly, if $\hat{u}(\theta)$ is derivable at 0 as the function from $C^1(\mathbb{R}^2, \mathbb{R}^2)$ to $L^1(\partial\Omega_0)$, then

$$K_2(\Omega) = \int_{\partial\Omega} u(\Omega) ds$$

is differentiable at Ω_0 and, for all $\theta \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, we have

$$K'_2(\Omega_0)(\theta) = \int_{\partial\Omega_0} (u(\Omega_0)(\operatorname{div} \theta - \nabla \theta n \cdot n) + Y(\theta)) ds.$$

Lemma 1 (G. Allaire [3] Remark 6.29 p.138) *The derivatives K'_1 and K'_2 can be also expressed using the Eulerian derivative U as:*

$$K'_1(\Omega_0)(\theta) = \int_{\Omega_0} (U(\theta) + \operatorname{div}(u(\Omega_0)\theta)) dx,$$

and

$$K'_2(\Omega_0)(\theta) = \int_{\partial\Omega_0} \left(U(\theta) + \theta \cdot n \left(\frac{\partial u(\Omega_0)}{\partial n} + Hu(\Omega_0) \right) \right) ds.$$

We prove the following theorem:

Theorem 8 *Let Ω_0 be a bounded domain in \mathbb{R}^2 with a connect boundary $\partial\Omega_0 \in C^3$, divided in three disjoint parts $\partial\Omega_0 = \Gamma_0 \sqcup \Gamma_D \sqcup \Gamma_N$. Let $\Omega \in \mathcal{C}(\Omega_0)$, defined in Eq. (16), and, such that $\partial\Omega = \Gamma \sqcup \Gamma_D \sqcup \Gamma_N$ with $\Gamma = (Id + \theta)\Gamma_0$ ($\theta \in W^{2,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ and $\|\theta\| < 1$). Let $u(\Omega_0) \in H^3(\Omega_0)$ be the solution of the problem (3) in Ω_0 with $g \in H^{\frac{5}{2}}(\Gamma_D)$ and*

$f \in H^1(\mathbb{R}^2)$ (see Theorem 1). Then the shape derivative of the objective function J_1 , defined in Eq. (19), is given by

$$\begin{aligned}
J_1'(\Omega_0)(\theta) &= \int_{\Gamma_0} \theta \cdot n (A|u(\Omega_0)|^2 + B|\nabla u(\Omega_0)|^2 + 2B|\alpha|^2|u(\Omega_0)|^2 \\
&\quad - 4C\text{Re}(\alpha)|u(\Omega_0)|^2 + CH|u(\Omega_0)|^2) ds \\
&\quad + \int_{\Gamma_0} \theta \cdot n \text{Re}(-\nabla u(\Omega_0) \cdot \nabla w + \omega^2 u(\Omega_0)w - fw - \alpha H u(\Omega_0)w + 2\alpha^2 u(\Omega_0)w) ds \\
&\quad + 2\mu \int_{\Gamma_0} \theta \cdot n (\text{Vol}(\Omega) - \text{Vol}(\Omega_0)) ds
\end{aligned} \tag{24}$$

with n the exterior normal vector on Γ_0 , H the curvature of the boundary Γ_0 , and $w \in V(\Omega_0)$ ($V(\Omega_0)$ is defined in Eq. (13)), the unique solution of the adjoint problem (see Eq. (29)) corresponding to u .

Proof. Since the data of the problem and the solution u are complex functions (except ω which is a positive constant), let us separate the imaginary and real parts, adopting the following notation: $u = u_R + iu_I$. Thus, the boundary value problem for the Helmholtz equation (3) takes the following form:

$$\Delta u_R + \omega^2 u_R = f_R(x) \quad x \in \Omega, \tag{25}$$

$$u_R = g_R(x) \quad \text{on } \Gamma_D, \quad \frac{\partial u_R}{\partial n} = 0 \quad \text{on } \Gamma_N, \quad \frac{\partial u_R}{\partial n} + \alpha_R u_R - \alpha_I u_I = 0 \quad \text{on } \Gamma,$$

$$\Delta u_I + \omega^2 u_I = f_I(x) \quad x \in \Omega, \tag{26}$$

$$u_I = g_I(x) \quad \text{on } \Gamma_D, \quad \frac{\partial u_I}{\partial n} = 0 \quad \text{on } \Gamma_N, \quad \frac{\partial u_I}{\partial n} + \alpha_I u_R + \alpha_R u_I = 0 \quad \text{on } \Gamma.$$

The objective functional is considered as a function of the real and the complex parts of u :

$$\begin{aligned}
J(\Omega, u_R, u_I) &= A \int_{\Omega} (|u_R|^2 + |u_I|^2) dx + B \int_{\Omega} (|\nabla u_R|^2 + |\nabla u_I|^2) dx \\
&\quad + C \int_{\Gamma} (|u_R|^2 + |u_I|^2) ds.
\end{aligned}$$

Variational formulation. We take the inner product in $L^2(\Omega)$ of (25) + i (26) by a test function $\phi = \phi_R + i\phi_I \in H^1(\Omega)$. With separation of the real and imaginary parts we obtain

$$\begin{aligned}
\int_{\Omega} (\Delta u_R + \omega^2 u_R - f_R) \phi_R dx - \int_{\Omega} (\Delta u_I + \omega^2 u_I - f_I) \phi_I dx &= 0, \\
\int_{\Omega} (\Delta u_I + \omega^2 u_I - f_I) \phi_R dx + \int_{\Omega} (\Delta u_R + \omega^2 u_R - f_R) \phi_I dx &= 0.
\end{aligned}$$

By integration by parts, we find

$$\begin{aligned}
\int_{\Gamma_D} \left(\frac{\partial g_R}{\partial n} \phi_R - \frac{\partial g_I}{\partial n} \phi_I \right) ds - \int_{\Gamma} ((\alpha_R u_R - \alpha_I u_I) \phi_R - (\alpha_I u_R + \alpha_R u_I) \phi_I) ds \\
+ \int_{\Omega} (\nabla u_I \nabla \phi_I - \nabla u_R \nabla \phi_R + \omega^2 (u_R \phi_R - u_I \phi_I) + f_I \phi_I - f_R \phi_R) dx = 0,
\end{aligned} \tag{27}$$

and

$$\begin{aligned} & \int_{\Gamma_D} \left(\frac{\partial g_I}{\partial n} \phi_R + \frac{\partial g_R}{\partial n} \phi_I \right) ds - \int_{\Gamma} ((\alpha_I u_R + \alpha_R u_I) \phi_R + (\alpha_R u_R - \alpha_I u_I) \phi_I) ds \\ & + \int_{\Omega} (-\nabla u_I \nabla \phi_R - \nabla u_R \nabla \phi_I + \omega^2 (u_I \phi_R + u_R \phi_I) - f_I \phi_R - f_R \phi_I) dx = 0. \end{aligned} \quad (28)$$

Formal derivation (Lagrangian). We define (see [3] p. 152) the Lagrangian of the optimization problem as the sum of the functional J and the variational formulations (27) and (28), by also adding the terms with Lagrangian multipliers λ_R and λ_I , in the aim to ensure a penalization the Dirichlet boundary condition on Γ_D :

$$\begin{aligned} L(\Omega, u_R, u_I, w_R, w_I, \lambda_R, \lambda_I) &= A \int_{\Omega} (|u_R|^2 + |u_I|^2) dx \\ &+ B \int_{\Omega} (|\nabla u_R|^2 + |\nabla u_I|^2) dx + C \int_{\Gamma} (|u_R|^2 + |u_I|^2) ds \\ &+ \int_{\Omega} (\nabla u_I \nabla w_I - \nabla u_R \nabla w_R + \omega^2 (u_R w_R - u_I w_I) + f_I w_I - f_R w_R) dx \\ &- \int_{\Gamma} ((\alpha_R u_R - \alpha_I u_I) w_R - (\alpha_I u_R + \alpha_R u_I) w_I) ds \\ &+ \int_{\Gamma_D} \left(\frac{\partial u_R}{\partial n} w_R - \frac{\partial u_I}{\partial n} w_I \right) ds + \int_{\Gamma_D} \lambda_R (u_R - g_R) ds + \int_{\Gamma_D} \lambda_I (u_I - g_I) ds, \end{aligned}$$

where $u_R, u_I, w_R, w_I, \lambda_R$ and λ_I are in $H^1(\mathbb{R}^2)$. As the functional J and the displacement vector θ of the boundary are real, for the optimization we need to consider only real part of variational form of the direct problem. Therefore the conjugate problem can be found from the system

$$\begin{aligned} \left\langle \frac{\partial L}{\partial u_R}(\Omega, u_R, u_I, w_R, w_I, \lambda_R, \lambda_I), \psi_R \right\rangle &= 0, \\ \left\langle \frac{\partial L}{\partial u_I}(\Omega, u_R, u_I, w_R, w_I, \lambda_R, \lambda_I), \psi_I \right\rangle &= 0, \end{aligned}$$

with

$$\begin{aligned} \left\langle \frac{\partial L}{\partial u_R}, \psi_R \right\rangle &= \int_{\Omega} (2A u_R \psi_R + 2B \nabla u_R \nabla \psi_R - \nabla w_R \nabla \psi_R + \omega^2 w_R \psi_R) dx \\ &- \int_{\Gamma} (\alpha_R w_R - \alpha_I w_I - 2C u_R) \psi_R ds + \int_{\partial\Omega} (2B \nabla u_R - \nabla w_R) \cdot n \psi_R ds \\ &+ \int_{\Gamma_D} \frac{\partial \psi_R}{\partial n} w_R ds + \lambda_R \int_{\Gamma_D} \psi_R ds, \end{aligned}$$

and

$$\begin{aligned}
\left\langle \frac{\partial L}{\partial u_I}, \psi_I \right\rangle &= \int_{\Omega} (2Au_I\psi_I + 2B\nabla u_I\nabla\psi_I + \nabla w_I\nabla\psi_I - \omega^2 w_I\psi_I) dx \\
&+ \int_{\Gamma} (\alpha_I w_R + \alpha_R w_I + 2Cu_I) \psi_I ds + \int_{\partial\Omega} (2B\nabla u_I + \nabla w_I) \cdot n \psi_I ds \\
&+ \int_{\Gamma_D} \frac{\partial\psi_I}{\partial n} w_I ds + \lambda_I \int_{\Gamma_D} \psi_I ds.
\end{aligned}$$

After the integration by parts, we obtain the following adjoint problem:

$$\left\{ \begin{array}{l} \Delta w_R + \omega^2 w_R = -2(Au_R(\Omega_0) - B\Delta u_R(\Omega_0)) \quad x \in \Omega_0, \\ w_R = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial w_R}{\partial n} = 0 \quad \text{on } \Gamma_N, \\ \frac{\partial w_R}{\partial n} + \alpha_R w_R - \alpha_I w_I = -2B[\alpha_R u_R(\Omega_0) - \alpha_I u_I(\Omega_0)] + 2Cu_R(\Omega_0) \quad \text{on } \Gamma_0, \\ \Delta w_I + \omega^2 w_I = 2(Au_I(\Omega_0) - B\Delta u_I(\Omega_0)) \quad x \in \Omega_0, \\ w_I = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial w_I}{\partial n} = 0 \quad \text{on } \Gamma_N, \\ \frac{\partial w_I}{\partial n} + \alpha_I w_R + \alpha_R w_I = 2B(\alpha_R u_I(\Omega_0) + \alpha_I u_R(\Omega_0)) - 2Cu_I(\Omega_0) \quad \text{on } \Gamma_0. \end{array} \right. \quad (29)$$

