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Wavelet analysis of turbulence reveals the multifractal nature of the Richardson cascade

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THE central problem of fully developed turbulence is the energy-cascading process. It has resisted all attempts at a full physical understanding or mathematical formulation. The main reasons for this failure are related to the large hierarchy of scales involved, the highly nonlinear character inherent in the Navier–Stokes equations, and the spatial intermittency of the dynamically active regions. Richardson¹ has described the interplay between large and small scales with the words ‘Big whirls have little whirls which feed on their velocity ... and so on to viscosity’, and the phenomenon so described is known as the Richardson cascade. This local interplay also forms the basis of a theory by Kolmogorov². It was later realized that the cascade ought to be intermittent, and Mandelbrot³ has given a fractal description of the intermittency of the fine structure. A particular case that emphasizes the dynamical aspect of the fractal models is the β -model⁴, in which the flux of energy is transferred to only a fixed

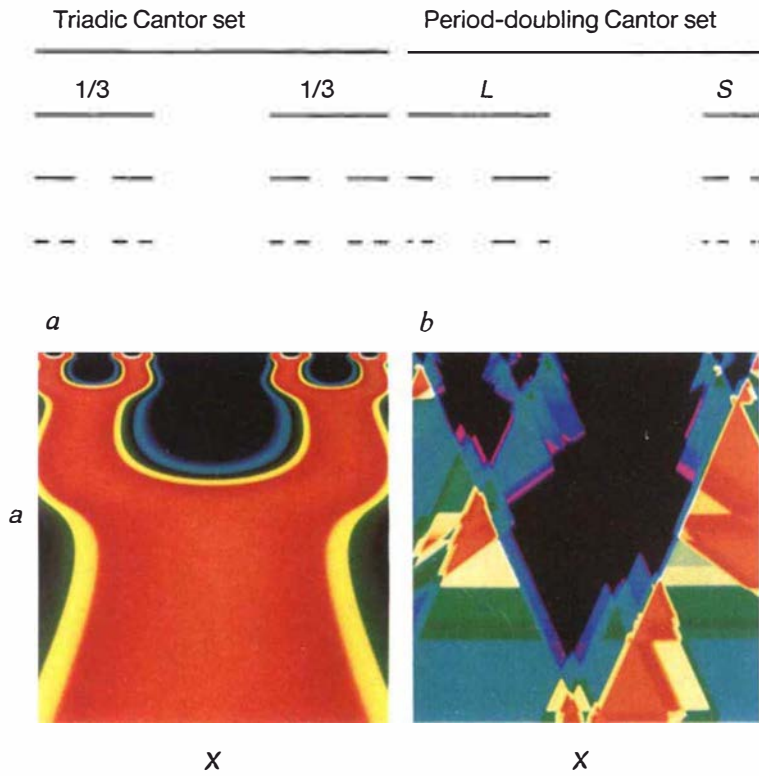


FIG. 1 Wavelet analysis of Cantor sets^{9,15}. Coordinates correspond to space (x) and scale (a); both are linear (small scales at the top). The intensity of the wavelet transform is colour-coded according to the natural light spectrum from black (low intensity) ($T_g \leq 0$) to red ($\max(T_g) > 0$) on each line $a = \text{constant}$. *a*, Uniform triadic Cantor set analysed using a 'Mexican hat' wavelet: $g(x) = (1-x^2)e^{-x^2/2}$; $x \in [0, 1]$, $a \in [2^{-9}, 2^0]$. *b*, Period-doubling Cantor set analysed using 'French hat' wavelets: $g(x) = 1(|x| < 1)$, $-1/2(1 \leq |x| \leq 3)$, $0(|x| > 3)$; $x \in [0.40115, \dots, 1]$, $a \in [2^{-10}, 2^{-1}]$. Above each figure we illustrate the process of subdivision leading to each Cantor set.

