Harmonic measure for biased random walk in a supercritical Galton-Watson tree

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Abstract

We consider random walks λ-biased towards the root on a Galton–Watson tree, whose offspring distribution $(p_k)_{k\geq 1}$ is non-degenerate and has finite mean $m > 1$. In the transient regime $0 < \lambda < m$, the loop-erased trajectory of the biased random walk defines the λ-harmonic ray, whose law is the λ-harmonic measure on the boundary of the Galton–Watson tree. We answer a question of Lyons, Pemantle and Peres [8] by showing that the λ-harmonic measure has a.s. strictly larger Hausdorff dimension than that of the visibility measure. We also prove that the average number of children of the vertices visited by the λ-harmonic ray is a.s. bounded below by m and bounded above by $m^{-1}\sum k^2 p_k$. Moreover, the average number of children along the λ-harmonic ray is a.s. strictly larger than the average number of children along the λ-biased random walk trajectory. We observe that the latter is not monotone in the bias parameter λ.

Keywords. random walk, harmonic measure, Galton–Watson tree, stationary measure.

AMS 2010 Classification Numbers. 60J15, 60J80.

1 Introduction

Consider a Galton–Watson tree $\mathbb{T}$ rooted at e with a non-degenerate offspring distribution $(p_k)_{k\geq 0}$. We suppose that $p_0 = 0$, $p_k < 1$ for all $k \geq 1$, and the mean offspring number $m = \sum_{k\geq 1} kp_k \in (1, \infty)$. So the Galton–Watson tree $\mathbb{T}$ is supercritical and leafless. Let $\mathcal{T}$ be the space of all infinite rooted trees with no leaves. The law of $\mathbb{T}$ is called the Galton–Watson measure **GW** on $\mathcal{T}$. For every vertex x in $\mathbb{T}$, let $\nu(x)$ stand for its number of children. We denote by x_* the parent of x and by $xi, 1 \leq i \leq \nu(x)$, the children of x.

For $\lambda \geq 0$, conditionally on $\mathbb{T}$, the λ-biased random walk $(X_n)_{n\geq 0}$ on $\mathbb{T}$ is a Markov chain starting from the root e, such that, from the vertex e all transitions to its children are equally likely, whereas for every vertex $x \in \mathbb{T}$ different from e,

$$P_{\mathbb{T}}(X_{n+1} = x_* \mid X_n = x) = \frac{\lambda}{\nu(x) + \lambda},$$

$$P_{\mathbb{T}}(X_{n+1} = xi \mid X_n = x) = \frac{1}{\nu(x) + \lambda}, \quad \text{for every } 1 \leq i \leq \nu(x).$$

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Note that $\lambda = 1$ corresponds to the simple random walk on $\mathbb{T}$, and $\lambda = 0$ corresponds to the simple *forward* random walk with no backtracking. Lyons established in [5] that $(X_n)_{n\geq 0}$ is almost surely transient if and only if $\lambda < m$. Throughout this work, we assume $\lambda < m$ and hence the λ-biased random walk is always transient.

For a vertex $x \in \mathbb{T}$, let $|x|$ stand for the graph distance from the root e to x. Let $\partial\mathbb{T}$ denote the boundary of $\mathbb{T}$, which is defined as the set of infinite rays in $\mathbb{T}$ emanating from the root. Since $(X_n)_{n\geq 0}$ is transient, its loop-erased trajectory defines a unique infinite ray $\Xi_\lambda \in \partial\mathbb{T}$, whose distribution is called the λ-harmonic measure. We call Ξ_λ the λ-harmonic ray in $\mathbb{T}$.

For different rays $\xi, \eta \in \partial\mathbb{T}$, let $\xi \wedge \eta$ denote the vertex common to both ξ and η that is farthest from the root. We define the metric

$$d(\xi, \eta) := \exp(-|\xi \wedge \eta|) \text{ for } \xi, \eta \in \partial\mathbb{T}, \xi \neq \eta.$$

Under this metric, the boundary $\partial\mathbb{T}$ has a.s. Hausdorff dimension $\log m$. Lyons, Pemantle and Peres [6, 7] showed the dimension drop of harmonic measure: for all $0 \leq \lambda < m$, the Hausdorff dimension of the λ-harmonic measure is a.s. a constant $d_\lambda < \log m$. The 0-harmonic measure associated with the simple forward random walk was called *visibility measure* in [6]. Its Hausdorff dimension is a.s. equal to the constant $\sum_{k\geq 1}(\log k)p_k = \mathbf{GW}[\log \nu]$, where we write $\nu = \nu(e)$ for the offspring number under $\mathbf{GW}$.

Recently, Berestycki, Lubetzky, Peres and Sly [4] applied the dimension drop result $d_1 < \log m$ to show cutoff for mixing time of simple random walk on a random graph starting from a typical vertex. The Hausdorff dimension of the 0-harmonic measure was similarly used in [4] and independently used by Ben-Hamou and Salez in [3] to determine the mixing time of the non-backtracking random walk on a random graph.

The primary result of this work answers a question of Ledrappier posed in [8]. This question is also stated as Question 17.28 in Lyons and Peres's book [9].

Theorem 1. *For all $\lambda \in (0, m)$, we have $d_\lambda > \mathbf{GW}[\log \nu]$, meaning that the Hausdorff dimension of the λ-harmonic measure is a.s. strictly larger than the Hausdorff dimension of the 0-harmonic measure. Moreover,*

$$\lim_{\lambda\to 0^+} d_\lambda = \mathbf{GW}[\log \nu] \quad \text{and} \quad \lim_{\lambda\to m^-} d_\lambda = \log m.$$

When λ increases to the critical value m, it is non-trivial that the support of the λ-harmonic measure has its Hausdorff dimension tending to that of the whole boundary. Besides, Jensen's inequality implies $\mathbf{GW}[\log \nu] > -\log \mathbf{GW}[\nu^{-1}]$. The preceding theorem thus improves the lower bound $d_\lambda > -\log \mathbf{GW}[\nu^{-1}]$ shown by Virág in Corollary 7.2 of [10].

Our proof of Theorem 1 originates from the construction of a probability measure $\mu_{\mathsf{HARM}_\lambda}$ on $\mathcal{T}$ that is stationary and ergodic for the harmonic flow rule. In Section 4 below, its Radon–Nikodým derivative with respect to $\mathbf{GW}$ is given by (7). We derive afterwards an explicit expression for the dimension d_λ, and prove Theorem 1 in Section 5. The way to find this harmonic-stationary measure $\mu_{\mathsf{HARM}_\lambda}$ is inspired by a recent work of Aïdékon [1], in which he found explicitly the stationary measure of the environment seen from a λ-biased random walk. It renders possible an application of the ergodic theory on Galton–Watson trees developed in [6] to the biased random walk. After introducing the escape probability of λ-biased random walk on a tree in Section 2, we will give a precise description of Aïdékon's stationary measure in Section 3.

Apart from the Hausdorff dimension of harmonic measure, another quantity of interest is the average number of children of vertices visited by the harmonic ray Ξ_λ. For an infinite path

$\overrightarrow{x} = (x_0, x_1, \ldots)$ in $\mathbb{T}$, if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \nu(x_k)$$

exists, we call it the average number of children of the vertices along the path $\overrightarrow{x}$. Section 7 will be devoted to comparing the average number of children of vertices along different random paths in $\mathbb{T}$. The main results in this direction are summarized in the following way.

Theorem 2. *For all $\lambda \in (0, m)$,*

(i) the average number of children of the vertices visited by the λ-harmonic ray Ξ_λ in $\mathbb{T}$ is a.s. strictly larger than m, and strictly smaller than $m^{-1}\sum k^2 p_k$;

(ii) the average number of children of the vertices visited by the λ-harmonic ray Ξ_λ in $\mathbb{T}$ is a.s. strictly larger than the average number of children seen by the λ-biased random walk $(X_n)_{n\geq 0}$ in $\mathbb{T}$.

Assertion (i) above was suggested by some numerical calculations concerning the case $\lambda = 1$ mentioned at the end of Section 17.10 in [9]. By the strong law of large numbers, the average number of children seen by the simple forward random walk is a.s. equal to m. On the other hand, the uniform measure on the boundary of $\mathbb{T}$ can be obtained by putting mass 1 uniformly on the vertices of level n in $\mathbb{T}$ and taking the weak limit as $n \to \infty$. We say that a random ray in $\mathbb{T}$ is uniform if it is distributed according to the uniform measure on $\partial\mathbb{T}$. When $\sum(k \log k)p_k < \infty$, the uniform measure on $\partial\mathbb{T}$ has a.s. Hausdorff dimension $\log m$, and the uniform ray in $\mathbb{T}$ can be identified with the distinguished infinite path in a size-biased Galton–Watson tree. In particular, the average number of children seen by the uniform ray in $\mathbb{T}$ is equal to $m^{-1}\sum k^2 p_k$. For more details we refer the reader to Section 6 of [6] or Chapter 17 of [9].

Concerning assertion (ii) in Theorem 2, it is worth pointing out that the average number of children seen by the λ-biased random walk is *not* monotone with respect to λ: Proposition 9 in Section 7 will show that this number is strictly smaller than m when $\lambda \in (0, 1)$, equal to m when $\lambda = 0$ or 1, and strictly larger than m when $\lambda \in (1, m)$. Hence, its right continuity at 0 (established in Proposition 10) implies that the average number of children seen by the λ-biased random walk cannot be monotonic nondecreasing for all $\lambda \in (0, m)$. This lack of monotonicity might be explained by two opposing effects of having a small bias λ: on one hand, it helps the random walk to escape faster to infinity, and we know that a high-degree path is in favour of the escape of the λ-biased random walk, but on the other hand, small bias implies less backtracking, so the λ-biased random walk spends less time on high-degree vertices.

