Oscillating singularities in locally self-similar functions

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Oscillating Singularities in Locally Self-Similar Functions

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Singularities induced by oscillating behavior are analyzed using the wavelet transform. We define two local exponents which allow us to characterize both the singularity strength (Hölder exponent) and the instantaneous frequency of the oscillations. Such oscillating singularities are shown to appear generically in local self-similar functions which are invariant under a nonhyperbolic mapping. We illustrate our results on both isolated singularities and nonisolated singularities appearing in fractal signals generated by nonhyperbolic iterative function systems.

During the past few years, there has been increasing interest in the study of irregular objects [1]. Fractal and multifractal concepts have been developed to describe the singular nature of such distributions through their scale invariance (self-similarity) properties [1,2]. They have been applied successfully in various physical situations ranging from the characterization of ramified patterns observed in growth experiments [3] to the study of the statistics of the velocity and the dissipation fields in fully developed turbulence [4,5]. In the latter context, some authors [6] advocate the use of models of eddy structures involving "spiral-type" behavior of the generic form

$$f(x) = |x - x_0|^{\gamma} \sin\left(\frac{2\pi}{|x - x_0|^{\beta}}\right) \qquad (\beta > 0, \gamma > -1). \tag{1}$$

Unfortunately, the multifractal formalism introduced in Ref. [2(a)] does not take into consideration these oscillating singular behaviors. The aim of this Letter is to study such locally accumulating oscillating behavior using a tool that has proven particularly powerful for analyzing singular functions, namely, the wavelet transform (WT) [7]. Within the general framework of WT analysis, we introduce two local exponents $\alpha(x_0)$ and $\varphi(x_0)$. The exponent $\alpha(x_0)$ can be seen as a generalization of the local singularity exponent commonly used in the multifractal description, whereas the exponent $\varphi(x_0)$ accounts for the behavior of the "instantaneous frequency." For singular behavior induced by "fast oscillations" [e.g., Eq. (1)], we show that $\varphi(x_0)$ plays an important role in the characterization of the strength of the singularity. This Letter is mainly devoted to the calculation of α and φ for locally self-similar functions. We demonstrate that oscillating (nonoscillating) singularities are generically associated to invariance properties under nonhyperbolic (hyperbolic) mappings. We illustrate our purpose on particular examples including the fractal coding function recently introduced by Gutzwiller and Mandelbrot [8].

The WT of a signal permits an analysis both in physical space and in scale space. It consists in decomposing a signal in terms of *wavelets* which are constructed from one single function, the analyzing wavelet ψ, by means of dilations and translations. The WT of a function $f(x)$ is defined as [7]

$$T_{\psi}(b,a) = \frac{1}{a}\int_{-\infty}^{+\infty} f(x)\psi\left(\frac{x-b}{a}\right)dx\,, \tag{2}$$

where b is a space parameter ($b \in R$) and a a scale parameter ($a \in R^{+*}$). The analyzing wavelet $\psi(x)$ is chosen well localized around $x = 0$ and with a vanishing integral so that a large value of $|T_{\psi}(b,a)|$ corresponds to a large variation of $f(x)$ over a distance a from the point $x = b$. Thus, as proved by Mallat and Hwang [9], if $f(x)$ is singular at $x = x_0$, then, at any scale a (arbitrarily small), the function $|T_{\psi}(\cdot,a)|$ is locally maximum at a point $b(a)$ in the neighborhood of x_0. The points $(b(a),a)$ in the space-scale half plane are generally referred to as WT modulus maxima (WTMM); they are lying on connected curves called maxima lines. These lines generally converge to the points where f is singular [9,10] [i.e., $b(a) \to x_0$ when $a \to 0^+$]. Moreover, provided some first moments of ψ are zero, one can recover [9–11] the Hölder exponent $h(x_0)$ of f at the point x_0 by studying the power-law behavior of the WT along a maxima line converging to x_0:

$$T_{\psi}((b(a),a)) \sim a^{h(x_0)}, \qquad a \to 0^+. \tag{3}$$

Let us recall that this exponent characterizes the singularity strength of $f(x)$ at the point $x = x_0$:

$$f(x_0 + l) = f(x_0) + lf^{(1)}(x_0) + \cdots + (l^n/n!)f^{(n)}(x_0) + O(|l|^{h(x_0)}), \tag{4}$$

where $n < h(x_0) < n + 1$ and $f^{(k)}$ is the kth derivative of f.

