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# Effective Description of Higher-Order Scalar-Tensor Theories

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## Abstract

Most existing theories of dark energy and/or modified gravity, involving a scalar degree of freedom, can be conveniently described within the framework of the Effective Theory of Dark Energy, based on the unitary gauge where the scalar field is uniform. We extend this effective approach by allowing the Lagrangian in unitary gauge to depend on the time derivative of the lapse function. Although this dependence generically signals the presence of an extra scalar degree of freedom, theories that contain only one propagating scalar degree of freedom, in addition to the usual tensor modes, can be constructed by requiring the initial Lagrangian to be degenerate. Starting from a general quadratic action, we derive the dispersion relations for the linear perturbations around Minkowski and a cosmological background. Our analysis directly applies to the recently introduced Degenerate Higher-Order Scalar-Tensor (DHOST) theories. For these theories, we find that one cannot recover a Poisson-like equation in the static linear regime except for the subclass that includes the Horndeski and so-called “beyond Horndeski” theories. We also discuss Lorentz-breaking models inspired by Horava gravity.

## 1 Introduction

The observation of the present cosmological acceleration has spurred the study of a wide range of theories of dark energy and modified gravity. The number of existing models is now so large that an effective approach encompassing as many models as possible is an efficient way to synthesize the various predictions and to confront theoretical models with present and forthcoming data. Since many models of dark energy and modified gravity, although not all of them, involve a scalar field in an explicit or implicit way, an effective description based on ADM treatment in the so-called unitary gauge where the scalar field is spatially uniform, is particularly useful and has been actively developed in the last few years.

Often called Effective Theory of Dark Energy [1–7], this approach (see also [8–12] for other effective approaches to scalar-tensor theories) is inspired by the Effective Field Theory of Inflation [13, 14] and is based on an action whose building blocks are the lapse  $N$ , the shift  $N^i$  and the spatial metric  $h_{ij}$ , which all appear in the ADM metric,

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (1.1)$$

The shift  $N^i$  and the spatial metric  $h_{ij}$  appear in the Lagrangian in combinations that behave as three-dimensional tensors under time-dependent spatial diffeomorphisms. One such combination is the “velocity” of the spatial metric, expressed by the extrinsic curvature tensor  $K_{ij}$ . Another one is the 3-dimensional Ricci scalar  $R$ .

The time derivative of the lapse is usually not included in the initial action because the presence of  $\dot{N}$  generically leads to an additional propagating degree of freedom. However, there are special cases where the action depends on  $\dot{N}$  without leading to an extra degree of freedom.<sup>1</sup> For instance, starting from an action whose ADM form in the unitary gauge does not contain any  $\dot{N}$  and making a conformal transformation of the metric that depends on the scalar field gradient leads to an action with an  $\dot{N}$  dependence. In that case, the presence of  $\dot{N}$  terms is not problematic because there is a degeneracy in the kinetic terms, which prevents the existence of a ghost-like degree of freedom (see [7] and [15]).

In the present work, we consider systematically Lagrangians quadratic in linear perturbations that contain time (and space) derivatives of  $\delta N$ , such as to include all possible terms containing at most two (space or time) derivatives.<sup>2</sup> The corresponding quadratic action, in an expansion around the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric  $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$ , can be written in the form

$$S^{\text{quad}} = \int d^3x dt a^3 \frac{M^2}{2} \left\{ \delta K_{ij} \delta K^{ij} - \left( 1 + \frac{2}{3} \alpha_L \right) \delta K^2 + (1 + \alpha_T) \left( R \frac{\delta \sqrt{h}}{a^3} + \delta_2 R \right) + H^2 \alpha_K \delta N^2 + 4H \alpha_B \delta K \delta N + (1 + \alpha_H) R \delta N + 4\beta_1 \delta K \delta \dot{N} + \beta_2 \delta \dot{N}^2 + \frac{\beta_3}{a^2} (\partial_i \delta N)^2 \right\}, \quad (1.2)$$

where  $H \equiv \dot{a}/a$  is the Hubble rate and  $\delta_2 R$  stands for the second order term in the perturbative expansion of  $R$ . Although the spatially diff-invariant combination denoting the “velocity” of the lapse is  $\dot{N} - N^i \partial_i N$ , the action above contains only  $\delta \dot{N}$ , to which the full combination reduces at linear order.

The above quadratic action extends the one derived in [4] and written in terms of the dimensionless time-dependent functions  $\alpha_A$  (introduced in [16] and [17]) in [7], with the addition of four new functions of time: the parameter  $\alpha_L$ , and the three parameters  $\beta_A$  that characterize the terms containing (time or space) derivatives of  $\delta N$ . These parameters can be given the following interpretation:

- $\alpha_L$  corresponds to a detuning of the extrinsic curvature terms. When  $\alpha_L = 0$  one recovers the combination  $K_{ij} K^{ij} - K^2$ , which is part of the four dimensional Ricci scalar (via the Gauss-Codazzi identity). This detuning appears in theories that already in their original formulation assume a preferred time slicing, such as Horava gravity [18] and its extensions [19–22].
- $\beta_1$  is analogous to the kinetic braiding  $\alpha_B$  for the additional degree of freedom present in higher-order theories.
- $\beta_2$ , similarly, is the analogue of the kineticity  $\alpha_K$ .
- $\beta_3$  is associated to the gradient energy of the additional degree of freedom. This comes from the acceleration of the unit vector normal to the uniform scalar field hypersurfaces, which in unitary gauge is given by  $a_i = \partial_i N / N$ .

<sup>1</sup>When higher time derivative terms can be treated perturbatively below some energy scale, the extra degree of freedom is not excited. Here we consider higher time derivatives at the same level as the other terms.

<sup>2</sup>For instance, since  $R$  contains two spatial derivatives, we do not include a term such as  $R \delta \dot{N}$ , which depends on three derivatives.

As will be shown explicitly in Sec. 2, one can obtain an action of the form (1.2) by starting from a covariant scalar-tensor action and choosing a slicing where the scalar field  $\phi$  depends only on time. Whereas usual scalar-tensor Lagrangians, which depend only on  $\phi$  and its first order gradient  $\nabla_\mu\phi$ , lead to effective perturbative actions where only  $\alpha_K$  can be nonzero, scalar-tensor Lagrangians that depend as well on second-order derivatives  $\nabla_\mu\nabla_\nu\phi$ , lead to a much richer phenomenology. Allowing for higher-order derivatives in the Lagrangian is potentially dangerous as, in general, this yields higher-order equations of motion requiring extra initial conditions, thus signalling the presence of an extra scalar degree of freedom associated with instabilities.

However, it is possible to find higher-order scalar-tensor theories that contain a single scalar degree of freedom (in addition to the tensor modes associated with gravity) by imposing some appropriate restrictions on the initial Lagrangian. For instance, requiring that the associated Euler-Lagrange equations are second order leads to Horndeski theories [23], associated with nonzero  $\alpha_K$ ,  $\alpha_B$  and  $\alpha_T$  (while  $\alpha_L$ ,  $\alpha_H$  and the  $\beta_A$  vanish). Overcoming the prejudice that second order equations of motion were necessary to get only one propagating scalar mode, the introduction of a larger class of models, often called “beyond Horndeski” theories, showed that the absence of an extra scalar mode is compatible with third order Euler-Lagrange equations [17, 24].<sup>3</sup> These “beyond Horndeski” theories give a nonzero  $\alpha_H$ , but  $\alpha_L$  and the  $\beta_A$  are still vanishing.

In [26, 27], it was realized that all higher-order scalar-tensor theories that contain a single scalar mode can be understood as degenerate theories, dubbed Degenerate Higher-Order Scalar-Tensor (DHOST) theories.<sup>4</sup> Here, degenerate means that the Hessian matrix obtained by taking the second derivatives of the Lagrangian with respect to velocities<sup>5</sup> is a degenerate matrix (see [30–32] for recent considerations on the notion of degeneracy). All DHOST theories up to quadratic order, i.e. whose Lagrangian depends quadratically on  $\nabla_\mu\nabla_\nu\phi$ , were identified in [26]. The systematic classification of DHOST theories up to cubic order was recently completed in [33].

Horndeski and “beyond Horndeski” theories are included in the class of DHOST theories. In fact, they belong to the same subclass of DHOST theories and can be related to each other via disformal transformations [17, 28, 29, 34, 35]. But, along this special subclass that contains Horndeski and “beyond Horndeski” theories, DHOST theories include many other subclasses of theories: six other subclasses in the purely quadratic case, eight in the purely cubic case and 24 other subclasses for theories with both quadratic and cubic terms.