The FKG inequality for product measures (also known as the Harris inequality) will be extremely useful in proving Theorem 2. For assertion (ii), we need some extra knowledge on the speed of the λ-biased random walk, which will be presented in Section 6.

We close this introduction by mentioning that the following question from [8] remains open.

Question 1. *Is the dimension d_λ of the λ-harmonic measure nondecreasing for $\lambda \in (0, m)$?*

Taking into account the previous discussion, we find it intriguing to ask the same question for the average number of children along the λ-harmonic ray.

Question 2. *Is the average number of children of the vertices visited by the λ-harmonic ray in $\mathbb{T}$ nondecreasing for $\lambda \in (0, m)$?*

2 Escape probability and the effective conductance

For a tree $T \in \mathcal{T}$ rooted at e, we define T_* as the tree obtained by adding to e an extra adjacent vertex e_*, called the parent of e. The new tree T_* is naturally rooted at e_*. For a vertex $u \in T$, the descendant tree T_u of u is the subtree of T formed by those edges and vertices which become disconnected from the root of T when u is removed. By definition, T_u is rooted at u.

Unless otherwise stated, we assume $\lambda \in (0, m)$ in the rest of the paper. Under the probability measure P_T, let $(X_n)_{n\geq 0}$ denote a λ-biased random walk on T_*. For any vertex $u \in T$, define $\tau_u := \min\{n \geq 0 \colon X_n = u\}$ the hitting time of u, with the usual convention that $\min \emptyset = \infty$. Let

$$\beta_\lambda(T) := P_T(\tau_{e_*} = \infty \mid X_0 = e) = P_T(\forall n \geq 1, X_n \neq e_* \mid X_0 = e)$$

be the probability of never visiting the parent e_* of e when starting from e. For notational ease, we will make implicit the dependency in λ of the escape probability by writing $\beta(T) = \beta_\lambda(T)$. For $\mathbf{GW}$-a.e. T, $0 < \beta(T) < 1$. By coupling with a biased random walk on $\mathbb{N}$, we see that $\beta(T) > 1 - \lambda$. Moreover, Lemma 4.2 of [1] shows that

$$0 < \mathbf{GW}\left[\frac{1}{\lambda - 1 + \beta(T)}\right] < \infty. \tag{1}$$

For a vertex $u \in T$, $|u| = 1$, the probability that a λ-harmonic ray in T passes through u is

$$\frac{\beta(T_u)}{\sum_{|w|=1} \beta(T_w)}.$$

If the tree T is viewed as an electric network, and if the conductance of an edge linking vertices of level n and $n+1$ is λ^{-n}, then $\mathcal{C}_\lambda(T)$ denotes the effective conductance of T from its root to infinity. As for the escape probability, we will write $\mathcal{C}(T)$ for $\mathcal{C}_\lambda(T)$ to simplify the notation. Using the link between reversible Markov chains and electric networks, we know that

$$\beta(T) = \mathcal{C}(T_*) = \frac{\mathcal{C}(T)}{\lambda + \mathcal{C}(T)} \quad \text{and} \quad \mathcal{C}(T) = \frac{\lambda \beta(T)}{1 - \beta(T)}. \tag{2}$$

This relationship between $\beta(T)$ and $\mathcal{C}(T)$ will be used repeatedly. Since $\mathcal{C}(T) > \beta(T)$, the lower bound $\mathcal{C}(T) > 1 - \lambda$ also holds. Moreover, for $x \in \mathbb{R}$,

$$\frac{\lambda^{-1} x\, \mathcal{C}(T)}{(\lambda - 1 + \mathcal{C}(T))(1 + \lambda^{-1}x) + \lambda^{-1}x} = \frac{\beta(T)x}{\lambda - 1 + \beta(T) + x}.$$

Taking $x = \mathcal{C}(T')$ for another tree T' yields the following identity

$$\frac{\beta(T')\mathcal{C}(T)}{\lambda - 1 + \beta(T') + \mathcal{C}(T)} = \frac{\beta(T)\mathcal{C}(T')}{\lambda - 1 + \beta(T) + \mathcal{C}(T')}. \tag{3}$$

Using (2) we can also verify that

$$\begin{aligned}(\lambda - 1 + \beta(T) + \mathcal{C}(T'))(1 + \lambda^{-1}\mathcal{C}(T)) &= \lambda(1 + \lambda^{-1}\mathcal{C}(T))(1 + \lambda^{-1}\mathcal{C}(T')) - 1 \\ &= (\lambda - 1 + \beta(T') + \mathcal{C}(T))(1 + \lambda^{-1}\mathcal{C}(T')). \end{aligned} \tag{4}$$

The following integrability result will be used to prove the inequality $d_\lambda > \mathbf{GW}[\log \nu]$.

Lemma 3. *For $0 < \lambda < m$, we have $\mathbf{GW}\big[\log \beta(T)^{-1}\big] < \infty$.*

Proof. For a λ-biased random walk $(X_k)_{k\geq 0}$ on the tree T, let

$$\tau_n := \min\{k \geq 0 \colon |X_k| = n\}$$

be its hitting time of level n. For a vertex $u \in T$, we define

$$\beta_n(u) := P_T(\tau_n < \tau_{u_*} \mid X_0 = u)$$

the probability to hit level n before u_* when starting from u. If the vertex u is at level n, clearly $\beta_n(u) = 1$. We write $1, 2, \ldots, \nu(e)$ for the children of the root e. The Markov property of random walk implies that

$$\beta_n(e) = \frac{\sum_{i=1}^{\nu(e)} \beta_n(i)}{\lambda + \sum_{i=1}^{\nu(e)} \beta_n(i)}.$$

See e.g. [1, Section 4.1] for the derivation. Taking $x = \lambda(\sum_{i=1}^{\nu(e)} \beta_n(i))^{-1}$ and $x_0 = \lambda\nu(e)^{-1} \leq x$ in the inequality $\log(1+x) \leq \log x + \log(1 + x_0^{-1})$, we deduce that

$$\log \frac{1}{\beta_n(e)} = \log\left(1 + \frac{\lambda}{\sum_{i=1}^{\nu(e)} \beta_n(i)}\right) \leq \log(1 + \lambda^{-1}\nu(e)) + \log \lambda + \log \frac{1}{\sum_{i=1}^{\nu(e)} \beta_n(i)}.$$

Let $\epsilon < 1$ be some positive number. For the descendant tree T_i, we have $\beta(T_i) \leq \beta_n(i)$. Then,

$$\log \frac{1}{\sum_{i=1}^{\nu(e)} \beta_n(i)} \leq \left(\log \frac{1}{\beta_n(1)}\right) \mathbf{1}_{\{\beta_n(i) \leq \epsilon, \forall i \geq 2\}} + \log \epsilon^{-1} \leq \left(\log \frac{1}{\beta_n(1)}\right) \mathbf{1}_{\{\beta(T_i) \leq \epsilon, \forall i \geq 2\}} + \log \epsilon^{-1}.$$

By convention, the indicator functions involved above are equal to 1 on the event $\{\nu(e) = 1\}$. Taking expectation gives

$$\mathbf{GW}\left[\log \frac{1}{\beta_n(e)}\right] \leq \mathbf{GW}\left[\log(\lambda + \nu(e))\right] + \log \epsilon^{-1} + \mathbf{GW}\left[\log \frac{1}{\beta_{n-1}(e)}\right] \mathbf{GW}\left[q_\epsilon^{\nu(e)-1}\right],$$

where $q_\epsilon := \mathbf{GW}(\beta(T) \leq \epsilon)$. Notice that $q_\epsilon \to 0$ as $\epsilon \to 0$. So we can take ϵ small enough such that

$$A_\epsilon := \mathbf{GW}[q_\epsilon^{\nu(e)-1}] < 1.$$

Hence, writing $B_\epsilon := \mathbf{GW}[\log(\lambda + \nu(e))] + \log \epsilon^{-1} < \infty$, we obtain

$$\mathbf{GW}\left[\log \frac{1}{\beta_n(e)}\right] \leq \frac{B_\epsilon}{1 - A_\epsilon}.$$

Letting $n \to \infty$, we have $\beta_n(e) \to \beta(T)$. An application of Fatou's lemma finishes the proof. $\square$

3 Stationary measure of the tree seen from random walk

We set up some notation before presenting Aïdékon's stationary measure. For a rooted tree $T \in \mathcal{T}$, its boundary ∂T is the set of all rays starting from the root. Clearly, one can identify ∂T_* with ∂T. Let

$$\mathcal{T}^* := \{(T, \xi) \mid T \in \mathcal{T}, \xi = (\xi_n)_{n \geq 0} \in \partial T\}$$

denote the space of trees with a marked ray. By definition, ξ_0 coincides with the root vertex of T. If T_1 and T_2 are two trees rooted respectively at e_1 and e_2, we define $T_1 \multimap\!\!\bullet T_2$ as the tree rooted at the root e_2 of T_2 formed by joining the roots of T_1 and T_2 by an edge. The root e_2 is the parent of e_1 in $T_1 \multimap\!\!\bullet T_2$, thus we will not distinguish e_2 from $(e_1)_*$. Given a ray $\xi \in \partial T$, there is a unique tree T^+ such that $T = T_{\xi_1} \multimap\!\!\bullet T^+$. Therefore, $\mathcal{T}^*$ is in bijection with the space

$$\{(T \multimap\!\!\bullet T^+, \xi) \mid T, T^+ \in \mathcal{T}, \xi = (\xi_n)_{n \geq 0} \in \partial T\}.$$

Introducing a marked ray helps us to keep track of the past trajectory of the biased random walk. In particular, the initial starting point of the random walk, towards which the bias is exerted, would be represented by the marked ray at infinity. To be more precise, if we assign a vertex $u \in T$ to be the new root of the tree T, the re-rooted tree will be written as $\mathsf{ReRoot}(T, u)$. Given $\xi = (\xi_n)_{n \geq 0} \in \partial T$, we say that x is the ξ-parent of y in T if x becomes the parent of y in the tree $\mathsf{ReRoot}(T, \xi_n)$ for all sufficiently large n. A random walk on T is *λ-biased towards ξ* if the random walk always moves to its ξ-parent with probability λ times that of moving to one of the other neighbors.