However, Eq. (3) is not relevant in the case of *oscillating singularities* [9], i.e., such that the Hölder exponent is increased by more than 1 when the function is integrated (this is generally induced by the presence of an

infinite number of accumulating oscillations in any neighborhood of the singularity). Let us illustrate this irrelevance on the particular class of "infinitely oscillating" functions defined in Eq. (1). These functions are singular at x_0 and correspond, according to the definition [Eq. (4)], to the Hölder exponents $h(x_0) = \gamma$. The graph of such a function is displayed in Fig. 1(a). As one can see in Fig. 1(b), the WTMM of this function are lying on an infinite (countable) number of maxima lines $\{l_n\}_{n\in\mathbb{Z}}$ appearing at smaller and smaller scales. Each maxima line corresponds to an oscillation in the signal, and the scale at which it appears is proportional to the distance between two successive maxima lines. This distance can be seen as an "instantaneous period." It is clear that the WT along any maxima line, for a small enough, will not account for the singularity behavior of f at $x_0 = 0$ since none of these lines converge towards the point $x_0 = 0$. Thus, instead of following a maxima line, as in Eq. (3), one must "jump" from one line to another in order to converge towards x_0 when going from large to small scales. Let us consider the maxima (b_n, a_n) which correspond to the greatest value of $|T_\psi|$ on each line l_n (if it is not unique we consider the one

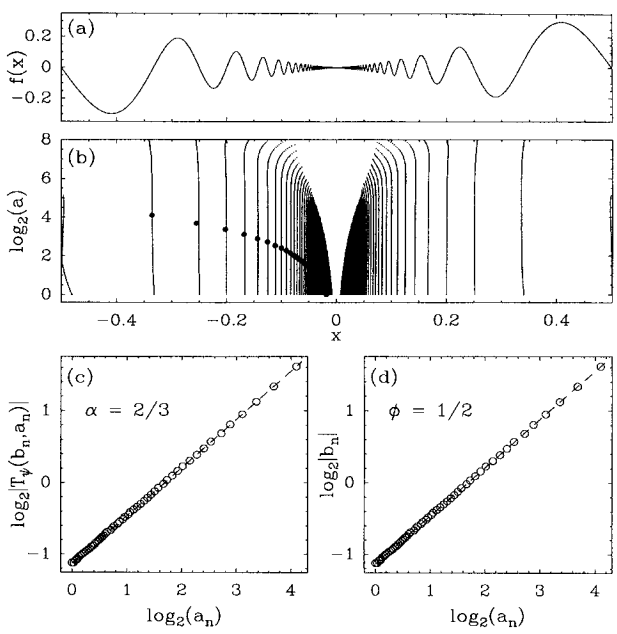

FIG. 1. Detection of the local exponents $\alpha(x_0)$ and $\varphi(x_0)$ associated to an oscillating singularity. (a) Graph of the function $f(x) = |x|^\gamma \sin(2\pi/|x|^\beta)$ for $\gamma = \frac{4}{3}$ and $\beta = 1$. The point $x_0 = 0$ corresponds to an oscillating singularity of f. (b) WT skeleton showing the positions of the modulus maxima for the signal in (a). The analyzing wavelet ψ is the first derivative of the Gaussian function. The maxima are lying on maxima lines. Along each line l_n, the symbol (•) marks the point (a_n, b_n) where the WT is the greatest. (c) $\log_2 |T\psi(b_n, a_n)|$ vs $\log_2 a_n$. The slope provides an estimate of $\alpha(x_0) = \gamma/(\beta + 1)$. (d) $\log_2 |b_n|$ vs $\log_2 a_n$. The slope gives an estimate of $\varphi(x_0) = 1/(\beta + 1)$.

appearing at the largest scale). As shown in Fig. 1(b), b_n goes to x_0 when a_n goes to 0. [It can be proven that the points (b_n, a_n) belong to the "ridge" defined in Ref. [12] as the instantaneous frequency curve in the (b, a) space-scale half plane.] Moreover, as illustrated in Fig. 1(c), one can prove that, along this sequence, the WT modulus follows a power-law scaling

$$|T_\psi(b_n, a_n)| \sim a_n^{\alpha(x_0)}, \tag{5}$$

where $\alpha(x_0) = \gamma/(\beta + 1)$. The so-obtained local exponent $\alpha(x_0)$ is smaller than the Hölder exponent $h(x_0) = \gamma$. It thus appears that, in order to extract $h(x_0)$, we must characterize the oscillations of $f(x)$, i.e., evaluate the exponent β. From α and β we will then naturally recover $\gamma = h(x_0)$.