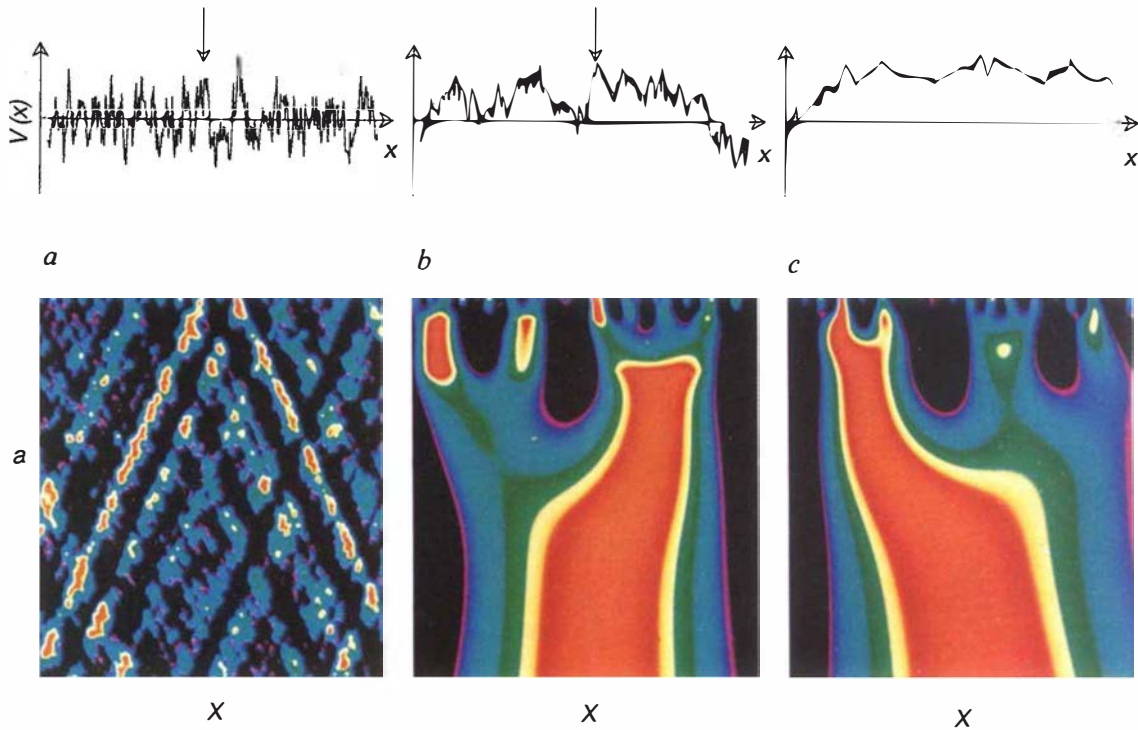


FIG. 2 Wavelet analysis of fully developed turbulence from wind-tunnel data. Coordinates and intensities are as in Fig. 1. The top graphs show the signals being analysed. Panel *a* is a 852-m-long sample, showing the large scales (from $28l_0$ to $l_0/10$) analysed using a 'French top hat' wavelet. In *b* the

'Mexican hat' wavelet was used for magnification $\times 20$ of the central position indicated by the arrow in the top graph of *a*. Panel *c* bears an analogous relation to *b*, again using the 'Mexican hat' wavelet. The successive forkings reveal the fractal nature of the Richardson cascade.

fraction β of the eddies of smaller scales. More recently, a multifractal model of the fine-scale intermittency has been introduced by Parisi and Frisch⁵; this accounts for the more complex cascading process suggested by the experimental data on inertial-range structure functions obtained by Anselmet *et al.*⁶. These statistical models are insufficient, however, because they cannot give any topological information about the intermittent structures in real space. Here we use the wavelet transform⁷ to analyse the velocity field of wind-tunnel turbulence at very high Reynolds numbers⁸. This 'space-scale' analysis is shown to provide the first visual evidence of the celebrated Richardson cascade¹, and reveals in particular its fractal character³. The results also indicate that the energy-cascading process has remarkable similarities with the deterministic construction rules of non-homogeneous Cantor sets^{9,10}.

Wavelet analysis⁷ is a mathematical technique introduced recently for analysing seismic data and acoustic signals^{11,12}. It provides a two-dimensional unfolding of one-dimensional signals, resolving both the position and the scale as independent variables. The method comprises an expansion of an arbitrary real-valued function $s(x)$ over wavelets that are constructed from a single function g by means of dilatations and translations^{7,13}. The wavelet transform of $s(x)$ with respect to the wavelet g is defined as:

$$T_g(a, x) = w(a) \int g((x-y)/a)s(y) dy \quad a > 0, x \in \mathfrak{R}$$

where the weight function $w(a)$ is introduced for visual enhancement.

The wavelet g is a regular function, localized around $x=0$. For a large class of functions $s(x)$, the wavelet transform may be inverted, provided that g is admissible ($\int g(y) dy = 0$) and satisfies others conditions^{7,14}. The wavelet transformation can be regarded as a mathematical microscope⁹, for which position and magnification correspond to x and a^{-1} , respectively, and the performance of the optics is determined by the choice of the analysing wavelet g . Wavelet analysis has become a powerful tool for locating singularities^{9,10}: a singularity of $s(x)$ at x_0 produces a cone-like structure in T_g , pointing towards the point ($a=0, x=x_0$). The nature of the singularities may be inferred from the power law behaviour of $T_g(a, x=x_0)$ on increasing the magnification a^{-1} .