We consider the Markov chain on $\mathcal{T}^*$ that, starting from some fixed tree T with a marked ray $\xi = (\xi_n)_{n \geq 0}$, is isomorphic to a random walk on T λ-biased towards ξ. Recall that $\nu(\xi_0)$ is the number of edges incident to the root. The transition probabilities $\mathbf{p}_{\mathsf{RW}_\lambda}$ of this Markov chain are defined as follows:

- If $T' = \mathsf{ReRoot}(T, x)$ and $\xi' = (x, \xi_0, \xi_1, \xi_2, \ldots)$ with a vertex x adjacent to ξ_0 being different from ξ_1,
$$\mathbf{p}_{\mathsf{RW}_\lambda}((T, \xi), (T', \xi')) = \frac{1}{\nu(\xi_0) - 1 + \lambda};$$
- If $T' = \mathsf{ReRoot}(T, \xi_1)$ and $\xi' = (\xi_1, \xi_2, \ldots)$,
$$\mathbf{p}_{\mathsf{RW}_\lambda}((T, \xi), (T', \xi')) = \frac{\lambda}{\nu(\xi_0) - 1 + \lambda};$$
- Otherwise, $\mathbf{p}_{\mathsf{RW}_\lambda}((T, \xi), (T', \xi')) = 0$.

Now we are ready to define the environment that is invariant under re-rooting along a λ-biased random walk. Let $\mathbb{T}$ and $\mathbb{T}^+$ be two independent Galton–Watson trees of offspring distribution $(p_k)_{k \geq 0}$. We write e for the root vertex of $\mathbb{T}$, and e^+ for the root vertex of $\mathbb{T}^+$. Let ν^+ denote the number of children of e^+ in $\mathbb{T}^+$. Similarly, let ν denote the number of children of e in $\mathbb{T}$. Note that the number of children of e^+ in $\mathbb{T} \multimap\!\!\bullet \mathbb{T}^+$ is $\nu^+ + 1$. Conditionally on $(\mathbb{T}, \mathbb{T}^+)$, let $\mathcal{R}$ be a random ray in $\mathbb{T}$ distributed according to the λ-harmonic measure on $\partial \mathbb{T}$. We assume that $(\mathbb{T} \multimap\!\!\bullet \mathbb{T}^+, \mathcal{R})$ is defined under the probability measure P.

Definition 1. *The λ-augmented Galton–Watson measure $\mathbf{AGW}_\lambda$ is defined as the probability measure on $\mathcal{T}^*$ that is absolutely continuous with respect to the law of $(\mathbb{T} \multimap\!\!\bullet \mathbb{T}^+, \mathcal{R})$ with density*

$$c_\lambda^{-1} \frac{(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}, \tag{5}$$

where

$$c_\lambda = E\left[\frac{(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\right]$$

is the normalizing constant.

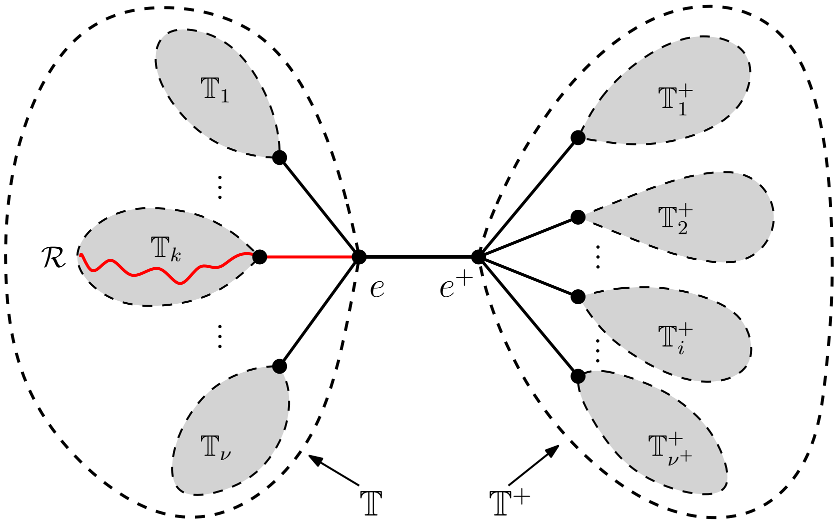

Figure 1: The random tree $\mathbb{T}\!\!-\!\!\bullet\mathbb{T}^+$ rooted at e^+ with a marked ray $\mathcal{R}$

It follows from the inequality $\lambda - 1 + \mathcal{C}(\mathbb{T}^+) > 0$ that

$$c_\lambda = E\Big[\frac{(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] < E[\lambda + \nu^+] = \lambda + m.$$

Let $\mathbb{T}_1^+, \dots, \mathbb{T}_{\nu^+}^+$ denote the descendant trees of the children of e^+ in $\mathbb{T}^+$. With a slight abuse of notation, let $\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_\nu$ denote the descendant trees of the children of e inside $\mathbb{T}$. See Fig. 1 for a schematic illustration. By the parallel law of conductances,

$$\mathcal{C}(\mathbb{T}^+) = \sum_{i=1}^{\nu^+} \beta(\mathbb{T}_i^+) \quad \text{and} \quad \mathcal{C}(\mathbb{T}) = \sum_{i=1}^{\nu} \beta(\mathbb{T}_i). \tag{6}$$

We will frequently use the branching property that conditionally on ν^+, the collection of trees $\{\mathbb{T}, \mathbb{T}_1^+, \dots, \mathbb{T}_{\nu^+}^+\}$ are independent and identically distributed according to $\mathbf{GW}$.

According to Theorem 4.1 in [1], the λ-augmented Galton–Watson measure $\mathbf{AGW}_\lambda$ is the asymptotic distribution of the environment seen from the λ-biased random walk on $\mathbb{T}$.

Proposition 4. *The Markov chain with transition probabilities $\mathsf{p}_{\mathsf{RW}_\lambda}$ and initial distribution $\mathbf{AGW}_\lambda$ is stationary.*

Proof. Let $F\colon \mathcal{T} \to \mathbb{R}^+$ and $G\colon \mathcal{T}^* \to \mathbb{R}^+$ be nonnegative measurable functions. Let $(\tilde{\mathbb{T}}\!\!-\!\!\bullet\tilde{\mathbb{T}}^+, \tilde{\mathcal{R}})$ denote the tree with a marked ray obtained from $(\mathbb{T}\!\!-\!\!\bullet\mathbb{T}^+, \mathcal{R})$ by performing a one-step transition according to $\mathbf{p}_{\mathsf{RW}_\lambda}$. It suffices to show that

$$E\Big[\frac{(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)} F(\tilde{\mathbb{T}}^+) G(\tilde{\mathbb{T}}, \tilde{\mathcal{R}})\Big] = E\Big[\frac{(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)} F(\mathbb{T}^+) G(\mathbb{T}, \mathcal{R})\Big].$$

To compute the left-hand side, we need to distinguish two different situations.

Case I: There exists $1 \le i \le \nu^+$ such that the root of $\mathbb{T}_i^+$ becomes the new root of $\tilde{\mathbb{T}}\!\!-\!\!\bullet\tilde{\mathbb{T}}^+$. For each $i \in [1, \nu^+]$, it happens with probability $1/(\nu^+ + \lambda)$. In this case,

$$\tilde{\mathbb{T}}^+ = \mathbb{T}_i^+ \quad \text{and} \quad \tilde{\mathbb{T}} = \mathbb{T}\!\!-\!\!\bullet\mathbb{T}_{\neq i}^+,$$

where $\mathbb{T}^+_{\neq i}$ stands for the tree rooted at e^+ containing only the descendant trees $\{\mathbb{T}^+_j, 1 \le j \le \nu^+, j \neq i\}$ together with the edges connecting their roots to e^+. It is easy to see that $\mathbb{T}^+_i$ and $\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq i}$ are two i.i.d. Galton–Watson trees. Meanwhile, $\tilde{\mathcal{R}} \in \partial\tilde{\mathbb{T}}$ is the ray $\mathcal{R}^+$ obtained by adding the vertex e^+ to the beginning of the sequence $\mathcal{R}$. We set accordingly

$$\begin{aligned}
I &:= E\Big[\frac{(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\sum_{i=1}^{\nu^+}\frac{1}{\nu^++\lambda}F(\mathbb{T}^+_i)G(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq i},\mathcal{R}^+)\Big]\\
&= E\Big[\frac{\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\sum_{i=1}^{\nu^+}F(\mathbb{T}^+_i)G(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq i},\mathcal{R}^+)\Big].
\end{aligned}$$