Then, $\lambda_R = (-2B\nabla u_R + \nabla w_R) \cdot n$ and $\lambda_I = (-2B\nabla u_I - \nabla w_I) \cdot n$. We notice that the adjoint problem (29) can be more compactly rewritten for the complex-valued functions $w \in V(\Omega_0)$ ($w = w_R + iw_I$), $u(\Omega_0)$ and α :

$$\left\{ \begin{array}{l} \Delta w + \omega^2 w = -2(A\bar{u}(\Omega_0) - B\Delta\bar{u}(\Omega_0)) \quad x \in \Omega_0, \\ w = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_N, \\ \frac{\partial w}{\partial n} + \alpha w = -2B\bar{\alpha}\bar{u}(\Omega_0) + 2C\bar{u}(\Omega_0) \quad \text{on } \Gamma_0. \end{array} \right. \quad (30)$$

Since the Dirichlet and Neumann boundaries are supposed to be fixed, we impose $\theta = 0$ on Γ_D and Γ_N . Consequently, functions λ_R and λ_I do not contribute to the final formula of the shape derivative. Hence, $J'(\Omega_0)(\theta)$ is given by

$$\begin{aligned}
J'(\Omega_0)(\theta) &= \frac{\partial L}{\partial \Omega}(\Omega_0, u_R, u_I, w_R, w_I)(\theta) \\
&= \int_{\Gamma_0} \theta \cdot n (A|u|^2 + B|\nabla u|^2 - 2C\text{Re}(\alpha)|u|^2 + CH|u|^2) ds \\
&+ \int_{\Gamma_0} \theta \cdot n \text{Re} \left(-\nabla u \cdot \nabla w + \omega^2 uw - fw - \alpha H uw - \alpha \frac{\partial(uw)}{\partial n} \right) ds, \quad (31)
\end{aligned}$$

where n is the outward normal on Γ_0 and H is the curvature of Γ_0 . Using the boundary conditions and adding the volume constraint, we directly obtain (24).

Derivation. Since Γ_D does not move in our assumption, and thus, the value g does not have any influence to the shape derivative $J'(\Omega_0)$, in what following, in the aim to simplify the notations, we take $g \equiv 0$ on Γ_D .

We take the variational form of the problem over Ω and we transform it in a variational form over Ω_0 . For this we change the variables $x = y + \theta(y)$ and consider $u(\theta) = u((Id + \theta)(\Omega_0)) \circ (Id + \theta)$ with a test function of the form $\phi(x) = \psi \circ (Id + \theta)^{-1}(x) = \psi(y)$. We use the formulas (see Ref. [3] Lemmas 6.21 and 6.23)

$$\begin{aligned} \int_{\Omega} f dx &= \int_{(Id+\theta)\Omega_0} f dx = \int_{\Omega_0} f \circ (Id + \theta) |\det \nabla (Id + \theta)| dx, \\ (\nabla f) \circ (Id + \theta) &= ((\nabla (Id + \theta))^{-1})^t \nabla (f \circ (Id + \theta)), \\ \int_{\partial(\Omega_0+\theta)} f ds &= \int_{\Gamma_0} f \circ (Id + \theta) |\det(I + \nabla \theta)| |(I + \nabla \theta)^{-1})^t n|_{\mathbb{R}^2} ds \end{aligned}$$

with $I = \nabla Id$ (*i.e.*, I is the identity matrix) to obtain

$$\begin{aligned} & - \int_{\Gamma_0} \alpha u(\theta) \psi |\det(I + \nabla \theta)| |(I + \nabla \theta)^{-1})^t n|_{\mathbb{R}^2} ds \\ & - \int_{\Omega_0} (A(\theta) \nabla u(\theta)) \cdot \nabla \psi dy + \int_{\Omega_0} \omega^2 u(\theta) \psi |\det(I + \nabla \theta)| dy \\ & = \int_{\Omega_0} f \circ (Id + \theta) \psi |\det(I + \nabla \theta)| dy - \int_{\Gamma_D} \frac{\partial u}{\partial n} \psi ds, \end{aligned}$$

where

$$A(\theta) = |\det(I + \nabla \theta)| (I + \nabla \theta)^{-1} ((I + \nabla \theta)^{-1})^t.$$

With the notation $Y = \langle u'(0), \theta \rangle$, being the Lagrangian derivative at 0 of $u(\theta)$ in the direction θ , and $u(0) = u(\Omega_0)$, we find that for all $\psi \in V(\Omega_0)$

$$\begin{aligned} & \int_{\Omega_0} (-\nabla Y \cdot \nabla \psi + \omega^2 Y \psi) dy + \int_{\Omega_0} (-\operatorname{div} \theta I + \nabla \theta + (\nabla \theta)^t) \nabla u(0) \cdot \nabla \psi dy \\ & + \int_{\Omega_0} \omega^2 u(0) \psi \operatorname{div} \theta dy = \int_{\Omega_0} \operatorname{div}(f \theta) \psi dy + \int_{\Gamma_0} \alpha (Y + u(0) (\operatorname{div} \theta - (\nabla \theta)^t n \cdot n)) \psi ds. \end{aligned}$$

Consequently, the Lagrangian derivative $Y \in V(\Omega_0)$ is the unique solution of the following problem

$$\begin{aligned} \Delta Y + \omega^2 Y &= \operatorname{div} (-\operatorname{div} \theta I + \nabla \theta + (\nabla \theta)^t) \nabla u(\Omega_0) \\ &\quad - \omega^2 u(\Omega_0) \operatorname{div} \theta + \operatorname{div}(f \theta) \quad \text{in } \Omega_0, \\ Y &= 0 \quad \text{on } \Gamma_D, \quad \frac{\partial Y}{\partial n} = 0 \quad \text{on } \Gamma_N, \\ \frac{\partial Y}{\partial n} + \alpha Y &= -\alpha u(\Omega_0) (\operatorname{div} \theta - (\nabla \theta)^t n \cdot n) \\ &\quad + (-\operatorname{div} \theta I + \nabla \theta + (\nabla \theta)^t) \nabla u(\Omega_0) \cdot n \quad \text{on } \Gamma_0. \end{aligned} \tag{32}$$

The well-posedness of the system (32) follows from the regularity of θ , f and u on Ω_0 (see Theorem 4).

The equation for the Lagrangian derivative Y can be simplified to

$$\Delta Y + \omega^2 Y = \Delta(\theta \cdot \nabla u(\Omega_0)) + \omega^2 \theta \cdot \nabla u(\Omega_0).$$

Since the Eulerian derivative $U = Y - \theta \cdot \nabla u(\Omega_0)$ and $\theta = 0$ on Γ_D and Γ_N , we obtain that $U \in V(\Omega_0)$ is solution of the following problem

$$\begin{cases} \Delta U + \omega^2 U = 0 & \text{in } \Omega_0, \\ U = 0 & \text{on } \Gamma_D, \quad \frac{\partial U}{\partial n} = 0 & \text{on } \Gamma_N, \\ \frac{\partial U}{\partial n} + \alpha U = \theta \cdot n \left(\alpha^2 u(\Omega_0) - \frac{\partial^2 u(\Omega_0)}{\partial n^2} \right) + \nabla_t(\theta \cdot n) \cdot \nabla_t u(\Omega_0) & \text{on } \Gamma_0, \end{cases} \quad (33)$$

with the notation of the tangential gradient $\nabla_t \phi = \nabla \phi - (\nabla \phi \cdot n)n$. Let us show it. Thanks to the regularity of the boundary $\partial\Omega$, the elements of $H^1(\Omega)$ can be considered as the restrictions of the corresponding elements of $H^1(\mathbb{R}^2)$. Thus, we can reformulate the variational form (27)–(28) by “to find $u(\Omega) \in V(\mathbb{R}^2)$, such that for all $\phi \in V(\mathbb{R}^2)$ it holds

$$\int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - \omega^2 \int_{\Omega} u \bar{v} dx + \int_{\Gamma} \alpha u \bar{v} d\sigma = \int_{\Omega} f \bar{v} dx.”$$

We derive the last equality at Ω_0 , using Theorem 7 and the facts, that $\theta = 0$ on Γ_D and Γ_N . Hence, we find that $u'(\Omega_0)(\theta) = U$, where U satisfies

$$\begin{aligned} & \int_{\Omega_0} (-\nabla U \cdot \nabla \phi + \omega^2 U \phi) dx - \int_{\Gamma_0} \alpha U \phi ds \\ &= \int_{\Gamma_0} \theta \cdot n \left(\nabla u \cdot \nabla \phi - \omega^2 u \phi + f \phi + \alpha H u \phi + \alpha \frac{\partial(u\phi)}{\partial n} \right) ds. \end{aligned} \quad (34)$$

From Eq. (34), for the test function ϕ of a compact support in Ω , we find (see for example Ref. [3] p. 144) that it holds the equation $\Delta U + \omega^2 U = 0$.

To prove the boundary conditions of system (33), we use the following relations on Γ_0

$$\nabla u \cdot \nabla \phi - \frac{\partial u}{\partial n} \frac{\partial \phi}{\partial n} = \nabla_t u \cdot \nabla_t \phi, \quad \frac{\partial u}{\partial n} = -\alpha u \text{ on } \Gamma_0,$$

to transform Eq. (34) to the equality

$$\int_{\Omega_0} (-\nabla U \nabla \phi + \omega^2 U \phi) dx - \int_{\Gamma_0} \alpha U \phi ds = \int_{\Gamma_0} \theta \cdot n (\nabla_t u \cdot \nabla_t \phi + (f - \omega^2 u) \phi + \alpha H u \phi - \alpha^2 u \phi) ds.$$

Noticing, as in Ref. [22] Proposition 5.4.12 and Theorem 5.4.13 p. 196, that for $u \in \{H^3(\Omega_0) \mid u = 0 \text{ on } \Gamma_D\}$ (using the regularity property for $f \in H^1(\mathbb{R}^2)$, $g \in H^3(\mathbb{R}^2)$ and $\partial\Omega_0 \in C^3$) and $\phi \in \{C^\infty(\mathbb{R}^2) \mid \phi = 0 \text{ on } \Gamma_D\}$, there hold the relations

$$\begin{aligned} f - \omega^2 u &= \Delta u = \Delta_t u + H \frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2} \text{ on } \Gamma_0, \\ \int_{\Gamma_0} \theta \cdot n \phi \Delta_t u ds &= - \int_{\Gamma_0} \nabla_t u \nabla_t (\phi \theta \cdot n) ds = - \int_{\Gamma_0} (\theta \cdot n \nabla_t u \nabla_t \phi + \phi \nabla_t u \nabla_t (\theta \cdot n)) ds. \end{aligned}$$

Since, in addition,

$$\int_{\Omega_0} (-\nabla U \nabla \phi + \omega^2 U \phi) dx = - \int_{\Gamma_0} \phi \frac{\partial U}{\partial n} ds + \int_{\Omega_0} (\Delta U + \omega^2 U) \phi dx = - \int_{\Gamma_0} \phi \frac{\partial U}{\partial n},$$

we finally obtain the boundary conditions for the system (33).

