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FLAT PARABOLIC VECTOR BUNDLES ON ELLIPTIC CURVES

THIAGO FASSARELLA AND FRANK LORAY

ABSTRACT. We describe the moduli space of logarithmic rank 2 connections on elliptic curves with 2 poles.

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1. INTRODUCTION

In this paper, we investigate the geometry of certain moduli spaces of connections on curves C . We consider pairs (E, ∇) where $E \rightarrow C$ is a rank 2 vector bundle and $\nabla : E \rightarrow E \otimes \Omega_C^1(D)$ is a logarithmic connection with (reduced) polar divisor $D = t_1 + \dots + t_n$. Once we have prescribed the base curve (C, D) , the trace

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connection $(\det(E), \text{tr}(\nabla))$ and the eigenvalues $\nu = (\nu_1^\pm, \dots, \nu_n^\pm)$ of the residual connection matrix at each pole, then we can define the moduli space $\text{Con}^\nu(C, D)$ of those pairs (E, ∇) up to isomorphism. For a generic choice of ν (compatible with $\text{tr}(\nabla)$) all connections (E, ∇) are irreducible and the moduli $\text{Con}^\nu(C, D)$ can be constructed as a GIT quotient (see [17]): it is a smooth irreducible quasi-projective variety of dimension $2N$ where $N = 3g - 3 + n$ is the dimension of deformation of the base curve and g denotes the genus of the curve. Moreover, the variety $\text{Con}^\nu(C, D)$ admits a holomorphic symplectic structure (see [5]) which turns to be algebraic (see [10, 8]): there is a rational 2-form ω which is regular and having maximal rank N on $\text{Con}^\nu(C, D)$.

It is natural to consider the forgetful map $\pi : (E, \nabla) \mapsto (E, \mathbf{p})$ which to a connection associates the underlying parabolic bundle: the parabolic data $\mathbf{p} = (p_1, \dots, p_n)$ consists of the ν_i^\pm -eigenline $p_i \subset E|_{t_i}$ for each pole. The moduli space $\text{Bun}(C, D)$ of those parabolic bundles admitting a connection is N -dimensional and the map π above turns to be Lagrangian, i.e. its fibers are Lagrangian submanifolds. However, $\text{Bun}(C, D)$ is not a variety, but a non Hausdorff scheme; it is a finite union of projective varieties patched together along Zariski open sets. Over the open subset of simple bundles (i.e. without automorphisms), the Lagrangian fibration π is an affine \mathbb{A}^N -bundle whose linear part is the cotangent bundle $T^*\text{Bun}(C, D)$, and the symplectic structure comes from Liouville form. It is mainly this heuristic picture that we want to describe in a particular case.

The picture is very well known in the case $(g, n) = (0, 4)$, since $\text{Con}^\nu(C, D)$ corresponds to the Okamoto space of initial conditions for Painlevé VI equation in this case (see [9]). The case $(0, n)$ has been studied in [1, 18, 13, 11] and corresponds to Garnier systems. The case $(1, 1)$ has been studied in [12], where it is shown to be equivalent to the Painlevé $(0, 4)$ case with particular exponents, due to hyperellipticity of the curve. Similarly, the case $(2, 0)$ is studied in [7] and turns to be the quotient of the Garnier case $(0, 6)$ by an involution, again by hyperellipticity. The case $(1, 2)$ involved in the present paper is the first one that does not reduce to genus zero case: for generic eigenvalues ν , the hyperelliptic involution does not preserve the spectral data.

Results. We fix C to be the elliptic curve with affine equation $y^2 = x(x - 1)(x - \lambda)$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$, and denote by $w_\infty \in C$ the point at infinity. In Section 2, the number $n = \deg(D)$ of poles and eigenvalues ν are arbitrary. There, we study which parabolic bundles (E, \mathbf{p}) over (C, D) are ν -flat, i.e. admit a connection ∇ with prescribed trace and exponents, compatible with parabolics. This has been done for a general curve in [4] in the orbifold case (i.e. with rational eigenvalues) and we extend his criterion for general eigenvalues in Theorem 2.2.

Theorem 1. *A parabolic bundle (E, \mathbf{p}) over an elliptic curve (C, D) is ν -flat if, and only if, it satisfies*

$$\text{Fuchs relation: } \deg(E) + \sum_{k=1}^n (\nu_k^+ + \nu_k^-) = 0$$

and for each decomposition $(E, \mathbf{p}) = (L', \mathbf{p}') \oplus (L'', \mathbf{p}'')$, we have

$$\deg(L) + \sum_{p_k \in L} \nu_k^+ + \sum_{p_k \notin L} \nu_k^- = 0$$

for $L = L'$ and L'' .

For generic ν satisfying Fuchs relation (and $n > 0$), the second identity cannot occur and in that case we have:

$$\nu\text{-flat} \iff \text{indecomposable.}$$

For $n = 2$, the moduli space of indecomposable bundles has been recently described in [6]. It is a non Hausdorff scheme X whose Hausdorff quotient is $\mathbb{P}^1 \times \mathbb{P}^1$. More precisely, there is an embedding $C \xrightarrow{\sim} \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ as a bidegree $(2, 2)$ curve, such that X is the union of two copies $X_<$ and $X_>$ of $\mathbb{P}^1 \times \mathbb{P}^1$ patched together outside the curve Γ . For instance, if we fix $\det(E) = \mathcal{O}_C(w_\infty)$, then $X_<$ will correspond to those parabolic bundles (E, \mathbf{p}) such that $E = E_1$ is the unique non trivial extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_1 \longrightarrow \mathcal{O}_C(w) \longrightarrow 0.$$

Then, the missing ν -flat bundles occurring in $X_>$ are those decomposable E with indecomposable parabolic structure. More details are recalled in Section 3.1.

We denote by $\text{Con}_<^\nu(C, D)$ the open subset of $\text{Con}^\nu(C, D)$ over $X_<$. In Section 3.4, an explicit universal family of connections is given for $\text{Con}_<^\nu(C, D)$: through a birational trivialization $E_1 \dashrightarrow \mathcal{O}_C \times \mathcal{O}_C$, it is given by an explicit family of Fuchsian system with 3 additional apparent singular points. To set our main result, given a parabolic bundle (E, \mathbf{p}) , let us introduce the parabolic structure $\mathbf{p}^- = (p_1^-, p_2^-)$ associated to ν_k^- -eigenspaces, and denote $\mathbf{p}^+ := \mathbf{p}$. Then, we have a natural map

$$\text{Par} : \begin{cases} \text{Con}_<^\nu(C, D) & \rightarrow & X_< \times X_< \\ (E_1, \nabla) & \mapsto & ((E_1, \mathbf{p}^+), (E_1, \mathbf{p}^-)) \end{cases}$$

Since ν is generic, we can assume $\nu_k^+ \neq \nu_k^-$ for $k = 1, 2$, and therefore $p_k^+ \neq p_k^-$. This implies that the image of Par in $X_< \times X_<$ avoid the ‘‘incidence variety’’

$$I := \{z_1 = \zeta_1\} \cup \{z_2 = \zeta_2\} \subset \underbrace{(\mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1) \times (\mathbb{P}_{\zeta_1}^1 \times \mathbb{P}_{\zeta_2}^1)}_{X_< \times X_<}$$

Setting $\nu_k := \nu_k^+ - \nu_k^-$, our main result is (see Theorem 3.8 and Section 3.8)

Theorem 2. *If $\nu_1 \cdot \nu_2 \neq 0$, then the map*

$$\text{Par} : \text{Con}_<^\nu(C, D) \rightarrow X_< \times X_<$$

induces an isomorphism onto the complement of the incidence variety $X_< \times X_< \setminus I$ and the image of the symplectic structure is given by

$$\omega = -\frac{1}{2} \left\{ \nu_1 \frac{dz_1 \wedge d\zeta_1}{(z_1 - \zeta_1)^2} + \nu_2 \frac{dz_2 \wedge d\zeta_2}{(z_2 - \zeta_2)^2} \right\}.$$

In the spirit of classical Torelli Theorem, this shows that exponents (difference of eigenvalues) can be read off from the moduli space, see Proposition 3.20. This completes the result of [6, Theorem B] where it is shown that the moduli space of ν -flat bundles keeps track of the punctured curve (C, D) .

We investigate in Section 3.6.3 how to cover the full moduli space $\text{Con}^\nu(C, D)$ by three charts like the one in Theorem 2, see Theorem 3.14. Finally, in Section 3.7 (see also Section 3.8), we study the ‘‘apparent map’’ which, to a connection $(E, \nabla) \in \text{Con}_<^\nu(C, D)$, associates the position of the apparent singular points of the corresponding scalar equation via the ‘‘cyclic vector’’ $\mathcal{O}_C \subset E_1$ (see [10, 13] for instance). This gives us a map

$$\text{App} : \text{Con}_<^\nu(C, D) \rightarrow |\mathcal{O}_C(w_\infty + t_1 + t_2)| \simeq \mathbb{P}^2$$

which turns out to be Lagrangian. Similarly as [13, Theorem 1.1] in genus zero case, we have

Theorem 3. *If $\nu_1 + \nu_2 + 1 \neq 0$, then the rational map*

$$\text{Bun} \times \text{App} : \text{Con}_{<}^{\nu}(C, D) \longrightarrow (\mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1) \times \mathbb{P}^2$$

is birational. More precisely, the restriction

$$\text{App} : \text{Bun}^{-1}(\mathbf{p}) \longrightarrow \mathbb{P}^2$$

is injective if, and only if, $\mathbf{p} \in X_{<} \simeq \mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1$ is lying outside $\{z_1 = t\} \cup \{z_2 = t\}$, where t is the first coordinate of t_1 .

Notation and conventions. Curves are always assumed to be irreducible and defined over the field \mathbb{C} of complex numbers. Given a projective smooth curve C and a holomorphic vector bundle $E \longrightarrow C$ over C we make no difference in notation between the total space E and its locally free sheaf defined by holomorphic sections. We denote by E^* the sheaf $\mathcal{H}om_{\mathcal{O}_C}(E, \mathcal{O}_C)$. If E and F are two vector bundles over C we denote by $\text{Hom}(E, F)$ the space of global sections of the sheaf $\mathcal{H}om_{\mathcal{O}_C}(E, F)$. In particular, $\text{End}(E) = \text{Hom}(E, E)$ denotes the set of endomorphisms of E . We use the notation \mathbb{P}_z^1 to denote the projective line \mathbb{P}^1 equipped with an affine coordinate $z \in \mathbb{C}$; this notation will be used to distinguish between several occurrences of \mathbb{P}^1 .

2. EXISTENCE OF LOGARITHMIC CONNECTIONS

In this section, we shall investigate the existence of logarithmic connections on a given rank 2 quasi-parabolic vector bundle (E, \mathbf{p}) over an elliptic curve C . We give a criterion in Theorem 2.2 which extends the famous Weil criterion [21] for holomorphic connections, and Biswas criterion [4] for logarithmic connections with rational residual eigenvalues.

2.1. Logarithmic connections. Let us fix a set of n distinct points $\mathbf{t} = \{t_1, \dots, t_n\}$ on a smooth projective complex curve C and let $D = t_1 + \dots + t_n$ be the reduced effective divisor associated to it. A *logarithmic connection* on a vector bundle E over C with polar divisor D is a \mathbb{C} -linear map

$$\nabla : E \longrightarrow E \otimes \Omega_C^1(D)$$

satisfying the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for any local section s of E and function f on C . If E is of rank r , then it is given by a cocycle $\{G_{ij}\} \in H^1(C, GL(r, \mathcal{O}_C))$ defined by an atlas of trivializations $C = \cup U_i$ where $E|_{U_i} \simeq U_i \times \mathbb{C}^r$. Locally, over each U_i , the connection writes $\nabla = d_C + A_i$ where $d_C : \mathcal{O}_C \longrightarrow \Omega_C^1$ is the differential operator on C and A_i is a $r \times r$ matrix with coefficients in $\Omega_C^1(D)$ that glue together through the transition map. That is, A_i is a $r \times r$ matrix of meromorphic 1-forms having at most simple poles on \mathbf{t} and this collection of matrices must satisfies

$$A_j = G_{ij} \cdot A_i \cdot G_{ij}^{-1} + dG_{ij} \cdot G_{ij}^{-1}$$

over intersections $U_i \cap U_j$. For each pole $t_k \in U_i$, the residue homomorphism $\text{Res}_{t_k}(\nabla) = \text{Res}_{t_k}(A_i) \in \text{End}(E|_{t_k})$ is well defined. If E is of rank 2, then let ν_k^+

and ν_k^- be the eigenvalues of $\text{Res}_{t_k}(\nabla)$, called *local exponents* of ∇ over t_k . The data

$$\nu = (\nu_1^\pm, \dots, \nu_n^\pm) \in \mathbb{C}^{2n}$$

is called *local exponent* of ∇ . We note that the connection ∇ induces a logarithmic connection $\text{tr}(\nabla)$ on the determinant line bundle $\det(E)$ with

$$\text{Res}_{t_k}(\text{tr}(\nabla)) = \nu_k^+ + \nu_k^-.$$

By Residue Theorem we obtain the Fuchs relation

$$(2.1) \quad d + \sum_{k=1}^n (\nu_k^+ + \nu_k^-) = 0.$$

where $d = \deg(E)$.

When E admits a holomorphic connection, then it is called *flat*. There is a similar notion for quasi-parabolic vector bundles. We fix $(E, \mathbf{p} = \{p_k\})$ a quasi-parabolic rank 2 vector bundle over (C, \mathbf{t}) , see Section 4.1. We shall say that (E, \mathbf{p}) is ν -*flat* if it admits a logarithmic connection ∇ with given local exponent ν satisfying

$$\text{Res}_{t_k}(\nabla)(p_k) = \nu_k^+ \cdot p_k.$$

We also say that (E, ∇, \mathbf{p}) is a ν -*parabolic connection*.

The following flatness criterion for vector bundles over curves is due to A. Weil (see [21, 2]): a vector bundle is flat if, and only if, it is the direct sum of indecomposable vector bundles of degree zero. A generalization to the parabolic context of the Weil's criterion was obtained in [4] where the local exponents are supposed to be rational numbers. Besides that, when $C = \mathbb{P}^1$ and ν is generic

$$(2.2) \quad \sum_{k=1}^n \nu_k^{\epsilon_k} \notin \mathbb{Z}; \quad \epsilon_k \in \{+, -\}$$

the following equivalent conditions are known (cf. [1]):

- (1) (E, \mathbf{p}) is ν -*flat*;
- (2) $\text{End}_0(E, \mathbf{p}) = 0$;
- (3) (E, \mathbf{p}) is indecomposable.

Remark 2.1. We note that if C is an elliptic curve then the above conditions (1) and (2) are not equivalents. For instance, let E_0 be the unique non trivial extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_0 \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

If \mathbf{p} is a parabolic structure with all parabolics lying on $\mathcal{O}_C \hookrightarrow E_0$, then from Proposition 4.4:

$$\text{End}_0(E_0, \mathbf{p}) = \mathbb{C}.$$

But (E_0, \mathbf{p}) is indecomposable and, as we will see in Theorem 2.2, it is ν -flat for any ν satisfying the Fuchs Relation.

2.2. Flatness criterion over elliptic curves. In this section, we will obtain a parabolic version of Weil's criterion over elliptic curves, in the same spirit of [1, 4] (see Theorem 2.2).

Before that, let us recall the definition of direct summand as well as parabolic degree. Let (E, \mathbf{p}) be a rank 2 quasi-parabolic bundle. We say that (L, \mathbf{q}) is a

direct summand of (E, \mathbf{p}) if either $(E, \mathbf{p}) = (L, \mathbf{q})$ or L is a line bundle and there exists another parabolic line bundle, say (M, \mathbf{r}) , such that

$$(E, \mathbf{p}) = (L, \mathbf{q}) \oplus (M, \mathbf{r}).$$

Its *parabolic degree*, denoted by $\deg_\nu(L, \mathbf{q})$, with respect to $\nu = (\nu_1^\pm, \dots, \nu_n^\pm)$ is defined as follows:

- if $(E, \mathbf{p}) = (L, \mathbf{q})$ then we set

$$\deg_\nu(L, \mathbf{q}) := \deg(E) + \sum_{k=1}^n (\nu_k^+ + \nu_k^-);$$

- and if $(E, \mathbf{p}) = (L, \mathbf{q}) \oplus (M, \mathbf{r})$ then it is defined by

$$\deg_\nu(L, \mathbf{q}) := \deg(L) + \sum_{k=1}^n \nu_k^{\epsilon_k}$$

where $\epsilon_k = +$ if p_k is contained in L and $\epsilon_k = -$ if p_k is contained in M .

Theorem 2.2. *Given a quasi-parabolic bundle (E, \mathbf{p}) over an elliptic curve C , the following conditions are equivalent*

- (1) (E, \mathbf{p}) is ν -flat;
- (2) every direct summand of (E, \mathbf{p}) is of parabolic degree zero with respect to ν .

The proof will be given in Section 2.2.2. As a consequence, one obtains the following corollary for generic exponents ν satisfying the Fuchs Relation

$$d + \sum_{k=1}^n (\nu_k^+ + \nu_k^-) = 0.$$

Corollary 2.3. *Assume $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$ for any $\epsilon_k \in \{+, -\}$. Given a quasi-parabolic bundle (E, \mathbf{p}) over an elliptic curve C , with $\deg(E) = d$, the following conditions are equivalent*

- (1) (E, \mathbf{p}) is ν -flat;
- (2) (E, \mathbf{p}) is indecomposable.

Recall that over \mathbb{P}^1 , the above conditions (1) and (2) are equivalent to (E, \mathbf{p}) be simple, i.e. the only automorphisms of E preserving parabolics are scalar. Here, over an elliptic curve it is no more true. As in Remark 2.1, if \mathbf{p} is a parabolic structure on E_0 with all parabolics lying on $\mathcal{O}_C \hookrightarrow E_0$ then $\text{End}_0(E_0, \mathbf{p}) = \mathbb{C}$. Hence, (E_0, \mathbf{p}) is not simple:

$$\text{Aut}(E_0, \mathbf{p}) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} ; a \in \mathbb{C}^*, b \in \mathbb{C} \right\}.$$

2.2.1. *Preliminary lemmas.* The easy part of Theorem 2.2 is (1) \Rightarrow (2). As we shall see in the following lemma, it is a consequence of Fuchs Relation.

Lemma 2.4. *If (E, \mathbf{p}) is ν -flat, then every direct summand of (E, \mathbf{p}) is of parabolic degree zero.*

Proof. Let us suppose $(E, \mathbf{p}) = (L, \mathbf{q}) \oplus (M, \mathbf{r})$. The bundle E is defined by gluing local charts $U_i \times \mathbb{C}^2$ with transition matrices

$$M_{ij} = \begin{pmatrix} a_{ij} & 0 \\ 0 & b_{ij} \end{pmatrix}$$

where the subbundles L and M are respectively generated by e_1 and e_2 .