Given $\mathbb{T}$ and $\mathbb{T}^+$, we let $\mathcal{R}_{\neq i}$ be a random ray in the tree $\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq i}$ distributed according to the λ-harmonic measure on the tree boundary. Then $\mathcal{R}^+$ can be identified with $\mathcal{R}_{\neq i}$ conditionally on $\{\mathcal{R}_{\neq i}\in\partial\mathbb{T}\}$. We see that I is equal to

$$\begin{aligned}
&E\Big[\frac{\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\sum_{i=1}^{\nu^+}F(\mathbb{T}^+_i)G(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq i},\mathcal{R}_{\neq i})\mathbf{1}_{\{\mathcal{R}_{\neq i}\in\partial\mathbb{T}\}}\frac{\mathcal{C}(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq i})}{\beta(\mathbb{T})}\Big]\\
&= E\Big[\frac{1}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\sum_{i=1}^{\nu^+}F(\mathbb{T}^+_i)G(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq i},\mathcal{R}_{\neq i})\mathbf{1}_{\{\mathcal{R}_{\neq i}\in\partial\mathbb{T}\}}\mathcal{C}(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq i})\Big].
\end{aligned}$$

By symmetry, we deduce further that

$$\begin{aligned}
I &= E\Big[\frac{\mathcal{C}(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq 1})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\mathbb{T}^+_1)G(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq 1},\mathcal{R}_{\neq 1})\Big(\mathbf{1}_{\{\mathcal{R}_{\neq 1}\in\partial\mathbb{T}\}}+\sum_{i=2}^{\nu^+}\mathbf{1}_{\{\mathcal{R}_{\neq 1}\in\partial\mathbb{T}^+_i\}}\Big)\Big]\\
&= E\Big[\frac{\mathcal{C}(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq 1})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\mathbb{T}^+_1)G(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq 1},\mathcal{R}_{\neq 1})\Big].
\end{aligned}$$

As $\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+) = \beta(\mathbb{T})+\sum_{i=1}^{\nu^+}\beta(\mathbb{T}^+_i) = \beta(\mathbb{T}^+_1)+\mathcal{C}(\mathbb{T}\text{-}\!\bullet\mathbb{T}^+_{\neq 1})$, we obtain from the previous display that

$$I = E\Big[\frac{\mathcal{C}(\mathbb{T})}{\lambda-1+\beta(\mathbb{T}^+)+\mathcal{C}(\mathbb{T})}F(\mathbb{T}^+)G(\mathbb{T},\mathcal{R})\Big].$$

Using (3) and (2), we get therefore

$$\begin{aligned}
I &= E\Big[\frac{\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\frac{\mathcal{C}(\mathbb{T}^+)}{\beta(\mathbb{T}^+)}F(\mathbb{T}^+)G(\mathbb{T},\mathcal{R})\Big]\\
&= E\Big[\frac{\beta(\mathbb{T})(\lambda+\mathcal{C}(\mathbb{T}^+))}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\mathbb{T}^+)G(\mathbb{T},\mathcal{R})\Big].
\end{aligned}$$

Case II: The vertex e becomes the new root of $\tilde{\mathbb{T}}\text{-}\!\bullet\tilde{\mathbb{T}}^+$, which happens with probability $\lambda/(\nu^++\lambda)$. In this case, if $\mathcal{R}$ passes through the root of $\mathbb{T}_k$ for some integer $k\in[1,\nu]$, then

$$\tilde{\mathbb{T}} = \mathbb{T}_k \quad \text{and} \quad \tilde{\mathbb{T}}^+ = \mathbb{T}^+\text{-}\!\bullet\mathbb{T}_{\neq k},$$

where $\mathbb{T}_{\neq k}$ stands for the tree rooted at e formed by all descendant trees $\{\mathbb{T}_\ell, 1\le \ell\le\nu, \ell\neq k\}$ together with the edges connecting their roots to e. As in the previous case, $\mathbb{T}_k$ and $\mathbb{T}^+\text{-}\!\bullet\mathbb{T}_{\neq k}$

are two independent Galton–Watson trees. But $\tilde{\mathcal{R}}$ is now the ray $\mathcal{R}^-$ obtained by deleting e from the beginning of the sequence $\mathcal{R}$. We set thus

$$
\begin{aligned}
II &:= E\Big[\frac{(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\frac{\lambda}{\nu^++\lambda}\sum_{k=1}^{\nu}F(\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k})G(\mathbb{T}_k,\mathcal{R}^-)\mathbf{1}_{\{\mathcal{R}^-\in\partial\mathbb{T}_k\}}\Big]\\
&= E\Big[\frac{\lambda\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\sum_{k=1}^{\nu}F(\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k})G(\mathbb{T}_k,\mathcal{R}^-)\mathbf{1}_{\{\mathcal{R}^-\in\partial\mathbb{T}_k\}}\Big].
\end{aligned}
$$

Given $\mathbb{T}$ and $\mathbb{T}^+$, we let $\mathcal{R}_k$ be a random ray in the tree $\mathbb{T}_k$ distributed according to the λ-harmonic measure. It follows that

$$
\begin{aligned}
II &= E\Big[\sum_{k=1}^{\nu}\frac{\lambda\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k})G(\mathbb{T}_k,\mathcal{R}_k)\frac{\beta(\mathbb{T}_k)}{\mathcal{C}(\mathbb{T})}\Big]\\
&= E\Big[\sum_{k=1}^{\nu}\frac{\beta(\mathbb{T}_k)}{(\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+))(1+\lambda^{-1}\mathcal{C}(\mathbb{T}))}F(\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k})G(\mathbb{T}_k,\mathcal{R}_k)\Big].
\end{aligned}
$$

Using the identity (4), we see that

$$
\begin{aligned}
(\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+))(1+\lambda^{-1}\mathcal{C}(\mathbb{T})) &= (\lambda-1+\beta(\mathbb{T}^+)+\mathcal{C}(\mathbb{T}))(1+\lambda^{-1}\mathcal{C}(\mathbb{T}^+))\\
&= (\lambda-1+\beta(\mathbb{T}_k)+\mathcal{C}(\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k}))(1+\lambda^{-1}\mathcal{C}(\mathbb{T}^+)).
\end{aligned}
$$

Together with (2), it implies

$$
\begin{aligned}
II &= E\Big[\sum_{k=1}^{\nu}\frac{\beta(\mathbb{T}_k)(1+\lambda^{-1}\mathcal{C}(\mathbb{T}^+))^{-1}}{\lambda-1+\beta(\mathbb{T}_k)+\mathcal{C}(\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k})}F(\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k})G(\mathbb{T}_k,\mathcal{R}_k)\Big]\\
&= E\Big[\sum_{k=1}^{\nu}\frac{\beta(\mathbb{T}_k)(1-\beta(\mathbb{T}^+))}{\lambda-1+\beta(\mathbb{T}_k)+\mathcal{C}(\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k})}F(\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k})G(\mathbb{T}_k,\mathcal{R}_k)\Big].
\end{aligned}
$$

Observe that the root of $\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k}$ has ν children. For any integer $m\geq k$, the conditional law of $(\mathbb{T}_k,\mathbb{T}^{+}\!-\!\!\bullet\mathbb{T}_{\neq k})$ given $\{\nu=m\}$ is the same as that of $(\mathbb{T},\mathbb{T}^+)$ conditionally on $\{\nu^+=m\}$. Hence, we obtain

$$
\begin{aligned}
II &= E\Big[\sum_{k=1}^{\nu^+}\frac{\beta(\mathbb{T})(1-\beta(\mathbb{T}^+_k))}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\mathbb{T}^+)G(\mathbb{T},\mathcal{R})\Big]\\
&= E\Big[\frac{\beta(\mathbb{T})(\nu^+-\mathcal{C}(\mathbb{T}^+))}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\mathbb{T}^+)G(\mathbb{T},\mathcal{R})\Big].
\end{aligned}
$$

Finally, adding up Cases I and II, we have

$$
\begin{aligned}
&E\Big[\frac{(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\tilde{\mathbb{T}}^+)G(\tilde{\mathbb{T}},\tilde{\mathcal{R}})\Big]\\
&= E\Big[\frac{\beta(\mathbb{T})(\lambda+\mathcal{C}(\mathbb{T}^+))}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\mathbb{T}^+)G(\mathbb{T},\mathcal{R})\Big]+E\Big[\frac{\beta(\mathbb{T})(\nu^+-\mathcal{C}(\mathbb{T}^+))}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\mathbb{T}^+)G(\mathbb{T},\mathcal{R})\Big]\\
&= E\Big[\frac{(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}F(\mathbb{T}^+)G(\mathbb{T},\mathcal{R})\Big],
\end{aligned}
$$

which completes the proof of the stationarity. $\square$

We write $\vec{x}$ for an infinite path $(x_n)_{n\geq 0}$ in T. Let $\mathsf{RW}_\lambda \times \mathbf{AGW}_\lambda$ be the probability measure on the space

$$\{(\vec{x}, (T,\xi)) \mid (T,\xi) \in \mathcal{T}^*, \vec{x} \subset T\}$$

that is associated to the Markov chain considered in Proposition 4. It is given by choosing a tree T with a marked ray ξ according to $\mathbf{AGW}_\lambda$, and then independently running on T a random walk λ-biased towards ξ.

4 Harmonic-stationary measure

Let HARM_λ^T be the flow on the vertices of T in correspondence with the λ-harmonic measure on ∂T, so that $\mathrm{HARM}_\lambda^T(u)$ coincides with the mass given by the λ-harmonic measure to the set of all rays passing through the vertex u. We denote by HARM_λ the transition probabilities for a Markov chain on $\mathcal{T}$, that goes from a tree T to the descendant tree T_u, $|u| = 1$, with probability

$$\mathrm{HARM}_\lambda^T(u) = \frac{\beta(T_u)}{\sum_{|w|=1} \beta(T_w)} = \frac{\beta(T_u)}{\mathcal{C}(T)}.$$

The existence of a HARM_λ-stationary probability measure $\mu_{\mathsf{HARM}_\lambda}$ that is absolutely continuous with respect to $\mathbf{GW}$ was established in Lemma 5.2 of [7]. Taking into account the stationary measure of the environment $\mathbf{AGW}_\lambda$, we can construct $\mu_{\mathsf{HARM}_\lambda}$ as an induced measure by considering the λ-biased random walk at the exit epochs. See [6, Section 8] and [7, Section 5] for more details.