The effective description of dark energy models is a powerful tool to confront models with present and future observations, see for instance [36–49]. It is also an efficient way to classify the phenomenology of various theories [50, 51]. So far, the effective approach has mainly been used for Horndeski and beyond Horndeski theories, although it has also been extended to include models such as Horava gravity [39, 52–54]. The purpose of this work is to generalize this effective approach in order to include DHOST theories.

The layout of this paper and our main results can be summarized as follows. In the next section, we briefly present the DHOST theories (up to cubic order) and derive their Lagrangian in the unitary gauge. The degeneracy of DHOST theories implies that the parameters in action (1.2) cannot be arbitrary but must satisfy some consistency relations. We find that there are two such sets of degeneracy conditions, given in Sec. 2.3, which we name  $\mathcal{C}_I$  and  $\mathcal{C}_{II}$ : they relate the

<sup>3</sup>See also [25] for an earlier example based on the disformal transformation of the Einstein-Hilbert Lagrangian.

<sup>4</sup>The theories discovered in [26] were also named Extended Scalar-Tensor (EST) theories in [28]. We prefer to use the more specific terminology of DHOST theories, introduced in [29].

<sup>5</sup>More precisely, as explained in detail in [26], one first introduces an auxiliary variable that includes the time derivative of the scalar field  $\dot{\phi}$ , as well as the lapse and the shift, so that all second-order time derivatives  $\ddot{\phi}$  are absorbed by the velocity of this auxiliary variable and there are no longer time derivatives of the lapse and of the shift. Thus, the kinetic part of the resulting Lagrangian does not depend anymore on an acceleration but just on velocities.

parameters  $\beta_1, \beta_2, \beta_3$ , with  $\alpha_L, \alpha_H$  and  $\alpha_T$ . The first set of conditions is characterized by  $\alpha_L = 0$ , while  $\beta_1$  remains arbitrary. By contrast, in the second set,  $\alpha_L$  is arbitrary while all  $\beta$ 's are fully determined in terms of  $\alpha_H, \alpha_T$  and  $\alpha_L$ . This implies that all the DHOST theories we investigate can be regrouped into three main families: those satisfying only  $\mathcal{C}_I$ , those verifying only  $\mathcal{C}_{II}$ , and finally the theories for which both sets of conditions  $\mathcal{C}_I$  and  $\mathcal{C}_{II}$  are valid.

In Sec. 2.4, we also study how the action (1.2) transforms under the most general conformal-disformal transformation, allowing the conformal and disformal factors to depend on the scalar field, as well as on  $X \equiv \nabla_\mu \phi \nabla^\mu \phi$ . After the transformation, the action takes the same form as (1.2), with its parameters related to those of eq. (1.2) by the transformations given in eq. (2.22). In general, all the parameters, except  $\alpha_L$ , are modified but we show that both sets of conditions  $\mathcal{C}_I$  and  $\mathcal{C}_{II}$  are preserved under these transformations. The two sets of conditions  $\mathcal{C}_I$  and  $\mathcal{C}_{II}$  share a common condition, which implies that only one scalar mode appears *in the unitary gauge*, but this condition is not enough to guarantee that this remains true in an arbitrary gauge.

The family of DHOST theories that satisfy  $\mathcal{C}_I$  but not  $\mathcal{C}_{II}$  coincides with theories that are related to Horndeski via (conformal-) disformal transformations. For all the other theories, i.e. those satisfying  $\mathcal{C}_{II}$ , we find that the effective Newton constant in the analog of the Poisson equation becomes infinite, as a direct consequence of one of the conditions in  $\mathcal{C}_{II}$ . Therefore, one cannot recover a Poisson-like equation in the static linear regime for these theories, in contrast with theories verifying only  $\mathcal{C}_I$ , where  $\beta_1$  is unconstrained. If this peculiar behaviour persists at the nonlinear level, this would indicate that only theories that are related to Horndeski via conformal or disformal transformations are phenomenologically viable.

We examine the dispersion relation for scalar modes around Minkowski in Sec. 3 and in a cosmological background in Sec. 4. In both cases, we observe that the dispersion relation  $\omega^2 = \omega^2(k^2)$  is in general a rational function of  $k^2$ . This drastically simplifies to a linear dispersion relation  $\omega^2 = c_s^2 k^2$  when the degeneracy conditions  $\mathcal{C}_I$  or  $\mathcal{C}_{II}$  are satisfied. In the cosmological context, we also derive the quadratic action for the curvature perturbation on uniform field hypersurfaces  $\zeta$  and show that it is conserved on super-Hubble scales. Then, in Sec. 5 we discuss two classes of Lorentz-breaking theories that have been introduced in the literature. Finally, we present some conclusions in the final section. We have also added several appendices, where more technical details are provided.

## 2 DHOST theories

In this section, we present a large class of scalar-tensor theories whose action, which depends on a metric  $g_{\mu\nu}$  and a scalar field  $\phi$ , leads to a quadratic action of the form (1.2) when written in the unitary gauge. More precisely, we assume that the Lagrangian depends not only on  $\phi$  and its gradient  $\phi_\mu \equiv \nabla_\mu \phi$  as usual, but also on its second derivatives  $\phi_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \phi$ .

### 2.1 Covariant action

Allowing for a dependence on  $\phi_{\mu\nu}$  up to cubic order, we consider an action of the form

$$S[g, \phi] = \int d^4x \sqrt{-g} \left[ P(X, \phi) + Q(X, \phi) \square \phi + f_2(X, \phi) {}^{(4)}R + C_{(2)}^{\mu\nu\rho\sigma} \phi_{\mu\nu} \phi_{\rho\sigma} + f_3(X, \phi) {}^{(4)}G_{\mu\nu} \phi^{\mu\nu} + C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\alpha\beta} \right], \quad (2.1)$$

where the functions  $f_2$  and  $f_3$  depend only on the scalars  $\phi$  and  $X \equiv \phi_\mu \phi^\mu$ ;  ${}^{(4)}R$  and  ${}^{(4)}G_{\mu\nu}$  denote, respectively, the usual Ricci scalar and Einstein tensor associated with the metric  $g_{\mu\nu}$ .

The tensors  $C_{(2)}$  and  $C_{(3)}$  are the most general tensors constructed from the metric  $g_{\mu\nu}$  and the first derivative of the scalar field  $\phi_\mu$ . It is easy to see that the quadratic terms can be written as

$$C_{(2)}^{\mu\nu\rho\sigma} \phi_{\mu\nu} \phi_{\rho\sigma} = \sum_{A=1}^5 a_A(X, \phi) L_A^{(2)}, \quad (2.2)$$

with

$$\begin{aligned} L_1^{(2)} &= \phi_{\mu\nu} \phi^{\mu\nu}, & L_2^{(2)} &= (\square\phi)^2, & L_3^{(2)} &= (\square\phi) \phi^\mu \phi_{\mu\nu} \phi^\nu, \\ L_4^{(2)} &= \phi^\mu \phi_{\mu\rho} \phi^{\rho\nu} \phi_\nu, & L_5^{(2)} &= (\phi^\mu \phi_{\mu\nu} \phi^\nu)^2. \end{aligned} \quad (2.3)$$

Similarly, the cubic terms can be written as

$$C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\alpha\beta} = \sum_{A=1}^{10} b_A(X, \phi) L_A^{(3)}, \quad (2.4)$$

where

$$\begin{aligned} L_1^{(3)} &= (\square\phi)^3, & L_2^{(3)} &= (\square\phi) \phi_{\mu\nu} \phi^{\mu\nu}, & L_3^{(3)} &= \phi_{\mu\nu} \phi^{\nu\rho} \phi_\rho^\mu, \\ L_4^{(3)} &= (\square\phi)^2 \phi_\mu \phi^{\mu\nu} \phi_\nu, & L_5^{(3)} &= \square\phi \phi_\mu \phi^{\mu\nu} \phi_{\nu\rho} \phi^\rho, & L_6^{(3)} &= \phi_{\mu\nu} \phi^{\mu\nu} \phi_\rho \phi^{\rho\sigma} \phi_\sigma, \\ L_7^{(3)} &= \phi_\mu \phi^{\mu\nu} \phi_{\nu\rho} \phi^{\rho\sigma} \phi_\sigma, & L_8^{(3)} &= \phi_\mu \phi^{\mu\nu} \phi_{\nu\rho} \phi^\rho \phi_\sigma \phi^{\sigma\lambda} \phi_\lambda, \\ L_9^{(3)} &= \square\phi (\phi_\mu \phi^{\mu\nu} \phi_\nu)^2, & L_{10}^{(3)} &= (\phi_\mu \phi^{\mu\nu} \phi_\nu)^3. \end{aligned} \quad (2.5)$$

In general, theories with an action of the form (2.1), which depends on second-order derivatives of  $\phi$ , contain two tensor modes and two scalar modes, one of which is associated with a so-called Ostrogradsky instability [55, 56]. However, it is possible to choose special functions  $a_A$  and  $b_A$  in the terms of the Lagrangian (2.2) and (2.4) so that the corresponding theory is degenerate and contains at most one propagating scalar mode. This class of theories, also known as DHOST theories, has originally been identified at quadratic order in  $\phi_{\mu\nu}$  (i.e. with the functions  $f_2$  and  $a_A$  only) in [26] and further studied in [27–29, 57] (see also [58] for an approach to scalar-tensor theories based on differential forms).