Therefore, thanks to Theorem 7, we find the shape derivative of J as

$$\begin{aligned} J'(\Omega_0)(\theta) &= \int_{\Omega_0} \operatorname{div} (\theta (A|u(\Omega_0)|^2 + B|\nabla u(\Omega_0)|^2)) dx \\ &+ C \int_{\Gamma_0} \theta \cdot n \left(\frac{\partial |u(\Omega_0)|^2}{\partial n} + H|u(\Omega_0)|^2 \right) ds + \int_{\Omega_0} (2A(u_R(\Omega_0)U_R + u_I(\Omega_0)U_I) \\ &+ 2B(\nabla u_R(\Omega_0)\nabla U_R + \nabla u_I(\Omega_0)\nabla U_I)) dx + 2C \int_{\Gamma_0} (u_R(\Omega_0)U_R + u_I(\Omega_0)U_I) ds \\ &= \int_{\Gamma_0} \theta \cdot n \left(A|u(\Omega_0)|^2 + B|\nabla u(\Omega_0)|^2 + C \frac{\partial |u(\Omega_0)|^2}{\partial n} + CH|u(\Omega_0)|^2 \right) ds \\ &+ \int_{\Omega_0} \operatorname{Re}(2A\bar{u}(\Omega_0)U + 2B\nabla \bar{u}(\Omega_0)\nabla U) dx + 2C \int_{\Gamma_0} \operatorname{Re}(\bar{u}(\Omega_0)U) ds. \end{aligned}$$

On the other hand, we also have relation (34). So, we would like to find ϕ , such that

$$\begin{aligned} \int_{\Omega_0} (-\nabla U \cdot \nabla \phi + \omega^2 U \phi) dx - \int_{\Gamma_0} \alpha U \phi ds + \int_{\Gamma_D} \nabla U \cdot n \phi ds = \\ - \int_{\Omega_0} (2A\bar{u}(\Omega_0)U + 2B\nabla \bar{u}(\Omega_0)\nabla U) - \int_{\Gamma_0} 2C\bar{u}(\Omega_0)U ds. \end{aligned} \quad (35)$$

Let w be the solution of the adjoint problem (30), then Eq. (35) is satisfied for $\phi = w$. Hence,

$$\begin{aligned} \int_{\Omega_0} \operatorname{Re}(2A\bar{u}(\Omega_0)U + 2B\nabla \bar{u}(\Omega_0)\nabla U) dx + \int_{\Gamma_0} 2C\operatorname{Re}(\bar{u}(\Omega_0)U) ds \\ = \int_{\Gamma_0} \theta \cdot n \operatorname{Re} \left(-\nabla u(\Omega_0) \cdot \nabla w + \omega^2 u(\Omega_0)w - fw - \alpha H u(\Omega_0)w - \alpha \frac{\partial (u(\Omega_0)w)}{\partial n} \right) ds. \end{aligned}$$

Finally, we obtain the shape derivative $J'(\Omega_0)(\theta)$ given by formula (31). Using the Robin boundary condition, we obtain

$$\alpha \nabla(uw) \cdot n = \alpha (w \nabla u \cdot n + u \nabla w \cdot n) = -2\alpha^2 uw - 2B|\alpha|^2 |u|^2 + 2C\alpha |u|^2.$$

Then, we can calculate the shape derivative of J as

$$\begin{aligned} J'(\Omega_0)(\theta) &= \int_{\Gamma_0} \theta \cdot n (A|u(\Omega_0)|^2 + B|\nabla u(\Omega_0)|^2 + 2B|\alpha|^2 |u(\Omega_0)|^2 \\ &- 4C\operatorname{Re}(\alpha)|u(\Omega_0)|^2 + CH|u(\Omega_0)|^2) ds \\ &+ \int_{\Gamma_0} \theta \cdot n \operatorname{Re} (-\nabla u(\Omega_0) \cdot \nabla w + \omega^2 u(\Omega_0)w - fw - \alpha H u(\Omega_0)w + 2\alpha^2 u(\Omega_0)w) ds. \end{aligned}$$

Now, if we add to the objective function the volume constraint with the Lagrange coefficient μ (see Eq. (19))

$$J_1(\Omega, u) = J(\Omega, u) + \mu (\text{Vol}(\Omega) - \text{Vol}(\Omega_0))^2,$$

the shape derivative of objective function J_1 is given by

$$J'_1(\Omega_0)(\theta) = J'(\Omega_0)(\theta) + 2\mu \int_{\Gamma_0} \theta \cdot n (\text{Vol}(\Omega) - \text{Vol}(\Omega_0)) ds,$$

which concludes the proof. \square

6 Shape optimization algorithm

We want to solve numerically, using the descend gradient method, the following minimization problem: for $\omega > 0$ and Ω_0 find $\Omega^{\text{opt}} \in U_{ad}(\lambda, \Omega_0)$, such that

$$J_1(\Omega^{\text{opt}}) = \min_{\Omega \in U_{ad}(\lambda, \Omega_0)} J_1(\Omega).$$

We rewrite the shape derivative of J_1 , given by Eq. (24), in the form

$$J'_1(\Omega_0)(\theta) = \int_{\Gamma_0} (\theta \cdot n)(-V) ds, \quad (36)$$

where by the velocity $-V$ is denoted

$$\begin{aligned} -V = & (A|u|^2 + B|\nabla u|^2 + 2B|\alpha|^2|u|^2 - 4C\text{Re}(\alpha)|u|^2 + CH|u|^2) \\ & + \text{Re}(-\nabla u \cdot \nabla w + \omega^2 uw - fw - \alpha H uw + 2\alpha^2 uw) \\ & + 2\mu (\text{Vol}(\Omega) - \text{Vol}(\Omega_0)). \end{aligned} \quad (37)$$

If the velocity V follows the outward normal direction, or equivalently, if $\theta \cdot n = V$, then Eq. (36) implies that

$$J'_1(\Omega_0)(\theta) = - \int_{\Gamma_0} V^2 ds < 0,$$

which ensures the decreasing behavior of the objective function. To calculate it, we need to know u , the solution of the Helmholtz equation in Ω_0 , but also w , the solution of the adjoint problem and the curvature H . Inspired by Refs. [3, 38, 39], we construct a shape optimization algorithm composed of the following steps:

- (i) Solving the Helmholtz problem (3) by a center finite difference schema in a fixed domain D , which is assumed to contain all domains Ω during the shape optimization process.
- (ii) Calculating the velocity V of the Robin boundary Γ , based on the shape derivative, and then extending a such velocity towards the normal vector on the whole domain D , or at least around the Robin boundary.

(iii) Solving the level set equation to obtain a new shape.

If $J'_1(\Omega)(\theta) \geq 0$, then Ω is an optimal domain, and the algorithm stops. In order to describe the shape of the domain, we use a concept of level sets. More precisely, the level set function ψ of the domain $\Omega \subset D$ is defined by

$$\begin{cases} \psi(x) = 0 & \text{iff } x \in (\partial\Omega \cap D), \\ \psi(x) < 0 & \text{iff } x \in \Omega, \\ \psi(x) > 0 & \text{iff } x \in (D \setminus \Omega). \end{cases}$$

The level set method, initially devised by S. Osher and J-A. Sethian in Ref. [38], allows, not only to define implicitly the domain, but also to follow easily the propagation of the boundary during the evolution process. Let us take into account a particle $x(t)$ on the boundary, which propagates in time, hence it has the zero-level set all time, *i.e.*, $\psi(x(t), t) = 0$. By the chain rule, it yields that

$$\psi_t + x'(t) \cdot \nabla\psi(x(t), t) = 0. \quad (38)$$

If V is the velocity in the outward normal direction of the boundary, *i.e.* $x'(t) \cdot n = V$, with $n = \frac{\nabla\psi}{|\nabla\psi|}$, then from Eq. (38), we obtain a so-called level set equation

$$\psi_t + V|\nabla\psi| = 0, \quad (39)$$

associated with the initial condition $\psi(x, t = 0) = 0$. This equation is of Hamilton-Jacobi type, and in what follows we call it the Hamilton-Jacobi equation.

To calculate the shape derivative of the objective function and velocity's extension, we use formula (24). More precisely, we calculate $-V$ (see Eq. (37)) and use the descend gradient method, taking $\theta = Vn$, in the aim to obtain a new domain, which has a smaller value of the objective function. After solving the Helmholtz equation (3) and the adjoint problem (30), we find numerically the corresponding solutions u and w , and evaluate ∇u and ∇w . The curvature H is calculated, basing on the level set function ψ , by the following equation

$$H = \nabla \cdot \frac{\nabla\psi}{|\nabla\psi|} = \frac{\psi_{yy}\psi_x^2 - 2\psi_x\psi_y\psi_{xy} + \psi_{xx}\psi_y^2}{(\psi_x^2 + \psi_y^2)^{3/2}}.$$

To apply the level set method, we need to know the velocity V not only on the boundary, but in both sides of the Robin boundary Γ . Naturally, V is defined inside the domain Ω , so we have to extend V outside the domain to be able to solve the Hamilton-Jacobi equation (39). The method to extend V outside the domain [37, 39], on one way, is to solve until the stationary state the equation

$$\phi_t + \beta(x, y)\nabla\phi \cdot n = 0,$$

with the initial condition $\phi(t = 0)$ equal to V inside the domain Ω and zero elsewhere. Here by n is denoted the outward normal vector to Γ . The mesh, used to solve the Hamilton-Jacobi equation, is coarser than the mesh, used to solve the Helmholtz equation.

We use an upwind schema for solving the Hamilton-Jacobi equation [37, 39] and discretize Eq. (39) as follows

$$\frac{\psi_{ij}^{n+1} - \psi_{ij}^n}{\Delta t} + [\max(V_{ij}, 0) \nabla^+ + \min(V_{ij}, 0) \nabla^-] = 0, \quad (40)$$

where

$$\begin{aligned} \nabla^+ &= \left[\max(D_{ij}^{-x}, 0)^2 + \min(D_{ij}^{+x}, 0)^2 + \max(D_{ij}^{-y}, 0)^2 + \min(D_{ij}^{+y}, 0)^2 \right]^{1/2}, \\ \nabla^- &= \left[\max(D_{ij}^{+x}, 0)^2 + \min(D_{ij}^{-x}, 0)^2 + \max(D_{ij}^{+y}, 0)^2 + \min(D_{ij}^{-y}, 0)^2 \right]^{1/2}, \\ D_{ij}^{-x} &= \frac{\psi^n(i, j) - \psi^n(i-1, j)}{\Delta x}, \quad D_{ij}^{+x} = \frac{\psi^n(i+1, j) - \psi^n(i, j)}{\Delta x}, \\ D_{ij}^{-y} &= \frac{\psi^n(i, j) - \psi^n(i, j-1)}{\Delta y}, \quad D_{ij}^{+y} = \frac{\psi^n(i, j+1) - \psi^n(i, j)}{\Delta y}, \end{aligned}$$

and $\psi_0 = \psi(t=0)$ is the signed distance function, defined by

$$\psi_0(x, y) = \pm \text{dist}[(x, y), \Gamma], \quad (x, y) \in D. \quad (41)$$

In the last formula, Γ is the Robin boundary, and the sign plus (or minus) corresponds to outside (or inside) of the domain Ω . Schema (40) is stable under the CFL condition

$$\Delta t \leq \frac{\Delta x}{\max(|V(x, y)|)\sqrt{2}} \quad (42)$$

with a space-step $\Delta x = \Delta y$.

7 Numerical experiments

For all numerical tests, presented below, we consider the rectangle $\overline{D} = [0, 3] \times [0, 1]$, and suppose that D always contains the domain Ω , on which we solve the Helmholtz equation. The boundaries Γ_N and Γ_D are fixed, as it is shown on Fig. 2, and Γ is the moving boundary inside of $\overline{G} = [\frac{3}{2}, 3] \times [0, 1]$. Here $G \cap \Omega$ is the $\frac{1}{2}$ -neighborhood of the flat Γ , fixed at $x = 2$. If Ω has the flat boundary Γ , then $\overline{\Omega} = [0, 2] \times [0, 1]$ with the characteristic lengths $\ell = 1$ and $L = 2\ell$.