According to Proposition 5.2 of [6], $\mu_{\mathsf{HARM}_\lambda}$ is equivalent to $\mathbf{GW}$ and the associated HARM_λ-Markov chain is ergodic. Ergodicity implies further that $\mu_{\mathsf{HARM}_\lambda}$ is the unique HARM_λ-stationary probability measure absolutely continuous with respect to $\mathbf{GW}$. Due to uniqueness, we can identify $\mu_{\mathsf{HARM}_\lambda}$ via the next result.

Lemma 5. *For every $x > 0$, set*

$$\kappa_\lambda(x) := \mathbf{GW}\Big[\frac{\beta(T)x}{\lambda - 1 + \beta(T) + x}\Big] = E\Big[\frac{\beta(\mathbb{T})x}{\lambda - 1 + \beta(\mathbb{T}) + x}\Big].$$

The finite measure $\kappa_\lambda(\mathcal{C}(T))\mathbf{GW}(\mathrm{d}T)$ is HARM_λ-stationary.

Proof. The function $\kappa_\lambda \colon \mathbb{R}^+ \to \mathbb{R}^+$ is bounded and strictly increasing. In fact, for $\mathbf{GW}$-a.e. T, $\lambda - 1 + \beta(T) > 0$. The function

$$\frac{\beta(T)x}{\lambda - 1 + \beta(T) + x}$$

is strictly increasing in x, and it is bounded above by $\beta(T)$. Thus, $\kappa_\lambda(x) < \mathbf{GW}[\beta(T)] < 1$.

We write ν for the offspring number of the root of T. Conditionally on the event $\{\nu = k\}$, let $T_1, \dots, T_k$ denote the descendant trees of the children of the root. In order to prove the HARM_λ-stationarity, we must verify that for any bounded measurable function F on $\mathcal{T}$, the integral $\int F(T)\kappa_\lambda(\mathcal{C}(T))\mathbf{GW}(\mathrm{d}T)$ is equal to

$$\begin{aligned} I &:= \sum_{k=1}^{\infty} p_k \sum_{i=1}^{k} \int F(T_i)\kappa_\lambda(\mathcal{C}(T)) \frac{\beta(T_i)}{\beta(T_1) + \cdots + \beta(T_k)} \mathbf{GW}(\mathrm{d}T \mid \nu = k) \\ &= \sum_{k=1}^{\infty} k p_k \int F(T_1)\kappa_\lambda(\mathcal{C}(T)) \frac{\beta(T_1)}{\beta(T_1) + \cdots + \beta(T_k)} \mathbf{GW}(\mathrm{d}T \mid \nu = k). \end{aligned}$$

Using the definition of κ_λ and the branching property, we see that I is given by

$$\begin{aligned}
&\sum_{k=1}^{\infty} kp_k \int F(T_1)\frac{\beta(T_0)\beta(T_1)}{\lambda-1+\beta(T_0)+\beta(T_1)+\cdots+\beta(T_k)}\mathbf{GW}(\mathrm{d}T\mid \nu=k)\mathbf{GW}(\mathrm{d}T_0)\\
&=\sum_{k=1}^{\infty} kp_k \int F(T_1)\frac{\beta(T_0)\beta(T_1)}{\lambda-1+\beta(T_0)+\beta(T_1)+\cdots+\beta(T_k)}\mathbf{GW}(\mathrm{d}T_0)\mathbf{GW}(\mathrm{d}T_1)\cdots\mathbf{GW}(\mathrm{d}T_k)\\
&=\sum_{k=1}^{\infty} p_k \int F(T_1)\frac{\beta(T_1)(\beta(T_0)+\beta(T_2)+\cdots+\beta(T_k))}{\lambda-1+\beta(T_0)+\beta(T_1)+\cdots+\beta(T_k)}\mathbf{GW}(\mathrm{d}T_0)\mathbf{GW}(\mathrm{d}T_1)\cdots\mathbf{GW}(\mathrm{d}T_k)\\
&=\int F(T_1)\frac{\beta(T_1)\mathcal{C}(T)}{\lambda-1+\beta(T_1)+\mathcal{C}(T)}\mathbf{GW}(\mathrm{d}T)\mathbf{GW}(\mathrm{d}T_1).
\end{aligned}$$

Hence, it follows from (3) that

$$I=\int F(T_1)\frac{\beta(T)\mathcal{C}(T_1)}{\lambda-1+\beta(T)+\mathcal{C}(T_1)}\mathbf{GW}(\mathrm{d}T)\mathbf{GW}(\mathrm{d}T_1)=\int F(T_1)\kappa_\lambda(\mathcal{C}(T_1))\mathbf{GW}(\mathrm{d}T_1),$$

which finishes the proof. □

We deduce from the preceding lemma that the Radon–Nikodým derivative of $\mu_{\mathsf{HARM}_\lambda}$ with respect to $\mathbf{GW}$ is a.s.

$$\frac{\mathrm{d}\mu_{\mathsf{HARM}_\lambda}}{\mathrm{d}\mathbf{GW}}(T)=\frac{1}{h_\lambda}\kappa_\lambda(\mathcal{C}(T))=\frac{1}{h_\lambda}\int\frac{\beta(T')\mathcal{C}(T)}{\lambda-1+\beta(T')+\mathcal{C}(T)}\mathbf{GW}(\mathrm{d}T'), \tag{7}$$

where the normalizing constant

$$h_\lambda=\int\frac{\beta(T')\mathcal{C}(T)}{\lambda-1+\beta(T')+\mathcal{C}(T)}\mathbf{GW}(\mathrm{d}T)\mathbf{GW}(\mathrm{d}T')=E\Big[\frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big].$$

Writing $\mathcal{R}(T)=\mathcal{C}(T)^{-1}$ for the effective resistance, one can reformulate (7) as

$$\frac{\mathrm{d}\mu_{\mathsf{HARM}_\lambda}}{\mathrm{d}\mathbf{GW}}(T)=\frac{1}{h_\lambda}\int\frac{\lambda^{-1}}{(\lambda-1)\mathcal{R}(T)\mathcal{R}(T')+\mathcal{R}(T)+\mathcal{R}(T')+\lambda^{-1}}\mathbf{GW}(\mathrm{d}T').$$

When $\lambda=1$, it coincides with the expression of the same density in Section 8 of [6].

Using the proof of Lemma 5, we see that the mesure $\mu_{\mathsf{HARM}_\lambda}$ defined by (7) is still HARM_λ-stationary when $p_0>0$ is allowed. We also point out that the proof of Proposition 17.31 in [9] (corresponding to the case $\lambda=1$) can be adapted to derive (7) from the construction of $\mu_{\mathsf{HARM}_\lambda}$ by inducing.

To finish this section, let us mention a work in progress of Rousselin containing some general condition for a Markov chain on trees to have a stationary measure. His result also applies to the HARM_λ-Markov chain considered above.

5 Dimension of the harmonic measure

Let T be a random tree distributed as $\mu_{\mathsf{HARM}_\lambda}$, and let Θ be the λ-harmonic ray in T. If we denote the vertices along Θ by $\Theta_0,\Theta_1,\ldots$, then according to the flow property of harmonic measure, the sequence of descendant trees $(\mathsf{T}_{\Theta_n})_{n\geq 0}$ is a stationary HARM_λ-Markov chain. In what follows, we write $\mathsf{HARM}_\lambda \times \mu_{\mathsf{HARM}_\lambda}$ for the law of (Θ, T) on the space $\{(\xi, T) \mid T \in \mathcal{T}, \xi \in \partial T\}$. Recall that the ergodicity of $\mathsf{HARM}_\lambda \times \mu_{\mathsf{HARM}_\lambda}$ results from Proposition 5.2 in [6].

As shown in [6, Section 5], the Hausdorff dimension d_λ of the λ-harmonic measure coincides with the entropy

$$\mathrm{Entropy}_{\mathsf{HARM}_\lambda}(\mu_{\mathsf{HARM}_\lambda}) := \int \log \frac{1}{\mathrm{HARM}_\lambda^T(\xi_1)} \mathsf{HARM}_\lambda \times \mu_{\mathsf{HARM}_\lambda}(\mathrm{d}\xi, \mathrm{d}T).$$

Thus, by (2) we have

$$\begin{aligned} d_\lambda &= \int \log \frac{\mathcal{C}(T)}{\beta(T_{\xi_1})} \mathsf{HARM}_\lambda \times \mu_{\mathsf{HARM}_\lambda}(\mathrm{d}\xi, \mathrm{d}T) \\ &= \int \log \frac{\lambda \beta(T)}{\beta(T_{\xi_1})(1-\beta(T))} \mathsf{HARM}_\lambda \times \mu_{\mathsf{HARM}_\lambda}(\mathrm{d}\xi, \mathrm{d}T). \end{aligned}$$

By stationarity,

$$d_\lambda = \int \log \frac{\lambda}{1-\beta(T)} \mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) = \int \log\big(\mathcal{C}(T) + \lambda\big) \mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T), \tag{8}$$

provided the integral $\int \log \beta(T)^{-1} \mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T)$ is finite. Using the explicit form (7) of $\mu_{\mathsf{HARM}_\lambda}$, we see that this integral is equal to

$$h_\lambda^{-1} E\Big[\frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)} \log \frac{1}{\beta(\mathbb{T}^+)} \Big],$$

in which the expectation is less than

$$E\Big[\frac{\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T})} \Big(\mathcal{C}(\mathbb{T}^+) \log \frac{1}{\beta(\mathbb{T}^+)} \Big) \Big] = E\Big[\frac{\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T})} \Big] \cdot E\Big[\frac{\lambda \beta(\mathbb{T}^+)}{1 - \beta(\mathbb{T}^+)} \log \frac{1}{\beta(\mathbb{T}^+)} \Big]. \tag{9}$$