The identification of DHOST theories has recently been extended up to cubic order, i.e. by including the second line of (2.1), in [33] and the interested reader will find the full classification there (see Table 1 for a short summary). The DHOST theories include all Horndeski theories but also new theories that lead to higher-order Euler-Lagrange equations even if no extra scalar mode propagates. In summary, there exist seven classes of purely quadratic theories (four classes with  $f_2 \neq 0$  and three classes with  $f_2 = 0$ ) and nine classes of purely cubic theories (two with  $f_3 \neq 0$  and seven with  $f_3 = 0$ ). These quadratic and cubic classes can be combined to yield hybrid theories, involving both quadratic and cubic terms, but all combinations are not possible: only 25 combinations (out of 63) lead to degenerate theories, often with extra conditions on the functions  $a_A$  and  $b_A$  in the Lagrangian (see [33] for details).

## 2.2 (3+1) decomposition in the unitary gauge

We now wish to reexpress the action (2.1) in ADM form in the unitary gauge. For simplicity, we discuss here only the quadratic case. More details about the calculations and their extension to cubic theories are given in App. A.

In order to write the  $(3 + 1)$  decomposition of the action (2.1), it is convenient to use the notation of [26, 27] and introduce the auxiliary variables<sup>6</sup>

$$A \equiv \frac{1}{N} \mathcal{D}_t \phi \equiv \frac{1}{N} (\dot{\phi} - N^i \partial_i \phi), \quad V \equiv \frac{1}{N} \mathcal{D}_t A. \quad (2.6)$$

The action (2.1) can then be expressed in terms of  $V$ , corresponding to the velocity of  $A$ , and of the extrinsic curvature tensor,

$$K_{ij} \equiv \frac{1}{2N} \left( \dot{h}_{ij} - D_i N_j - D_j N_i \right), \quad (2.7)$$

where  $D_i$  denotes the covariant derivative associated with the spatial metric  $h_{ij}$ . The full expression for the action in an arbitrary gauge can be found in [27].

Here, we restrict our derivation to the so-called unitary gauge, where the scalar field is uniform, i.e. such that

$$\partial_i \phi = 0 \quad (\text{unitary gauge}). \quad (2.8)$$

In the unitary gauge, the quantities  $A$  and  $V$  defined above reduce to

$$A = \frac{\dot{\phi}}{N}, \quad V \equiv \frac{1}{N} \left( \dot{A} + A \frac{N^i \partial_i N}{N} \right) \quad (\text{unitary gauge}). \quad (2.9)$$

Ignoring the  $P$  and  $Q$  terms of the Lagrangian, which do not play any role for the degeneracy, we can compute the ADM form of the elementary quadratic and cubic Lagrangians (2.3) and (2.5) in the unitary gauge. Their expressions are given in App. A. One can also obtain the analogous expression for the terms  $f_2 {}^{(4)}R$  and  $f_3 {}^{(4)}G_{\mu\nu} \phi^{\mu\nu}$  by using the Horndeski Lagrangians, as explained in the appendix.

In the quadratic case, we find that the total ADM action in the unitary gauge is given by

$$S = \int d^3x dt N \sqrt{h} \mathcal{L}, \quad (2.10)$$

with

$$\begin{aligned} \mathcal{L} = & f_2 R - 2f_{2\phi} A K + (f_2 + a_1 A^2) K_{ij} K^{ij} - (f_2 - a_2 A^2) K^2 \\ & + [a_1 + a_2 - (a_3 + a_4) A^2 + a_5 A^4] V^2 + A(4f_{2X} + 2a_2 - a_3 A^2) K V \\ & + [4f_{2X} A^2 - (2a_1 - a_4 A^2) A^2] \frac{\partial_i N \partial^i N}{N^2}. \end{aligned} \quad (2.11)$$

The full expression including the cubic terms is much more involved and is given in (A.13) of App. A.

We can further simplify the above expressions by assuming that the scalar field is proportional to the time coordinate  $t$ ,

$$\phi = \mu^2 t, \quad (2.12)$$

where  $\mu$  is some mass scale, so that

$$A = \frac{\mu^2}{N}, \quad V = -\frac{\mu^2}{N^3} \mathcal{D}_t N. \quad (2.13)$$

In this case, the dynamical quantities are the lapse  $N$  and the spatial metric.

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<sup>6</sup>We have slightly changed the notation by using  $A$  and  $V$  instead of  $A_*$  and  $V_*$ .

Upon expanding the above action (2.10) around a cosmological background up to quadratic order in perturbations, one obtains an expression of the form (1.2), with

$$\begin{aligned} \frac{M^2}{2} &= f_2 - a_1 X, & \frac{M^2}{2}(1 + \alpha_T) &= f_2, & \frac{M^2}{2}(1 + \alpha_H) &= f_2 - 2X f_{2X}, \\ \frac{M^2}{2} \left(1 + \frac{2}{3}\alpha_L\right) &= f_2 + a_2 X, & \frac{M^2}{2}\beta_2 &= -X(a_1 + a_2 + (a_3 + a_4)X + a_5 X^2), \\ 2M^2\beta_1 &= X(4f_{2X} + 2a_2 + a_3 X), & \frac{M^2}{2}\beta_3 &= -X(4f_{2X} - 2a_1 - a_4 X), \end{aligned} \quad (2.14)$$

where the right-hand side quantities are evaluated on the homogeneous and isotropic background (so that  $X = -\mu^4$ ). Let us stress that the coefficients  $\beta_1$  and  $\beta_2$  correspond to the terms in front of  $KV$  and  $V^2$ , respectively, in the unitary action. This means that all the  $\dot{N}$  terms disappear when  $\beta_1 = \beta_2 = 0$ .

The first two relations of (2.14) can be used to express  $M^2$  and  $\alpha_T$  in terms of  $f_2$  and  $a_1$ . Substituting into the other relations, one easily gets the other parameters,  $\alpha_H$ ,  $\alpha_L$ ,  $\beta_1$ ,  $\beta_2$  and  $\beta_3$ , in terms of  $f_2$ ,  $f_{2X}$  and  $a_A$  evaluated on the background. In App. A, we also give the full expressions of the effective parameters when the action also contains the cubic terms, thus in terms of  $f_3$  and the  $b_A$ .

Let us briefly discuss the values of these parameters for the quadratic DHOST theories. The classes Ib, IIb and IIIc are pathological, as noted in [27], because they do not contain propagating gravitons. Indeed, one sees immediately that if  $a_1 = f_2/X$ , which is the case for these three classes, the coefficient of the kinetic term for the gravitons  $K_{ij}K^{ij}$  disappears (since  $X = -A^2$  in the unitary gauge), i.e.  $M = 0$  and the theory does not contain tensor degrees of freedom.

The remaining theories, IIIa and IIIb, are also problematic, as pointed out in [57]. Indeed,  $f = 0$  implies that there is no gradient term for the gravitons since the spatial curvature  $R$  disappears. This means that the propagation speed for gravitational waves is zero, or equivalently,  $\alpha_T = -1$ . Note that the classes IIIa and IIIb also verify the property  $\alpha_H = -1$ .

From a phenomenological point of view, the classes Ia and IIa therefore appear to be the most interesting.