The Helmholtz equation is considered with a wave number $k = \frac{\omega}{c_0}$, *i.e.*,

$$\Delta u + k^2 u = -f,$$

where c_0 is the sound speed in the air. We take $f = 0$, $g = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-1/2)^2}{2\sigma^2}\right)$ with $\sigma = 1$ in the Helmholtz boundary value problem. For the chosen σ , the smallest wavelength, excited by g , is $\lambda = \frac{\ell}{2}$. The parameter α in the Robin boundary condition depends on the value of the frequency ω . It is calculated for ISOREL, using a minimization of the difference between the solution of the problem with a volume dissipation (described by a damped wave equation) and the solution of the problem with

the boundary dissipation for the flat shape of Γ (see Appendix A and Fig. 1). We solve the Helmholtz boundary value problem on the fine mesh with the size $h = \frac{\ell}{64}$, and we perform the level set approach for the optimization algorithm on the coarse mesh of the size $\kappa = 2h = \frac{\ell}{32}$ (in the aim of a penalization of too much complicated shapes of Γ). However, we notice that $\kappa \ll \lambda$.

7.1 Illustrations for Assumption 1 and Theorem 6

Time depending energy decay. We consider the three shallow cavities

$$\Omega = \Omega_0 \sqcup \Omega_1 =]0, 1[\times] - 2, 2[,$$

shown on Fig. 5 with two homogeneous media, air (lower part) and a porous material (upper part), separated by an internal boundary Γ_i , $i = 0, 1, 2$. To preserve the volume of each medium and to model the increasing irregularity of the interface, as compared to the plane Γ_0 (at $y = 0$), we chose as Γ_1 and Γ_2 as the first two fractal generations of a symmetric element. The external boundary $\partial\Omega$ is supposed to be perfectly rigid (Neumann boundary condition). Air is considered as a loss-less medium, and the porous medium (ISOREL) is considered as a dissipative homogeneous medium. As it was mentioned in Section 2, using the ideas of Hamet [21], the wave propagation in Ω can be modeled by problem (1) (which is the wave equation in the air ($a = 0$), and is the damped wave equation [4, 13] ($a \neq 0$) in the porous medium), where we take an initial data as a Gaussian, centered in a fixed point $x_0 = (0.75, -1.5)$ of Ω_0 :

$$u|_{t=0} = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{|x-x_0|^2}{2\delta^2}}, \quad \partial_t u|_{t=0} = 0$$

with $\delta = 0.1$. Such choice of δ ensures that $\text{supp}(u|_{t=0}) \subset \Omega_0$. We discretize Eq. (1) in a way which mimics the energy dissipation (2), and which is an adaptation to damped acoustic waves of the finite volume method presented in Ref. [23]. Let u_i^n be the discretized pressure in the control volume i at time $n\Delta t$, then we write

$$\xi \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} + a \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - [\nabla \cdot (\mu \nabla u^n)]_i = 0,$$

so that the energy $E^{n+1/2} := \frac{1}{2} \left(\int_{\Omega} \xi \left(\frac{u^{n+1} - u^n}{\Delta t} \right)^2 dx + \int_{\Omega} \mu \nabla u^n \cdot \nabla u^{n+1} dx \right)$ is damped as

$$\frac{1}{\Delta t} (E^{n+1/2} - E^{n-1/2}) = - \int_{\Omega_1} a \left(\frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)^2 dx.$$

Fig. 5 shows that an irregular shape of the internal boundary can significantly increase the dissipation properties of the porous medium ($\Gamma_{1,2}$ as compared to Γ_0). The efficient energy damping by Γ_1 , compared to the damping performances of Γ_0 , are much better and we notice that the wavelength λ of the wave, created by the initial data, is compared (two times bigger) to the characteristic length scale size of the geometry Γ_1 . At the same time, the small difference in the energy decays, corresponding to the internal boundaries Γ_1 and Γ_2 , confirms the hypothesis of Assumption 1: the wave does not penetrate in the

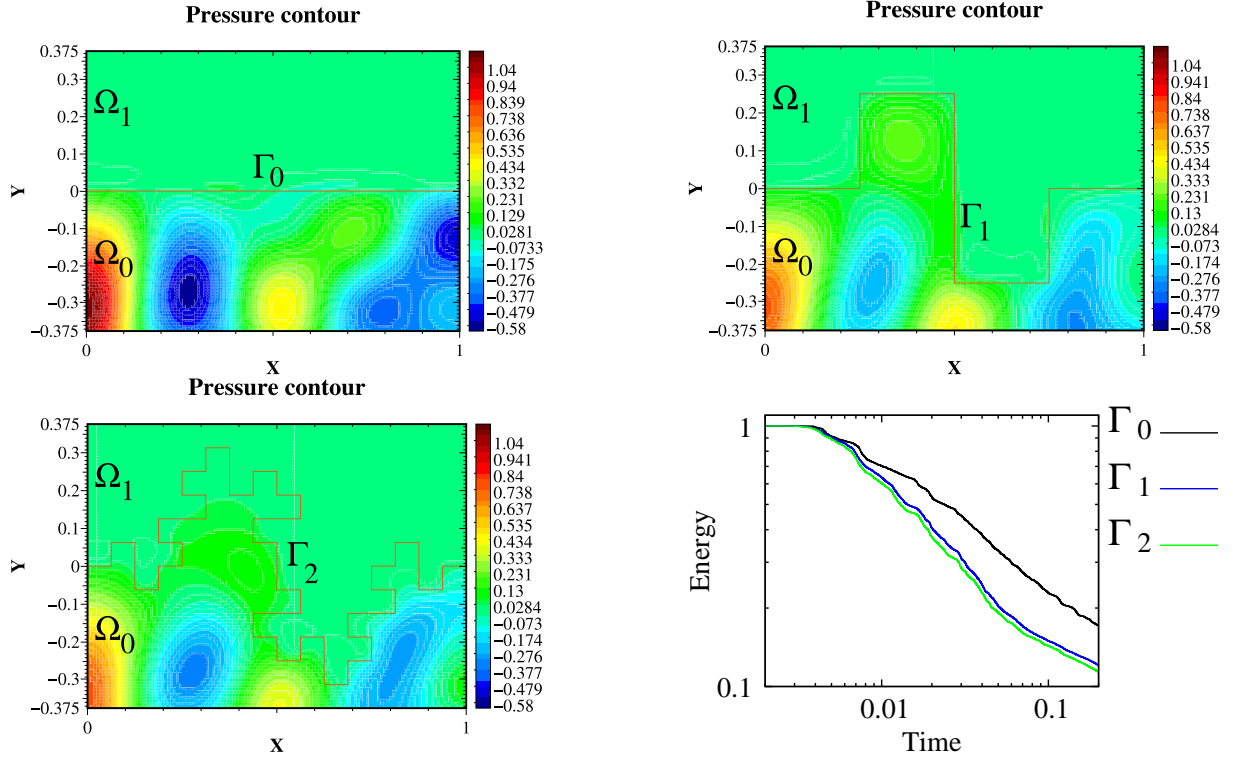


Figure 5: Pressure contours at $t = 0.01$ in cavities with an internal boundary of different Minkowski fractal generations (from left to right and top to bottom: Γ_0 (flat), Γ_1 and Γ_2) and the corresponding energy damping. The size of the mesh is 128×512 .

smallest geometry parts of the size $\lambda/8$, but the wave still keeps a good penetration for the scales of the order $\lambda/2$ as for Γ_1 . This finally implies that the shape of the internal boundary does not need to be “too complicated” for being an efficient acoustic absorbent for a fixed frequency.

Frequency optimization results. Taking $\bar{\Omega}_0 = [0, 2] \times [0, 1] \subset D$, we firstly calculate the values of J

$$J = \int_{\Omega} |u|^2 dx + \int_{\Omega} |\nabla u|^2 dx + \operatorname{Re}(\alpha) \int_{\Gamma} |u|^2 ds$$

for a range of frequencies for the flat shape Ω_0 , for instance for $\omega \in [2400, 4000]$. Let us fixe a frequency $\omega_0 = 3170$, corresponding to a local maximum of J . For this fixed ω_0 we have $\alpha = 23.7699 - 24.8367i$ (see Appendix A).

We perform two numerical tests, taking different initial domains Ω_0 in the shape optimization algorithm: the flat geometry of Γ (see Fig. 6) and a non-flat Γ (see Fig. 7) with the smallest characteristic geometric size $\ell_{\min}(\Omega_0)$ much smaller than the wavelength $\lambda = \frac{\ell}{2}$. The optimal shape on Fig. 6 has a mean value of shape’s scale length of the order of $\frac{\ell}{4}$, *i.e.* $\ell(\Omega_{16}) = \frac{\lambda}{2}$. The optimal shape $\hat{\Omega}_{10} = \Omega_{\text{opt}}^*$ on Fig. 7 keeps the largest characteristic geometrical size of the order of $\frac{\lambda}{2}$ ($\ell(\Omega_{\text{opt}}^*) = \frac{\lambda}{2}$) and for smaller scales Ω_{opt}^* is in a small neighborhood of $\Omega_{\text{opt}}^{\text{flat}}$.

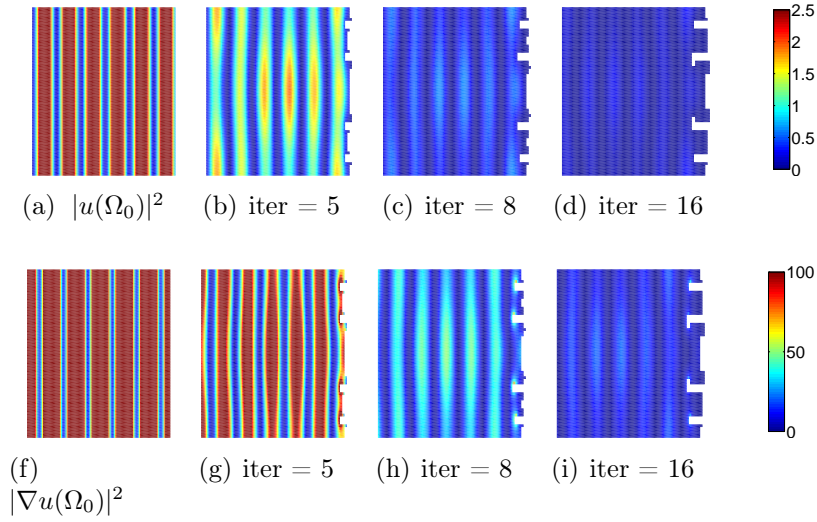


Figure 6: From the top to the bottom the values of $|u|^2$ and $|\nabla u|^2$ on the domains (from the left to right) Ω_0 (with the flat Γ), Ω_5 , Ω_8 and Ω_{16} respectively with the same scale of colors in each row (the red values are maximal and the blue values are minimal (near zero)). The domain Ω_0 is the initial shape and the domain Ω_{16} is the optimal shape for $\omega = 3170$.