Notice that for $x \in (0,1)$,

$$0 < \frac{x}{1-x} \log \frac{1}{x} < 1.$$

Hence, the product in (9) is bounded by

$$\mathbf{GW}\Big[\frac{\lambda \beta(T)}{\lambda - 1 + \beta(T)} \Big],$$

which is finite according to (1). Therefore, the formula (8) is justified. By (7) again, we obtain

$$\begin{aligned} d_\lambda &= h_\lambda^{-1} \int \log\big(\mathcal{C}(T) + \lambda\big) \frac{\beta(T')\mathcal{C}(T)}{\lambda - 1 + \beta(T') + \mathcal{C}(T)} \mathbf{GW}(\mathrm{d}T)\mathbf{GW}(\mathrm{d}T') \\ &= h_\lambda^{-1} E\Big[\log(\mathcal{C}(\mathbb{T}^+) + \lambda) \frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)} \Big]. \end{aligned}$$

Now we proceed to show that $d_\lambda > \mathbf{GW}[\log \nu]$. Recall that the function κ_λ is strictly increasing. The FKG inequality implies that

$$E\Big[\log(\mathcal{C}(\mathbb{T}^+) + \lambda) \frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)} \Big] > E\big[\log(\mathcal{C}(\mathbb{T}^+) + \lambda) \big] \times E\Big[\frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)} \Big].$$

In view of the previous formula for d_λ, it suffices to prove

$$\mathbf{GW}[\log(\mathcal{C}(T)+\lambda)] \geq \mathbf{GW}[\log \nu].$$

In fact, the strict inequality holds. Recall the notation that $T_1, \dots, T_\nu$ stand for the descendant trees of the children of the root in T. It is justified by Lemma 3 that

$$\begin{aligned}\mathbf{GW}[\log(\mathcal{C}(T)+\lambda)] = \mathbf{GW}\left[\log \frac{\lambda}{1-\beta(T)}\right] &= \mathbf{GW}\left[\log \frac{\lambda\beta(T)}{\beta(T_1)(1-\beta(T))}\right] \\ = \mathbf{GW}\left[\log \frac{\mathcal{C}(T)}{\beta(T_1)}\right] &= \mathbf{GW}\left[\log \frac{\sum_{i=1}^{\nu}\beta(T_i)}{\beta(T_1)}\right].\end{aligned}$$

By strict concavity of the log function,

$$\sum_{j=1}^{\nu}\frac{1}{\nu}\log\frac{\sum_{i=1}^{\nu}\beta(T_i)}{\beta(T_j)} \geq \sum_{j=1}^{\nu}\frac{1}{\nu}\log\nu = \log\nu,$$

with equality if and only if for all $1 \leq j \leq \nu$,

$$\frac{\sum_{i=1}^{\nu}\beta(T_i)}{\beta(T_j)} = \nu.$$

But this condition for equality cannot hold for $\mathbf{GW}$-a.e. T. Therefore,

$$\begin{aligned}\mathbf{GW}[\log(\mathcal{C}(T)+\lambda^{-1})] &= \mathbf{GW}\left[\log \frac{\sum_{i=1}^{\nu}\beta(T_i)}{\beta(T_1)}\right] \\ &= \mathbf{GW}\left[\sum_{j=1}^{\nu}\frac{1}{\nu}\log\frac{\sum_{i=1}^{\nu}\beta(T_i)}{\beta(T_j)}\right] > \mathbf{GW}[\log\nu].\end{aligned}$$

To complete the proof of Theorem 1, it remains to examine the asymptotic behaviors of d_λ: When $\lambda \to 0^+$, a.s. $\beta(\mathbb{T}) \to 1$ and $\mathcal{C}(\mathbb{T}^+) = \sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+) \to \nu^+$. Since

$$\frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)} \leq \beta(\mathbb{T}) \leq 1, \tag{10}$$

we can use Lebesgue's dominated convergence to get $\lim_{\lambda\to 0^+} h_\lambda = 1$. Similarly, it follows from

$$\log(\mathcal{C}(\mathbb{T}^+)+\lambda)\frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)} \leq \log(\mathcal{C}(\mathbb{T}^+)+\lambda) \leq \log(\nu^+ + m)$$

that $\lim_{\lambda\to 0^+} d_\lambda = E[\log\nu^+] = \mathbf{GW}[\log\nu]$.

When $\lambda \to m^-$, a.s. $\beta(\mathbb{T}) \to 0$ and $\mathcal{C}(\mathbb{T}^+) \to 0$. We have seen that the FKG inequality yields the lower bound

$$d_\lambda > E[\log(\mathcal{C}(\mathbb{T}^+)+\lambda)].$$

Using again dominated convergence, we obtain

$$\lim_{\lambda\to m^-} E[\log(\mathcal{C}(\mathbb{T}^+)+\lambda)] = \log m.$$

On the other hand, recall that $d_\lambda < \log m$. Consequently, $d_\lambda \to \log m$ when $\lambda \to m^-$.

6 Speed of the biased random walk

For the λ-biased random walk $(X_n)_{n\geq 0}$ on $\mathbb{T}$ starting from the root, Lyons, Pemantle and Peres [7] showed that a.s. the limit

$$\ell_\lambda := \lim_{n\to\infty} \frac{|X_n|}{n}$$

exists, and is a strictly positive deterministic constant provided $\lambda < m$. We call ℓ_λ the speed (or rate of escape) of the λ-biased random walk.

Given the asymptotic distribution $\mathbf{AGW}_\lambda$ of the environment, the speed ℓ_λ is calculated in [1, Theorem 1.1] as

$$\begin{aligned}\ell_\lambda &= c_\lambda^{-1} E\Big[\frac{(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big(\sum_{i=1}^{\nu^+}\frac{1}{\nu^++\lambda}-\frac{\lambda}{\nu^++\lambda}\Big)\Big]\\ &= c_\lambda^{-1} E\Big[\frac{(\nu^+-\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big].\end{aligned}$$

Another way to compute ℓ_λ comes from the observation that the speed equals the probability to be at an exit point under the stationary measure $\mathsf{RW}_\lambda \times \mathbf{AGW}_\lambda$ (see Proposition 8.2 in [6]), which yields the formula

$$\begin{aligned}\ell_\lambda &= c_\lambda^{-1} E\Big[\frac{(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\sum_{i=1}^{\nu^+}\frac{1}{\nu^++\lambda}\beta(\mathbb{T}_i^+)\Big]\\ &= c_\lambda^{-1} E\Big[\frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] = \frac{h_\lambda}{c_\lambda}.\end{aligned}$$

This alternative interpretation of ℓ_λ can be shown using the Kac lemma in ergodic theory. Here, we would rather establish the equality

$$E\Big[\frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] = E\Big[\frac{(\nu^+-\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] \tag{11}$$

by a direct verification. In fact, the right-hand side of (11) is given by

$$E\Big[\frac{(\nu^++1)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] - E\Big[\frac{(1+\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big].$$

By symmetry and (2), this difference is equal to

$$\begin{aligned}&E\Big[\frac{\mathcal{C}(\mathbb{T}^+)+\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] - E\Big[\frac{(1+\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]\\ &= E\Big[\frac{\mathcal{C}(\mathbb{T}^+)-\lambda\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]\\ &= E\Big[\frac{\mathcal{C}(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{(\lambda-1+\mathcal{C}(\mathbb{T}^+))(\lambda+\mathcal{C}(\mathbb{T}))+\mathcal{C}(\mathbb{T})}\Big] + E\Big[\frac{\lambda(\mathcal{C}(\mathbb{T}^+)-\mathcal{C}(\mathbb{T}))}{(\lambda-1+\mathcal{C}(\mathbb{T}^+))(\lambda+\mathcal{C}(\mathbb{T}))+\mathcal{C}(\mathbb{T})}\Big].\end{aligned}$$

The second expectation in the last line vanishes, because

$$
\begin{aligned}
E\Big[\frac{\lambda\mathcal{C}(\mathbb{T}^+)}{(\lambda-1+\mathcal{C}(\mathbb{T}^+))(\lambda+\mathcal{C}(\mathbb{T}))+\mathcal{C}(\mathbb{T})}\Big] &= E\Big[\frac{\mathcal{C}(\mathbb{T}^+)}{\lambda-1+\mathcal{C}(\mathbb{T}^+)+\mathcal{C}(\mathbb{T})+\lambda^{-1}\mathcal{C}(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}\Big] \\
&= E\Big[\frac{\mathcal{C}(\mathbb{T})}{\lambda-1+\mathcal{C}(\mathbb{T}^+)+\mathcal{C}(\mathbb{T})+\lambda^{-1}\mathcal{C}(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}\Big] \\
&= E\Big[\frac{\lambda\mathcal{C}(\mathbb{T})}{(\lambda-1+\mathcal{C}(\mathbb{T}^+))(\lambda+\mathcal{C}(\mathbb{T}))+\mathcal{C}(\mathbb{T})}\Big] < \lambda.
\end{aligned}
$$

To finish the proof of (11), we use (2) again to see that the left-hand side of (11) also coincides with

$$
E\Big[\frac{\mathcal{C}(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{(\lambda-1+\mathcal{C}(\mathbb{T}^+))(\lambda+\mathcal{C}(\mathbb{T}))+\mathcal{C}(\mathbb{T})}\Big].
$$

The following lemma will be useful in the next section.