### 2.3 Degeneracy conditions for the effective parameters

Among theories of the form (2.1), DHOST theories play a very special role as their Lagrangian is degenerate, which implies that they contain at most three propagating degrees of freedom, i.e. two tensor modes and one scalar degree of freedom.

Interestingly, the fully nonlinear degeneracy conditions boil down to two sets of very simple conditions for the effective parameters appearing in the quadratic perturbative action. Depending on the DHOST theory under consideration, we find that the effective parameters satisfy either

$$\mathcal{C}_I: \quad \alpha_L = 0, \quad \beta_2 = -6\beta_1^2, \quad \beta_3 = -2\beta_1 [2(1 + \alpha_H) + \beta_1(1 + \alpha_T)], \quad (2.15)$$

or the set of conditions

$$\mathcal{C}_{II}: \quad \beta_1 = -(1 + \alpha_L) \frac{1 + \alpha_H}{1 + \alpha_T}, \quad \beta_2 = -6(1 + \alpha_L) \frac{(1 + \alpha_H)^2}{(1 + \alpha_T)^2}, \quad \beta_3 = 2 \frac{(1 + \alpha_H)^2}{1 + \alpha_T}, \quad (2.16)$$

where we have assumed that  $\alpha_T \neq -1$  in the latter case (otherwise<sup>7</sup> one should use a regular version of the conditions obtained by multiplying both sides of the equalities by the denominator

<sup>7</sup>A model for which  $\alpha_T = -1$  is very peculiar since the speed of gravitational waves vanishes.

of the right hand side). It is immediate to see that both sets of conditions share the common condition

$$\mathcal{C}_U : \quad (1 + \alpha_L)\beta_2 = -6\beta_1^2, \quad (2.17)$$

which plays a special role in the unitary gauge, as we will see later. In the second set of conditions,  $\mathcal{C}_{II}$ , the three parameters  $\beta_A$  are completely determined by the parameters  $\alpha_L$ ,  $\alpha_H$  and  $\alpha_T$ . By contrast, in the set of conditions  $\mathcal{C}_I$ ,  $\beta_1$  remains independent of the  $\alpha_A$ . Note that a theory that satisfies  $\alpha_L = 0$  and the conditions  $\mathcal{C}_{II}$  automatically verifies  $\mathcal{C}_I$ . One can also recover directly the conditions  $\mathcal{C}_I$  and  $\mathcal{C}_{II}$  by rewriting the three degeneracy conditions involving  $f_2$ ,  $f_{2X}$  and the five functions  $a_A$ , derived in [26], in terms of the seven parameters  $M^2$ ,  $\alpha_L$ ,  $\alpha_H$ ,  $\alpha_T$  and  $\beta_A$ , as we show in App. B.

The degeneracy conditions satisfied by each DHOST subclass are indicated in Table 1. Among purely quadratic theories, the subclass<sup>8</sup> Ia satisfies the conditions  $\mathcal{C}_I$ , while the subclass IIa satisfies the conditions  $\mathcal{C}_{II}$ . As mentioned earlier, the effective coefficients cannot be defined for the theories Ib, IIb and IIIc for which  $M^2 = 0$ . Theories IIIa and IIIb satisfy  $\alpha_T = \alpha_H = -1$  and verify the regular version of conditions  $\mathcal{C}_{II}$ .

The situation with cubic DHOST theories is subtler. The reason is that there are more than three degeneracy conditions for the 11 functions that parametrize the space of cubic scalar-tensor theories, as shown in [33]. However, for linear perturbations about a cosmological background, these degeneracy conditions simply “project” onto  $\mathcal{C}_I$  or  $\mathcal{C}_{II}$ . If we pushed the effective description of a cubic DHOST theory up to higher order, we would expect the emergence of new degeneracy conditions, which would be reminiscent of the full degeneracy conditions obtained in the complete theory.

For the purely cubic theories, one can discard six subclasses out of nine, because they lead to  $M^2 = 0$ . Among the remaining three subclasses, the subclass <sup>3</sup>N-I, which includes the quintic Horndeski Lagrangian, satisfies the conditions  $\mathcal{C}_I$ , while the other two, <sup>3</sup>M-I and <sup>3</sup>M-II, obey the conditions  $\mathcal{C}_{II}$  (see Table 1).

Finally, let us discuss the combinations of quadratic and cubic theories. As shown in [33], there exist 25 subclasses of degenerate theories. Only one subclass, Ia & <sup>3</sup>N-I, satisfies the conditions  $\mathcal{C}_I$  only: this subclass contains the full Horndeski theory as well as the beyond-Horndeski extensions. Leaving aside seven subclasses for which  $M^2 = 0$ , we are left with 17 subclasses that satisfy the conditions  $\mathcal{C}_{II}$ . Among these, one subclass also satisfies  $\alpha_L = 0$  (and therefore the conditions  $\mathcal{C}_I$  too): Ia & <sup>3</sup>M-III. The other subclasses satisfy only the conditions  $\mathcal{C}_{II}$ .

It was shown in [29] that all subclasses of quadratic DHOST theories are stable with respect to conformal-disformal transformations, by which we mean that any theory is mapped into another theory belonging to the same subclass. One can conjecture that this should remain true for the cubic DHOST theories, although it has been checked only for the subclass <sup>3</sup>N-I containing Horndeski. Given these considerations, it is instructive to explore how the effective coefficients are transformed under conformal-disformal transformations.

## 2.4 Disformal transformations

Let us consider general (conformal-)disformal transformations, which define a new metric by using the scalar field, according to the expression [59]

$$\tilde{g}_{\mu\nu} = C(\phi, X)g_{\mu\nu} + D(\phi, X)\phi_\mu\phi_\nu. \quad (2.18)$$

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<sup>8</sup>For quadratic theories, several names have been introduced in previous works. Here we use the names introduced in [29] for quadratic theories. Other names have been used in [33] and are reported in Table 1.

As shown explicitly in [29] for quadratic theories, any DHOST theory can be mapped into another DHOST theory via this transformation. More precisely, if we start from a theory defined by the action  $\tilde{S}[\tilde{g}, \phi]$ , one can define a new theory as

$$S[g_{\mu\nu}, \phi] := \tilde{S}[\tilde{g}_{\mu\nu} = Cg_{\mu\nu} + D\phi_\mu\phi_\nu, \phi]. \quad (2.19)$$

The explicit transformation of the functions  $f_2$  and  $a_A$  can be found in [33], where it is also shown that all subclasses of quadratic DHOST theories are stable under disformal transformations. If the disformal transformation is invertible, i.e. it satisfies the condition [59]

$$C - XC_X - X^2D_X \neq 0, \quad (2.20)$$

then two disformally related theories are equivalent, provided matter is ignored. However, including matter and assuming that it is minimally coupled to the metric that appears in the two disformally related DHOST actions, one gets two physically distinct theories.

In order to compute the transformation of the effective parameters of the quadratic action, it is convenient to introduce the dimensionless time-dependent parameters

$$\alpha_C \equiv \frac{\dot{\phi}}{2HC} \frac{\partial C}{\partial \phi}, \quad \alpha_Y \equiv -\frac{X}{C} \frac{\partial C}{\partial X}, \quad \alpha_D \equiv -\frac{D}{D + C/X}, \quad \alpha_X \equiv -\frac{X^2}{C} \frac{\partial D}{\partial X}, \quad (2.21)$$

where the right-hand sides are evaluated on the background. These four dimensionless functions characterize how the quadratic action (1.2) transforms under the transformation (2.18). The functions  $\alpha_C$  and  $\alpha_D$  were introduced in [40, 60] to characterize conformal and disformal transformations that depend only on the scalar field value. Indeed, the structure of the action restricted to  $\alpha_H = \alpha_L = \beta_A = 0$  is invariant under this subset of transformations [60, 61]. The function  $\alpha_X$  was introduced in [47] to describe the transformation of the action restricted only to  $\alpha_L = \beta_A = 0$ . The relations between the effective parameters  $\alpha_K, \alpha_B, \alpha_T, \alpha_M$  and  $\alpha_H$  in different frames were given in these references.