Let ∇ be a connection over (E, \mathbf{p}) with local exponent ν , given in those charts by $\nabla = d + A_i$ where

$$A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

First we note that $\nabla_1 := \nabla|_L = d + \alpha_i$ defines a connection over L . In fact, the compatibility conditions for ∇ imply

$$\alpha_j = \alpha_i + a_{ij}^{-1} \cdot da_{ij}.$$

Then under hypothesis of (E, \mathbf{p}) be decomposable one obtains

$$\text{Res}_{t_k}(\nabla_1) = \nu \frac{\epsilon_k}{k}$$

where $\epsilon_k = +$ if p_k lies in L and $\epsilon_k = -$ if p_k lies in M . The conclusion of the proof follows from Fuchs Relation of ∇_1 over L . \square

In order to prove that (2) implies (1) in Theorem 2.2, let us consider the set $\text{End}_0(E, \mathbf{p})$ of traceless endomorphisms of E leaving fixed the parabolics. In the context of \mathfrak{sl}_2 -connections, the vanishing of this set is a sufficient condition to a quasi-parabolic vector bundle be ν -flat. This has been proved in [1, Proposition 3] in the case $X = \mathbb{P}^1$, but the same proof works to the general case. Here we do the details for the reader's convenience.

Lemma 2.5. *Let C be a projective smooth curve of genus $g \geq 0$ and E be a rank 2 vector bundle over C with trivial determinant bundle. If*

$$\text{End}_0(E, \mathbf{p}) = \{0\}$$

then (E, \mathbf{p}) is ν -flat for any ν satisfying the Fuchs Relation.

Proof. To give a logarithmic connection ∇ on (E, \mathbf{p}) with local exponent ν it is equivalent to give an \mathfrak{sl}_2 -connection $\nabla \otimes \zeta$ on (E, \mathbf{p}) for suitable connection ζ on \mathcal{O}_C . Then let us assume $\nu = (\pm\nu_1, \dots, \pm\nu_n)$.

We must construct a logarithmic connection ∇ over (E, \mathbf{p}) with local exponent ν satisfying

$$(2.3) \quad \text{Res}_{t_k}(\nabla)(p_k) = \nu_k \cdot p_k$$

for any $k = 1, \dots, n$. We can do it locally on C . The obstruction to global existence is measured by a certain cohomology group. In fact, let ∇_i be a connection over $E|_{U_i}$ satisfying the desired condition for any $k = 1, \dots, n$ such that $t_k \in U_i$ for suitable open sets $U_i \subset C$ covering C . Note that $\theta_{ij} = \nabla_i - \nabla_j$ satisfies $\text{Res}_{t_k}(\theta_{ij})(p_k) = 0$, by (2.3). Then the differences $\nabla_i - \nabla_j$ define a cocycle

$$\{\nabla_i - \nabla_j\} \in H^1(C, \mathcal{E})$$

where

$$\mathcal{E} = \{\theta \in \mathcal{E}nd_0(E) \otimes \Omega_C^1(D) ; \text{Res}_{t_k}(\theta)(p_k) = 0\}.$$

Hence the global existence is insured by the vanishing of this cohomology group. Here $\mathcal{E}nd_0(E)$ is the sheaf of traceless endomorphisms. We will prove that there is an isomorphism

$$H^1(C, \mathcal{E}) \simeq \text{End}_0(E, \mathbf{p})^*.$$

In order to prove it, let us remark that, from Serre's duality theorem, $H^1(C, \mathcal{E})$ is dual to $H^0(C, \mathcal{E}^* \otimes \Omega_C^1)$. Let us consider the \mathcal{O}_C -bilinear symmetric map

$$\begin{aligned} \mathcal{E}nd_0(E) \times \mathcal{E}nd_0(E) &\longrightarrow \mathcal{O}_C. \\ (A, B) &\longmapsto \text{tr}(A \cdot B) \end{aligned}$$

It induces an isomorphism $\psi : \mathcal{E}nd_0(E) \longrightarrow \mathcal{E}nd_0(E)^*$. Let $\tilde{\mathcal{E}}$ be the subsheaf of $\mathcal{E}nd_0(E)$ defined by

$$\tilde{\mathcal{E}} = \{\theta \in \mathcal{E}nd_0(E) ; \theta(p_k) = 0\}.$$

We note that \mathcal{E} coincides with $\tilde{\mathcal{E}} \otimes \Omega_C^1(D)$. Since the image of $\tilde{\mathcal{E}}$ by ψ is $\tilde{\mathcal{E}}^*$ one obtains

$$\tilde{\mathcal{E}}^* \otimes \mathcal{O}_C(-D) = \{\alpha \in \mathcal{E}nd_0(E) ; \alpha(p_k) \subset p_k\}.$$

This implies that

$$H^0(C, \tilde{\mathcal{E}}^* \otimes \mathcal{O}_C(-D)) = \text{End}_0(E, \mathbf{p}).$$

and we get the desired isomorphism:

$$H^1(C, \mathcal{E}) \simeq \text{End}_0(E, \mathbf{p})^*.$$

□

Now we shall give the proof of Theorem 2.2.

2.2.2. Proof of Theorem 2.2. It follows from Lemma 2.4 that (1) implies (2). Now let (E, \mathbf{p}) be a quasi-parabolic bundle satisfying the hypothesis (2) of the statement. First, let us assume that (E, \mathbf{p}) is decomposable:

$$(E, \mathbf{p}) = (L_1, \mathbf{p}_1) \oplus (L_2, \mathbf{p}_2).$$

Since (L_i, \mathbf{p}_i) has parabolic degree zero for each $i \in \{1, 2\}$, there is a logarithmic connection α_i on L_i , satisfying

$$\text{Res}_{t_k}(\alpha_i) = \nu_k^{\epsilon_k}$$

where $\epsilon_k = +$ if p_k lies in L_i and $\epsilon_k = -$ if p_k does not lie in L_i . They define a diagonal connection

$$\nabla = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

on (E, \mathbf{p}) with local exponent ν .

Now let (E, \mathbf{p}) be indecomposable. After elementary transformations and twists, one can assume that E has trivial determinant bundle. Besides that, it is enough to show the existence of an \mathfrak{sl}_2 -connection on (E, \mathbf{p}) with local exponent $\nu = (\pm\nu_1, \dots, \pm\nu_n)$. It follows from Proposition 4.5 that

$$\text{End}_0(E, \mathbf{p}) = \{0\}$$

unless E equals E_0 , up to elementary transformations and twists, and all the parabolics lie in the maximal subbundle $\mathcal{O}_C \hookrightarrow E_0$. Then, from Lemma 2.5, to conclude the proof of Theorem 2.2, it is enough to consider the case E_0 with all the parabolics in $\mathcal{O}_C \hookrightarrow E_0$. We shall do it in the following lemma.

Lemma 2.6. *Let (E_0, \mathbf{p}) be a quasi-parabolic bundle where all the parabolics lie in the maximal subbundle $\mathcal{O}_C \hookrightarrow E_0$. Then there exists an \mathfrak{sl}_2 -connection on (E_0, \mathbf{p}) with given local exponent $(\pm\nu_1, \dots, \pm\nu_n)$.*

Proof. The idea of the proof is the following: E_0 can be obtained from $\mathcal{O}_C \oplus \mathcal{O}_C(t_1)$ by one negative elementary transformation at a direction $q \subset \mathcal{O}_C \oplus \mathcal{O}_C(t_1)|_{t_1}$ which is not contained in $\mathcal{O}_C|_{t_1}$ neither in $\mathcal{O}_C(t_1)|_{t_1}$. The maximal subbundle $\mathcal{O}_C \hookrightarrow E_0$ corresponds to the maximal subbundle $\mathcal{O}_C(t_1) \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C(t_1)$. Then we need to construct a logarithmic connection ∇ on $\mathcal{O}_C \oplus \mathcal{O}_C(t_1)$ with poles at $D = t_1 + \dots + t_n$, satisfying the following conditions:

- (1) $\text{Res}_{t_1}(\nabla)$ has $-\nu_1$ and $\nu_1 - 1$ as eigenvalues, where $-\nu_1$ is associated with the eigenspace q as above;
- (2) $\text{Res}_{t_k}(\nabla)$, $k = 2, \dots, n$, has $\pm\nu_k$ as eigenvalues, where ν_k is associated with the eigenspace $\mathcal{O}_C(t_1)|_{t_k}$.

We can assume that the vector bundle $\mathcal{O}_C \oplus \mathcal{O}_C(t_1)$ is defined by the cocycle

$$G_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & a_{ij} \end{pmatrix}$$

where $\{a_{ij}\}$ defines the line bundle $\mathcal{O}_C(t_1)$.

Now, to give a logarithmic connection ∇ on $\mathcal{O}_C \oplus \mathcal{O}_C(t_1)$ with poles at D is equivalent to give $\nabla = d + A_i$ in charts $U_i \subset C$ where

$$A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \in \text{GL}_2(\Omega_{U_i}^1(D))$$

with $\alpha = \{\alpha_i\}$, $\beta = \{\beta_i\}$, $\gamma = \{\gamma_i\}$ and $\delta = \{\delta_i\}$ satisfying the compatibility conditions:

$$A_i \cdot G_{ij} = G_{ij} \cdot A_j + dG_{ij} \cdot G_{ij}^{-1}$$

on each intersection $U_i \cap U_j$. Equivalently,

$$\begin{cases} d + \alpha & \text{is a connection on } \mathcal{O}_C \\ d + \delta & \text{is a connection on } \mathcal{O}_C(t_1) \\ \beta & \text{defines a global section } \{a_i \beta_i\} \in \text{H}^0(C, \Omega_C^1(D - t_1)) \\ \gamma & \text{defines a global section } \{a_i^{-1} \gamma_i\} \in \text{H}^0(C, \Omega_C^1(D + t_1)) \end{cases}$$

where $a_{ij} = \frac{a_i}{a_j}$ is a meromorphic resolution of the cocycle, the vector e_1 generates the subbundle \mathcal{O}_C and e_2 generates $\mathcal{O}_C(t_1)$ on each chart $U_i \subset C$.

In order to find α and δ we define λ^+ and λ^- as

$$\begin{aligned} \lambda^+ &= \nu_2 + \dots + \nu_n \\ \lambda^- &= -\nu_2 - \dots - \nu_n - 1. \end{aligned}$$

Let $d + \alpha$ be a logarithmic connection on \mathcal{O}_C satisfying

$$\text{Res}_{t_1}(\alpha) = \lambda^+ ; \text{Res}_{t_k}(\alpha) = -\nu_k, k \geq 2$$

and $d + \delta$ be a logarithmic connection on $\mathcal{O}_C(t_1)$ satisfying

$$\text{Res}_{t_1}(\delta) = \lambda^- ; \text{Res}_{t_k}(\delta) = \nu_k, k \geq 2.$$

It remains to find β and γ subject to the conditions (1) and (2) above. Since residues of α and δ have already been chosen, condition (2) is equivalent to the vanishing of β in t_k for $k \geq 2$, that is, β is induced by an element

$$\beta' \in \text{H}^0(C, \Omega_C^1(D - t_1) \otimes \mathcal{O}_C(-t_2 - \dots - t_n)) = \text{H}^0(C, \Omega_C^1).$$

In addition, as C is an elliptic curve, we can assume $\beta' \in \mathbb{C}$. Then we have one degree of freedom to choose β satisfying condition (2), which is determined by its residue at t_1 , still denoted by $\beta' = \text{Res}_{t_1}(\beta)$.

The choice of γ is independent of condition (2). For instance for each $k \geq 2$, $\text{Res}_{t_k}(\gamma)$ determines the eigenvector associated with the eigenvalue $-\nu_k$. But we have to choose γ satisfying condition (1). Given $\gamma \in H^0(C, \Omega_C^1(D+t_1))$ with residue γ' at t_1 , one gets

$$\text{Res}_{t_1}(\nabla) = \begin{pmatrix} \lambda^+ & \beta' \\ \gamma' & \lambda^- \end{pmatrix}.$$

Then saying that $-\nu_1$ and $\nu_1 - 1$ are eigenvalues of $\text{Res}_{t_1}(\nabla)$ is equivalent to

$$\lambda^+ \lambda^- - \beta' \gamma' = (-\nu_1)(\nu_1 - 1).$$

We leave the reader to verify that this last equality is equivalent to

$$\beta' \gamma' = (\nu_1 + \lambda^+) \cdot (\nu_1 - \lambda^+ - 1).$$

On one hand, if the term in the right side is non zero, we can find $\beta' \neq 0$ and $\gamma' \neq 0$ satisfying this equality. In fact, $\gamma' \neq 0$ if and only if γ does not vanish at t_1 . Then we need to find $\gamma \in H^0(C, \Omega_C^1(D+t_1))$ which is not contained in $H^0(C, \Omega_C^1(D))$. We can do it, because C is elliptic, hence the linear system which corresponds to $H^0(C, \Omega_C^1(D+t_1))$ has no base points. On the other hand, if the term in the right side equals zero, we set

$$\begin{cases} \beta' = 0, \gamma' \neq 0 & \text{if } \nu_1 + \lambda^+ = 0 \\ \beta' \neq 0, \gamma' = 0 & \text{if } \nu_1 - \lambda^+ - 1 = 0 \end{cases}$$

Finally, with these choices we see that the eigenspace associated to $-\nu_1$ is not contained in $\mathcal{O}_C|_{t_1}$ neither in $\mathcal{O}_C(t_1)|_{t_1}$. \square

2.3. Indecomposable quasi-parabolic bundles. In view of Corollary 2.3 it is interesting to characterize degree one indecomposable rank 2 quasi-parabolic bundles when there is at least one parabolic point. Fix $\det(E) = \mathcal{O}_C(w)$, $w \in C$.

Proposition 2.7. *Assume $n \geq 1$ and let L be a degree k line bundle over the elliptic curve C . If the parabolic bundle $(L \oplus L^{-1}(w), \mathbf{p})$ is indecomposable then*

$$-n + 1 < 2k < n + 1.$$

Proof. Firstly let us assume $2k \leq -n + 1$. Since C is elliptic, it follows from Riemann-Roch theorem that

$$\dim H^0(C, \mathcal{O} \oplus L^{-2}(w)) = -2k + 2.$$

Then the family of embeddings $L \hookrightarrow L \oplus L^{-1}(w)$ is of dimension $-2k + 1$. Our hypothesis on k implies that we can choose an embedding of L containing any parabolic lying outside $L^{-1}(w)$. In this case our quasi-parabolic bundle would be decomposable. The same argument works with $L^{-1}(w)$ instead of L if $2k \geq n + 1$. \square

The case $n = 2$ is of particular interest for us. Then we shall give a simple consequence of this proposition that will be useful in the sequel. The following proposition characterizes quasi-parabolic bundles (E, \mathbf{p}) , $\mathbf{p} = \{p_1, p_2\}$, of rank 2 and determinant $\mathcal{O}_C(w)$ arising in our moduli space of connections.

Proposition 2.8. *Let (E, \mathbf{p}) , $\mathbf{p} = \{p_1, p_2\}$, be a rank 2 quasi-parabolic bundle over an elliptic curve C with $\det(E) = \mathcal{O}_C(w)$. If (E, \mathbf{p}) is indecomposable then one of the following assertions hold true*

- (1) $E \simeq L \oplus L^{-1}(w)$ with $\deg(L) = 0$; moreover, the parabolic structure satisfies:

- p_1, p_2 do not lie neither on $L^{-1}(w)$, and nor on the same embedding $L \hookrightarrow E$ (a codimension one condition);
 - $L^2 = \mathcal{O}_C(w - t_k)$ for some $k = 1, 2$, and p_k lie outside of L and $L^{-1}(w)$;
- (2) $E = E_1$ is indecomposable, defined by the unique non trivial extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_1 \longrightarrow \mathcal{O}_C(w) \longrightarrow 0.$$

Proof. If E is decomposable with $\det(E) = \mathcal{O}_C(w)$ then Proposition 2.7 implies $E \simeq L \oplus L^{-1}(w)$ with $\deg(L)$ equals 0 or 1. If $\deg(L) = 0$ we are done. If $\deg(L) = 1$, then we set $M := L^{-1}(w)$ to get $E \simeq M^{-1}(w) \oplus M$. Since $\deg(M) = 0$, this gives the first assertion when E is decomposable. Now any decomposition of E is given by the unique destabilizing subsheaf $L^{-1}(w)$ together with any embedding $L \hookrightarrow E$. Such an embedding is given by a linear combination of the initial factor L with a global section of $\mathrm{Hom}(L, L^{-1}(w)) \simeq L^{-2}(w)$, so that we get a one parameter family of possible decompositions. We note that all embeddings coincide with L exactly at $t \in C$ where $L^{-2}(w) = \mathcal{O}_C(t)$. If $t \neq t_1$, then there is a unique $L \subset E$ passing through p_1 , provided that $p_1 \notin L^{-1}(w)$; then a generic p_2 , i.e. not belonging to either that L or $L^{-1}(w)$ will be an obstruction to decomposability. On the other hand, if $t = t_1$, and if p_1 is generic, then there is no L passing through and (E, \mathbf{p}) is indecomposable. After reasoning similarly at t_2 , one easily deduce that the only cases where we cannot find a decomposition of E compatible with parabolics are those two cases listed in the statement.

If E is indecomposable and $\det(E) = \mathcal{O}_C(w)$, then its well known that E contains \mathcal{O}_C as maximal subbundle. Moreover, there is only one indecomposable rank 2 bundle, up to isomorphism, having $\mathcal{O}_C(w)$ as determinant bundle (see for example [14, Lemma 4.4 and 4.5]). \square

3. MODULI SPACE OF CONNECTIONS ON ELLIPTIC CURVES WITH TWO POLES

Now we shall fix the data for our moduli space of logarithmic connections with two poles over an elliptic curve; see Section 4.5 for a more complete introduction to moduli spaces of connections. Let C be an elliptic curve, for computation we assume that $C \subset \mathbb{P}^2$ is the smooth projective cubic curve

$$(3.1) \quad zy^2 = x(x-z)(x-\lambda z)$$

with $\lambda \in \mathbb{C}$, $\lambda \neq 0, 1$. Let us denote by $w_\infty = (0 : 1 : 0) \in C$ the identity with respect to the group structure, and

$$(3.2) \quad w_0 = (0 : 0 : 1), \quad w_1 = (1 : 0 : 1), \quad w_\lambda = (\lambda : 0 : 1)$$

the 2-torsion points.

Let $D = t_1 + t_2$ be a reduced divisor on C , where $t_2 = -t_1$ with respect to the group structure of C , i.e. defined by say $x = t$. We assume $t_1 \neq t_2$, i.e. $x \neq 0, 1, \lambda, \infty$. Let us fix local exponents $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ and define eigenvalues

$$(3.3) \quad \nu = (\nu_1^\pm, \nu_2^\pm) := \left(\pm \frac{\nu_1}{2} - \frac{1}{2}, \pm \frac{\nu_2}{2} \right).$$

therefore satisfying Fuchs Relation

$$\nu_1^+ + \nu_1^- + \nu_2^+ + \nu_2^- + 1 = 0.$$

To avoid dealing with reducible connections, we assume moreover a generic condition

$$\nu_1 \pm \nu_2 \notin \mathbb{Z} \setminus 2\mathbb{Z} \quad (\text{i.e. not an odd integer})$$

so that $\nu_1^{\epsilon_1} + \nu_2^{\epsilon_2} \notin \mathbb{Z}$ for any $\epsilon_k \in \{+, -\}$.