Lemma 6. *We have the inequalities*

$$
\frac{E\Big[\frac{\nu^+\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]}{E\Big[\frac{\nu^+(\nu^++\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]} > \frac{E\Big[\frac{\nu^+(\nu^+-\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]}{E\Big[\frac{\nu^+(\nu^++\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]} > \frac{m-\lambda}{m+\lambda} > \ell_\lambda = \frac{E\Big[\frac{(\nu^+-\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]}{E\Big[\frac{(\nu^++\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]}.
$$

Before giving the proof, we point out that $\frac{m-\lambda}{m+\lambda}$ is the speed of the λ-biased random walk on an m-ary tree when $m \in \mathbb{N}$. The strict inequality

$$
\ell_\lambda < \frac{m-\lambda}{m+\lambda}
$$

indicates the slowing down of the walk due to randomness of the Galton–Watson tree. It has already been shown in Corollary 7.1 of Virág [10] by different arguments.

Proof. To establish the first inequality, we proceed as in the proof of (11) to show that

$$
E\Big[\frac{\nu^+\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] > E\Big[\frac{\nu^+(\nu^+-\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]. \tag{12}
$$

The left-hand side is equal to

$$
E\Big[\frac{\nu^+\mathcal{C}(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{(\lambda-1+\mathcal{C}(\mathbb{T}^+))(\lambda+\mathcal{C}(\mathbb{T}))+\mathcal{C}(\mathbb{T})}\Big],
$$

while the right-hand side is given by

$$
\begin{aligned}
&E\Big[\frac{\nu^+(\nu^++1)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] - E\Big[\frac{\nu^+(1+\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] \\
&= E\Big[\frac{\nu^+(\mathcal{C}(\mathbb{T}^+)+\beta(\mathbb{T}))}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] - E\Big[\frac{\nu^+(1+\lambda)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] \\
&= E\Big[\frac{\nu^+(\mathcal{C}(\mathbb{T}^+)-\lambda\beta(\mathbb{T}))}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] \\
&= E\Big[\frac{\nu^+\mathcal{C}(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{(\lambda-1+\mathcal{C}(\mathbb{T}^+))(\lambda+\mathcal{C}(\mathbb{T}))+\mathcal{C}(\mathbb{T})}\Big] + E\Big[\frac{\lambda\nu^+(\mathcal{C}(\mathbb{T}^+)-\mathcal{C}(\mathbb{T}))}{(\lambda-1+\mathcal{C}(\mathbb{T}^+))(\lambda+\mathcal{C}(\mathbb{T}))+\mathcal{C}(\mathbb{T})}\Big].
\end{aligned}
$$

Notice that

$$\frac{\mathcal{C}(\mathbb{T}^+) - \mathcal{C}(\mathbb{T})}{(\lambda - 1 + \mathcal{C}(\mathbb{T}^+))(\lambda + \mathcal{C}(\mathbb{T})) + \mathcal{C}(\mathbb{T})} = \frac{\sum_{i=1}^{\nu^+} \beta(\mathbb{T}_i^+) - \mathcal{C}(\mathbb{T})}{\sum_{i=1}^{\nu^+} \beta(\mathbb{T}_i^+)(\lambda + \mathcal{C}(\mathbb{T})) + \lambda(\lambda - 1 + \mathcal{C}(\mathbb{T}))}$$

is strictly increasing with respect to ν^+. Applying the FKG inequality, we see that

$$E\Big[\frac{\lambda\nu^+(\mathcal{C}(\mathbb{T}^+) - \mathcal{C}(\mathbb{T}))}{(\lambda - 1 + \mathcal{C}(\mathbb{T}^+))(\lambda + \mathcal{C}(\mathbb{T})) + \mathcal{C}(\mathbb{T})}\Big] > E[\nu^+] \cdot E\Big[\frac{\lambda(\mathcal{C}(\mathbb{T}^+) - \mathcal{C}(\mathbb{T}))}{(\lambda - 1 + \mathcal{C}(\mathbb{T}^+))(\lambda + \mathcal{C}(\mathbb{T})) + \mathcal{C}(\mathbb{T})}\Big],$$

where in the product the second expectation equals zero. The proof of (12) is thus finished.

The second inequality in question can be easily reduced to

$$E\Big[\frac{\nu^+(\nu^+ + \lambda)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] > E[\nu^+ + \lambda] \cdot E\Big[\frac{\nu^+\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big],$$

which results again from the FKG inequality, since

$$E\Big[\frac{\nu^+(\nu^+ + \lambda)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] = E\Big[\frac{(\nu^+ + \lambda)\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big]$$

and

$$E\Big[\frac{\nu^+\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] = E\Big[\frac{\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big].$$

Conversely, the FKG inequality implies that

$$E\Big[\frac{(\nu^+ + \lambda)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] < E[\nu^+ + \lambda] \cdot E\Big[\frac{\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big],$$

which yields the third inequality $\ell_\lambda < \frac{m-\lambda}{m+\lambda}$. □

7 Average number of children along a random path

Recall that for every vertex x in a tree T, we write $\nu(x)$ for its number of children. Birkhoff's ergodic theorem implies that for $\mathsf{HARM}_\lambda \times \mu_{\mathsf{HARM}_\lambda}$-a.e. (ξ, T),

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \nu(\xi_k) = \int \nu(e)\mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) = h_\lambda^{-1} E\Big[\frac{\nu^+\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big].$$

The last expectation is finite, as we derive from (10) that

$$\frac{\nu^+\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)} \le \nu^+.$$

Since $\mu_{\mathsf{HARM}_\lambda}$ is equivalent to **GW**, the convergence above also holds for $\mathsf{HARM}_\lambda \times$ **GW**-a.e. (ξ, T). Hence, the average number of children of the vertices visited by the λ-harmonic ray in a Galton–Watson tree is the same as the $\mu_{\mathsf{HARM}_\lambda}$-mean degree of the root.

For every $k \ge 1$, we set

$$A(k) := E\Big[\frac{\beta(\mathbb{T})\sum_{i=1}^{k} \beta(\mathbb{T}_i^+)}{\lambda - 1 + \beta(\mathbb{T}) + \sum_{i=1}^{k} \beta(\mathbb{T}_i^+)}\Big].$$

The sequence $(A(k))_{k\geq 1}$ is strictly increasing. Moreover,

$$\frac{A(k)}{k} = E\Big[\frac{\beta(\mathbb{T})\beta(\mathbb{T}_1^+)}{\lambda - 1 + \beta(\mathbb{T}) + \sum_{i=1}^{k}\beta(\mathbb{T}_i^+)}\Big]$$

is strictly decreasing with respect to k.

Proposition 7. *For $0 < \lambda < m$,*

$$\int \nu(e)\mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) > m.$$

Furthermore, $\int \nu(e)\mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) \to m$ as $\lambda \to 0^+$.

Proof. The first assertion, reformulated as

$$E\Big[\frac{\nu^+\beta(\mathbb{T})\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{\lambda - 1 + \beta(\mathbb{T}) + \sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}\Big] > E[\nu^+]\cdot E\Big[\frac{\beta(\mathbb{T})\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{\lambda - 1 + \beta(\mathbb{T}) + \sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}\Big],$$

is a simple consequence of the FKG inequality, since

$$\mathbf{GW}[\nu A(\nu)] > \mathbf{GW}[\nu]\cdot\mathbf{GW}[A(\nu)].$$

When $\lambda \to 0^+$, a.s. $\beta(\mathbb{T}) \to 1$ and $\mathcal{C}(\mathbb{T}^+) \to \nu^+$. Using Lebesgue's dominated convergence, we have seen at the end of Section 5 that $\lim_{\lambda\to 0^+} h_\lambda = 1$. The same argument applies to the convergence of

$$E\Big[\frac{\nu^+\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big]$$

towards $E[\nu^+] = m$. $\square$

Under $\mathbf{GW}$ we define a random variable $\hat{\nu}$ having the size-biased distribution of ν.

Proposition 8. *For $0 < \lambda < m$,*

$$\int \nu(e)\mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) < \mathbf{GW}[\hat{\nu}] = m^{-1}\sum k^2 p_k.$$

If we assume further that $\sum k^3 p_k < \infty$, then $\int \nu(e)\mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) \to \mathbf{GW}[\hat{\nu}]$ as $\lambda \to m^-$.

Proof. Since $\int \nu(e)\mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) < \infty$, we may assume $\sum k^2 p_k < \infty$ throughout the proof. The inequality in the first assertion can be written as

$$E[\nu^+]\cdot E\Big[\frac{\nu^+\beta(\mathbb{T})\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{\lambda - 1 + \beta(\mathbb{T}) + \sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}\Big] < E[(\nu^+)^2]\cdot E\Big[\frac{\beta(\mathbb{T})\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{\lambda - 1 + \beta(\mathbb{T}) + \sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}\Big].$$

By conditioning on ν^+, we see that it is equivalent to

$$\mathbf{GW}[A(\hat{\nu})] < \mathbf{GW}[\hat{\nu}]\cdot\mathbf{GW}\Big[\frac{A(\hat{\nu})}{\hat{\nu}}\Big],$$

which results from the FKG inequality.