Since each of the maxima line corresponds to an oscillation in the signal, the abscissa b_n behave like $|b_n - x_0| \sim n^{-1/\beta}$. Moreover, the scale a_n is of the order of the distance between this maxima line and the next one, i.e., $a_n \sim b_n - b_{n+1} \sim n^{-(\beta+1)/\beta}$. Thus the distance between the nth maxima line and the singularity x_0 scales like

$$|b_n - x_0| \sim a_n^{\varphi(x_0)}, \qquad n \to +\infty, \tag{6}$$

where $\varphi(x_0) = 1/(\beta + 1)$. As illustrated in Fig. 1(d), the exponent $\varphi(x_0)$ can thus be extracted from a simple log-log plot of the distance $|b_n - x_0|$ vs the scale a_n. The Hölder exponent $h(x_0)$ can thus be recovered through the particularly simple relation

$$h(x_0) = \alpha(x_0)/\varphi(x_0). \tag{7}$$

Let us note that Eqs. (5), (6), and (7) hold for a general class of analyzing wavelets including the successive derivatives of the Gaussian functions [2(c)].

For a general function f, we define the exponent $\alpha(x_0)$ as in Eq. (5) and $\varphi(x_0)$ as the minimum of $\varphi(x_0)$ [Eq. (6)] and 1. The so-defined local exponents allow us to characterize very precisely the strength and the nature of a singular behavior at a given point x_0. Indeed, for a very large class of functions, one can prove that the relation [Eq. (7)] still holds; i.e., the strength of the singularity at x_0 is directly measured by the ratio $\alpha(x_0)/\varphi(x_0)$. Moreover, $\varphi(x_0) = 1$ indicates that x_0 corresponds to a nonoscillating singularity. In that case, we thus get $\alpha(x_0) = h(x_0)$; i.e., Eq. (3) holds and becomes the same as Eq. (5). On the other hand, a value of $\varphi(x_0)$ smaller than 1 corresponds to an oscillating singularity and, consequently, to an exponent $\alpha(x_0)$ smaller than $h(x_0)$. Equation (3) does not hold anymore. The smaller $\varphi(x_0)$ the more oscillating f in the neighborhood of x_0. Moreover, one can prove that the derivative $f'(x) = df/dx$ corresponds generically to the exponents $\alpha'(x_0) = \alpha(x_0) - 1$ and $\varphi'(x_0) = \varphi(x_0)$. The Hölder exponent of f' at x_0 can thus be derived from the one of f using Eq. (7):

$$h'(x_0) = h(x_0) - 1/\varphi(x_0). \tag{8}$$

Hence, *the more oscillating the signal the more irregular its derivative.* Let us note that mathematicians have recently refined the notion of Hölder spaces by introducing a 2-microlocal spaces [13,14] that allows a better characterization of the action of integration or differentiation operators in functional analysis. Our exponents α and φ can actually be linked to the 2-microlocalization local exponents s and s' through the relation $\alpha = s + s'(1 - \varphi)$.

Let us now consider the class of functions (or distributions) $f(x)$ which are locally self-similar around a given point $x_0 = 0$. We suppose that there exists an increasing function $F(x)$ with a fixed point x_0 which is stable under the iteration of F and such that $f(x) = f(F(x))/P(x)$, where $P(x)$ is some smooth weight function satisfying $0 < P(x_0) \leq 1$. This invariance property of f can be easily transposed in terms of self-similarity of its WT [15]:

$$T_\psi(b,a) \simeq \frac{1}{P(b)} T(F(b), F'(b)a), \qquad (9)$$

where $F'(x)$ is the first derivative of $F(x)$. The same kind of relation naturally holds for the modulus maxima sequence (b_n, a_n) corresponding to the greatest values of the WT along the maxima lines. By iterating this relation, one gets $b_n = F^{[n]}(b_0)$ and $a_n = (F^{[n]})'(b_0)a_0$, where $F^{[n]}$ is the nth iterate of F. Moreover $T_n = |T_\psi(b_n, a_n)| \sim P(b_0)P(b_1)\cdots P(b_n)$. One can distinguish two cases.