Fix $\zeta : \mathcal{O}_C(w_\infty) \rightarrow \mathcal{O}_C(w_\infty) \otimes \Omega_C^1(t_1)$ any rank one logarithmic connection on $\mathcal{O}_C(w_\infty)$ satisfying

$$\text{Res}_{t_1}(\zeta) = -1$$

Since local ζ -horizontal sections have a simple zero at t_1 , the invertible sheaf generated by these sections is $\mathcal{O}_C(w_\infty - t_1)$, and ζ corresponds to a holomorphic connection on this latter bundle. In particular, the monodromy of ζ must be non trivial.

We denote by $\text{Con}^\nu(C, D)$ the moduli space of triples (E, ∇, \mathbf{p}) where

- (1) (E, \mathbf{p}) is a rank 2 quasi-parabolic vector bundle over (C, D) having $\mathcal{O}_C(w_\infty)$ as determinant bundle;
- (2) $\nabla : E \rightarrow E \otimes \Omega_C^1(D)$ is a logarithmic connection on E with polar divisor D , having ν as local exponents, $\text{Res}_{t_k}(\nabla)$ acts on p_k by multiplication by ν_k^+ and its trace is given by $\text{tr}(\nabla) = \zeta$;
- (3) two triples (E, ∇, \mathbf{p}) and $(E', \nabla', \mathbf{p}')$ are equivalent when there is an isomorphism between quasi-parabolic bundles (E, \mathbf{p}) and (E', \mathbf{p}') conjugating ∇ and ∇' .

The reason why we have chosen $\mathcal{O}_C(w_\infty)$ instead of \mathcal{O}_C as our fixed determinant bundle is the following: there is an open subset of $\text{Con}^\nu(C, D)$ formed by triples $(E_1, \nabla, \mathbf{p})$ where E_1 is the unique indecomposable rank 2 bundle over C , up to isomorphism, corresponding to the extension

$$0 \rightarrow \mathcal{O}_C \rightarrow E_1 \rightarrow \mathcal{O}_C(w_\infty) \rightarrow 0.$$

This allow us to avoid dealing with varying underlying vector bundle. Some consequences of this choice will be more clear in the next section.

In the next lines, we will study the forgetful map $Bun : (E, \nabla, \mathbf{p}) \mapsto (E, \mathbf{p})$ from $\text{Con}^\nu(C, D)$ to the moduli space $\text{Bun}(C, D)$ of *parabolic bundles* (E, \mathbf{p}) having $\mathcal{O}_C(w_\infty)$ as determinant bundle. Actually, the construction of the moduli space $\text{Con}^\nu(C, D)$ needs a choice of weights to impose a stability condition. But under our generic hypothesis on ν , all connections are stable and the construction does not depend of this choice. On the other hand, to obtain a good moduli space of quasi-parabolic bundles, we need to introduce a stability condition (see Section 4.1).

3.1. Moduli space of parabolic vector bundles. In this section we recall the construction of the moduli space of parabolic bundles over an elliptic curve, see [6] for details. Given weights $\mu = (\mu_1, \mu_2) \in [0, 1]^2$, let $\text{Bun}_{w_\infty}^\mu(C, D)$ be the moduli space of μ -semistable parabolic bundles with fixed determinant bundle $\mathcal{O}_C(w_\infty)$. The space of weights is divided in two chambers by a wall $\mu_1 + \mu_2 = 1$. Inside each chamber any point in $\text{Bun}_{w_\infty}^\mu(C, D)$ is represented by the same bundle and all μ -semistable bundles are μ -stable. Strictly μ -semistable bundles only occurs along the wall. It follows from [6, Theorem A] that $\text{Bun}_{w_\infty}^\mu(C, D)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ for any choice of weights. We recall briefly this construction in the next lines to the readers convenience.

Firstly, we describe parabolic bundles inside the chambers, see also [6, Proposition 4.4]. Let us denote by $X_{<} = \text{Bun}_{w_\infty}^{<}(C, D)$ the moduli space corresponding to the chamber $0 < \mu_1 + \mu_2 < 1$ and by $X_{>} = \text{Bun}_{w_\infty}^{>}(C, D)$ the other moduli space corresponding to $1 < \mu_1 + \mu_2 < 2$. All the μ -semistable parabolic bundles arising in $X_{<}$ are of the form $(E_1, \mathbf{p} = \{p_1, p_2\})$. Maximal subbundles of E_1 have degree zero, then any parabolic bundle in $X_{<}$ is μ -stable. Each parabolic bundle is completely determined by

$$(p_1, p_2) \in E_1|_{t_1} \times E_1|_{t_2} \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

Then, we get the identification $X_{<} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. When $1 < \mu < 2$, parabolic bundles $(E_1, \{p_1, p_2\})$ having a degree zero line bundle $L \hookrightarrow E_1$ passing through the two parabolic directions became unstable, where L is destabilizing:

$$\text{Stab}(L) = 1 - \mu_1 - \mu_2 < 0.$$

But we need to add parabolic bundles of the following form

$$(L \oplus L^{-1}(w_\infty), \{p_1, p_2\}), \quad \text{deg}(L) = 0$$

where no parabolic p_k lie on $L^{-1}(w_\infty)$ and not all p_k lie on the same embedding $L \hookrightarrow E$ (see Proposition 2.8). Since automorphisms group of $L \oplus L^{-1}(w_\infty)$ is two dimensional, all parabolic bundles

$$(L \oplus L^{-1}(w_\infty), \{p_1, p_2\})$$

with L fixed, represent the same element in $X_{>}$. Then one obtains an identification $X_{>} \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

When μ is inside the wall, the picture is described in [6, Proposition 4.1 and Theorem 4.2]. If $\mu_1 + \mu_2 = 1$ with $\mu_k \neq 0$ for $k = 1, 2$, respective parabolic bundles in $\Gamma_{<}$ and $\Gamma_{>}$ are identified and parabolic bundles $(L \oplus L^{-1}(w_\infty), \mathbf{p})$ with both direction on L appear. But if $\mu_k = 0$, then we also find bundles $L \oplus L^{-1}(w_\infty)$ with p_k lying on $L^{-1}(w_\infty)$.

Let us identify C with its Jacobian variety $\text{Jac}(C)$ of degree zero line bundles

$$\begin{aligned} C &\longrightarrow \text{Jac}(C). \\ p &\mapsto \mathcal{O}_C(w_\infty - p) \end{aligned}$$

The locus Γ of parabolic bundles inside $X_{>}$ that became unstable when we change the chamber are parametrized by C

$$\begin{aligned} C \simeq \text{Jac}(C) &\hookrightarrow \Gamma \subset X_{<} \\ L &\mapsto (E, \mathbf{p}_L) \end{aligned}$$

where $E = E_1$ and $\mathbf{p}_L = \{p_1^L, p_2^L\}$ corresponds to the parabolic directions inside L . Similarly, we still call Γ the respective locus inside $X_{>}$ parametrized by the following map

$$\begin{aligned} C \simeq \text{Jac}(C) &\hookrightarrow \Gamma \subset X_{>} \\ L &\mapsto (E, \mathbf{p}_L) \end{aligned}$$

where $E = L \oplus L^{-1}(w_\infty)$, one of the parabolics lying in L and the other one outside both L and $L^{-1}(w_\infty)$. This locus Γ corresponds to a curve of type $(2, 2)$ parametrized by C .

The two charts $X_{<}$ and $X_{>}$ are identified outside Γ and this provides a stratification of the coarse moduli space of *simple parabolic bundles*

$$(3.4) \quad \text{Bun}_{w_\infty}(C, D) = X_{<} \sqcup X_{>}.$$

That is, simple parabolic bundles are parametrized by two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ identified outside a $(2, 2)$ curve Γ . Now we would like to characterize indecomposable but not simple parabolic bundles, see also [6, Table 1].

Proposition 3.1. *Let (E, \mathbf{p}) be an indecomposable parabolic bundle over (C, D) having $\mathcal{O}_C(w_\infty)$ as determinant line bundle. If (E, \mathbf{p}) is not simple, then*

$$(E, \mathbf{p}) = (L \oplus L^{-1}(w_\infty), \mathbf{p})$$

where $L^2 = \mathcal{O}_C(w_\infty - t_k)$, $k \in \{1, 2\}$. Moreover, p_k does not lie in L neither in $L^{-1}(w_\infty)$ and the other parabolic lies in $L^{-1}(w_\infty)$.

Proof. If (E, \mathbf{p}) is indecomposable but not simple, then it follows from Proposition 2.8 that

$$(E, \mathbf{p}) = (L \oplus L^{-1}(w_\infty), \mathbf{p}), \quad \deg(L) = 0.$$

Since (E, \mathbf{p}) is not simple, at least one of the parabolics lies in $L^{-1}(w_\infty)$. Writing

$$L^2 = \mathcal{O}_C(w_\infty - w)$$

for some $w \in C$, one obtains

$$\mathbb{P}E = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(w)).$$

The family of sections corresponding to the family of embeddings $L \hookrightarrow E$ has a base point at the fiber over $w \in C$. Therefore if w is distinct of t_k , $k = 1, 2$, we can choose an L containing the parabolic outside $L^{-1}(w_\infty)$. In this situation, (E, \mathbf{p}) is decomposable. Now if $w = t_k$, then p_k must be outside both L and $L^{-1}(w_\infty)$, and the other parabolic must lie in $L^{-1}(w_\infty)$; indeed, otherwise (E, \mathbf{p}) would be decomposable (see Figure 1). \square

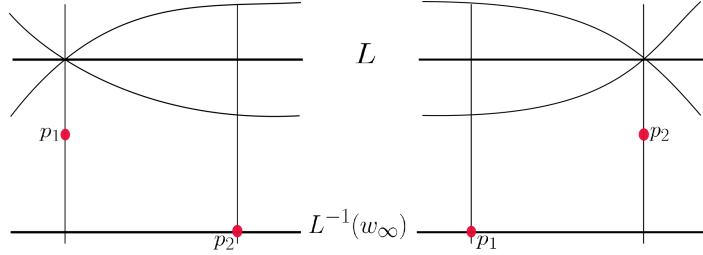


FIGURE 1. Indecomposable not simple, $L^2 = \mathcal{O}_C(w_\infty - t_k)$, $k \in \{1, 2\}$.

3.2. Fiber compactification of the moduli space and Higgs fields. Let us consider the following open subset of the moduli space of connections corresponding to stable parabolic bundles in $X_<$:

$$\text{Con}_<^\nu(C, D) = \{(E, \nabla, \mathbf{p}) \in \text{Con}^\nu(C, D) ; (E, \mathbf{p}) \in X_<\}.$$

Similarly, we define $\text{Con}_>^\nu(C, D)$ with $X_>$ instead of $X_<$. We note that $\text{Con}_<^\nu(C, D)$ corresponds to triples (E, ∇, \mathbf{p}) where $E = E_1$.

The union $\text{Con}_<^\nu(C, D) \cup \text{Con}_>^\nu(C, D)$ is the locus where the forgetful map with values at simple parabolic bundles $\text{Con}^\nu(C, D) \rightarrow \text{Bun}_{w_\infty}(C, D)$ is well defined.

Given a parabolic bundle $(E_1, \mathbf{p}) \in X_{<}$, any two connections ∇ and ∇' on it differ to each other by a parabolic Higgs field

$$\nabla' - \nabla = \Theta \in H^0(C, \text{End}(E_1, \mathbf{p}) \otimes \Omega_C^1(D)) =: \text{Higgs}_{<}^\nu(C, D).$$

Since endomorphisms of (E_1, \mathbf{p}) are scalars and $\Omega_C^1 \simeq \mathcal{O}_C$, this vector space is two dimensional. Then, the fiber of the forgetful map $Bun : \text{Con}_{<}^\nu(C, D) \rightarrow X_{<}$ over (E_1, \mathbf{p}) identifies with the two dimensional affine space

$$Bun^{-1}(E_1, \mathbf{p}) \simeq \nabla^0 + H^0(C, \text{End}(E_1, \mathbf{p}) \otimes \Omega_C^1(D)),$$

where $(E_1, \nabla^0, \mathbf{p})$ is any parabolic connection belonging to the fiber. We can compactify the affine \mathbb{C}^2 -bundle $Bun : \text{Con}_{<}^\nu(C, D) \rightarrow X_{<}$ by compactifying the fiber

$$\overline{Bun^{-1}(E_1, \mathbf{p})} = \mathbb{P}(\mathbb{C} \cdot \nabla^0 \oplus H^0(C, \text{End}(E_1, \mathbf{p}) \otimes \Omega_C^1(D))).$$

Varying $(E_1, \mathbf{p}) \in X_{<}$ and choosing a local section ∇^0 over local open sets of $X_{<}$ we construct a \mathbb{P}^2 -bundle

$$Bun : \overline{\text{Con}_{<}^\nu(C, D)} \rightarrow X_{<} \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

where

$$\mathbb{P}\text{Higgs}_{<}^\nu(C, D) := \overline{\text{Con}_{<}^\nu(C, D)} \setminus \text{Con}_{<}^\nu(C, D)$$

is the moduli space of projective Higgs fields.

3.3. From logarithmic connections to fuchsian systems with five poles. In order to study the \mathbb{P}^2 -bundle

$$Bun : \overline{\text{Con}_{<}^\nu(C, D)} \rightarrow X_{<}$$

we perform three elementary transformations on E_1 to arrive on the trivial vector bundle. Then, we use global coordinates to obtain an explicit universal family for our moduli space. The idea is the following. For any triple

$$(E, \nabla, \mathbf{p}) \in \overline{\text{Con}_{<}^\nu(C, D)}$$

the underlying rank 2 bundle is always $E = E_1$. It is well known that E_1 can be obtained from the trivial vector bundle $\mathcal{O}_C \oplus \mathcal{O}_C$ by three elementary transformations on distinct basis and distinct fibers (see for instance [14, Theorem 4.8]). Therefore, to give a logarithmic connection on E_1 is equivalent to give a fuchsian system on the trivial bundle having three apparent singular points. This will be explained in the next few lines.

We say that $t \in C$ is an **apparent singular** point for ∇ if

- (1) the residual part $\text{Res}_t \nabla$ has $\{\frac{1}{2}, -\frac{1}{2}\}$ as eigenvalues; and
- (2) the $\frac{1}{2}$ -eigenspace of $\text{Res}_t \nabla$ is also invariant by the constant part of the connection matrix.

These conditions does not depend on the choice of local trivialization for E ; condition (2) is equivalent to say that the local monodromy is semi-simple, i.e. $\pm I$.

Remark 3.2. We shall note that if ∇ is an \mathfrak{sl}_2 connection, requiring that t is an apparent singularity with a given direction p imposes three linear conditions on the coefficients of ∇ . In fact, if we denote by $A_{-1}, A_0 \in GL(2, \mathbb{C})$ the residual and the constant part of the connection matrix, respectively, then (i) and (ii) above means

$$\begin{cases} (A_{-1} - \frac{1}{2}I) \cdot p = 0 \\ (A_0 \cdot p) \wedge p = 0 \end{cases}$$

in which gives us three linear conditions.

We note that local monodromy does not change when we perform an elementary transformation over an apparent singular point, but the residual matrix becomes a multiple of the identity. In fact, let us assume that $x = t$ is an apparent singular point where the kernel of the residual part is

$$\mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then around $x = t$ the connection matrix writes

$$\nabla = d + \begin{pmatrix} \frac{1}{2} & \beta_{-1} \\ 0 & -\frac{1}{2} \end{pmatrix} \cdot \frac{dx}{(x-t)} + \begin{pmatrix} \alpha_0 & \beta_0 \\ 0 & -\alpha_0 \end{pmatrix} \cdot dx + o(x-t)$$

Applying an elementary transformation

$$elm^+(Y) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \cdot Y$$

we see that the residual matrix becomes

$$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

We now fix the data in order to define the moduli space of fuchsian systems with five poles on C and three apparent singular points. Consider the 2-torsion points $w_i = (i, 0) \in C$, for $i = 0, 1, \lambda$, defined in (3.2). The divisor defined by them is linearly equivalent to $3w_\infty$

$$w_0 + w_1 + w_\lambda \sim 3w_\infty.$$

Let us consider the reduced divisor

$$D' = w_0 + w_1 + w_\lambda + t_1 + t_2$$

and fix a local exponent

$$(3.5) \quad \theta = \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{\nu_1}{2}, \pm \frac{\nu_2}{2} \right)$$

We denote by $\text{Syst}^\theta(C, D')$ the moduli space of \mathfrak{sl}_2 -Fuchsian systems (i.e. logarithmic \mathfrak{sl}_2 -connections on the trivial bundle) having D' as divisor of poles, θ as local exponents and such that

- the three singular points w_0, w_1 and w_λ are apparent singular points;
- the corresponding $\frac{1}{2}$ -eigenspaces q_i are pairwise distinct.

In other words, up to isomorphism of the trivial vector bundle, one can assume $p_i = (w_i, (i, 0))$, for $i = 0, 1, \lambda$.

Proposition 3.3. *There is an isomorphism of moduli spaces*

$$(3.6) \quad \text{Syst}^\theta(C, D') \xrightarrow{\sim} \text{Con}_{<}^\nu(C, D)$$

Proof. Let us describe the isomorphism (3.6); let (E, ∇, \mathbf{p}) be a point in $\text{Syst}^\theta(C, D')$. Consider the composition of three positive elementary transformations

$$(3.7) \quad el = elem_{w_0, w_1, w_\lambda}^+ : \mathcal{O}_C \oplus \mathcal{O}_C \dashrightarrow \tilde{E}$$

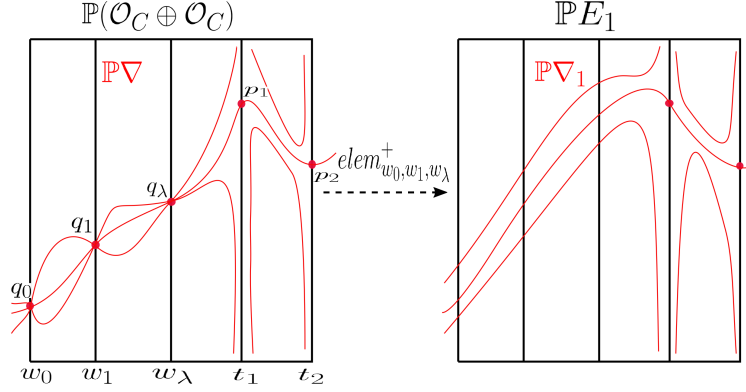


FIGURE 2. Elementary Transformation

on distinct basis points $\{w_0, w_1, w_\lambda\}$ and distinct fibers $\{q_0, q_1, q_\lambda\}$ (see Figure 2). After this birational modification of the trivial bundle, the corresponding indecomposable rank 2 vector bundle \tilde{E} has determinant

$$\det \tilde{E} = \mathcal{O}_C(w_0 + w_1 + w_\lambda).$$

Since $w_0 + w_1 + w_\lambda$ is linearly equivalent to $3w_\infty$ then

$$\det \tilde{E} = \mathcal{O}_C(3w_\infty).$$

Applying such composition of elementary transformations (3.7) to $(\mathcal{O}_C \oplus \mathcal{O}_C, \nabla)$ one obtains a new pair $(\tilde{E}, \tilde{\nabla}) = el_*(\mathcal{O}_C \oplus \mathcal{O}_C, \nabla)$. The eigenvalues $\{\frac{1}{2}, -\frac{1}{2}\}$ of ∇ over w_i , are changed by

$$\begin{aligned} \tilde{\nu}_i^+ &= \frac{1}{2} - 1 = -\frac{1}{2} \\ \tilde{\nu}_i^- &= -\frac{1}{2} \end{aligned}$$

and other eigenvalues $\{\frac{\nu_i}{2}, -\frac{\nu_i}{2}\}$ over t_i , for $i = 1, 2$, are left unchanged.