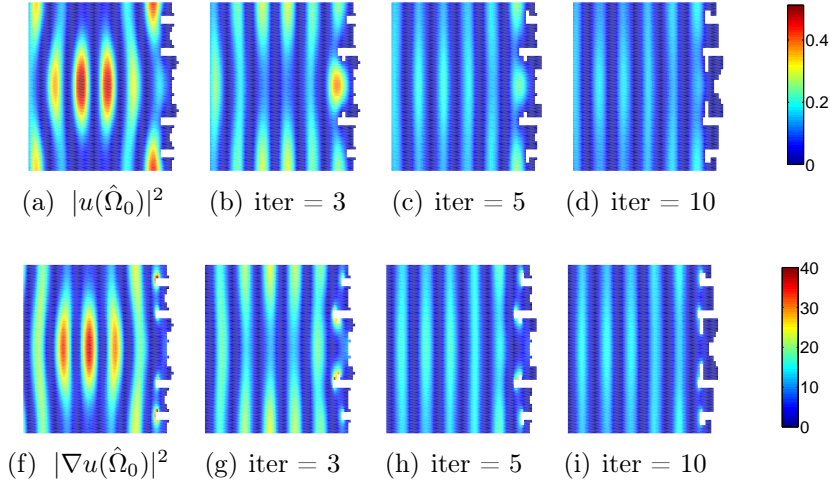


Figure 7: From the top to the bottom the values of $|u|^2$ and $|\nabla u|^2$ on the domains (from the left to right) $\hat{\Omega}_0$, $\hat{\Omega}_3$, $\hat{\Omega}_5$ and $\hat{\Omega}_{10}$ respectively with the same scale of colors in each row. The domain $\hat{\Omega}_0$ is the initial shape and the domain $\hat{\Omega}_{10}$ is the optimal shape for $\omega = 3170$.

Figure 8 confirms the concept of the ε -optimality of a domain form a small neighborhood of the optimal domain with the same characteristic geometric size, equal to $\lambda/2$ (see Corollary 1). In addition, Figure 8 shows the existence of the frequency interval $[\omega_1, \omega_2]$, including $\omega_0 = 3170$, for which the optimal shapes $\Omega_{\text{opt}}^{\text{flat}}$ and Ω_{opt}^* are ε -optimal by the

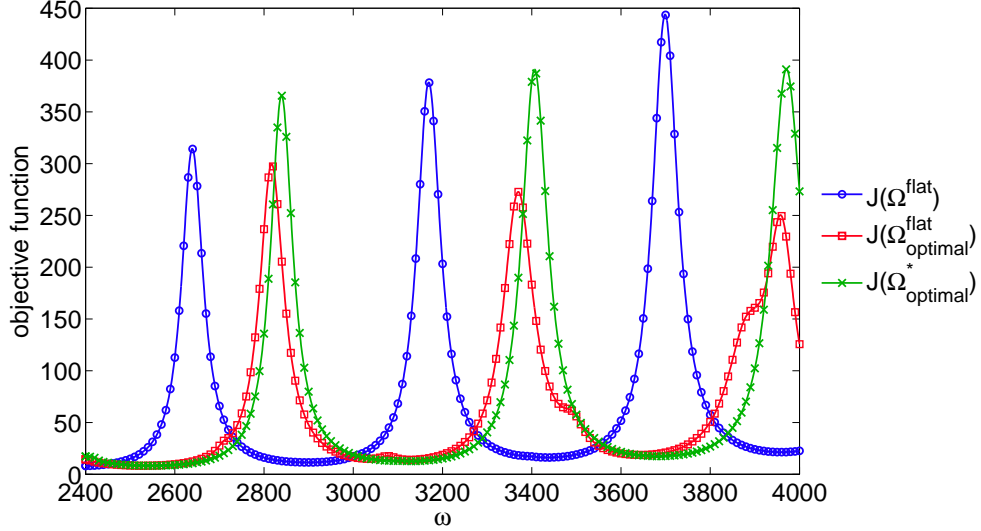


Figure 8: The objective function J as a function of ω for the flat shape $\Omega_0 = \Omega^{\text{flat}}$ given by the blue line, for the optimal shape $\Omega_{16} = \Omega_{\text{opt}}^{\text{flat}}$ (see Fig. 6) given by the red line, and for the optimal shape $\hat{\Omega}_{10} = \Omega_{\text{opt}}^*$ (see Fig. 7) given by the green line. The optimal domains $\Omega_{\text{opt}}^{\text{flat}}$ and Ω_{opt}^* are $J(\Omega_0)(\omega_0)/J(\Omega_{\text{opt}}^*)(\omega_0) = 27.5$ times better for the energy dissipation than the flat shape Ω_0 .

continuity of the functional J_1 on ω .

7.2 Properties of the optimization algorithm

In this Section we illustrate the stability properties of the optimization algorithm.

As in Subsection 7.1, we fix the frequency $\omega_0 = 3170$, which is a local maximum of $J(\Omega)(\omega) = \int_{\Omega} |u|^2 dx$, calculated for $\Omega_0 =]0, 2[\times]0, 1[$ in a range of frequencies, for instance, $\omega \in [3000, 6000]$. This time we chose $A = 1$ and $B = C = 0$ for the simulation of the acoustical energy.

If we start the optimization algorithm one time from $\Omega_0 = \Omega_0^a$ and the second time from $\Omega_0 = \Omega_0^b$, such that $d_H(\Omega_0^a, \Omega_0^b) < \varepsilon$ is small enough, then the optimal shapes Ω_{opt}^a and Ω_{opt}^b are “almost the same”, *i.e.* there exists $C > 0$, depending only on ε , such that the distance

$$d_H(\Omega_{\text{opt}}^a, \Omega_{\text{opt}}^b) < C(\varepsilon)d_H(\Omega_0^a, \Omega_0^b)$$

is also small enough. Hence, $|J(\Omega_{\text{opt}}^a)(\omega_0) - J(\Omega_{\text{opt}}^b)(\omega_0)| \ll 1$ is also small enough by the continuity of J as a function of the domain; see Fig. 9 for the numerical example.

Let us also notice, that, as for the question of Mark Kac “Can one hear the shape of a drum?”, we don’t have the uniqueness of the optimal shape Γ , since different shapes can have the same spectrum and be identically efficient in the dissipation of the energy in the fixed range of frequency. Fig. 10 illustrates the case, when the initial shape $\Omega_0 = \Omega_0^c$ is not in a small neighborhood of Ω_{opt}^a and $\ell(\Omega_0^c)$ is almost equal to $\ell(\Omega_{\text{opt}}^a)$. For this choice of Ω_0^c we obtain that Ω_{opt}^d is not in a small neighborhood of $\Omega_{\text{opt}}^{\text{flat}}$, but we still have $|J(\Omega_{\text{opt}}^c)(\omega_0) - J(\Omega_{\text{opt}}^a)(\omega_0)| \ll 1$. Moreover, Fig. 11 shows, that the values of the

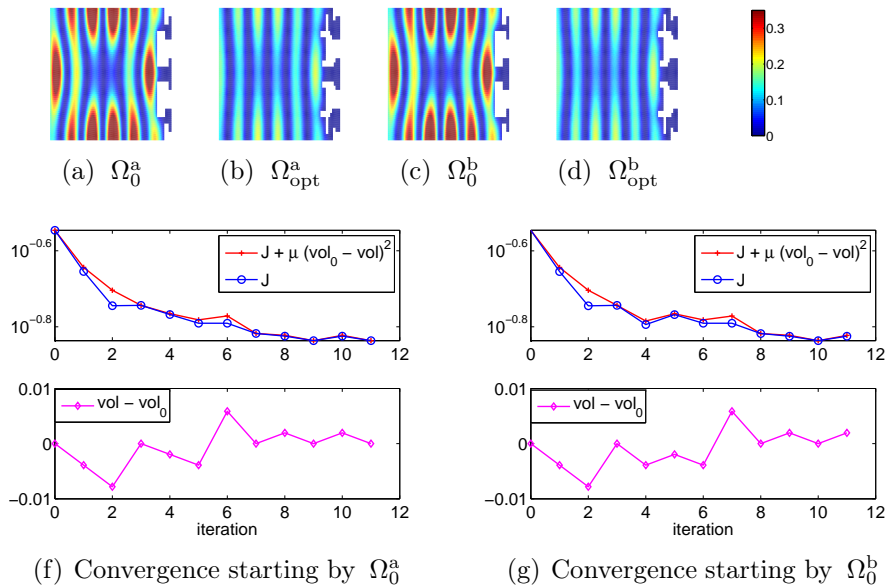


Figure 9: The values of $|u|^2$ are presented on two initial and optimal domains for the fixed frequency $\omega_0 = 3170$. From the left to the right: the initial domain Ω_0^a and the corresponding optimal domain $\Omega_{\text{opt}}^a = \Omega_{11}^a$, the initial domain Ω_0^b , taken in a small neighborhood of Ω_0^a , and the corresponding optimal domain $\Omega_{\text{opt}}^b = \Omega_{10}^b$. We see that Ω_{opt}^a is in a small neighborhood of Ω_{opt}^b (the shapes of Γ^a and Γ^b are almost the same). The values of J are also almost the same: $J(\Omega_{\text{opt}}^a)(\omega_0) \approx 0.1458$ and $J(\Omega_{\text{opt}}^b)(\omega_0) \approx 0.1458$. To compare to the flat shape $\bar{\Omega}_0 = [0, 2] \times [0, 1]$, for which $J(\Omega_0)(\omega_0) = 4.286$, $J(\Omega_0)(\omega_0)/J(\Omega_{\text{opt}}^a)(\omega_0) = 27.492$, hence the optimal shapes dissipate the energy 27.5 times better than the flat one. In the bottom, the convergence of the optimization algorithm for two cases of the initial domain: for Ω_0^a in the left and for Ω_0^b in the right.

functional $|J(\Omega_{\text{opt}}^c)(\omega) - J(\Omega_{\text{opt}}^a)(\omega)| \ll 1$ are almost the same for all ω in a rather large neighborhood of ω_0 .

7.3 Optimized “simple” wall for a large range of frequencies

In this Subsection, we are searching of an ε -optimal shape of the wall Ω , minimizing the acoustical energy

$$J(\Omega)(\omega) = \int_{\Omega} |u|^2 dx$$

in a large range of frequencies with a simplest possible design. Let us fix the range of frequencies for the energy dissipation: $\omega \in [3000, 6000]$.

As in Sections 7.1 and 7.2, we fix the frequency $\omega_0 = 3170$ of a local maximum of J on $\Omega_{\text{flat}} =]0, 2[\times]0, 1[$. We perform the shape optimization algorithm for this frequency, taking as the initial shape Ω_0 , given on Fig. 12, and we obtain Ω_1 , optimal at $\omega = 3170$. Noticing that all local maxima of $J(\Omega_1)$ are smaller than the local maxima of $J(\Omega_{\text{flat}})$

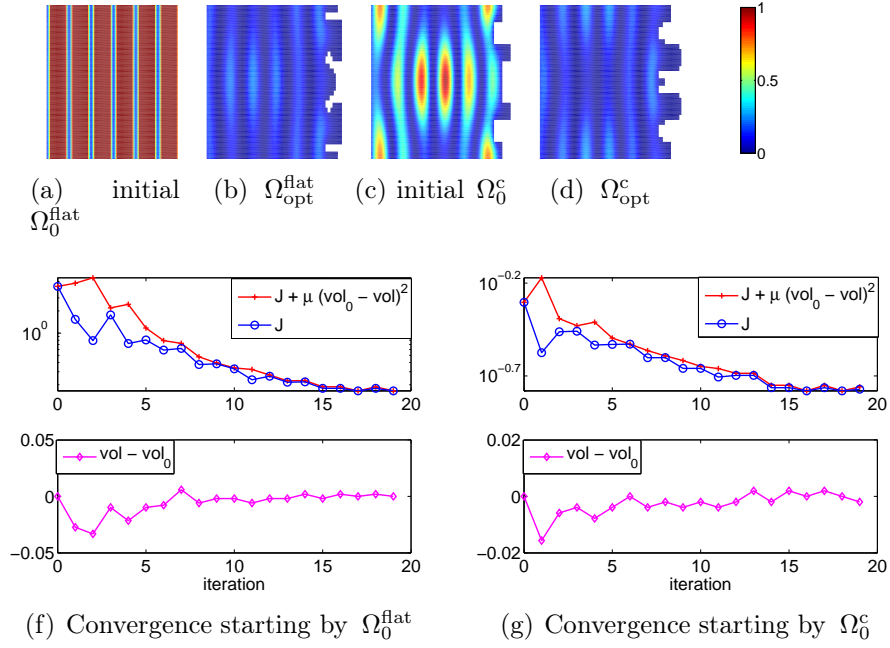


Figure 10: The values of $|u|^2$ are presented on two initial and optimal domains for the fixed frequency $\omega_0 = 3170$. From the left to the right: the initial domain Ω_0^{flat} and the corresponding optimal domain $\Omega_{\text{opt}}^{\text{flat}}$, the initial domain Ω_0^c , significantly different to Ω_0^{flat} and to $\Omega_{\text{opt}}^{\text{flat}}$, taken with $\ell(\Omega_0^c) \approx \ell(\Omega_{\text{opt}}^{\text{flat}})$, and the corresponding optimal domain Ω_{opt}^c . We see that $\Omega_{\text{opt}}^{\text{flat}}$ is not in a small neighborhood of Ω_{opt}^c (the shapes of Γ^a and Γ^b are really different). But the values of J for $\omega_0 = 3170$ are also almost the same: $J(\Omega_{\text{opt}}^{\text{flat}}) = 0.1654$ and $J(\Omega_{\text{opt}}^c) = 0.1659$.