(i) $F'(x_0) < 1$; i.e., x_0 is an hyperbolic fixed point. One can easily prove that $|b_n - x_0| \sim a_n \sim F'(x_0)^n$ when $n \to \infty$. One thus gets $\varphi(x_0) = 1$; i.e., the singularity located at x_0 is a nonoscillating singularity and $h(x_0) = \alpha(x_0)$. The value of the exponent $\alpha(x_0)$ depends upon the expression of $P(x)$. In the case where $P(x_0) < 1$, $[T_n \sim P(x_0)^n]$, we obtain $h(x_0) = \alpha(x_0) = -\ln(P)/\ln[F'(x_0)]$.

(ii) $F'(x_0) = 1$; i.e., x_0 is a nonhyperbolic fixed point. Generically, in the neighborhood of x_0, one can write $F(x) = F(x_0) + x - x_0 - C|x - x_0|^r + o[(x - x_0)^r]$ ($C > 0$ and $r > 1$). Using the former expressions of b_n and a_n, one gets $|b_n - x_0| \sim n^{1/(1-r)}$ and $a_n \sim n^{r/1-r}$. We thus obtain, from Eq. (6), $\varphi(x_0) = 1/r < 1$; i.e., x_0 corresponds to an oscillating singularity. Again, the value of the exponent $\alpha(x_0)$ depends on the expression of $P(x)$. The case where $P(x_0) < 1$ is degenerated in the sense that $\alpha(x_0) = +\infty$; i.e., $h(x_0) = +\infty$; $f(x)$ is not singular at $x = x_0$, it is C^∞ even though it is infinitely oscillating around x_0. However, in the same spirit as for SRB (Sinai-Ruelle-Bowen) measures [16], if we choose the weight function $P(x) = |F'(x)|^\eta$ for x in the neighborhood of x_0 (η is an arbitrary positive real number), then we get $T_n \sim [F^{[n]'}(b_0)]^\eta T_0$ and thus $\alpha(x_0) = \eta$. f is singular at x_0 with an Hölder exponent $h(x_0) = r\eta$.

Oscillating singularities thus appear generically in locally self-similar functions invariant under nonhyperbolic mappings. The functions $x^\gamma \sin(2\pi/x^\beta)$ [Eq. (1)] are actually good examples of such functions. Indeed, they are invariant under the mapping $F(x) = x/(1 + x^\beta)^{1/\beta}$ [$F'(0) = 1$ and $r = \beta + 1$] with $P(x) \sim F'(x)^\eta$ [where $\eta = \gamma/(\beta + 1)$]. According to the considerations just above, one easily recovers $\varphi(0) = 1/r = 1/(\beta + 1)$ and $\alpha(0) = \eta = \gamma/(\beta + 1)$. In Fig. 2(c), we have displayed a fractal measure which has been constructed on the interval [0,1] by iterating the iterative function system (IFS) [17] shown in Fig. 2(a). Each branch of the IFS, $F_1(x) = x/(1 + 3x)$ [$F_2(x) = (x + 1)/2$], is associated with the weight function $P_1(x) = |F_1'(x)|^{1/3}$ [$P_2(x) = (1/2)^{1/3}$]. Each point of the support of the measure can