Then the birational map el yields a new connection $\tilde{\nabla}$ on \tilde{E}

$$\tilde{\nabla} : \tilde{E} \longrightarrow \tilde{E} \otimes \Omega_C^1(w_0 + w_1 + w_\lambda + t_1 + t_2)$$

with local exponents

$$\tilde{\nu} = \left(\tilde{\nu}_1^\pm = -\frac{1}{2}, \tilde{\nu}_2^\pm = -\frac{1}{2}, \tilde{\nu}_3^\pm = -\frac{1}{2}, \pm\frac{\nu_1}{2}, \pm\frac{\nu_2}{2} \right).$$

Since singularities over w_i are apparent, we promptly deduce that $\tilde{\nabla}$ has local monodromy $-Id$ over them. That is, these singular points are projectively apparent. Its trace connection $\text{tr}(\tilde{\nabla})$ is the unique rank one connection $d - \frac{dy}{y}$ on

$$\mathcal{O}_C(w_0 + w_1 + w_\lambda) = \mathcal{O}_C(3w_\infty).$$

with trivial monodromy, poles on w_0, w_1, w_λ and exponents -1 . In order to restore a connection on E_1 without singularities over w_i , for $i = 0, 1, \lambda$, and with trace ζ , we have to twist $(\tilde{E}, \tilde{\nabla})$ by a suitable logarithmic rank one connection ξ on $\mathcal{O}_C(-w_\infty)$,

namely a square root ξ of

$$(3.8) \quad (\mathcal{O}_C(-w_\infty), \xi)^{\otimes 2} := (\mathcal{O}_C(w_\infty), \zeta) \otimes \left(\mathcal{O}_C(3w_\infty), \zeta \otimes \left(d - \frac{dy}{y} \right) \right)^{\otimes (-1)} \\ = \left(\mathcal{O}_C(-2w_\infty), \zeta \otimes \left(d + \frac{dy}{y} \right) \right)$$

where ζ is the fixed trace connection in the moduli space $\text{Con}_<^\nu(C, D)$. The resulting logarithmic connection $\nabla_1 = \tilde{\nabla} \otimes \xi$ on $E_1 = \tilde{E} \otimes \mathcal{O}_C(-w_\infty)$ is nonsingular over w_i , for $i = 0, 1, \lambda$, and has eigenvalues $\{\frac{\nu_1}{2} - \frac{1}{2}, -\frac{\nu_1}{2} - \frac{1}{2}\}$ over t_1 and $\{\frac{\nu_2}{2}, -\frac{\nu_2}{2}\}$ over t_2 . The parabolic structure \mathbf{p}_1 on E_1 over $\bar{D} = t_1 + t_2$ is defined by the image by el of the parabolic structure on $\mathcal{O}_C \oplus \mathcal{O}_C$ forgetting the directions over w_i , $i = 0, 1, \lambda$.

Then for each element (E, ∇, \mathbf{p}) in $\text{Syst}^\theta(C, D')$, we have associated an element $(E_1, \nabla_1, \mathbf{p}_1)$ in $\text{Con}^\nu(C, D)$. This process can be reversed, and one gets an isomorphism

$$\begin{aligned} \text{Syst}^\theta(C, D') &\xrightarrow{\sim} \text{Con}_<^\nu(C, D) \\ (E, \nabla, \mathbf{p}) &\mapsto (E_1, \nabla_1, \mathbf{p}_1) \end{aligned}$$

In fact, the inverse of transformation $E_1 \dashrightarrow \mathcal{O}_C \oplus \mathcal{O}_C$ obtained by reversing (3.7) and (3.8) can be described as follows. Let $L_i \subset E_1$ denotes the unique embedding of $\mathcal{O}_C(w_i - w_\infty)$ for $i = 0, 1, \lambda$. Then, L_i is transformed in $\mathcal{O}_C \oplus \mathcal{O}_C$ as the constant (trivial) subbundle generated by $(i, 0)$ for $i = 0, 1, \lambda$. This characterizes the inverse transformation. \square

Remark 3.4. As shown in Figure 3, the subbundles $L_0, L_1, L_\lambda \subset E_1$ previously defined correspond, after projectivization, to sections of $\mathbb{P}E_1$ having +1 self-intersection. We similarly define $L_\infty \subset E_1$ as the unique trivial subbundle $\mathcal{O}_C \hookrightarrow E_1$. Through the transformation $\mathcal{O}_C \oplus \mathcal{O}_C \dashrightarrow E_1$ obtained by composing birational modification el (3.7) and twist by (3.8), one easily check that the subbundle $L_\infty \simeq \mathcal{O}_C$ of E_1 corresponds to the subbundle $\mathcal{O}_C(-w_0 - w_1 - w_\lambda + w_\infty) \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C$ generated by the rational section

$$(x, y) \mapsto \begin{pmatrix} 1 \\ x \end{pmatrix}.$$

This subbundle corresponds, after projectivization, to the unique section

$$s : C \longrightarrow C \times \mathbb{P}^1$$

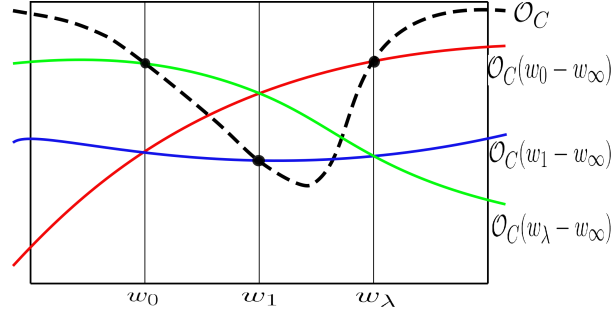
having +4 self-intersection, passing through the points

$$q_i = ((i, 0), (1 : i)), \quad i = 0, 1, \lambda$$

and which is invariant under the hyperelliptic involution.

3.4. Universal family of fuchsian systems. As we have seen in Proposition 3.3 the open subset $\text{Con}_<^\nu(C, D)$ of the moduli space $\text{Con}^\nu(X, D)$ is isomorphic to the moduli space of fuchsian systems $\text{Syst}^\theta(C, D')$ with three apparent singularities over w_i and local exponent θ as in (3.5). In this section, we exhibit an explicit universal family for $\text{Syst}^\theta(C, D')$, when restricted to two suitable open sets of parabolic bundles. As a consequence, we determine the \mathbb{P}^2 -bundle

$$\text{Bun} : \overline{\text{Con}_<^\nu(C, D)} \longrightarrow X_< \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$


 FIGURE 3. Sections of $\mathbb{P}E_1$

Firstly we would like to write $X_{<} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, minus two points, as a union of two copies $U_0 \cup U_\infty$ of \mathbb{C}^2 (see below). The \mathbb{P}^2 -bundle above will be trivial when restricted to each one of them.

Let U_0 and U_∞ be defined as follows

$$U_0 = \{z = ((z_1 : 1), (z_2 : 1)) ; z_1, z_2 \in \mathbb{C}\}$$

$$U_\infty = \{((1 : Z_1), (1 : Z_2)) ; Z_1, Z_2 \in \mathbb{C}\}.$$

Note that $U_0 \cup U_\infty$ cover $\mathbb{P}^1 \times \mathbb{P}^1$ minus two points:

$$((1 : 0), (0 : 1)) \text{ and } ((0 : 1), (1 : 0)).$$

Each point $z \in U_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ corresponds to a parabolic vector bundle (E_1, z) in $X_{<}$. By using the compositions of elementary transformations (3.7), one can identify each (E_1, z) in $X_{<}$ with a parabolic bundle $(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{p}_z)$ where the parabolic structure is

$$\mathbf{p}_z := \{q_0, q_1, q_\lambda, z_1, z_2\}.$$

Thanks to Proposition 3.3, we identify the fiber $Bun^{-1}(E_1, z)$ with the space of fuchsian systems in $\text{Syst}^\theta(C, D')$ over the same fixed parabolic bundle

$$(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{p}_z)$$

for particular θ as in (3.5).

Any fuchsian system $\nabla \in \text{Syst}^\theta(C, D')$ writes

$$\nabla = d + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \cdot \frac{dx}{y}$$

where $\omega = dx/y$ is a global regular 1-form on C and α, β, γ are rational functions on C with at most simple poles on

$$D' = w_0 + w_1 + w_\lambda + t_1 + t_2.$$

The vector space that parametrizes the space of such rational functions is five dimension:

$$\alpha, \beta, \gamma \in H^0(C, \mathcal{O}_C(w_0 + w_1 + w_\lambda + t_1 + t_2)).$$

Then, we have 15 parameters for α, β and γ . Each eigenvector q_i over the apparent singular point w_i imposes three linear conditions on the coefficients of ∇ , for each $i = 0, 1, \lambda$ (see Remark 3.2). Similarly, having z_1 (or z_2) as eigenvector with corresponding ν_1^+ (or ν_2^+) as eigenvalue imposes two linear conditions.

$\nabla^0 = d + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \cdot \frac{dx}{y}$
$\alpha = \frac{1}{4} \left(\frac{((\lambda+(x-\lambda-1)t)x(\nu_1+\nu_2)-((\lambda+1)x^2+(-\lambda-(\lambda+1)t)x+t\lambda))}{(x-t)y} \right)$ $+ \frac{1}{4} \left(\frac{(\lambda+(t-\lambda-1)x)t(\nu_1-\nu_2)}{(x-t)r} - \frac{2x(\nu_1 z_1 + \nu_2 z_2)}{y} + \frac{(\lambda-t+1)(\nu_1 z_1 - \nu_2 z_2)}{r} \right)$
$\beta = \frac{1}{4} \left(\frac{2(\nu_1 z_1 + \nu_2 z_2) - (\lambda+1-x)(\nu_1 + \nu_2 + 1) + 2x}{y} - \frac{2(t(\nu_2 - \nu_1) + \nu_1 z_1 - \nu_2 z_2)}{r} \right)$
$\gamma = \frac{x}{4} \left(\frac{2(\lambda-x(1+\lambda-t))(\nu_1 z_1 + \nu_2 z_2)}{y(x-t)} + \frac{\lambda(\nu_1 + \nu_2 - 1)}{y} + \frac{2r(\nu_1 z_1 - \nu_2 z_2)}{(x-t)t} \right)$
$\Theta_1^0 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} \cdot \frac{dx}{y}$
$\alpha_1 = \frac{-1}{2(x-t)} \left(\frac{2z_1 x((t-z_1)x+t(z_1-1)+\lambda(1-t))}{y} + \frac{(t\lambda+(2t-z_1)(t-\lambda-1)z_1)x}{r} \right)$ $+ \frac{1}{2(x-t)} \left(\frac{t((t-2z_1)\lambda-(t-\lambda-1)z_1^2)}{r} \right)$
$\beta_1 = \frac{x(t-z_1)(x+z_1-\lambda-1)+t(z_1-\lambda)(z_1-1)}{y(x-t)} + \frac{(z_1-t)^2 x+t((2t-z_1)z_1+\lambda-t(\lambda+1))}{r(x-t)}$
$\gamma_1 = \frac{z_1 x}{(x-t)} \left(\frac{x(z_1(1-t)+\lambda(z_1-1))+\lambda(t-z_1)}{y} - \frac{r z_1}{t} \right)$
$\Theta_2^0 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & -\alpha_2 \end{pmatrix} \cdot \frac{dx}{y}$
$\alpha_2 = \frac{-1}{2(x-t)} \left(\frac{2z_2 x((t-z_2)x+t(z_2-1)+\lambda(1-t))}{y} - \frac{((t\lambda+(2t-z_2)(t-\lambda-1)z_2)x)}{r} \right)$ $+ \frac{-1}{2(x-t)} \left(\frac{t((t-2z_2)\lambda-(t-\lambda-1)z_2^2)}{r} \right)$
$\beta_2 = \frac{x(t-z_2)(x+z_2-\lambda-1)+t(z_2-\lambda)(z_2-1)}{y(x-t)} - \frac{(z_2-t)^2 x+t((2t-z_2)z_2+\lambda-t(\lambda+1))}{r(x-t)}$
$\gamma_2 = \frac{z_2 x}{(x-t)} \left(\frac{x(z_2(1-t)+\lambda(z_2-1))+\lambda(t-z_2)}{y} + \frac{r z_2}{t} \right)$

TABLE 1. Basis for the universal family over U_0

In order to find a basis for the moduli space of fuchsian systems $\nabla(z) \in \text{Syst}^\theta(C, D')$ with given parabolic structure \mathbf{p}_z , we have solved this linear system with 13 equations and 15 variables.

Let us denote by $t_1 = (t, r)$ and $t_2 = (t, -r)$. For any $z \in U_0$, a fuchsian system $\nabla(z)$ like above writes

$$\nabla(z) = \nabla^0(z) + c_1 \cdot \Theta_1^0(z) + c_2 \cdot \Theta_2^0(z)$$

for $c_1, c_2 \in \mathbb{C}$, where $\nabla^0(z)$ is a particular fuchsian system as well as $\Theta_1^0(z)$, $\Theta_2^0(z)$ are particular Higgs fields given by Table 1.

In order to obtain also a universal family over the other open set U_∞ of $X_<$, for each $(Z_1, Z_2) = (\frac{1}{z_1}, \frac{1}{z_2}) \in U_\infty$ the respective fuchsian and Higgs fields are

$$\begin{cases} \nabla^\infty(Z_1, Z_2) = \nabla^0(\frac{1}{Z_1}, \frac{1}{Z_2}) + \frac{\nu_1}{2} \cdot Z_1 \cdot \Theta_1^0(\frac{1}{Z_1}, \frac{1}{Z_2}) + \frac{\nu_2}{2} \cdot Z_2 \cdot \Theta_2^0(\frac{1}{Z_1}, \frac{1}{Z_2}) \\ \Theta_1^\infty(Z_1, Z_2) = (Z_1)^2 \cdot \Theta_1^0(\frac{1}{Z_1}, \frac{1}{Z_2}) \\ \Theta_2^\infty(Z_1, Z_2) = (Z_2)^2 \cdot \Theta_2^0(\frac{1}{Z_1}, \frac{1}{Z_2}) \end{cases}$$

Here we shall explain why the parameters ν_1, ν_2 appear in the definition of ∇^∞ . Note that ∇^0 is a rational section of $\text{Con}_{<}^\nu(C, D) \rightarrow X_<$. In fact, $\nabla^0(\frac{1}{Z_1}, \frac{1}{Z_2})$ has $\{Z_1 Z_2 = 0\}$ as pole. This can be checked by using the explicit expression for ∇^0 given in Table 1. Then the coefficients $\frac{\nu_1}{2}, \frac{\nu_2}{2}$ appearing in the definition of ∇^∞ have been chosen to make ∇^∞ regular.

We summarize the above discussion in the next proposition. The intrinsic meaning of these basis will be given in the next section.

Proposition 3.5. *For each $j \in \{0, \infty\}$, the \mathbb{P}^2 -bundle $Bun : \overline{\text{Con}}_{<}^\nu(C, D) \rightarrow X_{<}$ is trivial when restricted to U_j :*

$$\begin{aligned} U_j \times \mathbb{P}^2 &\xrightarrow{\cong} \overline{\text{Con}}_{<}^\nu(C, D)|_{U_j} \\ (u_j, (C_0 : C_1 : C_2)) &\mapsto \mathbb{P}[C_0 \cdot \nabla^j(u_j) + C_1 \cdot \Theta_1^j(u_j) + C_2 \cdot \Theta_2^j(u_j)] \end{aligned}$$

and the open set $\text{Con}_{<}^\nu(C, D)|_{U_j}$ is given by $(C_0 : C_1 : C_2) = (1 : c_1 : c_2)$.

3.5. An open set of the moduli space. Without loss of generality, we can assume that ν is as in (3.3)

$$(3.9) \quad (\nu_1^\pm, \nu_2^\pm) = \left(\pm \frac{\nu_1}{2} - \frac{1}{2}, \pm \frac{\nu_2}{2} \right)$$

(see Section 4.5). The condition $\nu_k^+ = \nu_k^-$ is equivalent to $\nu_k = 0$ for each $k \in \{1, 2\}$.

3.5.1. Description of $\overline{\text{Con}}_{<}^\nu(C, D)$. We now apply results of Section 3.4 to study the \mathbb{P}^2 -bundle

$$Bun : \overline{\text{Con}}_{<}^\nu(C, D) \rightarrow X_{<} \simeq \mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1$$

as well as the projectivized moduli space of Higgs fields

$$\mathbb{P}\text{Higgs}_{<}^\nu(C, D) := \overline{\text{Con}}_{<}^\nu(C, D) \setminus \text{Con}_{<}^\nu(C, D).$$

Theorem 3.6. *With identification $X_{<} \simeq \mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1$, the following assertions hold true:*

(1) *The canonical isomorphism $\text{Higgs}_{<}^\nu(C, D) \xrightarrow{\sim} T^*X_{<}$ is given over charts U_0 and U_∞ by the identifications*

$$(3.10) \quad \begin{cases} \Theta_1^0 &\mapsto dz_1 \\ \Theta_2^0 &\mapsto dz_2 \end{cases} \quad \text{and} \quad \begin{cases} \Theta_1^\infty &\mapsto dZ_1 \\ \Theta_2^\infty &\mapsto dZ_2 \end{cases}$$

respectively. The Liouville form is $c_1 dz_1 + c_2 dz_2$ and its differential

$$\omega = dc_1 \wedge dz_1 + dc_2 \wedge dz_2$$

induces the canonical symplectic structure on $\text{Higgs}_{<}^\nu(C, D)|_{U_0}$.

(2) *We have $\overline{\text{Con}}_{<}^\nu(C, D) = \mathbb{P}(\mathcal{E}_1^\nu)$, where \mathcal{E}_1^ν is the extension of $\mathcal{O}_{X_{<}}$ by $T^*X_{<}$*

$$0 \rightarrow T^*X_{<} \rightarrow \mathcal{E}_1^\nu \rightarrow \mathcal{O}_{X_{<}} \rightarrow 0$$

determined by

$$\left(\frac{\nu_1}{2}, \frac{\nu_2}{2} \right) \in H^1(X_{<}, T^*X_{<}) \simeq H^1(\mathbb{P}_{z_1}^1, T^*\mathbb{P}_{z_1}^1) \oplus H^1(\mathbb{P}_{z_2}^1, T^*\mathbb{P}_{z_2}^1) \simeq \mathbb{C}^2.$$

Proof. The isomorphism of (1) comes from Serre duality (see [7, Section 6]):

$$\text{Higgs}_{<}^\nu(C, D) = H^0(C, \text{End}(E_1, \mathbf{p}) \otimes \Omega_C^1(D)) \simeq H^1(C, \text{End}(E_1, \mathbf{p})) \simeq T^*X_{<}$$

and is well-known. By the same argument as in [7, Prop 6.1], we can check the identification (3.10) therefore proving (1).