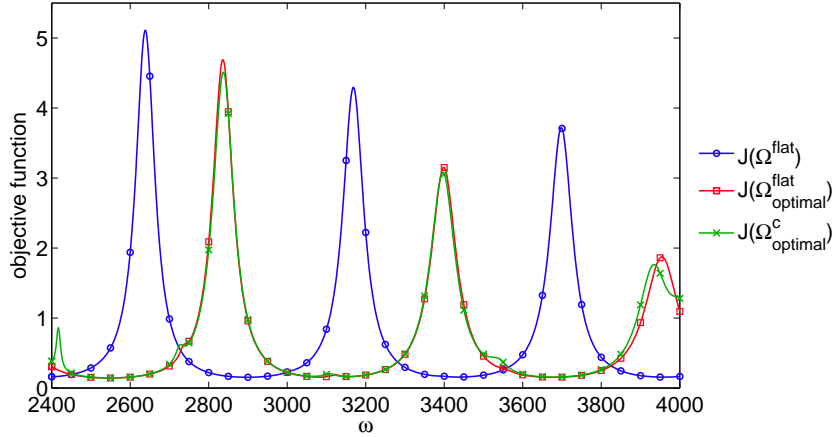


Figure 11: The objective function J as a function of ω for the flat shape Ω_0 is given by the blue line, for the optimal shape $\Omega_{\text{opt}}^{\text{flat}}$ (see Fig. 10) is given by the red line, and for the optimal shape Ω_{opt}^c (see Fig. 10) is given by the green line.

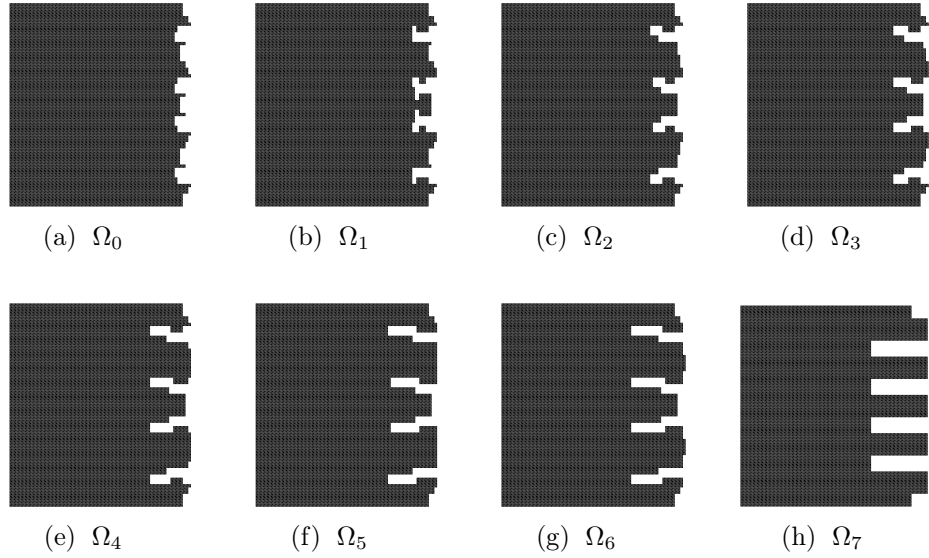


Figure 12: Shapes, which are used in the optimization algorithm process: from left to right in the top line- Ω_0 (the initial shape), Ω_k , $k = 1, 2, 3$, and from left to right in the bottom line - Ω_k , $k = 4, 5, 6, 7$. The domain Ω_7 is generated manually in the aim to simplify Ω_6 (the final ε -optimal shape).

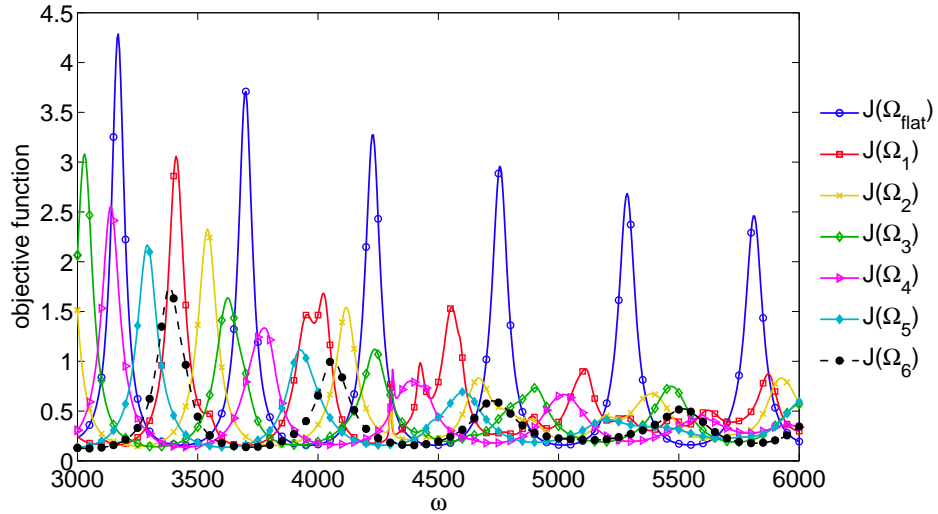


Figure 13: The values of the objective function ($A = 1, B = 0, C = 0$) on flat shape $J(\Omega_0)$ as a function on $\omega \in [3000, 6000]$ are presented by the blue line, the values of $J(\Omega_1)$ (see Fig.12 for the shape of Ω_1) are presented by the red line, of $J(\Omega_2)$ by the yellow line, of $J(\Omega_3)$ by the green line, of $J(\Omega_4)$ by the magenta line, of $J(\Omega_5)$ by the marine blue line, and of $J(\Omega_6)$ by the black dashed line.

(see Fig. 13), we choose Ω_1 as the initial domain and restart the optimization algorithm,

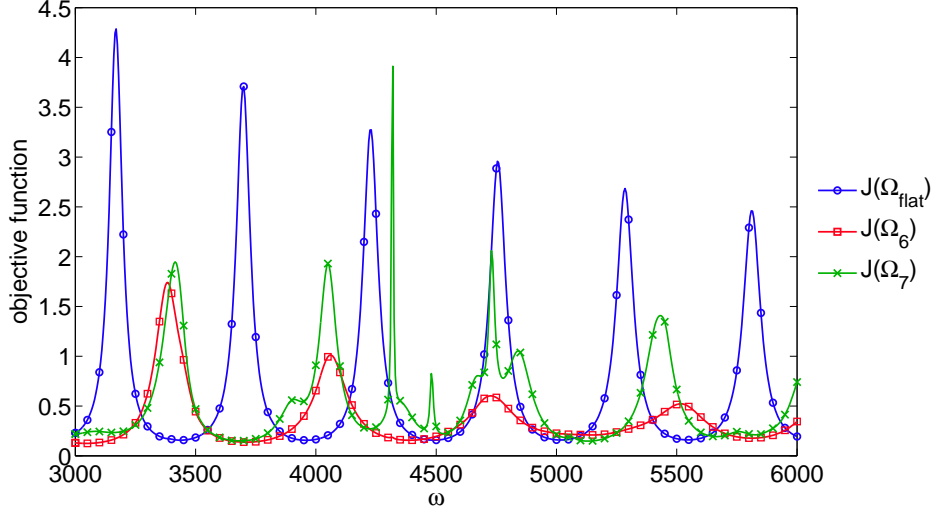


Figure 14: Comparison of the dissipative properties of the flat shape Ω_{flat} , the optimal Ω_6 and of its simplification Ω_7 . The values of $J(\Omega_{\text{flat}})$, of $J(\Omega_6)$ and of $J(\Omega_7)$ ($A = 1, B = 0, C = 0$) as functions of $\omega \in [3000, 6000]$ are given by the blue, red and green lines respectively.

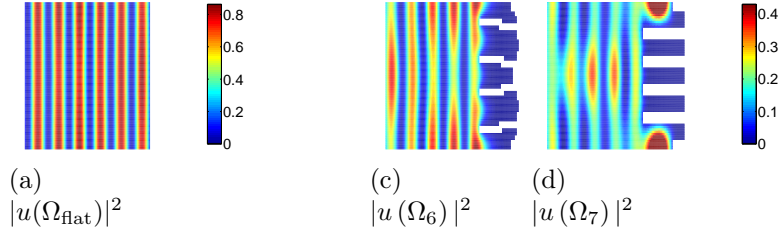


Figure 15: Energy distribution in Ω_{flat} , Ω_6 and Ω_7 respectively for $\omega = 3235$, corresponding to the case, when $J(\Omega_6) \approx J(\Omega_7)$ are almost the same (precisely $J(\Omega_6) = 0.2841$, $J(\Omega_7) = 0.2829$)

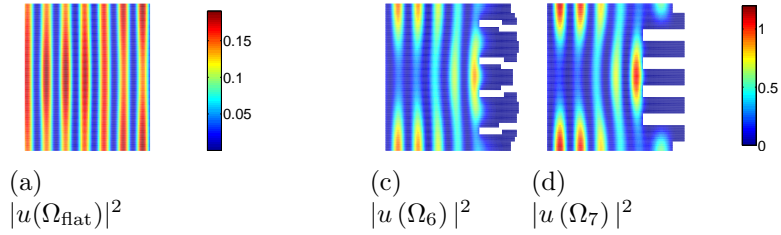


Figure 16: Energy distribution in Ω_{flat} , Ω_6 and Ω_7 respectively for $\omega = 3495$, corresponding to the case, when $J(\Omega_6) = 0.4767$ and $J(\Omega_7) = 0.5077$ take slight different values.