As emphasized by Arnéodo *et al.*⁹, the wavelet transform assists visualization of self-similar properties of fractal objects. In particular, it illustrates the complexity of the fractal under consideration, revealing the hierarchy that governs the relative positioning of the singularities. In Fig. 1a, the successive pitchfork branchings observed in $T_g(a, x)$ as a result of increasing the magnification a^{-1} illustrate how the uniform (single scale) triadic Cantor set is constructed^{9,10}. The unit interval $[0, 1]$ is initially divided into three sub-intervals, each of length $l=1/3$. The middle interval is removed and the same process is repeated on each of the two remaining sub-intervals, and so on *ad infinitum*. In Fig. 1b, the same analysis is performed for the multifractal (two-scale) Cantor set that arises at the onset of chaos through the period doubling cascade scenario^{9,15}. The pitchfork branchings observed on increasing the magnification a^{-1} are no longer symmetric, and occur not in pairs but successively, as the signature of two unequal scaling factors, one large and one small ($L(\alpha_{PD}^{-1}) \approx 1/2.5029 \dots$) and $S(\alpha_{PD}^{-2})$ respectively). Moreover, the internal self similarity of T_g (which is central to the renormalization-group study of the period doubling transition to chaos) illustrates the basic construction rule of the period-doubling Cantor set: from one stage to the next, the relative positions of the large and small sub-intervals is unchanged when dividing a small interval, but is exchanged when dividing a large interval.

The fractal structures proposed in fully developed turbulence are real-space structures, in contrast to fractal attractors, which reside in phase space. Thus the wavelet transform can be applied

directly to turbulent velocity fields obtained from experiment or from numerical simulations¹⁶. Here we report the first such analysis performed on the velocity data from high-Reynolds-number, three-dimensional turbulence. The data were obtained in the large wind tunnel S1 of ONERA in Modane, France. A hot wire probe was located on the axis of the return section which was 24 m in diameter and 150 m in length. The turbulence level, expressed by the ratio of the r.m.s. value of the longitudinal velocity fluctuations to the mean velocity, was less than 7%, which implies that time variations are equivalent to spatial variations (the Taylor hypothesis). The scale ratio of the velocity signal shown in Fig. 2 is about 4×10^4 , giving 15 m for the integral scale l_0 and 0.35 mm for the dissipation scale l_d . The Reynolds number based on the Taylor microscale is $R_\lambda = 2,720$. This turbulence has an inertial range of at least two decades^{8,17}.

A signal of total length 852 m was analysed hierarchically. Figure 2a corresponds to the total span and to a range of scales (that is, of parameter a) from $28l_0$ to $l_0/10$. Figure 2b shows the region around the arrow in Fig. 2a, with position and scale variables magnified by a factor of 20, and Fig. 2c is related to Fig. 2b in the same way. A 'French top hat' piecewise constant analysing wavelet was used in the case of Fig. 2a, emphasizing the cone like structures. A 'Mexican hat' wavelet was used in Figs 2b and c, because it provides a (marginally) better visualization of the Richardson cascade¹. Only positive values of the wavelet transform are shown. In the figures the weight function $w(a)$ was chosen such that the maximum of $T_g(a, x)$ on each $a = \text{const.}$ line is independent of a .

Large scale eddies are revealed in Fig. 2a by the superimposed cone like patterns of different intensities, indicating structures of scale $a \sim l_0$. These large-scale structures seem randomly distributed in space and are fairly space-filling. Very different small scale patterns are displayed in Figs 2b and c. These are created by a process in which typically an eddy divides asymmetrically (in intensity) into two smaller eddies, and this subdivision is repeated about eight to ten times before eddies reach a scale at which they are readily dissipated. These successive forkings produce a fractal structure, in which small-scale activity becomes increasingly sparse. The ratio between the scales of successive generations can take different values, which is another indication that a multifractal description is appropriate.

In summary, we have used wavelet analysis, which is already known to be a powerful tool for analysing multifractal attractors^{9,15}, to explore the spatial structure of fully developed turbulence. Our emphasis has been to identify new qualitative features. Similar techniques can be used to reveal fractal clustering of galaxies on very large scales (A. Bijaoui & E. Slezak, unpublished work).

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