Here we extend these results to the general action (1.2). In particular, as explicitly shown in App. C, the effective parameters in the quadratic action derived from  $\tilde{S}$  are related to those associated with  $S$  via the transformations:

$$\begin{aligned} \tilde{M}^2 &= \frac{M^2}{C\sqrt{1 + \alpha_D}}, \\ \tilde{\alpha}_L &= \alpha_L, \\ \tilde{\alpha}_T &= (1 + \alpha_T)(1 + \alpha_D) - 1, \\ \tilde{\alpha}_H &= \Xi(1 + \alpha_D)[1 + \alpha_H - \alpha_Y(1 + \alpha_T)] - 1, \\ \tilde{\beta}_1 &= \Xi[\alpha_Y(1 + \alpha_L) + \beta_1], \\ \tilde{\beta}_2 &= \Xi^2[\beta_2 - 6\alpha_Y(\alpha_Y(1 + \alpha_L) + 2\beta_1)], \\ \tilde{\beta}_3 &= \Xi^2(1 + \alpha_D)[\beta_3 + 2\alpha_Y^2(1 + \alpha_T) - 4\alpha_Y(1 + \alpha_H)], \end{aligned} \quad (2.22)$$

where we have introduced

$$\Xi \equiv \frac{1}{(1 + \alpha_D)(1 + \alpha_X + \alpha_Y)}. \quad (2.23)$$

This function is always finite for an invertible transformation (2.18). Indeed, in terms of the parameters (2.21), the condition (2.20) implies  $1 + \alpha_X + \alpha_Y \neq 0$ , while one must impose  $1 + \alpha_D > 0$  in order to conserve the metric signature. Here we focus our attention on the parameters that are

directly involved in the degeneracy constraints and do not show the analogous transformations for the other parameters  $\alpha_K$  and  $\alpha_B$ , whose explicit expressions are given in App. C. One can check that the two sets of degeneracy conditions (2.15) and (2.16), as well as the common condition (2.17), are all invariant under the above disformal transformations.

Interestingly, for degenerate theories satisfying either (2.15) or (2.16), it is possible to cancel simultaneously all three  $\beta_A$  via a conformal transformation verifying

$$\alpha_Y = -\frac{\beta_1}{1 + \alpha_L}, \quad (2.24)$$

provided  $\alpha_L \neq -1$ . One can also cancel  $\alpha_H$  via a disformal transformation such that

$$\alpha_X = \alpha_H - (2 + \alpha_T)\alpha_Y. \quad (2.25)$$

As a consequence, it is possible to cancel both  $\beta_A$  and  $\alpha_H$  via a disformal transformation characterized by

$$\alpha_Y = -\frac{\beta_1}{1 + \alpha_L}, \quad \alpha_X = \alpha_H + \frac{2 + \alpha_T}{1 + \alpha_L}\beta_1. \quad (2.26)$$

Such a transformation is well defined for theories satisfying the conditions (2.15) but not theories verifying (2.16) for which the quantity  $1 + \alpha_X + \alpha_Y$  vanishes. In the first case, one simply recovers the property that theories belonging to the same subclass as Horndeski can be related to Horndeski via a disformal transformation.

### 3 Dispersion relation for a Minkowski background

For simplicity, we first study the linear perturbations about a Minkowski background for theories of the form (2.1). We thus specialize the quadratic action (1.2) to the limit  $a = 1$  and  $H = 0$ : this is equivalent to assuming that the typical frequencies and wave numbers are much higher than the cosmological ones. For convenience, we redefine the coefficients of the  $\delta N^2$  and  $\delta K \delta N$  terms as

$$M_K^2 = H^2 \alpha_K, \quad M_B = H \alpha_B, \quad (3.1)$$

and we assume that the mass parameters  $M_K$  and  $M_B$  can take any finite value in the Minkowski limit.

The scalar type perturbations in the unitary gauge can be expressed in terms of the quantities  $\psi$  and  $\zeta$ , defined by

$$N^i = \delta^{ij} \partial_j \psi, \quad h_{ij} = e^{2\zeta} \delta_{ij}. \quad (3.2)$$

Substituting into the quadratic action, we thus obtain an action that depends on the three perturbations  $\zeta$ ,  $\delta N$ ,  $\psi$  and their derivatives. All the coefficients are constant since we are now in a Minkowski background.

#### 3.1 Dispersion relation and degeneracy

In order to derive the dispersion relation, one considers perturbations of the form

$$\begin{pmatrix} \delta N(t, \mathbf{x}) \\ \zeta(t, \mathbf{x}) \\ \psi(t, \mathbf{x}) \end{pmatrix} = e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} \begin{pmatrix} \delta N(\omega, \mathbf{k}) \\ \zeta(\omega, \mathbf{k}) \\ \psi(\omega, \mathbf{k}) \end{pmatrix} \equiv e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} U, \quad (3.3)$$

where  $U$  denotes the column vector of the three perturbations in Fourier space. The resulting quadratic Lagrangian is of the form

$$L^{(2)} = U^\dagger K_U U, \quad (3.4)$$

where  $K_U$  is the  $3 \times 3$  kinetic matrix with components

$$K_U = M^2 \begin{pmatrix} M_K^2 + \beta_2 \omega^2 + \beta_3 k^2 & 2(1 + \alpha_H)k^2 + 6\beta_1 \omega^2 + 6iM_B \omega & 2(M_B - i\beta_1 \omega)k^2 \\ 2(1 + \alpha_H)k^2 + 6\beta_1 \omega^2 - 6iM_B \omega & -6(1 + \alpha_L)\omega^2 + 2(1 + \alpha_T)k^2 & 2i(1 + \alpha_L)\omega k^2 \\ 2(M_B + i\beta_1 \omega)k^2 & -2i(1 + \alpha_L)\omega k^2 & -\frac{2}{3}\alpha_L k^4 \end{pmatrix}. \quad (3.5)$$

One finds the dispersion relation by imposing

$$\det K_U = 0, \quad (3.6)$$

which yields

$$\mathcal{E}_1 \omega^4 + (\mathcal{E}_2 k^2 + \mathcal{E}_3)\omega^2 + \mathcal{E}_4 k^4 + \mathcal{E}_5 k^2 = 0, \quad (3.7)$$

with the coefficients

$$\begin{aligned} \mathcal{E}_1 &= 3[(1 + \alpha_L)\beta_2 + 6\beta_1^2], \\ \mathcal{E}_2 &= 6[2(1 + \alpha_H) + (1 + \alpha_T)\beta_1]\beta_1 + \alpha_L(1 + \alpha_T)\beta_2 + 3(1 + \alpha_L)\beta_3, \\ \mathcal{E}_3 &= 3[(1 + \alpha_L)M_K^2 + 6M_B^2], \\ \mathcal{E}_4 &= -\alpha_L[2(1 + \alpha_H)^2 - (1 + \alpha_T)\beta_3], \\ \mathcal{E}_5 &= (1 + \alpha_T)(\alpha_L M_K^2 + 6M_B^2). \end{aligned} \quad (3.8)$$

In the general case, the dispersion relation is a quartic polynomial in  $\omega$  with only even powers, which means that there are two solutions for  $\omega^2$ , corresponding to the presence of two scalar modes, as expected. In particular, we note that the two parameters  $\beta_1$  and  $\beta_2$  contribute to the highest order coefficient in  $\omega$ , which is consistent with their interpretation given in the introduction. Interestingly, the structure of the coefficient  $\mathcal{E}_1$  is the same as that of  $\mathcal{E}_3$  with  $\beta_1$  and  $\beta_2$  playing the role of  $\alpha_B$  and  $\alpha_K$ , respectively (reminding that  $M_B \equiv H\alpha_B$  and  $M_K^2 \equiv H^2\alpha_K$ ). Note also that the highest term in spatial derivatives disappears when  $\alpha_L = 0$ .