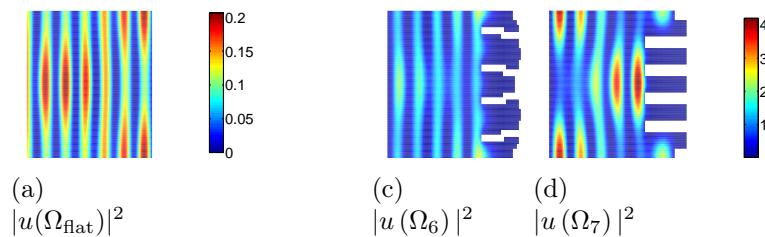


Figure 17: Energy distribution in Ω_{flat} , Ω_6 and Ω_7 respectively for $\omega = 3415$, the frequency, which yields a local maximum of the objective function on the domain Ω_7 .

minimizing in the neighborhood of Ω_1 the sum of functionals

$$\sum_{k=1}^3 J(\Omega)(\omega_k),$$

where $\omega_1 = 3410$, $\omega_2 = 4025$ and $\omega_3 = 4555$ are the local maxima of $J(\Omega_1)$. This minimization gives the optimal shape Ω_2 , such that

1. Ω_2 is ε -optimal in the neighborhood of ω_k for $k = 0, 1, 2, 3$;
2. all local maxima of $J(\Omega_2)$ are smaller than the local maxima of $J(\Omega_1)$.

Choosing $\omega_4 = 3625$ and $\omega_5 = 4240$, corresponding to the local maxima of $J(\Omega_2)$, we take Ω_2 as the initial domain and restart the optimization algorithm, minimizing

$$J(\Omega)(\omega_4) + J(\Omega)(\omega_5)$$

to obtain the optimal shape Ω_3 , such that

1. Ω_3 is ε -optimal in the neighborhood of ω_k for $k = 0, \dots, 5$;
2. all local maxima of $J(\Omega_3)$ are smaller than the local maxima of $J(\Omega_2)$.

We iterate this process up to Ω_6 and we are stopped by the restriction that Γ must be containing by the area $[\frac{3}{2}, 3] \times [0, 1]$.

The shape of Ω_6 contains multiscale geometry, which ensures the dissipative performances of the wall in a large range of frequencies (see Fig. 13). Thinking about the demolding process of wall's construction, we simplify the geometry of Ω_6 , deleting the multiscale and keeping only the biggest characteristic scale of Ω_6 (see the domain Ω_7 (generated by hand) on Fig. 12). As we can see from Fig. 14, since we have kept almost unchanged the biggest characteristic geometric size $\ell(\Omega_6) \approx \ell(\Omega_7)$, the energy dissipation is almost the same in the corresponding range of frequencies (see red and green lines for $[3000, 3700]$ on Fig. 14). As all smaller scale details have been deleted, the shape of Ω_7 is not so good as the shape of Ω_6 to dissipate higher frequencies (see red and green lines for $[3700, 6000]$ on Fig. 14). Hence, Fig. 14 shows that the compromises between two desired properties “to be the most dissipative” (as Ω_6 here) and “to be simple to construct” (on the example of Ω_7) is not too bad, especially if we know the most important frequencies to dissipate. Figures 15–17 show the energy distribution for three values of frequencies illustrating the three typical cases: $J(\Omega_6) \approx J(\Omega_7)$, $J(\Omega_6) < J(\Omega_7)$ and $J(\Omega_7)$ has its local maximum.

8 Conclusion

The well-posedness for the wave equation and the Helmholtz equation with a damping on the boundary was obtained in the class of n -sets, which generalizes the case of Lipschitz boundary to d -sets, which are fractal for $n - 1 < d < n$. In the framework of a noise barrier optimization, we have introduced the concept of the ε -optimal domains and have shown that for an efficient dissipation of the energy in a large band of frequencies, the ε -optimal domain must have a multiscale boundary geometry. More precisely, we have proved that an ε -optimal domain for all frequencies exists and, to be the most dissipative, it has a fractal boundary with a characteristic scale for a fractal generation $\lambda/2$. The shape derivation of a general energy functional was obtained for the complex-valued Helmholtz boundary value problem. With the purpose to find the most efficient and the simplest ε -optimal domain, easy to construct, we have developed a numerical algorithm, using the classical conjugate gradient method, combined with the level set method, which allows to find an efficient multiscale shape for a large range of frequency. We have illustrated the theoretical results by the numerical examples. In particular, we show that if, in addition, we simplify the obtained optimal shape, by deleting the smaller scales of the geometry, the new shape is efficient in the frequencies corresponding to its characteristic geometry scale length, but no more efficient in the higher frequencies.

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A Approximation of the damping parameter α in the Robin boundary condition by a model with dissipation in the volume

Theorem 9 *Let $\Omega =]-L, L[\times]-\ell, \ell[$ be a domain with a simply connected sub-domain Ω_0 , whose boundaries are $] -L, 0[\times \{\ell\}$, $\{-L\} \times]-\ell, \ell[$, $] -L, 0[\times \{-\ell\}$ and another boundary, denoted by Γ , which is the straight line starting in $(0, -\ell)$ and ending in $(0, \ell)$, denoting by Γ . In addition let Ω_1 be the supplementary domain of Ω_0 in Ω , so that Γ is the common boundary of Ω_0 and Ω_1 . The length L is supposed to be large enough.*

Then the following problem (the frequency version of the wave damped problem (1))

$$-\nabla \cdot (\mu_0 \nabla u_0) - \omega^2 \xi_0 u_0 = 0 \text{ in } \Omega_0, \quad (43)$$

$$-\nabla \cdot (\mu_1 \nabla u_1) - \omega^2 \tilde{\xi}_1 u_1 = 0 \text{ in } \Omega_1, \quad (44)$$

with

$$\tilde{\xi}_1 = \xi_1 \left(1 + \frac{ai}{\xi_1 \omega} \right),$$

together with boundary conditions on Γ

$$u_0 = u_1 \quad \text{and} \quad \mu_0 \nabla u_0 \cdot n = \mu_1 \nabla u_1 \cdot n, \quad (45)$$

and the condition on the left boundary

$$u_0(-L, y) = g(y), \quad (46)$$

and some other boundary conditions, can be replaced by the following model

$$-\nabla \cdot (\mu_0 \nabla u_2) - \omega^2 \xi_0 u_2 = 0 \quad \text{in } \Omega_0 \quad (47)$$

with boundary absorption condition on Γ

$$\mu_0 \nabla u_2 \cdot n + \alpha u_2 = 0 \quad (48)$$

and the condition on the left boundary

$$u_2(-L, y) = g(y) \quad (49)$$

with a complex parameter α , minimizing the following expression

$$A \|u_0 - u_2\|_{L^2(\Omega_0)}^2 + B \|\nabla(u_0 - u_2)\|_{L^2(\Omega_0)}^2.$$

More precisely, let u_0 , u_1 , u_2 and g be decomposed into Fourier modes in the y direction, denoting by k the associated wave number, and let

$$\begin{aligned} e_k(\alpha) &= A \|u_{0,k} - u_{2,k}\|_{L^2([-L,0])}^2 + B \|\nabla(u_{0,k} - u_{2,k})\|_{L^2([-L,0])}^2, \\ f(x) &= (\lambda_0 \mu_0 - x) \exp(-\lambda_0 L) + (\lambda_0 \mu_0 + x) \exp(\lambda_0 L), \\ \chi(k, \alpha) &= g_k \left(\frac{\lambda_0 \mu_0 - \lambda_1 \mu_1}{f(\lambda_1 \mu_1)} - \frac{\lambda_0 \mu_0 - \alpha}{f(\alpha)} \right), \\ \eta(k, \alpha) &= g_k \left(\frac{\lambda_0 \mu_0 + \lambda_1 \mu_1}{f(\lambda_1 \mu_1)} - \frac{\lambda_0 \mu_0 + \alpha}{f(\alpha)} \right), \end{aligned}$$

where

$$\begin{cases} \lambda_0 = \sqrt{k^2 - \frac{\xi_0}{\mu_0} \omega^2} & \text{if } k^2 \geq \frac{\xi_0}{\mu_0} \omega^2, \\ \lambda_0 = i \sqrt{\frac{\xi_0}{\mu_0} \omega^2 - k^2} & \text{if } k^2 \leq \frac{\xi_0}{\mu_0} \omega^2. \end{cases} \quad (50)$$

Then

$$\begin{aligned} e_k(\alpha) &= (A + B|k|^2) \left(\frac{1}{2\lambda_0} \{ |\chi|^2 [1 - \exp(-2\lambda_0 L)] \right. \\ &\quad \left. + |\eta|^2 [\exp(2\lambda_0 L) - 1] \} + 2L \operatorname{Re}(\chi \bar{\eta}) \right) \\ &\quad + B \frac{\lambda_0}{2} \{ |\chi|^2 [1 - \exp(-2\lambda_0 L)] + |\eta|^2 [\exp(2\lambda_0 L) - 1] \} - 2B \lambda_0^2 L \operatorname{Re}(\chi \bar{\eta}) \end{aligned}$$

if $k^2 \geq \frac{\xi_0}{\mu_0} \omega^2$ or

$$e_k(\alpha) = (A + B|k|^2) \left(L(|\chi|^2 + |\eta|^2) + \frac{i}{\lambda_0} \text{Im} \{ \chi \bar{\eta} [1 - \exp(-2\lambda_0 L)] \} \right) \\ + BL|\lambda_0|^2 (|\chi|^2 + |\eta|^2) + iB\lambda_0 \text{Im} \{ \chi \bar{\eta} [1 - \exp(-2\lambda_0 L)] \}$$

if $k^2 < \frac{\xi_0}{\mu_0} \omega^2$, and the parameter α can be found from the minimization of the error function

$$e(\alpha) := \sum_{k=\frac{n\pi}{L}, n \in \mathbb{Z}} e_k(\alpha).$$

Proof. We shall decompose u_0 , u_1 and u_2 into modes in the y direction, denoting by k the associated wave number.

The mode $u_{0,k}$ solves

$$\partial_{xx} u_{0,k} - \left(k^2 - \frac{\xi_0}{\mu_0} \omega^2 \right) u_{0,k} = 0,$$

and thus

$$u_{0,k}(x) = A_0 \exp(\lambda_0 x) + B_0 \exp(-\lambda_0 x), \quad (51)$$

where λ_0 is given in Eq. (50).

The mode $u_{1,k}$ solves

$$\partial_{xx} u_{1,k} - \left(k^2 - \frac{\tilde{\xi}_1}{\mu_1} \omega^2 \right) u_{1,k} = 0,$$

and thus

$$u_{1,k}(x) = A_1 \exp(\lambda_1 x) + B_1 \exp(-\lambda_1 x), \quad (52)$$

where

$$\lambda_1^2 = k^2 - \left(1 + \frac{ai}{\xi_1 \omega} \right) \frac{\xi_1}{\mu_1} \omega^2,$$

so that

$$\lambda_1 = \frac{1}{\sqrt{2}} \sqrt{k^2 - \frac{\xi_1}{\mu_1} \omega^2 + \sqrt{\left(k^2 - \frac{\xi_1}{\mu_1} \omega^2 \right)^2 + \left(\frac{a\omega}{\mu_1} \right)^2}} \\ - \frac{i}{\sqrt{2}} \sqrt{\frac{\xi_1}{\mu_1} \omega^2 - k^2 + \sqrt{\left(k^2 - \frac{\xi_1}{\mu_1} \omega^2 \right)^2 + \left(\frac{a\omega}{\mu_1} \right)^2}}.$$

For large L , since $Re(\lambda_1) > 0$, the value of A_1 tend to 0, so that we may neglect the first contribution in the right-hand side of (52). Consequently we consider the expression

$$u_{1,k}(x) = B_1 \exp(-\lambda_1 x). \quad (53)$$

Continuity conditions (45) and expressions (51) and (53) imply the following relations

$$A_0 + B_0 = B_1 \quad , \quad \mu_0 \lambda_0 (A_0 - B_0) = -\mu_1 \lambda_1 B_1,$$

from which we infer that

$$B_0 = \frac{\lambda_0 \mu_0 + \lambda_1 \mu_1}{\lambda_0 \mu_0 - \lambda_1 \mu_1} A_0,$$

and thus

$$u_{0,k}(x) = A_0 \left[\exp(\lambda_0 x) + \frac{\lambda_0 \mu_0 + \lambda_1 \mu_1}{\lambda_0 \mu_0 - \lambda_1 \mu_1} \exp(-\lambda_0 x) \right].$$