It follows by Proposition 3.5 that $\{\nabla^j, \Theta_1^j, \Theta_2^j\}$, for each $j \in \{0, \infty\}$, are sections trivializing $\overline{\text{Con}}_{<}^\nu(C, D)|_{U_j}$

$$\begin{aligned} U_j \times \mathbb{P}^2 &\xrightarrow{\cong} \overline{\text{Con}}_{<}^\nu(C, D)|_{U_j} \\ (u_j, (C_0 : C_1 : C_2)) &\mapsto [C_0 \cdot \nabla^j(u_j) + C_1 \cdot \Theta_1^j(u_j) + C_2 \cdot \Theta_2^j(u_j)]. \end{aligned}$$

By construction, in the intersection $U_0 \cap U_\infty$ they satisfy

$$\begin{cases} \nabla^\infty = \nabla^0 + \frac{\nu_1}{2} \cdot Z_1 \cdot \Theta_1^0 + \frac{\nu_2}{2} \cdot Z_2 \cdot \Theta_2^0 \\ \Theta_1^\infty = (Z_1)^2 \cdot \Theta_1^0 \\ \Theta_2^\infty = (Z_2)^2 \cdot \Theta_2^0 \end{cases}$$

Therefore, transition chart from $U_\infty \times \mathbb{C}^3$ to $U_0 \times \mathbb{C}^3$ give us the following cocycle

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\nu_1}{2} \cdot Z_1 & Z_1^2 & 0 \\ \frac{\nu_2}{2} \cdot Z_2 & 0 & Z_2^2 \end{pmatrix}.$$

Since $U_0 \cup U_\infty$ covers the base $X_{<} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ minus two points, this cocycle completely determines the extension. This proves that \mathcal{E}_1^ν corresponds to

$$\left(\frac{\nu_1}{2}, \frac{\nu_2}{2} \right) \in H^1(\mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1, T^*(\mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1)) \simeq H^1(\mathbb{P}_{z_1}^1, T^*\mathbb{P}_{z_1}^1) \oplus H^1(\mathbb{P}_{z_2}^1, T^*\mathbb{P}_{z_2}^1) \simeq \mathbb{C}^2.$$

This concludes item (2) of the statement. \square

We deduce the following corollary.

Corollary 3.7. *The \mathbb{P}^2 -bundle $Bun : \overline{\text{Con}}_{<}^\nu(C, D) \rightarrow X_{<}$ is the trivial extension*

$$\overline{\text{Con}}_{<}^\nu(C, D) \simeq \mathbb{P}(T^*X_{<} \oplus \mathcal{O}_{X_{<}})$$

if and only if $\nu_1 = \nu_2 = 0$.

In fact, when $\nu_1 \cdot \nu_2 \neq 0$, then one easily see that all these \mathbb{C}^2 -bundle are isomorphic between them, they correspond to the extension defined by the cocycle

$$\begin{pmatrix} 1 & 0 & 0 \\ z_1 & z_1^2 & 0 \\ z_2 & 0 & z_2^2 \end{pmatrix}.$$

This will be done in a geometric way in the next section.

3.5.2. *The moduli space $\text{Con}_{<}^\nu(C, D)$ for $\nu_1 \cdot \nu_2 \neq 0$.* When $\nu_1 \cdot \nu_2 \neq 0$ the kernel of the residual part of $\nabla \in \text{Con}_{<}^\nu(C, D)$ over each t_i has two distinct eigendirections

$$\ker(\text{Res}_{t_i} \nabla - \nu_i^\pm I) = p_i^\pm(\nabla) \in \mathbb{P}^1$$

where p_i^ϵ corresponds to the eigenvalue ν_i^ϵ for $\epsilon \in \{+, -\}$.

Let us denote by $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ the diagonal and $S := (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$ its complement. Then one gets a mapping which associates, to each $\nabla \in \text{Con}_{<}^\nu(C, D)$, the eigenvectors of its residual part

$$\begin{aligned} \text{Par} : \text{Con}_{<}^\nu(C, D) &\longrightarrow S^2 \\ \nabla &\mapsto ((p_1^+(\nabla), p_1^-(\nabla)), (p_2^+(\nabla), p_2^-(\nabla))) \end{aligned}$$

In the next theorem, we will prove that this map is an isomorphism. For each $\epsilon \in \{+, -\}$, we shall denote by τ^ϵ the projection corresponding to the directions $\{p_1^\epsilon, p_2^\epsilon\}$:

$$\begin{aligned} \tau^\epsilon : S^2 &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1. \\ ((p_1^+, p_1^-), (p_2^+, p_2^-)) &\mapsto (p_1^\epsilon, p_2^\epsilon) \end{aligned}$$

Similarly, for each $\epsilon \in \{+, -\}$, there is a projection Bun^ϵ corresponding to the respective parabolic structure associated to eigenvalues $\{\nu_1^\epsilon, \nu_2^\epsilon\}$.

$$\begin{aligned} Bun^\epsilon : \text{Con}_<^\nu(C, D) &\longrightarrow X_<. \\ (E_1, \nabla, \mathbf{p}) &\mapsto (E_1, \mathbf{p}^\epsilon(\nabla)) \end{aligned}$$

Note that $\text{Con}_<^\nu(C, D)$ has been defined by assuming that if $(E_1, \nabla, \mathbf{p})$ is a point inside it then \mathbf{p} corresponds to the eigenvalues $\{\nu_1^+, \nu_2^+\}$. This means that the above projection Bun^+ coincides with Bun of previous sections.

Theorem 3.8. *If $\nu_1 \cdot \nu_2 \neq 0$ then $\text{Par} : \text{Con}_<^\nu(C, D) \longrightarrow S^2$ is an isomorphism between \mathbb{C}^2 -affine bundles over $\mathbb{P}^1 \times \mathbb{P}^1$*

$$\begin{array}{ccc} \text{Con}_<^\nu(C, D) & \xrightarrow[\simeq]{\text{Par}} & S^2 \\ & \searrow^{Bun^\epsilon} \quad \swarrow_{\tau^\epsilon} & \\ & \mathbb{P}^1 \times \mathbb{P}^1 & \end{array}$$

for each $\epsilon \in \{+, -\}$.

Proof. In order to prove the theorem we shall use the isomorphism

$$\text{Syst}^\theta(C, D') \simeq \text{Con}_<^\nu(C, D)$$

given by Proposition 3.3. We need to prove that the four eigenvectors $\{\mathbf{p}_1^\pm(\nabla), \mathbf{p}_2^\pm(\nabla)\}$ corresponding to the eigenvalues $\{\pm \frac{\nu_1}{2}, \pm \frac{\nu_2}{2}\}$ determine the connection.

Since $\nu_1 \cdot \nu_2 \neq 0$, one has two distinct eigenvectors over each singular point t_1 and t_2 . Then we can assume, without loss of generality, that $\mathbf{p}_1^+(\nabla) = (1 : z)$ and $\mathbf{p}_2^+(\nabla) = (1 : w)$. If $\mathbf{p}_i^+(\nabla) = (0 : 1)$ for some $i = 1, 2$, the same argument works with $-\nu_i$ in place of ν_i .

The fiber of

$$Bun : \text{Syst}^\theta(C, D') \longrightarrow X_< \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

over $z \in U_0$ is the two dimensional affine space

$$(3.11) \quad \nabla_{\mathbf{c}}(z) = \nabla^0(z) + c_1 \cdot \Theta_1^0(z) + c_2 \cdot \Theta_2^0(z)$$

varying $\mathbf{c} = (c_1, c_2) \in \mathbb{C}^2$, where ∇_0 , Θ_1^0 and Θ_2^0 were given in Table 1. Using the explicit expressions for them, we can compute the two eigenvectors of the kernel of $\nabla_{\mathbf{c}}(z)$ at each fiber over t_1 and t_2 . They are

$$(3.12) \quad \text{Par} : \nabla_{\mathbf{c}} \mapsto \begin{cases} p_1^+ = (1 : z_1) & p_1^- = (c_1 : c_1 z_1 - \frac{\nu_1}{2}) \\ p_2^+ = (1 : z_2) & p_2^- = (c_2 : c_2 z_2 - \frac{\nu_2}{2}) \end{cases}$$

This proves that $\nabla_{\mathbf{c}}(z)$ is determined by its four residual eigendirections over t_1 and t_2 . \square

Remark 3.9. The section $\nabla_0(z)$ of Bun^+ is characterized by the property that

$$\text{Par}(\nabla_0) = \begin{cases} p_1^+ = (1 : z_1) & p_1^- = (0 : 1) \\ p_2^+ = (1 : z_2) & p_2^- = (0 : 1) \end{cases}$$

It thus coincides with a fiber of Bun^- .

Remark 3.10. Now we would like to remark that there is an isomorphism between S^2 and $(Q \times Q^*) \setminus \mathcal{I}$, where $Q \subset \mathbb{P}^3$ is the smooth quadric surface, Q^* its dual variety and \mathcal{I} is the incidence variety. As a consequence, we obtain the following isomorphism

$$\text{Con}_{<}^{\nu}(C, D) \simeq (Q \times Q^*) \setminus \mathcal{I}.$$

In fact, let $\check{\mathbb{P}}^3$ be the set of hyperplanes in \mathbb{P}^3 . The incidence variety \mathcal{I} is defined as

$$\mathcal{I} = \{(q, T_r Q) \in Q \times Q^* ; q \in T_r Q\}.$$

We note that Q and Q^* are isomorphic. For instance, we can assume that Q is defined by zero locus of $f := x_0x_3 - x_1x_2$. Its dual variety Q^* can be identified with the image of the polar map $\phi_f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, which is defined by the derivatives of f . That is, Q^* is just the image of Q by a linear map and it is determined by the same equation f as Q .

Via Segre embedding, we can identify $(Q \times Q^*) \setminus \mathcal{I}$ with $(\mathbb{P}^1 \times \mathbb{P}^1)^2 \setminus I$ where

$$I = \{((z_1, \zeta_1), (z_2, \zeta_2)) \in (\mathbb{P}^1 \times \mathbb{P}^1)^2 ; z_1 = z_2 \text{ or } \zeta_1 = \zeta_2\}.$$

In addition, this last variety is isomorphic to S^2 :

$$\begin{aligned} (\mathbb{P}^1 \times \mathbb{P}^1)^2 \setminus I &\longrightarrow S^2 \\ ((z_1, \zeta_1), (z_2, \zeta_2)) &\mapsto ((z_1, z_2), (\zeta_1, \zeta_2)). \end{aligned}$$

3.6. Whole moduli space of connections. In order to describe the whole moduli space, we have to introduce other open charts.

3.6.1. Description of $\overline{\text{Con}}_{>}^{\nu}(C, D)$. Recall that we have a stratification of the moduli space of simple parabolic bundles, see (3.4):

$$\text{Bun}_{w_{\infty}}(C, D) = X_{<} \sqcup X_{>}.$$

Given $(E, \mathbf{p}) \in X_{>}$, we perform two elementary transformations elm_{t_1, t_2} with center at \mathbf{p} , followed by a twisting by $\mathcal{O}_C(-w_{\infty})$ in order to obtain a parabolic vector bundle $(\tilde{E}, \tilde{\mathbf{p}}) \in X_{<}$. Such operation defines a mapping ϕ_D from $X_{>}$ to $X_{<}$ that sends the locus $\Gamma \subset X_{>}$ into the locus $\Gamma \subset X_{<}$. Identifying $X_{>}$ and $X_{<}$ with $\mathbb{P}^1 \times \mathbb{P}^1$ we can show that this map corresponds to the following one

$$\begin{aligned} \phi_D : \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ (z_1, z_2) &\mapsto (z_2, z_1) \end{aligned}$$

(see [6, Proposition 5.5] for details).

Theorem 3.11. *There is a fiber-preserving isomorphism Φ_D :*

$$\begin{array}{ccc} \overline{\text{Con}}_{>}^{\nu}(C, D) & \xrightarrow[\simeq]{\Phi_D} & \overline{\text{Con}}_{<}^{\lambda}(C, D) \\ \downarrow & & \downarrow \\ X_{>} & \xrightarrow{\phi_D} & X_{<} \end{array}$$

where $\lambda = (\lambda_1^{\pm}, \lambda_2^{\pm})$ with

$$\begin{cases} \lambda_k^+ = \nu_k^+ - 1/2 \\ \lambda_k^- = \nu_k^- + 1/2 \end{cases}$$

for each $k \in \{1, 2\}$.

Proof. The above isomorphism is given by performing a positive elementary transformation over $D = t_1 + t_2$. In fact, given $(E, \nabla, \mathbf{p}) \in \text{Con}_>^\nu(C, D)$, let us consider the composition of two positive elementary transformations

$$(3.13) \quad el = \text{elem}_{t_1, t_2}^+ : E \dashrightarrow E'$$

with center at p_1^+ and p_2^+ , respectively. The birational map el yields a new logarithmic connection ∇' on E' . We also shall fix a suitable rank one logarithmic connection on $\mathcal{O}_C(-w_\infty)$ with poles on D satisfying the prescribed condition on residues:

$$\text{Res}_{t_k}(\zeta) = \frac{1}{2}$$

for $k \in \{1, 2\}$. Twisting (E', ∇') by $(\mathcal{O}_C(-w_\infty), \zeta)$ one obtains a pair

$$(\tilde{E}, \tilde{\nabla}) = (E' \otimes \mathcal{O}_C(-w_\infty), \nabla' \otimes \zeta)$$

where \tilde{E} has $\mathcal{O}_C(w_\infty)$ as determinant line bundle and the resulting connection $\tilde{\nabla}$ has local exponents λ as stated.

Then we get a mapping $\Phi_D : \text{Con}_>^\nu(C, D) \longrightarrow \text{Con}_<^\lambda(C, D)$ in which can be reversed by the respective negative elementary transformation followed by a twisting by $\mathcal{O}_C(w_\infty)$. \square

Corollary 3.12. *The \mathbb{P}^2 -bundle $\text{Bun} : \overline{\text{Con}_>^\nu(C, D)} \longrightarrow X_>$ is the trivial extension*

$$\overline{\text{Con}_>^\nu(C, D)} = \mathbb{P}(T^*X_> \oplus \mathcal{O}_{X_>})$$

if and only if $\nu_k^+ - \nu_k^- = 1$ for $k \in \{1, 2\}$. If $\nu_k^+ - \nu_k^- \neq 1$ for $k \in \{1, 2\}$, then $\overline{\text{Con}_>^\nu(C, D)}$ is isomorphic to S^2 .

Proof. First part of the statement is a consequence of Theorem 3.11 and Corollary 3.7. In the second part, we use Theorem 3.8 to conclude the proof. \square

3.6.2. Fiber over each indecomposable bundle. In view of Corollary 2.3, any parabolic bundle arising in our moduli space $\text{Con}^\nu(C, D)$ is indecomposable. There are two types of indecomposable parabolic bundles: simple and not simple. As we have seen in Section 3.1, the moduli space of simple parabolic bundle having $\mathcal{O}_C(w_\infty)$ as determinant line bundle is a union of two copies of $\mathbb{P}^1 \times \mathbb{P}^1$

$$\text{Bun}_{w_\infty}(C, D) = X_< \cup X_>$$

identifying identical parabolic bundles outside a curve Γ of type $(2, 2)$. On the other hand, indecomposable not simple parabolic bundles were characterized in Proposition 3.1. There are eight different types: for each square root $L_{i,k}$ of $\mathcal{O}_C(w_\infty - t_k)$, $k \in \{1, 2\}$ and $i \in \{1, \dots, 4\}$, there is a unique parabolic bundle up to automorphism

$$(3.14) \quad \mathcal{E}_{i,k} = (L_{i,k} \oplus L_{i,k}^{-1}(w_\infty), \mathbf{p})$$

with parabolic directions as in Proposition 3.1 (see Figure 1).

Let $\mathcal{C}_\mathcal{E} \subset \text{Con}^\nu(C, D)$ be the set of ν -flat connections over a given parabolic bundle $\mathcal{E} = (E, \mathbf{p})$. The whole moduli space $\text{Con}^\nu(C, D)$ is the union of $\mathcal{C}_\mathcal{E}$ when \mathcal{E} runs over all indecomposable parabolic bundles.

Proposition 3.13. *Assume $\nu_k^+ \neq \nu_k^-$ for each $k = 1, 2$. The moduli space $\mathcal{C}_\mathcal{E}$ of ν -flat connections over a given indecomposable parabolic bundle $\mathcal{E} = (E, \mathbf{p})$ is a two dimensional affine space.*

Proof. If \mathcal{E} belongs to either $X_<$ or $X_>$ then \mathcal{E} is simple. Since only automorphisms preserving parabolics are scalar, $\mathcal{C}_{\mathcal{E}}$ can be identified with the two dimensional affine space

$$\mathcal{C}_{\mathcal{E}} \simeq \mathbb{H}^0(C, \text{End}\mathcal{E} \otimes \Omega_C^1(D)).$$

Now let $\mathcal{E} = \mathcal{E}_{i,k}$ as in (3.14) be an indecomposable parabolic bundle but not simple. For sake of simplicity, one may assume that $\mathcal{E} = (L \oplus L^{-1}(w_{\infty}), \mathbf{p})$ with

$$L^2 = \mathcal{O}_C(w_{\infty} - t_1).$$

Where the parabolic direction $\mathbf{p} = (p_1, p_2)$ satisfies

$$\begin{cases} p_1 \not\subseteq L_{t_1} \text{ and } p_1 \not\subseteq L^{-1}(w_{\infty})_{t_1} \\ p_2 \subset L^{-1}(w_{\infty})_{t_2} \end{cases}$$

like Figure 1. By our hypothesis on ν , any ν -flat connection $\nabla \in \mathcal{C}_{\mathcal{E}}$ has two distinct eigendirections over each t_k , for each $k = 1, 2$

$$\text{Par}(\text{Res}_{t_k} \nabla) = p_k^{\pm}(\nabla).$$

Besides that, the ‘‘positive’’ directions coincide with \mathbf{p} , that is,

$$\begin{cases} p_1^+(\nabla) = p_1 \\ p_2^+(\nabla) = p_2. \end{cases}$$

Also we note that since $p_2 \subset L^{-1}(w_{\infty})_{t_2}$ then $p_2^-(\nabla) \not\subseteq L^{-1}(w_{\infty})_{t_2}$. We set

$$q_2 = p_2^-(\nabla)$$

to simplify the notation. Therefore, each element $\nabla \in \mathcal{C}_{\mathcal{E}}$ defines a connection on $L \oplus L^{-1}(w_{\infty})$ satisfying

$$(3.15) \quad \begin{cases} \text{Res}_{t_1}(\nabla)(p_1) = \nu_1^+ \cdot p_1 \\ \text{Res}_{t_2}(\nabla)(q_2) = \nu_2^- \cdot q_2 \end{cases}$$

where

$$\begin{cases} p_1 \not\subseteq L_{t_1} \text{ and } p_1 \not\subseteq L^{-1}(w_{\infty})_{t_1} \\ q_2 \not\subseteq L^{-1}(w_{\infty})_{t_2}. \end{cases}$$

A priori $q_2 = p_2^-(\nabla)$ depends of ∇ . But since p_2 lies in the maximal subbundle $L^{-1}(w_{\infty})$, the automorphism group of $E = L \oplus L^{-1}(w_{\infty})$ fixing parabolic $\mathbf{p} = \{p_1, p_2\}$ is two dimensional

$$\text{Aut}(E, \mathbf{p}) = \left\{ \begin{pmatrix} a & 0 \\ \gamma & a \end{pmatrix} ; a \in \mathbb{C}^*, \gamma \in \mathbb{H}^0(C, \mathcal{O}_C(t_1)) \right\}.$$

Hence, we may suppose that all the connections $\nabla \in \mathcal{C}_{\mathcal{E}}$ have the same eigendirection q_2 outside $L^{-1}(w_{\infty})$.