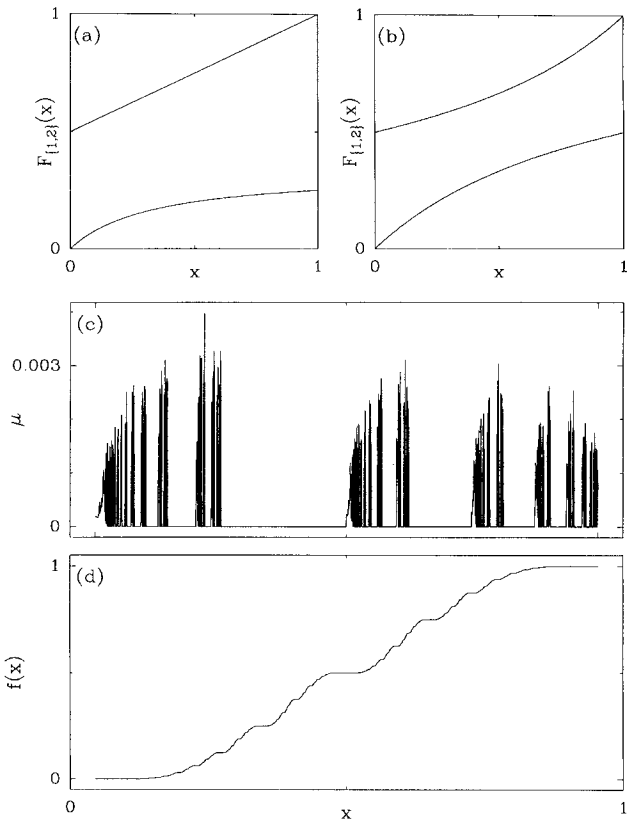

FIG. 2. Fractal distributions with oscillating behavior. (a) Two branch IFS, $F_1(x) = x/(1 + 3x)$ and $F_2(x) = (x + 1)/2$, used to build the measure in (c). Only the second branch is hyperbolic [$F_1'(0) = 1, F_2'(1) = \frac{1}{2}$]. (b) Two branch IFS, $F_1(x) = x/(1 + x)$ and $F_2(x) = 1/(2 - x)$, used to build the function in (d). Both branches are nonhyperbolic [$F_1'(0) = F_2'(1) = 1$]. (c) Fractal measure generated by the two branch IFS shown in (a) when using the respective weight functions $P_1(x) = |F_1'(x)|^{1/3}$ and $P_2(x) = (1/2)^{1/3}$. Any point of the support of the measure can be addressed through a symbolic sequence of 1's and 2's. As F_1 is nonhyperbolic, any symbolic sequence which ends with an infinite number of 1 corresponds to an oscillating singularity ($\varphi = \frac{1}{2}$). (d) Graph of the Gutzwiller-Mandelbrot function generated by the two branch nonhyperbolic IFS displayed in (b). Constant (<1) weight functions are used; the oscillating behaviors are thus degenerated in the sense that $\alpha = h = +\infty$; i.e., the signal is not singular although infinitely oscillating. These behaviors are dense in [0,1]. All the other points correspond to nonoscillating singularities.

be addressed by a symbolic sequence of 1's and 2's, e.g., $x = 0$ corresponds to the symbolic sequence $1111\ldots$ whereas $x = 1$ corresponds to $2222\ldots$. As the first branch is nonhyperbolic, one can show that all the symbolic sequences which end with an infinite number of 1's correspond to oscillating singularities. In the same way, the sequences ending with an infinite number of 2's correspond to nonoscillating singularities. Actually one can show that this is also the case for any periodic symbolic sequence which does not end by an infinite number of 1's. Thus, in the neighborhood of any oscillating singularity, there exists an infinite number of both oscillating and nonoscillating singularities.

As the last example, let us consider the coding function introduced by Gutzwiller and Mandelbrot [8] in the context of chaotic Hamiltonian systems. It has been proven by Bessis and Mantica [18] that this function can be obtained using a 2-branch IFS [Fig. 2(b)] whose two branches are nonhyperbolic. The graph of this coding function is shown in Fig. 2(d). Nonoscillating ($\varphi = 1$, e.g., periodic symbolic sequences) and oscillating behavior ($\varphi < 1$) are locally accumulating. However, the weight functions associated to the nonhyperbolic branches are strictly smaller than 1. Thus, the oscillating behaviors are "degenerated" in the sense that they do not induce any singularity ($\alpha = h = +\infty$). These "smooth" behaviors are responsible for the slow decay of the $f(\alpha)$ singularity spectrum estimated by Gutzwiller and Mandelbrot [8] using the classical multifractal formalism.

To summarize, we have thus defined two new local exponents which allow us to characterize very precisely both the nature and the strength of a singular behavior. These wavelet-based exponents [Eqs. (5) and (6)] can be seen as a generalization of the classical singularity exponent. They appear to be crucial for the understanding of the change of local regularity properties of a distribution when operating differentiation or integration. Let us note that the classical multifractal formalism [or its wavelet-based generalization [2(c),5,10]] accounts only for the fluctuations of the scaling exponent defined in Eq. (3), which has no meaning when oscillating singularities are present. We have shown that such oscillating behaviors generically appear in fractal objects that are self-similar under nonhyperbolic mappings. These situations are commonly encountered in mathematics or physics; let us mention, for example, the famous Riemann-Weierstrass function [11(b)] or the Farey-tree partitioning of rationals used to study the distribution of mode-locking intervals for critical circle maps [19]. A "grand-canonical" multifractal formalism that would account for the fluctuations of both exponents α and φ would be of fundamental interest in this context.

Note added.—After completion of this work, we have been aware of a work by S. Jaffard and Y. Meyer [14] who introduced similar exponents in the context of "shirp analysis."

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