If the condition  $\mathcal{E}_1 = 0$  is satisfied, which is equivalent to the condition  $\mathcal{C}_U$  identified in (2.17), then only a single scalar mode remains. However, the above statement is valid in the unitary gauge for linear perturbations. In an arbitrary gauge, unless the theory is fully degenerate, i.e. it satisfies the other conditions in (2.15) or (2.16), there still exists an extra scalar mode that simply does not show up at the linear level (see discussion in App. B.2). In order to ensure the absence of this extra scalar mode, it is sufficient to require either of the degeneracy conditions (2.15) and (2.16), which implies  $\mathcal{E}_2 = 0$  and  $\mathcal{E}_4 = 0$ . In this case, the dispersion relation takes the very simple form

$$\omega^2 - c_s^2 k^2 = 0, \quad c_s^2 \equiv -\frac{1}{3} \frac{(1 + \alpha_T)(6M_B^2 + \alpha_L M_K^2)}{M_K^2(1 + \alpha_L) + 6M_B^2} \quad (\text{degenerate}). \quad (3.9)$$

### 3.2 Newtonian limit

One can also use the quadratic action to explore how the usual Poisson equation is modified. To do so, we consider the static limit (i.e.  $\omega = 0$ ) of the quadratic action and we introduce a point

mass  $m$  which is minimally coupled to the metric. If we work in the unitary gauge, we find that the kinetic matrix (3.5) for the variables  $\delta N$ ,  $\zeta$  and  $\psi$  reads, in the limit  $\omega = 0$ ,

$$K_V = M^2 \begin{pmatrix} \beta_3 k^2 + M_K^2 & 2(1 + \alpha_H)k^2 & 2M_B k^2 \\ 2(1 + \alpha_H)k^2 & 2(1 + \alpha_T)k^2 & 0 \\ 2M_B k^2 & 0 & -\frac{2}{3}\alpha_L k^4 \end{pmatrix}. \quad (3.10)$$

The second line implies

$$\zeta = -\frac{1 + \alpha_H}{1 + \alpha_T} \delta N, \quad (3.11)$$

which is equivalent to the relation

$$\Psi = \frac{1 + \alpha_H}{1 + \alpha_T} \Phi, \quad (3.12)$$

between the gravitational potential  $\Phi$  (which in general is related to the unitary gauge variables by  $\Phi = \delta N + \dot{\psi}$ ) and the spatial gravitational potential  $\Psi = -\zeta$ .

We now distinguish the two cases  $\alpha_L \neq 0$  and  $\alpha_L = 0$ . If  $\alpha_L \neq 0$ , the last line of the kinetic matrix yields

$$k^2 \psi = 3 \frac{M_B}{\alpha_L} \delta N. \quad (3.13)$$

Substituting into the first line and going back into real space, one finds the generalized Poisson equation

$$M^2 \left[ 2 \frac{(1 + \alpha_H)^2}{1 + \alpha_T} - \beta_3 \right] \Delta \Phi + M^2 \left( M_K^2 + 6 \frac{M_B^2}{\alpha_L} \right) \Phi = m \delta^{(3)}(\mathbf{x}), \quad (3.14)$$

where  $\Delta \equiv \delta^{ij} \partial_i \partial_j$  denotes the Laplacian. The coefficient in front of  $\Delta \Phi$  in the generalized Poisson equation (3.14) corresponds to  $(4\pi G_N)^{-1}$ , where  $G_N$  is the effective Newton constant. For DHOST theories with  $\alpha_L \neq 0$ , we see immediately that the coefficient in front of the Laplacian in the Poisson equation vanishes, because of (2.16), which means that the effective Newton constant in the linear regime is infinite for these theories.

If  $\alpha_L = 0$ , one obtains the generalized Poisson equation

$$M^2 \left[ 2 \frac{(1 + \alpha_H)^2}{1 + \alpha_T} - \beta_3 \right] \Delta \Phi + M^2 M_K^2 \Phi = m \delta^{(3)}(\mathbf{x}). \quad (3.15)$$

For DHOST theories that satisfy the conditions (2.15) but not (2.16), one thus gets a finite Newton constant  $G_N$  in the linear regime

$$8\pi G_N = \frac{1}{M^2} \left[ \frac{(1 + \alpha_H)^2}{1 + \alpha_T} - \frac{\beta_3}{2} \right]^{-1}. \quad (3.16)$$

These results seem to indicate that only theories that are related to Horndeski via conformal or disformal transformations are phenomenologically viable. One should however investigate whether this peculiar behaviour persists at the nonlinear level.

We can check that for  $\beta_3 = 0$  this effective Newton constant agrees with the one found for Horndeski and beyond Horndeski theories in the quasi-static regime. In our notation, the full expression is given by eq. (3.24) of [47] (see also [62]). In the absence of background matter and for  $a$ ,  $M$  and  $\alpha_H = \text{constants}$ , this reads

$$8\pi G_N = \frac{1}{2M^2(1 + \alpha_H)^2} \left[ 1 + \alpha_T + \frac{2\xi^2}{c_s^2(\alpha_K + 6\alpha_B^2)} \right], \quad (3.17)$$

with  $\xi \equiv \alpha_B(1 + \alpha_T) + \alpha_T - \alpha_M - \alpha_H(1 + \alpha_M)$ . It is easy to check that, in the Minkowski limit, the last term in the brackets becomes  $1 + \alpha_T$  and eq. (3.16) with  $\beta_3 = 0$  is recovered.

## 4 Quadratic action in a cosmological background

In this section, we study the quadratic action for the propagating degrees of freedom in a cosmological background. Since none of the additional operators studied here contributes to tensor modes, we restrict our analysis to scalar perturbations. A derivation of the quadratic action for tensor perturbations, which depends only on the parameters  $M^2$  and  $\alpha_T$ , can be found for instance in [4, 7]. Using the usual expressions for the scalar perturbations in unitary gauge,

$$N^i = \delta^{ij} \partial_j \psi, \quad h_{ij} = a^2(t) e^{2\zeta} \delta_{ij}, \quad (4.1)$$

we can express the action (1.2) as a functional of  $\zeta$ ,  $\delta N$  and  $\psi$  and their derivatives. It is convenient to distinguish the two cases  $\alpha_L = 0$  and  $\alpha_L \neq 0$ .

### 4.1 Case $\alpha_L = 0$

Let us assume  $\alpha_L = 0$ , i.e. the first condition of eq. (2.15). Using eq. (4.1) and noticing that the terms quadratic in  $\Delta\psi$  cancel up to a total derivative, action (1.2) becomes

$$\begin{aligned} S^{\text{quad}} = \int d^3x dt a^3 \frac{M^2}{2} & \left\{ -6\dot{\zeta}^2 + 12\beta_1 \dot{\zeta} \delta\dot{N} + \beta_2 \delta\dot{N}^2 + 12H \left[ (1 + \alpha_B) \dot{\zeta} - \beta_1 \delta\dot{N} \right] \delta N \right. \\ & + H^2 (\alpha_K - 6 - 12\alpha_B) \delta N^2 + 4 \left[ \dot{\zeta} - \beta_1 \delta\dot{N} - H(1 + \alpha_B) \delta N \right] \Delta\psi \\ & \left. + \frac{1}{a^2} \left[ 2(1 + \alpha_T) (\partial_i \zeta)^2 + 4(1 + \alpha_H) \partial_i \zeta \partial_i \delta N + \beta_3 (\partial_i \delta N)^2 \right] \right\}. \end{aligned} \quad (4.2)$$

The kinetic Lagrangian is given by the first three terms on the right-hand side,

$$\mathcal{L}_{\text{kin}}^{\text{quad}} = a^3 \frac{M^2}{2} (-6\dot{\zeta}^2 + 12\beta_1 \dot{\zeta} \delta\dot{N} + \beta_2 \delta\dot{N}^2). \quad (4.3)$$

Thus, without any assumption on the time dependent functions  $\beta_A$ , the action above describes in general two propagating scalar modes,  $\zeta$  and  $\delta N$ , while  $\psi$ , which appears without time derivatives, can be treated as a Lagrange multiplier. In this general case, the full analysis, including matter, extends our analysis of the previous section and is discussed in App. D.

The expression (4.3) is degenerate if the determinant of the kinetic matrix vanishes, i.e. if

$$\beta_2 = -6\beta_1^2, \quad (4.4)$$

where one recognizes the second condition in eq. (2.15), or the condition (2.17) with  $\alpha_L = 0$ . In this case the kinetic Lagrangian can be written as

$$\mathcal{L}_{\text{kin}}^{\text{quad}} = -6a^3 \frac{M^2}{2} (\dot{\zeta} - \beta_1 \delta\dot{N})^2, \quad (4.5)$$

which suggests that the time derivatives of  $\delta\dot{N}$  can be eliminated by replacing the variable  $\zeta$  with the new variable

$$\tilde{\zeta} \equiv \zeta - \beta_1 \delta N. \quad (4.6)$$

This variable describes the propagating scalar degree of freedom in the degenerate case.