Reciprocally, given a connection ∇ on $L \oplus L^{-1}(w_{\infty})$ satisfying (3.15), we will show that $\nabla \in \mathcal{C}_{\mathcal{E}}$. In fact, suppose ∇ is a connection on $(E, \{p_1, q_2\})$ satisfying (3.15) and

$$\begin{cases} p_1 \not\subseteq L_{t_1} \text{ and } p_1 \not\subseteq L^{-1}(w_{\infty})_{t_1} \\ q_2 \not\subseteq L^{-1}(w_{\infty})_{t_2}. \end{cases}$$

We will prove that the second eigendirection of the residual part of ∇ at t_2 which corresponds to ν_2^+ lies in $L^{-1}(w_{\infty})$. In order to prove it, let us consider the apparent map with respect to $\mathcal{L} = L^{-1}(w_{\infty})$

$$\mathcal{L} \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega_C^1(t_1 + t_2) \longrightarrow (E/\mathcal{L}) \otimes \Omega_C^1(t_1 + t_2) = L(t_1 + t_2).$$

The zero divisor of the corresponding \mathcal{O}_C -linear map

$$\varphi_{\nabla} : \mathcal{O}_C \longrightarrow \mathcal{L}^{-1} \otimes L(t_1 + t_2) = \mathcal{O}_C(t_2)$$

defines an element of the linear system $\mathbb{P}H^0(C, \mathcal{O}_C(t_2)) = |t_2|$. This means that its zero divisor is exactly t_2 , because our curve is elliptic. Consequently, the residual part of ∇ has an eigendirection, say p_2 , lying in $L^{-1}(w_{\infty})$. Since q_2 corresponds to eigenvalue ν_2^- , then p_2 corresponds to ν_2^+ . It is enough to prove that $\nabla \in \mathcal{C}_{\mathcal{E}}$.

Let us denote by \mathcal{E}_- the parabolic bundle obtained by taking q_2 instead of p_2

$$\mathcal{E}_- = (L \oplus L^{-1}(w_{\infty}), \{p_1, q_2\}).$$

We have showed above that each $\nabla \in \mathcal{C}_{\mathcal{E}}$ can be seen as a connection on the parabolic bundle \mathcal{E}_- where

$$\begin{cases} \text{Res}_{t_1}(\nabla)(p_1) = \nu_1^+ \cdot p_1 \\ \text{Res}_{t_2}(\nabla)(q_2) = \nu_2^- \cdot q_2 \end{cases}$$

and vice versa. Then $\mathcal{C}_{\mathcal{E}}$ can be identified with the affine vector space

$$\mathcal{C}_{\mathcal{E}} \simeq H^0(C, \text{End}(\mathcal{E}_-) \otimes \Omega_C^1(D)).$$

As we know \mathcal{E}_- is simple, then the above cohomology group is a two dimensional vector space. \square

3.6.3. Patching open charts. Let us suppose $\nu_k^+ \neq \nu_k^-$ for each $k = 1, 2$. Given an element $(E, \nabla, \mathbf{p}) \in \text{Con}^{\nu}(C, D)$, the parabolic direction \mathbf{p} is nothing but the eigendirection for the residual part of ∇ with respect to ν_k^+ . Under our hypothesis on ν , the parabolic data is actually uniquely defined by the connection itself. Then we shall write just (E, ∇) instead of (E, ∇, \mathbf{p}) . Each ∇ has two pairs of ‘‘positive’’ and ‘‘negative’’ eigendirections

$$\begin{cases} \mathbf{p}_{\nabla}^+ = (p_1^+(\nabla), p_2^+(\nabla)) \\ \mathbf{p}_{\nabla}^- = (p_1^-(\nabla), p_2^-(\nabla)) \end{cases}$$

defined by the eigenvalues ν_k^+ and ν_k^- , respectively.

The moduli space $\text{Con}^{\nu}(C, D)$ admits two forgetful maps with values in the moduli space of simple parabolic bundles

$$\begin{aligned} Bun^+ : \text{Con}^{\nu}(C, D) &\dashrightarrow Bun(C, D) \\ (E, \nabla) &\mapsto (E, \mathbf{p}_{\nabla}^+) \end{aligned}$$

and

$$\begin{aligned} Bun^- : \text{Con}^{\nu}(C, D) &\dashrightarrow Bun(C, D) \\ (E, \nabla) &\mapsto (E, \mathbf{p}_{\nabla}^-) \end{aligned}$$

They are complementary in the following sense. Indetermination points of Bun^+ are not indetermination points of Bun^- and vice versa. Indeed, indetermination points of Bun^+ are of the form (E, ∇) such that $(E, \mathbf{p}_{\nabla}^+)$ is not simple. There are eight indecomposable not simple parabolic bundles. It follows from Proposition 3.13 that the indetermination points of Bun^+ is a union of eight two dimensional affine spaces. But we could define our moduli space of connections by using ν_k^- instead of ν_k^+ . This description for indetermination points of Bun^+ is completely symmetric for Bun^- . In fact, indetermination points of Bun^- are of the form (E, ∇) such that $(E, \mathbf{p}_{\nabla}^-)$ is not simple. Besides that, there is an open set of $\text{Con}^{\nu}(C, D)$ where fibers of Bun^+ and Bun^- are transverse to each other.

In order to describe our moduli space, one define the following open subsets

$$\begin{aligned} \mathcal{C} &:= \{(E, \nabla) \in \text{Con}^\nu(C, D); (E, \mathbf{p}_\nabla^+) \in X_{<}\} \\ &= \{(E, \nabla) \in \text{Con}^\nu(C, D); (E, \mathbf{p}_\nabla^-) \in X_{<}\} \\ \mathcal{C}_>^+ &:= \{(E, \nabla) \in \text{Con}^\nu(C, D); (E, \mathbf{p}_\nabla^+) \in X_{>}\} \\ \mathcal{C}_>^- &:= \{(E, \nabla) \in \text{Con}^\nu(C, D); (E, \mathbf{p}_\nabla^-) \in X_{>}\} \end{aligned}$$

Note that $\mathcal{C} = \text{Con}_{<}^\nu(C, D)$ and $\mathcal{C}_>^+ = \text{Con}_{>}^\nu(C, D)$. The union $\mathcal{C} \cup \mathcal{C}_>^+$ is the biggest open set where Bun^+ is well defined, i.e. its domain of definition is

$$\text{Dom}(Bun^+) = \mathcal{C} \cup \mathcal{C}_>^+$$

We would like to remark that $(E, \mathbf{p}_\nabla^+) \in X_{<}$ if, and only if $(E, \mathbf{p}_\nabla^-) \in X_{<}$, because $X_{<}$ corresponds to parabolic bundles with underlying vector bundle $E = E_1$. Then $\mathcal{C} \cup \mathcal{C}_>^-$ is the domain of definition of Bun^-

$$\text{Dom}(Bun^-) = \mathcal{C} \cup \mathcal{C}_>^-.$$

Applying Corollary 3.12 with ν_k^- in place of ν_k^+ , one obtains the same statement for $\mathcal{C}_>^-$ instead of $\mathcal{C}_>^+$. Hence, $Bun^- : \mathcal{C} \rightarrow X_{>}$ is an affine \mathbb{C}^2 -bundle over $X_{>}$. We summarize the above discussion and previous results in the following theorem. We still denote by $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ the diagonal and $S := (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$ its complement.

Theorem 3.14. *Assume $\nu_1^{\epsilon_1} + \nu_2^{\epsilon_2} \notin \mathbb{Z}$ for any $\epsilon_k \in \{+, -\}$. The moduli space $\text{Con}^\nu(C, D)$ is a union of three affine \mathbb{C}^2 -bundles over $\mathbb{P}^1 \times \mathbb{P}^1$*

$$\text{Con}^\nu(C, D) = \mathcal{C} \cup \mathcal{C}_>^+ \cup \mathcal{C}_>^-.$$

The intersection $\mathcal{U} := \mathcal{C} \cap \mathcal{C}_>^+ \cap \mathcal{C}_>^-$ is the set of pairs (E, ∇) where $E = E_1$ and there is no degree zero subbundle of E_1 passing through neither \mathbf{p}^+ nor \mathbf{p}^- . Moreover, there are fiber-preserving isomorphisms between affine \mathbb{C}^2 -bundle over $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Phi_D^+ : \mathcal{C}_>^+ \rightarrow \text{Con}_>^\lambda(C, D) \quad \text{and} \quad \Phi_D^- : \mathcal{C}_>^- \rightarrow \text{Con}_>^\gamma(C, D)$$

where $\lambda = (\lambda_1^\pm, \lambda_2^\pm)$ and $\gamma = (\gamma_1^\pm, \gamma_2^\pm)$ satisfy

$$\begin{cases} \lambda_k^+ = \nu_k^+ - 1/2 \\ \lambda_k^- = \nu_k^- + 1/2 \end{cases} \quad \text{and} \quad \begin{cases} \gamma_k^+ = \nu_k^+ + 1/2 \\ \gamma_k^- = \nu_k^- - 1/2 \end{cases}$$

for each $k \in \{1, 2\}$.

Proof. The hypothesis $\nu_1^{\epsilon_1} + \nu_2^{\epsilon_2} \notin \mathbb{Z}$ implies that only indecomposable parabolic bundles appears in our moduli space (see Corollary 2.3). If (E, ∇) does not belong to $\mathcal{C} \cup \mathcal{C}_>^+$, then $(E, \nabla, \mathbf{p}_\nabla^+)$ is not simple. That is, $(E, \mathbf{p}) = \mathcal{E}_{i,k}$ as in (3.14) and exactly one of the parabolics $p_1^+(\nabla)$ or $p_2^+(\nabla)$ lies in the maximal subbundle $L^{-1}(w_\infty)$ according with the square root $L_{i,k}$ of $\mathcal{O}_C(w_\infty - t_i)$ (see Figure 1). In particular, both parabolics $p_1^-(\nabla)$ and $p_2^-(\nabla)$ do not lie in $L^{-1}(w_\infty)$ and then

$$(E, \{p_1^-(\nabla), p_2^-(\nabla)\}) \in X_{>}.$$

This implies that $(E, \nabla) \in \mathcal{C}_>^-$ and therefore one concludes that

$$\text{Con}^\nu(C, D) = \mathcal{C} \cup \mathcal{C}_>^+ \cup \mathcal{C}_>^-.$$

Now, let us describe the intersection $\mathcal{U} = \mathcal{C} \cap \mathcal{C}_>^+ \cap \mathcal{C}_>^-$. If $(E, \nabla) \in \mathcal{C} \cap \mathcal{C}_>^+$, then $E = E_1$ and there is no degree zero line bundle passing through \mathbf{p}_∇^+ , because

$$\mathcal{C} \cap \mathcal{C}_>^+ = \text{Con}_{<}^\nu(C, D) \cap \text{Con}_{>}^\nu(C, D).$$

The same conclusion can be done for \mathbf{p}_∇^- in place of \mathbf{p}_∇^+ when $(E, \nabla) \in \mathcal{C}_>^-$.

The fiber-preserving isomorphism Φ_D^+ is obtained by taking two positive elementary transformations with center at \mathbf{p}_∇^+ followed by twisting by $\mathcal{O}_C(-w_\infty)$ (see Theorem 3.11 for details). The other fiber-preserving isomorphism Φ_D^- is defined similarly taking the elementary transformation at \mathbf{p}_∇^- instead of \mathbf{p}_∇^+ . \square

Recall that $S = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$ and there are projections

$$\begin{aligned} \tau^\epsilon : S^2 &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ ((p_1^\epsilon, p_1^-), (p_2^\epsilon, p_2^-)) &\mapsto (p_1^\epsilon, p_2^\epsilon) \end{aligned}$$

for $\epsilon \in \{+, -\}$, making S^2 a double \mathbb{C}^2 -affine bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. Fibers of τ^+ and τ^- are transverse between them.

Theorem 3.15. *Assume $\nu_1^{\epsilon_1} + \nu_2^{\epsilon_2} \notin \mathbb{Z}$ for any $\epsilon_k \in \{+, -\}$. If $\nu_k^+ - \nu_k^- \notin \{0, 1, -1\}$ for $k \in \{1, 2\}$, then the moduli space $\text{Con}^\nu(C, D)$ of ν -flat connections over the elliptic curve C , minus two points, is a union of three copies of S^2*

$$\text{Con}^\nu(C, D) = S^2 \cup_{\Psi^+} S^2 \cup_{\Psi^-} S^2.$$

Where $\Psi^\pm : S^2 \dashrightarrow S^2$ are fiber-preserving isomorphisms outside a $(2, 2)$ curve $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ isomorphic to C

$$\begin{array}{ccccc} S^2 & \overset{\Psi^-}{\dashrightarrow} & S^2 & \overset{\Psi^+}{\dashleftarrow} & S^2 \\ \tau^- \searrow & & \tau^- \swarrow & & \tau^+ \swarrow \\ & \mathbb{P}^1 \times \mathbb{P}^1 & & & \mathbb{P}^1 \times \mathbb{P}^1 \\ & \tau^+ \swarrow & & & \tau^- \searrow \end{array}$$

Proof. The three copies of S^2 are given by Theorem 3.6

$$\text{Con}^\nu(C, D) = \mathcal{C} \cup \mathcal{C}_>^+ \cup \mathcal{C}_>^-.$$

In fact, $\nu_k^+ - \nu_k^- \neq 0$ implies that $\mathcal{C} \simeq S^2$ (see Theorem 3.8). If $\nu_k^+ - \nu_k^- \notin \{1, -1\}$, by the same reason, the two moduli spaces $\text{Con}_<^\gamma(C, D)$ and $\text{Con}_<^\lambda(C, D)$ of Theorem 3.6, can be also identified with S^2 . Then we get the following fiber-preserving isomorphisms

$$\Phi_D^+ : \mathcal{C}_>^+ \longrightarrow \text{Con}_<^\gamma(C, D) \simeq S^2 \quad \text{and} \quad \Phi_D^- : \mathcal{C}_>^- \longrightarrow \text{Con}_<^\lambda(C, D) \simeq S^2.$$

We define Ψ^+ and Ψ^- as inverse of Φ_D^+ and Φ_D^- , respectively. To conclude the proof of theorem we remark that $\mathcal{C}_>^+ = \text{Con}_>^\nu(C, D)$ intersect $\mathcal{C} = \text{Con}_<^\nu(C, D)$ outside the locus of pairs (E_1, ∇) such that the respective parabolic bundle $(E_1, \mathbf{p}_\nabla^+)$ has two directions $\mathbf{p}_\nabla^+ = \{p_1^+, p_2^+\}$ lying in the same degree zero line bundle $L \in \text{Jac}^0(C) \simeq C$. This locus is a $(2, 2)$ curve $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ parametrized by C . The same argument works for $\mathcal{C}_>^-$ with \mathbf{p}_∇^- instead of \mathbf{p}_∇^+ (see Figure 4). \square

3.7. Apparent singularities. In this section, we will study the *apparent map* with respect to $\mathcal{O}_C \hookrightarrow E_1$ (see [13]). Each connection ∇ on E_1 defines an \mathcal{O}_C -linear map:

$$\mathcal{O}_C \xrightarrow{\iota} E_1 \xrightarrow{\nabla} E_1 \otimes \Omega_C^1(D) \longrightarrow (E_1/\mathcal{O}_C) \otimes \Omega_C^1(D)$$

where the last arrow is defined by the quotient map from E_1 to $E_1/\iota(\mathcal{O}_C)$ (denoted E_1/\mathcal{O}_C) for short. That is, we shall consider the mapping

$$\varphi_\nabla : \mathcal{O}_C \longrightarrow (E_1/\mathcal{O}_C) \otimes \Omega_C^1(D).$$

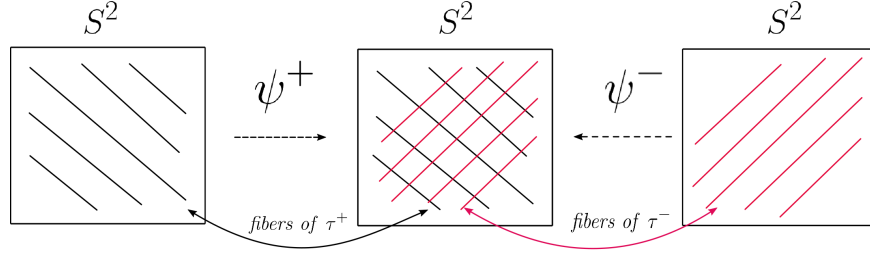


FIGURE 4. Description of the moduli space

Its zero divisor defines an element of the linear system

$$\mathbb{P}H^0(C, (E_1/\mathcal{O}_C) \otimes \Omega_C^1(D)).$$

Since $D = t_1 + t_2$, $\det(E_1) = \mathcal{O}_C(w_\infty)$ and $\Omega_C^1 \simeq \mathcal{O}_C$ then

$$\mathbb{P}H^0(C, (E_1/\mathcal{O}_C) \otimes \Omega_C^1(D)) = |\mathcal{O}_C(w_\infty + t_1 + t_2)| \simeq \mathbb{P}^2.$$

Then one gets a map

$$App : \text{Con}_{<}^\nu(C, D) \longrightarrow |\mathcal{O}_C(w_\infty + t_1 + t_2)| \simeq \mathbb{P}^2.$$

On the other hand, we have a mapping

$$Bun : \text{Con}_{<}^\nu(C, D) \longrightarrow X_{<} \simeq \mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1$$

which comes from the forgetful map.

Definition 3.16. Let $V \simeq \mathbb{C}^2 \subset X_{<} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ be the open subset corresponding to parabolic bundles (E_1, \mathbf{p}) , $\mathbf{p} = \{p_1, p_2\}$, such that $p_i \neq (1 : t)$ for $i = 1, 2$, where $t \in \mathbb{C}$ is the first coordinate of $t_1 = (t, r) \in C$. We have the decomposition:

$$\mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1 = V \sqcup (\{z_1 = t\} \cup \{z_2 = t\}) \quad \text{and} \quad \{z_1 = t\} \cap \{z_2 = t\} =: \{u_t\}.$$

Theorem 3.17. *If $\nu_1 + \nu_2 + 1 \neq 0$, then the map $Bun \times App$ defines a birational map*

$$Bun \times App : \overline{\text{Con}_{<}^\nu(C, D)} \longrightarrow (\mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1) \times \mathbb{P}^2$$

between \mathbb{P}^2 -bundles over $\mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1$. Moreover, the following assertions hold true

- (1) its restriction to $V \subset \mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1$ is an isomorphism

$$Bun \times App|_{Bun^{-1}(V)} : Bun^{-1}(V) \xrightarrow{\simeq} V \times \mathbb{P}^2;$$

- (2) if $u \in \{z_1 = t\} \cup \{z_2 = t\} \setminus \{u_t\}$, then the image of

$$Bun \times App|_{Bun^{-1}(u)} : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

is a line;

- (3) if $\nu_1 + \nu_2 - 1 \neq 0$ and $u = u_t$, then $Bun \times App|_{Bun^{-1}(u_t)}$ is constant and its image is the point

$$w_\infty + t_1 + t_2 \in |\mathcal{O}_C(w_\infty + t_1 + t_2)| \simeq \mathbb{P}^2;$$

- (4) if $\nu_1 + \nu_2 - 1 = 0$ and $u = u_t$, then $Bun^{-1}(u_t)$ lies in the indetermination locus of $Bun \times App$.