The decomposition of the boundary condition (46) into Fourier modes implies that $u_{0,k}(-L) = g_k$, which gives the final expression

$$u_{0,k}(x) = g_k \frac{[(\lambda_0 \mu_0 - \lambda_1 \mu_1) \exp(\lambda_0 x) + (\lambda_0 \mu_0 + \lambda_1 \mu_1) \exp(-\lambda_0 x)]}{[(\lambda_0 \mu_0 - \lambda_1 \mu_1) \exp(-\lambda_0 L) + (\lambda_0 \mu_0 + \lambda_1 \mu_1) \exp(\lambda_0 L)]}. \quad (54)$$

Let us now turn to the expression of $u_{2,k}$. Since the equation (47) is the same as that verified by $u_{0,k}$, both solutions have the same general form:

$$u_{2,k}(x) = A_2 \exp(\lambda_0 x) + B_2 \exp(-\lambda_0 x).$$

The Robin boundary condition (48) on Γ implies that

$$\mu_0 \lambda_0 (A_2 - B_2) + \alpha (A_2 + B_2) = 0,$$

which means that

$$u_{2,k}(x) = A_2 \left[\exp(\lambda_0 x) + \frac{\lambda_0 \mu_0 + \alpha}{\lambda_0 \mu_0 - \alpha} \exp(-\lambda_0 x) \right].$$

Application of the boundary condition (49) implies the final expression

$$u_{2,k}(x) = g_k \frac{[(\lambda_0 \mu_0 - \alpha) \exp(\lambda_0 x) + (\lambda_0 \mu_0 + \alpha) \exp(-\lambda_0 x)]}{[(\lambda_0 \mu_0 - \alpha) \exp(-\lambda_0 L) + (\lambda_0 \mu_0 + \alpha) \exp(\lambda_0 L)]}. \quad (55)$$

Using (54) and (55), we have that

$$(u_{0,k} - u_{2,k})(x) = \chi(k, \alpha) \exp(\lambda_0 x) + \eta(k, \alpha) \exp(-\lambda_0 x), \quad (56)$$

where the coefficients χ and η are computed from (54) and (55). In order to compute the L^2 norm of this expression, we must first compute the square of its modulus (by $\bar{\eta}$ is denoted the complex conjugate of η):

$$|u_{0,k} - u_{2,k}|^2(x) = |\chi|^2 |\exp(\lambda_0 x)|^2 + |\eta|^2 |\exp(-\lambda_0 x)|^2 + 2\text{Re} \left(\chi \bar{\eta} \exp(\lambda_0 x) \overline{\exp(-\lambda_0 x)} \right).$$

Note that, according to the values of k , the expression above may be simplified into

$$|u_{0,k} - u_{2,k}|^2(x) = |\chi|^2 \exp(2\lambda_0 x) + |\eta|^2 \exp(-2\lambda_0 x) + 2\text{Re}(\chi \bar{\eta}),$$

if $k^2 \geq \frac{\xi_0}{\mu_0} \omega^2$, or

$$|u_{0,k} - u_{2,k}|^2(x) = |\chi|^2 + |\eta|^2 + 2\operatorname{Re}(\chi\bar{\eta} \exp(2\lambda_0 x)),$$

if $k^2 < \frac{\xi_0}{\mu_0} \omega^2$. Thus, we have for $k^2 \geq \frac{\xi_0}{\mu_0} \omega^2$

$$\int_{-L}^0 |u_{0,k} - u_{2,k}|^2(x) dx = \frac{1}{2\lambda_0} \{ |\chi|^2 [1 - \exp(-2\lambda_0 L)] + |\eta|^2 [\exp(2\lambda_0 L) - 1] \} + 2L\operatorname{Re}(\chi\bar{\eta})$$

or, for $k^2 < \frac{\xi_0}{\mu_0} \omega^2$,

$$\int_{-L}^0 |u_{0,k} - u_{2,k}|^2(x) dx = L(|\chi|^2 + |\eta|^2) + \frac{i}{\lambda_0} \operatorname{Im} \{ \chi\bar{\eta} [1 - \exp(-2\lambda_0 L)] \}.$$

Now, we also have to compute the L^2 norm of the gradient of $(u_{0,k} - u_{2,k})$. Noting that

$$\nabla(u_{0,k} - u_{2,k}) = \begin{pmatrix} \partial_x(u_{0,k} - u_{2,k}) \\ ik(u_{0,k} - u_{2,k}) \end{pmatrix},$$

it holds that

$$|\nabla(u_{0,k} - u_{2,k})|^2 = |k|^2 |u_{0,k} - u_{2,k}|^2 + |\partial_x(u_{0,k} - u_{2,k})|^2.$$

With the expression (56), it follows that

$$|\partial_x(u_{0,k} - u_{2,k})|^2 = |\lambda_0|^2 [|\chi|^2 \exp(2\lambda_0 x) + |\eta|^2 \exp(-2\lambda_0 x) - 2\operatorname{Re}(\chi\bar{\eta})],$$

if $k^2 \geq \frac{\xi_0}{\mu_0} \omega^2$, or

$$|\partial_x(u_{0,k} - u_{2,k})|^2 = |\lambda_0|^2 [|\chi|^2 + |\eta|^2 - 2\operatorname{Re}(\chi\bar{\eta} \exp(2\lambda_0 x))],$$

if $k^2 < \frac{\xi_0}{\mu_0} \omega^2$, and thus

$$\int_{-L}^0 |\partial_x(u_{0,k} - u_{2,k})|^2(x) dx = \frac{\lambda_0}{2} \{ |\chi|^2 [1 - \exp(-2\lambda_0 L)] + |\eta|^2 [\exp(2\lambda_0 L) - 1] \} - 2\lambda_0^2 L \operatorname{Re}(\chi\bar{\eta}),$$

if $k^2 \geq \frac{\xi_0}{\mu_0} \omega^2$, or, if $k^2 < \frac{\xi_0}{\mu_0} \omega^2$,

$$\int_{-L}^0 |\partial_x(u_{0,k} - u_{2,k})|^2(x) dx = L|\lambda_0|^2 (|\chi|^2 + |\eta|^2) + i\lambda_0 \operatorname{Im} \{ \chi\bar{\eta} [1 - \exp(-2\lambda_0 L)] \}.$$

Therefore, we can find α as the solution of the mentioned minimization problem. \square

Since the minimization will be done numerically and since the sequence $(z, -z, z - z, \dots) = z(\exp(ij\Delta x)/\Delta x)$ is the highest frequency mode that can be reached on a grid of size Δx , then, in practice, the sum may be truncated to

$$e_{\Delta x}(\alpha) := \sum_{k=\frac{n\pi}{L}, n \in \mathbb{Z}, -\frac{L}{\Delta x} \leq n \leq \frac{L}{\Delta x}} e_k(\alpha).$$

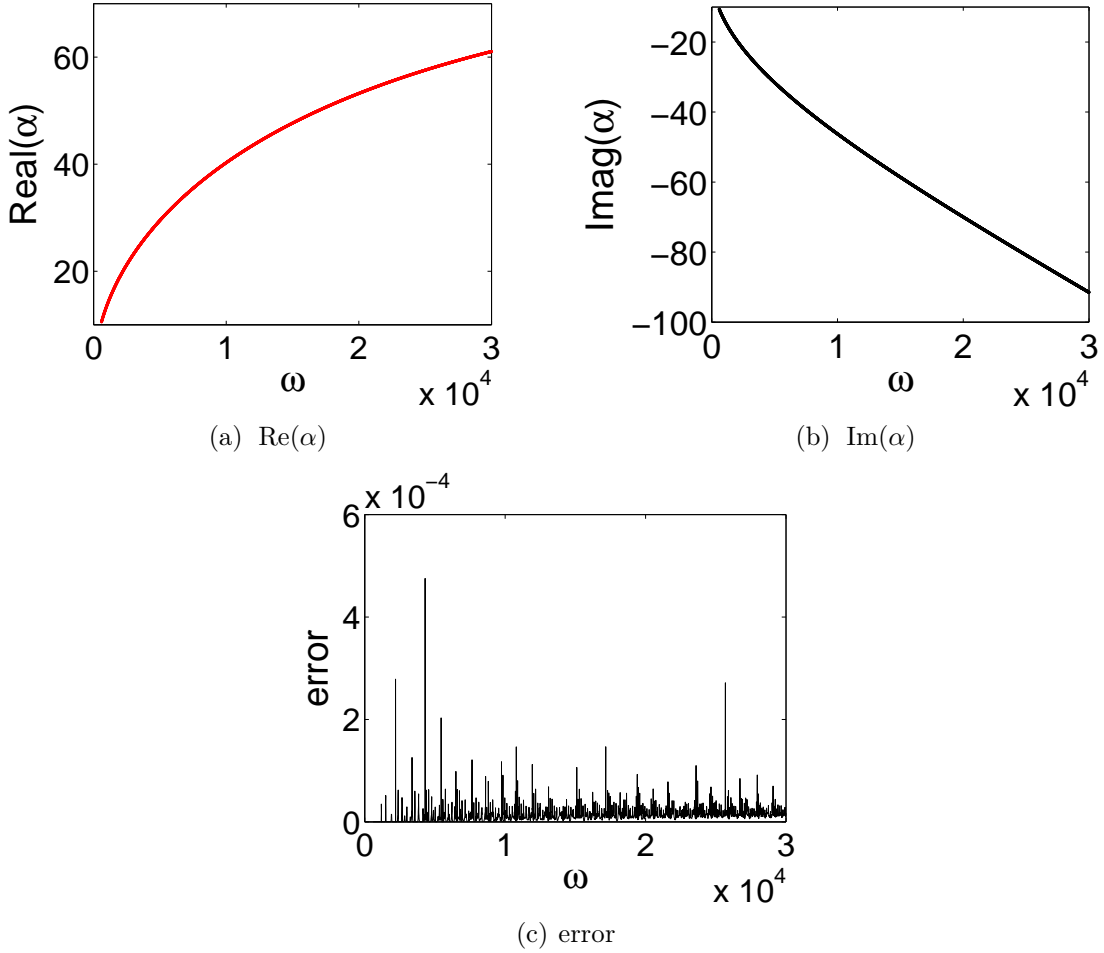


Figure 18: The real (top left) and imaginary (top right) parts of α and the sum of the errors $e_{\Delta x}$ (in the bottom) as function of frequencies $\omega \in [600, 30000]$ calculated for the ISOREL porous material.

For the equations (43)–(44), we use the same coefficients as for problem (1) and take the values corresponding to a porous medium, called ISOREL, using in the building isolation. More precisely we assume: $\phi = 0.7$, $\gamma_p = 1.4$, $\sigma = 142300 N \cdot m^{-4} \cdot s$, $\rho_0 = 1.2 kg/m^3$, $\alpha_h = 1.15$, $c_0 = 340 m \cdot s^{-1}$.

Using the function `fminsearch` (in Matlab), we find the value of α presented in Fig. A.

Remark 10 *Fig. A allows us to compare the difference between two considered time-dependent models for the damping in the volume and for the damping on the boundary. We see that $\text{Re}(\alpha)$ is not a constant in general, but for $\omega \rightarrow +\infty$ $\text{Im}(\alpha)$ is a linear function of ω . In this sense, the damping properties of two models are almost the same, but the reflection is more accurately considered by the damping wave equation in the volume.*

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