To find the associated quadratic action, we can proceed similarly to what was done in [4, 6]. Varying the action with respect to  $\psi$  yields the scalar component of the momentum constraint. In terms of the new variable, it reads

$$\delta N = \frac{\dot{\tilde{\zeta}}}{H(1 + \alpha_B) - \dot{\beta}_1}. \quad (4.7)$$

Substituting this expression for  $\delta N$  into the action and performing some integration by parts, we finally find the quadratic action for the propagating degree of freedom  $\tilde{\zeta}$ ,

$$S^{\text{quad}} = \int d^3x dt a^3 \frac{M^2}{2} \left[ A_{\tilde{\zeta}} \dot{\tilde{\zeta}}^2 + B_{\tilde{\zeta}} \frac{(\partial_i \tilde{\zeta})^2}{a^2} + C_{\tilde{\zeta}} \frac{(\partial_i \dot{\tilde{\zeta}})^2}{a^2} \right]. \quad (4.8)$$

Here  $A_{\tilde{\zeta}}$ ,  $B_{\tilde{\zeta}}$  and  $C_{\tilde{\zeta}}$  are background-dependent functions whose explicit expressions are

$$A_{\tilde{\zeta}} = \frac{1}{(1 + \alpha_B - \dot{\beta}_1/H)^2} \left[ \alpha_K + 6\alpha_B^2 - \frac{6}{a^3 H^2 M^2} \frac{d}{dt} (a^3 H M^2 \alpha_B \beta_1) \right], \quad (4.9)$$

$$B_{\tilde{\zeta}} = 2(1 + \alpha_T) - \frac{2}{a M^2} \frac{d}{dt} \left[ \frac{a M^2 (1 + \alpha_H + \beta_1 (1 + \alpha_T))}{H(1 + \alpha_B) - \dot{\beta}_1} \right], \quad (4.10)$$

$$C_{\tilde{\zeta}} = \frac{4(1 + \alpha_H)\beta_1 + 2(1 + \alpha_T)\beta_1^2 + \beta_3}{(1 + \alpha_B - \dot{\beta}_1/H)^2}. \quad (4.11)$$

As the above action implies that  $\tilde{\zeta}$  is conserved in the long wavelength limit, i.e.  $\dot{\tilde{\zeta}} \approx 0$  for  $k \ll aH$ , and eq. (4.7) implies that  $\delta N$  vanishes in the same limit, it follows that  $\zeta$  is conserved on large scales,

$$\dot{\zeta} \approx 0 \quad (k \ll aH), \quad (4.12)$$

as in the more standard case of Horndeski and beyond Horndeski theories [7].

Up to now we have imposed only the first two conditions in eq. (2.15). Since only  $\beta_1$  enters in the above expressions for  $A_{\tilde{\zeta}}$  and  $B_{\tilde{\zeta}}$ , these definitions remain unchanged when the full degeneracy conditions  $\mathcal{C}_1$  are imposed, while the function  $C_{\tilde{\zeta}}$  vanishes. In this case the action takes the usual form,

$$S^{\text{quad}} = \int d^3x dt a^3 \frac{M^2}{2} \left[ A_{\tilde{\zeta}} \dot{\tilde{\zeta}}^2 + B_{\tilde{\zeta}} \frac{(\partial_i \tilde{\zeta})^2}{a^2} \right]. \quad (4.13)$$

Absence of instabilities requires that the coefficients in the action satisfy

$$A_{\tilde{\zeta}} \geq 0, \quad B_{\tilde{\zeta}} \leq 0. \quad (4.14)$$

In particular, the first condition is equivalent to

$$\alpha \equiv \alpha_K + 6\alpha_B^2 - \frac{6}{a^3 H^2 M^2} \frac{d}{dt} (a^3 H M^2 \alpha_B \beta_1) \geq 0. \quad (4.15)$$

The dispersion relation is standard,  $\omega^2 = c_s^2 k^2 / a^2$ , with a sound speed given by

$$c_s^2 = -B_{\tilde{\zeta}} / A_{\tilde{\zeta}}. \quad (4.16)$$

It is straightforward to check that for  $\beta_1 = 0$  one recovers the sound speed derived for theories belonging to the Horndeski (see eg. [63]) and beyond Horndeski classes in [4, 7, 17]. Moreover, in the Minkowski limit, we recover eq. (3.9) in the case  $\alpha_L = 0$ . One can also verify that under a general transformation (2.18), the above expression for the sound speed transforms like the lightcone, i.e.  $\tilde{c}_s^2 = (1 + \alpha_D) c_s^2$ , as expected.

## 4.2 Case $\alpha_L \neq 0$

We now consider the case  $\alpha_L \neq 0$ . Varying the action with respect to  $\psi$  yields the scalar component of the momentum constraint, which reads

$$\frac{\alpha_L}{3} \frac{\Delta\psi}{a^2} + (1 + \alpha_B + \alpha_L)H\delta N + \beta_1\delta\dot{N} - (1 + \alpha_L)\dot{\zeta} = 0. \quad (4.17)$$

Using this expression to eliminate  $\psi$  from the action, one obtains

$$\begin{aligned} S^{\text{quad}} = & \int d^3x dt a^3 \frac{M^2}{2} \left\{ \frac{1}{\alpha_L} \left[ 6(1 + \alpha_L)\dot{\zeta}^2 - 12\beta_1\delta\dot{N}\dot{\zeta} + (6\beta_1^2 + \alpha_L\beta_2)\delta\dot{N}^2 \right. \right. \\ & + 12H \left( \beta_1(1 + \alpha_B)\delta\dot{N} - (1 + \alpha_B + \alpha_L)\dot{\zeta} \right) \delta N \\ & + H^2 \left( 6\alpha_B^2 + 12\alpha_B + \alpha_L\alpha_K + 6(1 + \alpha_L) \right) \delta N^2 \left. \right] \\ & + \frac{1}{a^2} \left[ 2(1 + \alpha_T)(\partial_i\zeta)^2 + 4(1 + \alpha_H)\partial_i\zeta\partial^i\delta N + \beta_3(\partial_i\delta N)^2 \right] \left. \right\}. \end{aligned} \quad (4.18)$$

Once again, in the absence of any assumption on the  $\beta_A$ , this action describes in general two propagating scalar modes,  $\zeta$  and  $\delta N$ .

The kinetic matrix is degenerate for

$$\beta_2 = -6 \frac{\beta_1^2}{1 + \alpha_L}, \quad (4.19)$$

which corresponds to the condition of eq. (2.17). It can be diagonalized by introducing the variable

$$\tilde{\zeta} = \zeta - \frac{\beta_1}{1 + \alpha_L} \delta N, \quad (4.20)$$

which represents the propagating degree of freedom in this case.

Using these two relations in action (4.18) and performing some integrations by parts, the action takes the form

$$\begin{aligned} S^{\text{quad}} = & \int d^3x dt a^3 \frac{M^2}{2} \left\{ 6 \frac{1 + \alpha_L}{\alpha_L} \dot{\tilde{\zeta}}^2 + 2\mathcal{V}_{12}\dot{\tilde{\zeta}}\delta N + \mathcal{M}_{22}\delta N^2 \right. \\ & + \frac{1}{a^2} \left[ 2(1 + \alpha_T)(\partial_i\tilde{\zeta})^2 + 2\mathcal{S}_{12}\partial_i\tilde{\zeta}\partial^i\delta N + \mathcal{S}_{22}(\partial_i\delta N)^2 \right] \left. \right\}, \end{aligned} \quad (4.21)$$

with

$$\begin{aligned} \mathcal{S}_{12} &= 2 \left[ 1 + \alpha_H + \beta_1 \frac{1 + \alpha_T}{1 + \alpha_L} \right], \\ \mathcal{S}_{22} &= 4\beta_1 \frac{1 + \alpha_H}{1 + \alpha_L} + 2\beta_1^2 \frac{1 + \alpha_T}{(1 + \alpha_L)^2} + \beta_3. \end{aligned} \quad (4.22)$$

The explicit expressions of  $\mathcal{V}_{12}$  and  $\mathcal{M}_{22}$  above are not relevant for this discussion and can be found in App. D. Variation of action (4.21) with respect to  $\delta N$  gives then the Hamiltonian constraint equation,

$$\mathcal{V}_{12}\dot{\tilde{\zeta}} + \mathcal{M}_{22}\delta N = \frac{1}{a^2} \left[ \mathcal{S}_{12}\Delta\tilde{\zeta} + \mathcal{S}_{22}\Delta\delta N \right]. \quad (4.23)$$

The constraint equation (4.23) can be solved for  $\delta N$  as a function of  $\tilde{\zeta}$  and  $\dot{\tilde{\zeta}}$  and plugged back into eq. (4.21), which yields the quadratic action for the propagating degree of freedom  $\tilde{\zeta}$ . If  $\mathcal{S}_{12}$

and  $\mathcal{S}_{22}$  do not vanish, this action contains higher spatial derivatives and Laplacian operators in the denominator, resulting in a nonstandard dispersion relation for  $\tilde{\zeta}$ . We have written its explicit expression in App. D, including also matter perturbations for completeness. Analogously to what happens in the previous section, there is no mass term in the action and thus  $\tilde{\zeta}$  and  $\zeta$  are conserved in the long wavelength limit.