Remark 3.18. A phenomenon similar to case (3) was observed in the genus zero case by Szilárd Szabó in [20].

Proof. Firstly, let us fix $\mathbf{z} \in U_0 \subset X_{<}$. We will use the explicit basis $\{\nabla^0, \Theta_1^0, \Theta_2^0\}$ given in Table 1 to compute the mapping

$$(3.16) \quad App_{(\mathbf{z})} : Bun^{-1}(\mathbf{z}) \simeq \mathbb{P}^2 \longrightarrow |\mathcal{O}_C(w_\infty + t_1 + t_2)| \simeq \mathbb{P}^2$$

with respect to the subbundle $\mathcal{O}_C(-2w_\infty) \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C$ (see Remark 3.4).

In order to give explicit coordinates for App , we fix a basis of rational functions

$$\left\{ \frac{1}{(x-t)}, \frac{x}{(x-t)}, \frac{y}{(x-t)} \right\}$$

for the vector space $H^0(C, \mathcal{O}_C(w_\infty + t_1 + t_2))$ and denote by $\mathbf{a} = (a_0 : a_1 : a_2)$ projective coordinates with respect to that basis.

Let $\mathbf{c} = (c_0 : c_1 : c_2) \in \mathbb{P}^2$ be coordinates of $Bun^{-1}(\mathbf{z})$ with respect to the basis $\{\nabla^0(\mathbf{z}), \Theta_1^0(\mathbf{z}), \Theta_2^0(\mathbf{z})\}$ (see Section 3.4), that is, each $\nabla(\mathbf{z}) \in Bun^{-1}(\mathbf{z})$ writes

$$\nabla(\mathbf{z}) = c_0 \cdot \nabla^0(\mathbf{z}) + c_1 \cdot \Theta_1^0(\mathbf{z}) + c_2 \cdot \Theta_2^0(\mathbf{z}).$$

Since the subbundle $\mathcal{O}_C(-2w_\infty) \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C$ corresponds to the explicit section

$$s : C \longrightarrow \mathcal{O}_C \oplus \mathcal{O}_C ; (x, y) \mapsto \begin{pmatrix} 1 \\ x \end{pmatrix}$$

as in Remark 3.4, we can compute the coordinates of the mapping $App_{\mathbf{z}} : \mathbb{P}_{\mathbf{c}}^2 \longrightarrow \mathbb{P}_{\mathbf{a}}^2$:

$$App_{\mathbf{z}}(\mathbf{c}) = a_0 \cdot \frac{1}{(x-t)} + a_1 \cdot \frac{x}{(x-t)} + a_2 \cdot \frac{y}{(x-t)}$$

where

$$\begin{aligned} a_0(\mathbf{c}) &= r(\nu_1(2z_1 - t) + \nu_2(2z_2 - t) - t) \cdot c_0 & -4rz_1(z_1 - t) \cdot c_1 & -4rz_2(z_2 - t) \cdot c_2 \\ a_1(\mathbf{c}) &= -r(\nu_1 + \nu_2 - 1) \cdot c_0 & +4r(z_1 - t) \cdot c_1 & +4r(z_2 - t) \cdot c_2 \\ a_2(\mathbf{c}) &= 2(\nu_1(z_1 - t) - \nu_2(z_2 - t)) \cdot c_0 & -4(z_1 - t)^2 \cdot c_1 & +4(z_2 - t)^2 \cdot c_2 \end{aligned}$$

In other words, $App_{\mathbf{z}} : \mathbb{P}_{\mathbf{c}}^2 \longrightarrow \mathbb{P}_{\mathbf{a}}^2$ is a projective transformation defined by a matrix with determinant

$$\det(App_{\mathbf{z}}) = -32r^2(t - z_1)^2(t - z_2)^2(\nu_1 + \nu_2 + 1).$$

We promptly deduce that the rank drop exactly when $z_1 = t$ or $z_2 = t$. In addition, if $z_1 = t$ and $z_2 \neq t$, then $App_{\mathbf{z}}$ does not depend of c_1 and its matrix has rank two. The same happens if $z_1 \neq t$ and $z_2 = t$. If $z_1 = z_2 = t$, then the matrix has rank one. If $\nu_1 + \nu_2 - 1 \neq 0$, a straightforward computation shows that the image point is $(t : -1 : 0)$, which corresponds to a non-vanishing constant function in $H^0(C, \mathcal{O}_C(w_\infty + t_1 + t_2))$. This shows that the image point is $w_\infty + t_1 + t_2 \in |\mathcal{O}_C(w_\infty + t_1 + t_2)|$ via the identification $|\mathcal{O}_C(w_\infty + t_1 + t_2)| \simeq \mathbb{P}H^0(C, \mathcal{O}_C(w_\infty + t_1 + t_2))$. Assertion (4) can easily be verified.

We now consider the case where

$$\mathbf{Z} = (Z_1, Z_2) = (1/z_1, 1/z_2) \in U_\infty \subset X_{<}$$

Using the basis $\{\nabla^\infty, \Theta_1^\infty, \Theta_2^\infty\}$ instead of $\{\nabla^0, \Theta_1^0, \Theta_2^0\}$ one can find the coordinates of $App_{\mathbf{z}} : \mathbb{P}_{\mathbf{c}}^2 \longrightarrow \mathbb{P}_{\mathbf{a}}^2$ in terms of this basis:

$$App_{\mathbf{z}}(C_0 \cdot \nabla^\infty + C_1 \cdot \Theta_1^\infty + C_2 \cdot \Theta_2^\infty) = a_0 \cdot \frac{1}{(x-t)} + a_1 \cdot \frac{x}{(x-t)} + a_2 \cdot \frac{y}{(x-t)}$$

where

$$\begin{aligned} a_0(\mathbf{C}) &= tr(1 - \nu_1 - \nu_2) \cdot C_0 & +4r(1 - tZ_1) \cdot C_1 & +4r(1 - tZ_2) \cdot C_2 \\ a_1(\mathbf{C}) &= r(\nu_1(2tZ_1 - 1) + \nu_2(2tZ_2 - 1) - 1) \cdot C_0 & +4rZ_1(tZ_1 - 1) \cdot C_1 & +4rZ_2(tZ_2 - 1) \cdot C_2 \\ a_2(\mathbf{C}) &= 2t(\nu_1(tZ_1 - 1) - \nu_2(tZ_2 - 1)) \cdot C_0 & +4(tZ_1 - 1)^2 \cdot C_1 & -4(tZ_2 - 1)^2 \cdot C_2 \end{aligned}$$

It follows that

$$\det(App_{\mathbf{z}}) = -32(tZ_1 - 1)^2(tZ_2 - 1)^2(\nu_1 + \nu_2 + 1).$$

Since $U_0 \cup U_\infty = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \{(\infty, 0), (0, \infty)\}$, to conclude the proof of theorem, it remains to consider the case where $u = (\infty, 0)$ and $u = (0, \infty)$; we do not detail. \square

Theorem 3.19. *If $\nu_1 + \nu_2 + 1 = 0$, then the rational map*

$$Bun \times App : \overline{\text{Con}}_1^\nu(C, D) \longrightarrow \mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1 \times \mathbb{P}^2$$

is non-dominant. Moreover, $(Bun \times App)^{-1}(y)$ is one dimensional for general y lying on the image of $Bun \times App$.

Proof. Assume $\nu_1 + \nu_2 + 1 = 0$. If $z_1 \neq t$ and $z_2 \neq t$ then

$$App_{\mathbf{z}}(\nabla^0(\mathbf{z})) = -\frac{\nu_1}{2(z_1 - t)} App_{\mathbf{z}}(\Theta_1^0(\mathbf{z})) + \frac{\nu_1 + 1}{2(z_2 - t)} App_{\mathbf{z}}(\Theta_2^0(\mathbf{z})).$$

In addition, $App_{\mathbf{z}}(\Theta_1^0(\mathbf{z}))$ and $App_{\mathbf{z}}(\Theta_2^0(\mathbf{z}))$ do not coincide in \mathbb{P}^2 . This shows that the image of $App_{\mathbf{z}} : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ is the line spanned by $App_{\mathbf{z}}(\Theta_1^0(\mathbf{z}))$ and $App_{\mathbf{z}}(\Theta_2^0(\mathbf{z}))$. \square

3.8. Symplectic structure and Torelli phenomenon. Recall that the moduli space of connections $\text{Con}^\nu(C, D)$ admits a canonical holomorphic symplectic structure, i.e. a holomorphic non degenerate 2-form $\omega : \omega \wedge \omega \neq 0$ at any point of $\text{Con}^\nu(C, D)$ (see for instance [19]). Given any Lagrangian rational (or local) section $\nabla_0 : \text{Bun}^\nu(C, D) \rightarrow \text{Con}^\nu(C, D)$, then the reduction map

$$\nabla \longmapsto \nabla - \nabla_0$$

$$\begin{array}{ccc} \text{Con}^\nu(C, D) & \text{-----} & \text{Higgs}^\nu(C, D) \\ & \searrow & \swarrow \\ & \text{Bun}^\nu(C, D) & \end{array}$$

is symplectic. Moreover, the following three maps are Lagrangian

$$\begin{array}{ccc} \text{Con}_{<}^\nu(C, D) & \xrightarrow{App} & |\mathcal{O}_C(w_\infty + t_1 + t_2)| \\ \swarrow^{Bun^+} & & \searrow^{Bun^-} \\ X_{<} & & X_{<} \end{array}$$

i.e. with Lagrangian fibers (see [19]). In particular, the section ∇^0 defined in Section 3.4 is Lagrangian since it is a fiber of Bun^- (see Remark 3.9). We promptly deduce from the reduction map $\nabla_{\mathbf{c}} \mapsto \nabla_{\mathbf{c}} - \nabla^0$ applied to the universal connection $\nabla_{\mathbf{c}}$ defined by (3.11) that the symplectic structure on $\text{Con}^\nu(C, D)$ is given by $\omega = dc_1 \wedge dz_1 + dc_2 \wedge dz_2$ as for Higgs bundles in Theorem 3.6.

The image of ω by the map $\text{Par} : \text{Con}_{<}^\nu(C, D) \rightarrow S$ defined in (3.12) is given by setting $c_i = \nu_i/2(z_i - \zeta_i)$, $i = 1, 2$, i.e. by

$$\omega = -\frac{1}{2} \left\{ \nu_1 \frac{dz_1 \wedge d\zeta_1}{(z_1 - \zeta_1)^2} + \nu_2 \frac{dz_2 \wedge d\zeta_2}{(z_2 - \zeta_2)^2} \right\}.$$

In particular, we see that, taking into account the symplectic structure of the bundle $Bun : \text{Con}_{<}^\nu(C, D) \rightarrow \text{Bun}_{<}^\nu(C, D)$, we can recover the eigenvalues ν_1, ν_2 in the spirit of Torelli Theorem.

Proposition 3.20. *If we have an equivariant bundle symplectic isomorphism*

$$\begin{array}{ccc} (\text{Con}_{<}^\nu(C, D), \omega) & \xrightarrow[\sim]{\Phi} & (\text{Con}_{<}^{\tilde{\nu}}(C, D), \tilde{\omega}) \\ \downarrow \text{Bun} & & \downarrow \text{Bun} \\ \text{Bun}_{<}(C, D) & \xrightarrow[\sim]{\phi} & \text{Bun}_{<}(C, D) \end{array}$$

Then $(\tilde{\nu}_1, \tilde{\nu}_2) = (\nu_1, \nu_2)$ or (ν_2, ν_1) .

Proof. Any automorphism ϕ of $\text{Bun}_{<}(C, D) = X_{<} = \mathbb{P}_{z_1}^1 \times \mathbb{P}_{z_2}^1$ writes

$$(z_1, z_2) \mapsto (\varphi(z_1), \psi(z_2)) \quad \text{or} \quad (\varphi(z_2), \psi(z_1))$$

for some Moebius transformations φ, ψ . The map $\phi_D : (z_1, z_2) \mapsto (z_2, z_1)$ admits a lifting

$$\Phi_D : ((z_1, \zeta_1), (z_2, \zeta_2)) \mapsto ((z_2, \zeta_2), (z_1, \zeta_1))$$

satisfying

$$\Phi_D^* \omega = -\frac{1}{2} \left\{ \nu_2 \frac{dz_1 \wedge d\zeta_1}{(z_1 - \zeta_1)^2} + \nu_1 \frac{dz_2 \wedge d\zeta_2}{(z_2 - \zeta_2)^2} \right\}$$

(it permutes ν_1 and ν_2). Now, considering ϕ isotopic to the identity, we will prove that its lifting Φ preserves coefficients ν_1 and ν_2 . First, observe that ϕ admits a lifting preserving ν_1 and ν_2 , namely

$$\Phi : ((z_1, \zeta_1), (z_2, \zeta_2)) \mapsto ((\varphi(z_1), \varphi(\zeta_1)), (\psi(z_2), \psi(\zeta_2))).$$

Indeed, we can decompose φ and ψ as a combination of $z \mapsto \alpha z$, $z \mapsto z + 1$ and $z \mapsto 1/z$, and easily check that, for each of these transformations, $\frac{dz \wedge d\zeta}{(z - \zeta)^2}$ is invariant, and therefore ω as well. We end up the proof by showing that there are no other equivariant bundle isomorphisms, i.e. if ϕ is the identity, then so is Φ . Indeed, when we fix z_2 , the Φ has to preserve the polar locus of ω (being symplectic), namely $\zeta_2 = z_2$ and $\zeta_1 = z_1$. In restriction to $\zeta_2 = z_2$ (recall we have fixed z_2) Φ induces an automorphism of $\mathbb{P}_{z_1}^1 \times \mathbb{P}_{\zeta_1}^1$ preserving the diagonal $\zeta_1 = z_1$ and fixing z_1 . It is therefore the identity. We easily conclude that Φ is the identity everywhere. \square

It is interesting to make the link with the approach of [13]. There, a birational model of the moduli of parabolic bundles was introduced by using apparent map in restriction to Higgs fields (see [13, Corollary 4.5]). Precisely, the restriction of apparent map to Higgs fields is given by

$$\text{App}_{\mathbf{z}}(c_1 \cdot \Theta_1^0(\mathbf{z}) + c_2 \cdot \Theta_2^0(\mathbf{z})) = a_0 \cdot \frac{1}{(x-t)} + a_1 \cdot \frac{x}{(x-t)} + a_2 \cdot \frac{y}{(x-t)}$$

where

$$\begin{aligned} a_0 &= -r(c_1 z_1(t - z_1) + c_2 z_2(t - z_2)) \\ a_1 &= r(c_1(t - z_1) + c_2(t - z_2)) \\ a_2 &= c_1(t - z_1)^2 + c_2(t - z_2)^2 \end{aligned}$$

Therefore, the image $\text{App}_{\mathbf{z}}(\text{Higgs}_{<}^\nu(C, D)) \subset |\mathcal{O}_C(w_\infty + t_1 + t_2)|$ is given by $a_0 b_0 + a_1 b_1 + a_2 b_2 = 0$ where

$$\begin{aligned} b_0 &= 2t - z_1 - z_2 \\ b_1 &= t(z_1 + z_2) - 2z_1 z_2 \\ b_2 &= r(z_1 - z_2) \end{aligned}$$

and we get a natural birational map

$$Bun' : Bun_{<}(C, D) \rightarrow |\mathcal{O}_C(w_\infty + t_1 + t_2)|^* \simeq \mathbb{P}_{\mathbf{b}}^2 ; (z_1, z_2) \mapsto (b_0 : b_1 : b_2).$$

This map blow-up the point $(z_1, z_2) = (t, t)$ and contracts the strict transform of the two lines $z_1 = t$ and $z_2 = t$. Another birational model of $\text{Con}'_{<}(C, D)$, more in the spirit of [13], is given by

$$(3.17) \quad App \times Bun' : \text{Con}'_{<}(C, D) \xrightarrow{\sim} \mathbb{P}_{\mathbf{a}}^2 \times \mathbb{P}_{\mathbf{b}}^2$$

and we get (compare with [13, Theorem 1.1])

Proposition 3.21. *If $\nu_1 + \nu_2 + 1 \neq 0$, then the map (3.17) is birational and the image of the symplectic form is given by*

$$\omega = d\eta \quad \text{where} \quad \eta = \left(\frac{\nu_1 + \nu_2 + 1}{4} \right) \frac{a_0 db_0 + a_1 db_1 + a_2 db_2}{a_0 b_0 + a_1 b_1 + a_2 b_2}.$$

The polar locus of ω is supported by the incidence variety $a_0 b_0 + a_1 b_1 + a_2 b_2 = 0$.

4. APPENDIX

4.1. Parabolic vector bundles. Let C be a smooth irreducible projective complex curve and $D = t_1 + \dots + t_n$ be a reduced divisor supported on n distinct points $\{t_1, \dots, t_n\} \subset C$. A rank two *quasi-parabolic vector bundle* on (C, D) is the data (E, \mathbf{p}) where E is a rank two holomorphic vector bundle over C and $\mathbf{p} = \{p_1, \dots, p_n\}$ are given 1-dimensional subspaces $p_k \subset E_{t_k}$ for $k \in \{1, \dots, n\}$. An isomorphism between quasi-parabolic vector bundles is, by definition, an isomorphism between underlying vector bundles preserving parabolic directions. A *parabolic vector bundle* is a quasi-parabolic vector bundle together with a collection of weights $\mu = (\mu_1, \dots, \mu_n) \in [0, 1]^n$. It allows us to introduce a notion of stability in order to introduce a good moduli space. Given a line bundle $L \subset E$, the μ -stability index of L is the real number

$$\text{Stab}(L) := \deg E - 2 \deg L + \sum_{p_k \neq L_{t_k}} \mu_k - \sum_{p_k = L_{t_k}} \mu_k.$$

Definition 4.1. A parabolic vector bundle (E, \mathbf{p}) is called μ -stable (resp. μ -semistable) if for any rank one subbundle $L \subset E$, the following inequality holds

$$\text{Stab}(L) > 0 \quad (\text{resp. } \text{Stab}(L) \geq 0).$$

It is well known, see [16], that the moduli space of μ -semistable parabolic vector bundles with fixed determinant line bundle is a normal irreducible projective variety. The open subset of μ -stable parabolic bundles is smooth. We note that the stability index of $L \subset E$ is zero if, and only if, the weights lie along the following hyperplane in $[0, 1]^n$

$$\deg(E) - 2 \deg L + \sum_{p_k \neq L_{t_k}} \mu_k - \sum_{p_k = L_{t_k}} \mu_k = 0.$$

Each one of these hyperplanes is called a *wall*. If we cut out $[0, 1]^n$ by all possible walls, one gets in the complement of finitely many irreducible connected components, which are called *chambers*. In each chamber, any μ -semistable parabolic vector bundle is μ -stable, and the moduli space is constant, that is, it is independent of μ . Nevertheless, it can be empty. When we have two adjacent chambers separated by a wall, then there is a locus of μ -stable parabolic bundles in each

chamber that became unstable when we cross the wall. Along the wall, we identify strictly semistable parabolic bundles (E, \mathbf{p}) and (E', \mathbf{p}') with $\text{gr}(E, \mathbf{p}) = \text{gr}(E', \mathbf{p}')$, see [16, Section 4]. Over the projective line, the description of the moduli space of quasi-parabolic bundles has been done in [3].