For the second assertion, remark that

$$\begin{aligned} E\Big[\frac{\nu^+\beta(\mathbb{T})\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{m-1}\Big] &= \frac{\mathbf{GW}[\nu^2]\cdot\mathbf{GW}[\beta(T)]^2}{m-1}, \\ E\Big[\frac{\beta(\mathbb{T})\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{m-1}\Big] &= \frac{\mathbf{GW}[\nu]\cdot\mathbf{GW}[\beta(T)]^2}{m-1}. \end{aligned}$$

When the offspring distribution p admits a second moment, Proposition 3.1 of [2] shows that

$$\frac{\beta(T)}{\mathbf{GW}[\beta(T)]}$$

is uniformly bounded in $L^2(\mathbf{GW})$. Using this fact, we can verify that

$$\lim_{\lambda\to m^-} h_\lambda \cdot E\Big[\frac{\beta(\mathbb{T})\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{m-1}\Big]^{-1} = 1.$$

With the third moment condition $\sum k^3 p_k < \infty$, we similarly have

$$\lim_{\lambda\to m^-} E\Big[\frac{\nu^+\beta(\mathbb{T})\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}\Big]\cdot E\Big[\frac{\nu^+\beta(\mathbb{T})\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{m-1}\Big]^{-1} = 1.$$

Consequently,

$$\int \nu(e)\mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) \to \frac{\mathbf{GW}[\nu^2]}{\mathbf{GW}[\nu]} = \mathbf{GW}[\hat{\nu}]$$

as $\lambda \to m^-$. □

Now we turn to investigate the average number of children seen by λ-biased random walk on the Galton–Watson tree $\mathbb{T}$. First of all, as remarked in [6, Section 8], the ergodicity of $\mathsf{HARM}_\lambda \times \mu_{\mathsf{HARM}}$ implies that $\mathsf{RW}_\lambda \times \mathbf{AGW}_\lambda$ is also ergodic. For a tree T rooted at e, let $\nu^+(e)$ denote the number of children of the root minus 1. Since

$$E\Big[\frac{\nu^+(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] = E\Big[\frac{(\lambda+\nu^+)\mathcal{C}(\mathbb{T}^+)}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big] \le \lambda + E[\nu^+] < \infty,$$

it follows from Birkhoff's ergodic theorem that for $\mathsf{RW}_\lambda \times \mathbf{AGW}_\lambda$-a.e. $(\vec{x}, (T,\xi))$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\nu(x_k) = \int \nu^+(e)\mathbf{AGW}_\lambda(\mathrm{d}T,\mathrm{d}\xi) = c_\lambda^{-1}E\Big[\frac{\nu^+(\lambda+\nu^+)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\mathcal{C}(\mathbb{T}^+)}\Big]. \tag{13}$$

Using arguments similar to those in the remark on page 600 of [6], we deduce that the average number of children seen by the λ-biased random walk on $\mathbb{T}$ is a.s. given by the same integral $\int \nu^+(e)\mathbf{AGW}_\lambda(\mathrm{d}T,\mathrm{d}\xi)$.

Proposition 9. *We have*

$$\int \nu^+(e)\mathbf{AGW}_\lambda(\mathrm{d}T,\mathrm{d}\xi) \quad \begin{cases} < m & \text{when } 0<\lambda<1; \\ = m & \text{when } \lambda\in\{0,1\}; \\ > m & \text{when } 1<\lambda<m. \end{cases}$$

Proof. For every integer $k \geq 1$ we set

$$B_\lambda(k) := E\Big[\frac{(\lambda+k)\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{k}\beta(\mathbb{T}_i^+)}\Big].$$

Clearly, we have

$$\int \nu^+(e)\mathbf{AGW}_\lambda(\mathrm{d}T,\mathrm{d}\xi) = m\frac{\mathbf{GW}[B_\lambda(\hat{\nu})]}{\mathbf{GW}[B_\lambda(\nu)]}.$$

When $\lambda \in \{0,1\}$, $B_\lambda(k) \equiv 1$ for all k. We will show that the sequence $(B_\lambda(k))_{k\geq 1}$ is strictly decreasing when $0 < \lambda < 1$, and strictly increasing when $1 < \lambda < m$. Therefore, by the FKG inequality, $\mathbf{GW}[B_\lambda(\hat{\nu})] > \mathbf{GW}[B_\lambda(\nu)]$ when $1 < \lambda < m$, and $\mathbf{GW}[B_\lambda(\hat{\nu})] < \mathbf{GW}[B_\lambda(\nu)]$ when $0 < \lambda < 1$.

To get the claimed monotonicity of the sequence $(B_\lambda(k))_{k\geq 1}$, notice that

$$B_\lambda(k+1) = E\Big[\frac{(\lambda+k)\beta(\mathbb{T})+\beta(\mathbb{T}_{k+1}^+)}{\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{k+1}\beta(\mathbb{T}_i^+)}\Big].$$

Simple calculations give

$$\begin{aligned}
B_\lambda(k+1)-B_\lambda(k) &= E\Big[\frac{-(\lambda+k)\beta(\mathbb{T})\beta(\mathbb{T}_{k+1}^+)+\beta(\mathbb{T}_{k+1}^+)(\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{k}\beta(\mathbb{T}_i^+))}{(\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{k+1}\beta(\mathbb{T}_i^+))(\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{k}\beta(\mathbb{T}_i^+))}\Big]\\
&= E\Big[\frac{\beta(\mathbb{T}_{k+1}^+)(\lambda-1)(1-\beta(\mathbb{T}))}{(\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{k+1}\beta(\mathbb{T}_i^+))(\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{k}\beta(\mathbb{T}_i^+))}\Big]\\
&\quad + E\Big[\frac{-k\beta(\mathbb{T}_{k+1}^+)\beta(\mathbb{T})+\beta(\mathbb{T}_{k+1}^+)\sum_{i=1}^{k}\beta(\mathbb{T}_i^+)}{(\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{k+1}\beta(\mathbb{T}_i^+))(\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{k}\beta(\mathbb{T}_i^+))}\Big].
\end{aligned}$$

Since the last expectation vanishes, $B_\lambda(k+1)-B_\lambda(k) < 0$ if and only if $\lambda < 1$. ☐

The next result, together with the preceding one, shows that $\int \nu^+(e)\mathbf{AGW}_\lambda(\mathrm{d}T,\mathrm{d}\xi)$ is not monotone with respect to λ.

Proposition 10. *As $\lambda \to 0^+$, $\int \nu^+(e)\mathbf{AGW}_\lambda(\mathrm{d}T,\mathrm{d}\xi)$ converges to m.*

Proof. Note that

$$\frac{\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)} \leq 1.$$

By Lebesgue's dominated convergence it follows that $\lim_{\lambda\to 0^+} c_\lambda = 1$. Similarly, we have

$$\lim_{\lambda\to 0^+} E\Big[\frac{\lambda\nu^+\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}\Big] = 0.$$

On the other hand,

$$E\Big[\frac{(\nu^+)^2\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}\Big] = E\Big[\frac{\nu^+\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}{\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}\Big],$$

to which we can apply Lebesgue's dominated convergence again to get

$$\lim_{\lambda\to 0^+} E\Big[\frac{(\nu^+)^2\beta(\mathbb{T})}{\lambda-1+\beta(\mathbb{T})+\sum_{i=1}^{\nu^+}\beta(\mathbb{T}_i^+)}\Big] = E[\nu^+] = m.$$

In view of (13), the proof is thus finished. ☐

Proposition 11. *Assume that* $\sum k^3 p_k < \infty$. *Then,*

$$\lim_{\lambda \to m-} \int \nu^+(e) \mathbf{AGW}_\lambda(\mathrm{d}T, \mathrm{d}\xi) = \frac{m^2 + \sum k^2 p_k}{2m}.$$

Proof. As for the analogous result in Proposition 8, we can use the uniform boundedness in $L^2(\mathbf{GW})$ of $\beta(T)/\mathbf{GW}[\beta(T)]$ to see that

$$\lim_{\lambda \to m^-} c_\lambda \cdot E\Big[\frac{(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1}\Big]^{-1} = 1 = \lim_{\lambda \to m^-} E\Big[\frac{\nu^+(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] \cdot E\Big[\frac{\nu^+(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1}\Big]^{-1}.$$

Hence, it follows from

$$E\Big[\frac{(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1}\Big]^{-1} E\Big[\frac{\nu^+(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1}\Big] = \frac{E[\nu^+(\lambda + \nu^+)]}{E[\lambda + \nu^+]} = \frac{\lambda m + \sum k^2 p_k}{\lambda + m}$$

that

$$\lim_{\lambda \to m^-} c_\lambda^{-1} E\Big[\frac{\nu^+(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] = \lim_{\lambda \to m^-} \frac{\lambda m + \sum k^2 p_k}{\lambda + m} = \frac{m^2 + \sum k^2 p_k}{2m},$$

which finishes the proof by (13). □

Proposition 12. *For all* $0 < \lambda < m$, *we have* $\int \nu(e) \mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) > \int \nu^+(e) \mathbf{AGW}_\lambda(\mathrm{d}T, \mathrm{d}\xi)$.

Proof. In view of (11), the required inequality

$$\begin{aligned} &E\Big[\frac{\nu^+\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] E\Big[\frac{(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] \\ &> E\Big[\frac{\nu^+(\lambda + \nu^+)\beta(\mathbb{T})}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] E\Big[\frac{\beta(\mathbb{T})\mathcal{C}(\mathbb{T}^+)}{\lambda - 1 + \beta(\mathbb{T}) + \mathcal{C}(\mathbb{T}^+)}\Big] \end{aligned}$$

directly results from Lemma 6. □

Combining Propositions 7, 8, 10 and 11, we see that

$$\lim_{\lambda \to 0^+} \Big(\int \nu(e) \mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) - \int \nu^+(e) \mathbf{AGW}_\lambda(\mathrm{d}T, \mathrm{d}\xi) \Big) = 0,$$

and if $\sum k^3 p_k < \infty$,

$$\lim_{\lambda \to m^-} \Big(\int \nu(e) \mu_{\mathsf{HARM}_\lambda}(\mathrm{d}T) - \int \nu^+(e) \mathbf{AGW}_\lambda(\mathrm{d}T, \mathrm{d}\xi) \Big) = \frac{\sum k^2 p_k - m^2}{2m} > 0.$$

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