4.2. Elementary transformations. In the construction of the moduli space of quasi-parabolic bundles, the determinant line bundle is fixed. Actually, up to twists and elementary transformations, we can choose the determinant bundle arbitrarily, for instance the trivial line bundle \mathcal{O}_C . Twists preserve the parity of the determinant and elementary transformations change it. We start by recalling what is an elementary transformation as well as its main properties. Given $t \in C$ and a direction $p \subset E_t$ the vector bundle E^- is defined by the following exact sequence of sheaves

$$0 \longrightarrow E^- \longrightarrow E \longrightarrow E/p \longrightarrow 0.$$

where p appearing above is considered as a sky-scraper sheaf. The new parabolic direction $p^- \subset E^-$ is the kernel of the morphism $E^- \longrightarrow E$. By identifying sections of E and E^- outside of t , one obtains a birational bundle transformation

$$\text{elem}_t^- : E \dashrightarrow E^-$$

with center at p , which is an isomorphism outside t . We shall say that it is a *negative elementary transformation*. At a neighborhood of t , it can be described as follows. We can choose a local trivialization $E|_U \simeq U \times \mathbb{C}^2$ such that

$$p = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \subset E_t.$$

and $\text{elem}_t^- : E|_U \dashrightarrow E^-|_U$ is given by

$$\text{elem}_t^-(x, Y) = \begin{pmatrix} 1/x & 0 \\ 0 & 1 \end{pmatrix} \cdot Y.$$

From the point of view of ruled surfaces, it corresponds to a flip with center at $[p] \in \mathbb{P}E$, that is, a blow up at $[p]$ followed by a contraction of the old fiber. If the direction p is contained in a line bundle $L \subset E$, then it is left unchanged and one obtains a line bundle $L \subset E^-$. If $L \subset E$ does not contain p , then we get $L^- \subset E^-$ where

$$L^- = L \otimes \mathcal{O}_C(-t).$$

In addition, we have the following property

$$\det(E^-) = \det(E) \otimes \mathcal{O}_C(-t).$$

When we perform an elementary transformation, the stability condition is preserved after an appropriate modification of weights. If (E, \mathbf{p}) is μ -stable and we perform an elementary transformation $\text{elem}_{t_k}^-$, then (E^-, \mathbf{p}^-) is μ' -stable where

$$\begin{cases} \mu'_k = 1 - \mu_k \\ \mu'_j = \mu_j \quad j \neq k. \end{cases}$$

If $\text{Bun}_L^\mu(C, D)$ denotes the moduli space of μ -semistable parabolic vector bundles with fixed determinant line bundle L , then $\text{elem}_{t_k}^-$ defines an isomorphism between moduli spaces

$$\begin{aligned} \text{elem}_{t_k}^- : \text{Bun}_L^\mu(C, D) &\longrightarrow \text{Bun}_{L(-t_k)}^{\mu'}(C, D) \\ (E, \mathbf{p}) &\mapsto \text{elem}_{t_k}^-(E, \mathbf{p}). \end{aligned}$$

We can define a *positive elementary transformation* $elem_t^+$ as

$$elem_t^+ := \mathcal{O}_C(t) \otimes elem_t^- : E \dashrightarrow E^+$$

where $E^+ = E^- \otimes \mathcal{O}_C(t)$. It is the inverse of $elem_t^-$. As before, stability condition is preserved by elementary positive transformations, with the same modification of weights.

4.3. Endomorphisms of quasi-parabolic vector bundles. The space of global endomorphisms of a rank two vector bundle as well as the automorphism group are well known, see for example [15, Theorem 1]. In this section, we study the space of traceless endomorphisms of a quasi-parabolic bundles over an elliptic curve.

Let (E, \mathbf{p}) be a quasi-parabolic bundle. We say that an endomorphism $\sigma \in \text{End}(E)$ fixes the parabolic structure \mathbf{p} if $\sigma(p_k) \subset p_k$, for all $k = 1, \dots, n$. Let $\text{End}(E, \mathbf{p})$ be the vector space of endomorphisms fixing the parabolic structure \mathbf{p} and $\text{End}_0(E, \mathbf{p})$ its subspace of traceless endomorphisms. We notice that we have a canonical decomposition:

$$\text{End}(E, \mathbf{p}) = \langle Id \rangle \oplus \text{End}_0(E, \mathbf{p})$$

where $Id \in \text{End}(E, \mathbf{p})$ is the identity. This follows from the following simple remark: if A is a 2×2 matrix then

$$A = \frac{\text{tr}(A)}{2} \cdot Id + \left(A - \frac{\text{tr}(A)}{2} \cdot Id \right).$$

Lemma 4.2. *Let $E = \mathcal{O}_C \oplus L$ be a rank 2 bundle over a projective smooth curve C with \mathcal{L} of nonnegative degree. The following statements hold true.*

(a) *If $L \not\cong \mathcal{O}_C$ then*

$$\text{End}_0(E) = \left\{ \begin{pmatrix} a & 0 \\ \gamma & -a \end{pmatrix} ; a \in \mathbb{C}, \gamma = \{\gamma_i\} \in \mathbf{H}^0(C, L) \right\}$$

(b) *If $\mathcal{L} = \mathcal{O}_C$ then*

$$\text{End}_0(E) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} ; a, b, c \in \mathbb{C} \right\}$$

Proof. We leave the proof to the reader; see for example [15, Theorem 1]. \square

We recall that a quasi-parabolic bundle (E, \mathbf{p}) is decomposable if there exists a decomposition $E = L_1 \oplus L_2$ such that each parabolic direction is contained in L_1 or L_2 . In this case, we write

$$(E, \mathbf{p}) = (L_1, \mathbf{p}_1) \oplus (L_2, \mathbf{p}_2).$$

4.4. The case of elliptic curves. In what follows we will determine the traceless endomorphisms of an indecomposable quasi-parabolic bundle over an elliptic curve. This will be useful to assure existence of logarithmic connections. Before that, we shall give one example.

Example 4.1. Let $(\mathcal{O}_C \oplus \mathcal{O}_C(t_1), \{p_1, p_2\})$ be a quasi-parabolic bundle over an elliptic curve $(C, \{t_1, t_2\})$ with

- (1) p_1 outside $\mathcal{O}_C|_{t_1}$ and $\mathcal{O}_C(t_1)|_{t_1}$; and
- (2) $p_2 \subset \mathcal{O}_C(t_1)|_{t_2}$.

It is indecomposable because any subbundle given by an embedding of the trivial bundle $\mathcal{O}_C \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C(t_1)$ corresponds to a section of $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(t_1))$ which has $(1 : 0)$ as a base point over t_1 . In fact, since C is elliptic $h^0(\mathcal{O}_C(t_1)) = 1$. On the other hand, if $\gamma \in H^0(C, \mathcal{O}_C(t_1))$ is a section which corresponds to the divisor $D = t_1$ then

$$\text{End}_0(E, \mathbf{p}) = \left\{ \begin{pmatrix} 0 & 0 \\ c\gamma & 0 \end{pmatrix} ; c \in \mathbb{C} \right\}.$$

Proposition 4.3. *Let $E = \mathcal{O}_C \oplus L$, $\deg(L) \geq 0$, be a decomposable rank 2 bundle over an elliptic curve C . Assume (E, \mathbf{p}) is indecomposable but not simple, i.e. $\text{End}_0(E, \mathbf{p}) \neq \{0\}$. Then we are in the following case*

- (1) $\text{End}_0(E, \mathbf{p}) = \mathbb{C}$,
- (2) the support D of parabolics splits as $D = D_0 + D_1$ with $\deg(D_1) > 0$,
- (3) $L \simeq \mathcal{O}_C(D_1)$ (and has > 0 degree),
- (4) parabolics over D_0 are lying on L ,
- (5) parabolics over D_1 are outside L and generic.

Here, generic means that there is no embedding $\mathcal{O}_C \hookrightarrow E$ passing through all parabolics over D_1 , i.e. (E, \mathbf{p}) is indecomposable.

Observe that, by Lemma 4.2, one can find an embedding $\mathcal{O}_C \hookrightarrow E$ passing through all but one parabolics over D_1 . In particular, given the decomposition $D = D_0 + D_1$, there is a unique such parabolic bundle (E, \mathbf{p}) up to isomorphism.

Proof. If $L = \mathcal{O}_C$ and (E, \mathbf{p}) is indecomposable, then there are at least 3 parabolics which not two of them lies in the same embedding of $\mathcal{O}_C \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C$. Then it follows from Lemma 4.2 item (b) that

$$\text{End}_0(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{p}) = \{0\}.$$

Let us suppose $L \not\cong \mathcal{O}_C$ and let ϕ be a traceless endomorphism that fixes the parabolics. Lemma 4.2 item (a) implies that we can choose a covering of C and trivializations such that the vector e_1 generates \mathcal{O}_C , e_2 generates L and

$$\phi = \begin{pmatrix} a & 0 \\ \gamma & -a \end{pmatrix}$$

where $a \in \mathbb{C}$, $\gamma \in H^0(C, L)$. If $a \neq 0$, then we see that the locus of fixed points of ϕ outside L defines a section of E

$$\begin{pmatrix} 1 \\ \frac{\gamma}{2a} \end{pmatrix}$$

which generates a subbundle $\mathcal{O}_C \hookrightarrow E$ containing all parabolics outside L , showing that (E, \mathbf{p}) must be decomposable. We can thus assume $a = 0$.

Since ϕ preserves the parabolic structure, we have for each parabolic p_k

- either p_k is in the subbundle $L \subset E$,
- or the support t_k of p_k is a zero of γ .

Therefore, we can decompose $D = D_0 + D_1$ where D_0 is the support of those p_k 's lying in L .

Since (E, \mathbf{p}) is indecomposable, there is no embedding $\mathcal{O}_C \hookrightarrow E$ passing through all parabolics over D_1 . Assume that our decomposition $E = \mathcal{O}_C \oplus L$ maximize the number of parabolics lying on the first factor. Set $D_1 = D'_1 + D''_1$ with D'_1 supporting those parabolics in \mathcal{O}_C and $\deg(D''_1) > 0$. By maximality, we have that

each section $\varphi \in H^0(C, L)$ which vanishes on D'_1 automatically vanishes on D''_1 . In other words, each section of $L \otimes \mathcal{O}_C(-D'_1)$ automatically vanishes on D''_1 . On the other hand, we know that $L \otimes \mathcal{O}_C(-D'_1)$ admits a non zero section defined by γ . But on the elliptic curve C , the only linear systems with base points are of the form $|t_k|$, meaning that $L \otimes \mathcal{O}_C(-D'_1) \simeq \mathcal{O}_C(t_k)$ and $D''_1 = t_k$ reduces to a single point. We therefore deduce that $L \simeq \mathcal{O}_C(D_1)$ and all parabolics but p_k , over D_1 , are lying on the first factor $\mathcal{O}_C \subset E$. \square

Let E_0 be the unique indecomposable rank 2 bundle, over an elliptic curve, with trivial determinant and having \mathcal{O}_C as maximal subbundle. It corresponds to the non trivial extension defined by the following exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_0 \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Proposition 4.4. *Let (E_0, \mathfrak{p}) be a quasi-parabolic bundle, where E_0 is the indecomposable bundle as above.*

(1) *If all parabolics lie in the maximal subbundle $\mathcal{O}_C \hookrightarrow E_0$ then*

$$\text{End}_0(E_0, \mathfrak{p}) \simeq \mathbb{C}$$

(2) *If there exists at least one parabolic outside $\mathcal{O}_C \hookrightarrow E_0$ then*

$$\text{End}_0(E_0, \mathfrak{p}) = \{0\}.$$

Proof. The traceless endomorphism space of E_0 is given by

$$\text{End}_0(E_0) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} ; b \in \mathbb{C} \right\}.$$

Here we are considering that in local charts $U_i \subset C$, e_1 generates the maximal subbundle \mathcal{O}_C . Then any traceless endomorphism leaves \mathcal{O}_C invariant. Also if a parabolic direction outside \mathcal{O}_C is fixed, one gets $b = 0$. This is enough to conclude the proof of proposition. \square

Proposition 4.5. *Let (E, \mathfrak{p}) be indecomposable but not simple rank 2 quasi-parabolic bundle over an elliptic curve C . Assume E has trivial determinant line bundle. Then, up to elementary transformations and twists, we can assume $E = E_0$ with all the parabolic lying in the maximal subbundle $\mathcal{O}_C \hookrightarrow E_0$.*

Proof. The proof follows essentially from Proposition 4.3 and Proposition 4.4. Suppose E is decomposable, $E = M^{-1} \oplus M$, $\deg M = k \geq 0$. Then $\mathbb{P}E = \mathbb{P}(\mathcal{O}_C \oplus M^2)$. From Proposition 4.3, the support D of parabolics splits as $D = D_0 + D_1$ with $\deg(D_1) > 0$ and $\mathcal{O}_C(D_1) = M^2$. Parabolics over D_1 are outside $M \hookrightarrow E$ (which corresponds to $M^2 \hookrightarrow \mathcal{O}_C \oplus M^2$) and generic. Parabolics over D_0 are lying on M . After a composition $elem_{D_1}$ of $2k$ elementary transformation over D_1 and twist (to get trivial determinant), we arrive in E_0 with all the parabolics lying in the maximal subbundle. If E is indecomposable, the conclusion follows from Proposition 4.4. \square

4.5. Moduli space of connections. Let C be a smooth projective curve and $D = t_1 + \dots + t_n$ be a reduced divisor on C , $n \geq 1$. We will fix the data in order to introduce the moduli space of connections. Firstly, let us fix a degree d line bundle L_0 over C . We also set a local exponent $\nu \in \mathbb{C}^{2n}$ satisfying the Fuchs relation

$$\sum_{k=1}^n (\nu_k^+ + \nu_k^-) + d = 0$$

and the generic condition $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$ for any $\epsilon_k \in \{+, -\}$, to avoid reducible connections. Let $\zeta : L_0 \rightarrow L_0 \otimes \Omega_C^1(D)$ be any fixed rank one logarithmic connection on L_0 satisfying

$$\text{Res}_{t_k}(\zeta) = \nu_k^+ + \nu_k^-$$

for all $k = 1, \dots, n$. We denote by $\text{Con}^\nu(C, D)$ the moduli space of triples (E, ∇, \mathbf{p}) where

- (1) (E, \mathbf{p}) is a rank 2 quasi-parabolic vector bundle over (C, D) having L_0 as determinant bundle;
- (2) $\nabla : E \rightarrow E \otimes \Omega_C^1(D)$ is a logarithmic connection on E with polar divisor D , having ν as local exponents and $\text{tr}(\nabla) = \zeta$;
- (3) two triples (E, ∇, \mathbf{p}) and $(E', \nabla', \mathbf{p}')$ are equivalent when there is an isomorphism between quasi-parabolic bundles (E, \mathbf{p}) and (E', \mathbf{p}') conjugating ∇ and ∇' .

Actually, in order to obtain a good moduli space we need a stability condition. A triple (E, ∇, \mathbf{p}) is called μ -stable (resp. μ -semistable) if for any ∇ -invariant line bundle $L \subset E$, we have

$$\text{Stab}(L) > 0 \quad (\text{resp. } \text{Stab}(L) \geq 0)$$

(see Definition 4.1). But an invariant line bundle L would force a relation

$$\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} + \deg(L) = 0$$

which is obtained by applying Fuchs relation to the restriction $\nabla|_L$. This contradicts our hypothesis on ν . Therefore under generic condition on the local exponent all the connections arising in our moduli space are stable. It follows from [17, Theorem 3.5] that $\text{Con}^\nu(C, D)$ is a quasi-projective variety.

A priori, $\text{Con}^\nu(C, D)$ depends on the choice of L_0 . But up to twists, we can go into either the even case $L_0 = \mathcal{O}_C$ or the odd case $L_0 = \mathcal{O}_C(t)$. In fact, given a rank one logarithmic connection $\eta : L \rightarrow L \otimes \Omega_C^1(D)$ with local exponents (k_1, \dots, k_n) , we can define a twisting map

$$\begin{aligned} \otimes(L, \eta) : \text{Con}^\nu(C, D) &\longrightarrow \text{Con}^{\nu'}(C, D) \\ (E, \nabla, \mathbf{p}) &\mapsto (E \otimes L, \nabla \otimes \eta, \mathbf{p}) \end{aligned}$$

where $\nu' = (\nu_1^\pm + k_1, \dots, \nu_n^\pm + k_n)$. Such map is an isomorphism between moduli spaces, in particular our moduli space only depend on differences $\nu_k^+ - \nu_k^-$. In the even case, we can assume that $(L_0, \zeta) = (\mathcal{O}_C, d)$ where d means the trivial rank one connection. In the odd case, one may suppose $(L_0, \zeta) = (\mathcal{O}_C(t), d - \frac{dx}{x-t})$.

Now let us deal with elementary transformations. When we perform a transformation $\text{elem}_{t_k}^- : (E, \mathbf{p}) \dashrightarrow (E^-, \mathbf{p}^-)$ the new connection ∇^- on E^- has local exponents

$$(\nu_k^+, \nu_k^-)' = (\nu_k^- + 1, \nu_k^+)$$

and the other ν_j , $j \neq k$, are left unchanged. Finally, we can go from the odd to the even case by performing one negative elementary transformation $\text{elem}_{t_n}^-$.

In the case we are interested in, C is supposed to be an elliptic curve and $D = t_1 + t_2$. For computation, we can assume $C \subset \mathbb{P}^2$ is the smooth projective cubic curve

$$(4.1) \quad zy^2 = x(x-z)(x-\lambda z)$$

with $\lambda \in \mathbb{C}$, $\lambda \neq 0, 1$. And by the above digression one can set $L = \mathcal{O}_C(w_\infty)$, $w_\infty = (0 : 1 : 0)$. As local exponents, we can take

$$(4.2) \quad (\nu_1^\pm, \nu_2^\pm) = \left(\pm \frac{\nu_1}{2} - \frac{1}{2}, \pm \frac{\nu_2}{2} \right).$$

We note that for each $k \in \{1, 2\}$, the condition $\nu_k^+ = \nu_k^-$ is equivalent to $\nu_k = 0$.

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UNIVERSIDADE FEDERAL FLUMINENSE, RUA MÁRIO SANTOS BRAGA S/N, NITERÓI, RJ, BRASIL
E-mail address: `tfassarella@id.uff.br`

UNIV RENNES 1, CNRS, INSTITUT DE RECHERCHE EN MATHÉMATIQUE DE RENNES, IRMAR,
UMR 6625, RENNES, FRANCE
E-mail address: `frank.loray@univ-rennes1.fr`