We now derive the action for  $\tilde{\zeta}$  when the full set of degeneracy conditions (2.16) is imposed. In this case,  $\mathcal{S}_{12} = 0 = \mathcal{S}_{22}$  and the constraint equation (4.23) simplifies,

$$\delta N = -\frac{\mathcal{V}_{12}}{\mathcal{M}_{22}}\dot{\tilde{\zeta}} \quad (\mathcal{C}_{\text{II}}), \quad (4.24)$$

while the expressions for  $\mathcal{V}_{12}$  and  $\mathcal{M}_{22}$  take the following form

$$\begin{aligned} \mathcal{V}_{12} &= -\frac{6}{\alpha_{\text{L}}}\left[H(1+\alpha_{\text{B}}+\alpha_{\text{L}})+(1+\alpha_{\text{L}})\frac{d}{dt}\left(\frac{1+\alpha_{\text{H}}}{1+\alpha_{\text{T}}}\right)\right] \quad (\mathcal{C}_{\text{II}}), \\ \mathcal{M}_{22} &= \frac{\alpha_{\text{L}}}{6(1+\alpha_{\text{L}})}\mathcal{V}_{12}^2 + H^2\left(\alpha_{\text{K}} + \frac{6\alpha_{\text{B}}^2}{1+\alpha_{\text{L}}}\right) + \frac{6}{a^3M^2}\frac{d}{dt}\left(a^3M^2H\alpha_{\text{B}}\frac{1+\alpha_{\text{H}}}{1+\alpha_{\text{T}}}\right) \quad (\mathcal{C}_{\text{II}}). \end{aligned} \quad (4.25)$$

Replacing  $\delta N$  into action (4.21) using the above constraint, one obtains a quadratic action of the form of eq. (4.13), with

$$A_{\tilde{\zeta}} = 6\frac{1+\alpha_{\text{L}}}{\alpha_{\text{L}}}-\frac{\mathcal{V}_{12}^2}{\mathcal{M}_{22}}, \quad B_{\tilde{\zeta}} = 2(1+\alpha_{\text{T}}) \quad (\mathcal{C}_{\text{II}}). \quad (4.26)$$

In the limit  $\alpha_{\text{L}} = 0$  these expressions reduce to those in eqs. (4.9) and (4.10), with the function  $\beta_1$  given by eq. (2.16), i.e.

$$A_{\tilde{\zeta}} = \frac{\alpha_{\text{K}} + 6\alpha_{\text{B}}^2 + \frac{6}{a^3H^2M^2}\frac{d}{dt}\left(a^3HM^2\alpha_{\text{B}}\frac{1+\alpha_{\text{H}}}{1+\alpha_{\text{T}}}\right)}{\left[1+\alpha_{\text{B}}+\frac{d}{dt}\left(\frac{1+\alpha_{\text{H}}}{1+\alpha_{\text{T}}}\right)\right]^2}, \quad B_{\tilde{\zeta}} = 2(1+\alpha_{\text{T}}) \quad (\mathcal{C}_{\text{I}} \cap \mathcal{C}_{\text{II}}). \quad (4.27)$$

## 5 Lorentz-breaking theories

In this section, we show how our analysis can also be applied to theories already proposed in the literature, such as Lorentz-breaking theories inspired by Horava's gravity and khronometric theories.

### 5.1 Lorentz-breaking theories

In our analysis of DHOST theories, we have started from a covariant formulation of the action and then derived the Lagrangian in the unitary gauge. Several models that explicitly break Lorentz invariance have been proposed in the literature and their action is often given directly in the unitary gauge. An illustrative example is Horava gravity [18], with several of its extensions. A general presentation of these models can be found in [21], which we will follow in our discussion below. The actions studied in [21] are of the form<sup>9</sup>

$$\begin{aligned} S_{\text{unitary}} &= \int d^3x dt \sqrt{h} N \left\{ \frac{M^2}{2} \left( K_{ij}K^{ij} - \lambda K^2 + \frac{\lambda_2}{N^6} \left( \dot{N} - N^i \partial_i N \right)^2 - \lambda_3 \frac{\partial_i N \partial^i N}{N^2} - \mathcal{V} \right) \right. \\ &\quad \left. + \frac{\lambda_0}{4} \mu^4 \left( \frac{1}{N^2} - 1 \right)^2 \right\}, \end{aligned} \quad (5.1)$$

<sup>9</sup>The notation for the coefficients in the action differs from the one of [21].

and  $\mathcal{V}$  is some potential term which is not relevant for us and will be ignored below. As one can see, the action contains an explicit dependence on  $\dot{N}$ . In [21], different cases are considered, depending on the values of the parameters in the Lagrangian. The case  $\lambda_2 = 0$  and  $\lambda_3 = 0$  corresponds to Horava's projectable model in the limit  $\lambda_0 \rightarrow \infty$ . The healthy extension of Horava's non projectable model [19] corresponds to  $\lambda_0 = \lambda_2 = 0$ . Finally, [21] also considers an extension with  $\lambda_0$  and  $\lambda_2$  non zero.

It is always possible to rewrite the unitary action (5.1) as a covariant action by using the Stueckelberg formalism, where the time coordinate of the unitary gauge is replaced by a scalar field  $\phi$ . This leads to the covariant action [21]

$$S_{\text{cov}} = \int d^4x \sqrt{-g} \left\{ \frac{M^2}{2} \left[ {}^{(4)}R + \frac{\lambda-1}{X} \left( \square\phi - \frac{1}{X} \phi^\mu \phi_{\mu\nu} \phi^\nu \right)^2 + \frac{\lambda_2}{\mu^4 X^2} (\phi^\mu \phi_{\mu\nu} \phi^\nu)^2 + \frac{\lambda_3}{X^2} \left( \phi^\mu \phi_{\mu\nu} \phi^{\nu\lambda} \phi_\lambda - \frac{1}{X} (\phi^\mu \phi_{\mu\nu} \phi^\nu)^2 \right) \right] + \frac{\lambda_0}{4\mu^4} (X + \mu^4)^2 \right\}. \quad (5.2)$$

One immediately sees that this action is of the form (2.1) with no cubic terms and

$$\begin{aligned} a_1 &= 0, & a_2 &= \frac{(\lambda-1)M^2}{2X}, & a_3 &= \frac{(1-\lambda)M^2}{X^2}, \\ a_4 &= \frac{\lambda_3 M^2}{2X^2}, & a_5 &= M^2 \left( \frac{\lambda - \lambda_3 - 1}{2X^3} + \frac{\lambda_2}{2\mu^2 X^2} \right), \end{aligned} \quad (5.3)$$

$P(X) = \lambda_0(X + \mu^4)^2/(4\mu^4)$  and  $Q(X) = 0$ . This corresponds to the following effective parameters

$$\alpha_T = \alpha_H = 0, \quad \alpha_L = 3(\lambda-1)/2, \quad \beta_1 = 0, \quad \beta_2 = \lambda_2, \quad \beta_3 = \lambda_3. \quad (5.4)$$

and  $\alpha_K = \lambda_0 \mu^4 / (2H^2 M^2)$ .

When  $\lambda_2 = 0$ , one has  $\beta_1 = \beta_2 = 0$  and the condition (2.17) is verified. This means that one finds a single propagating scalar mode in the unitary gauge. However, even if  $\lambda_2 = 0$ , the parameters do not satisfy the full degeneracy conditions, unless  $\lambda_3 = 2$ . The covariant theory (5.2) (with  $\lambda_2 = 0$ ) is not a DHOST theory and thus contains an extra degree of freedom, although it is not directly visible in the unitary gauge (see discussion in App. B.2). This is an example of theory that looks degenerate in the unitary gauge but is not degenerate. In order to get rid of this dangerous extra degree of freedom, one can either define the theory directly in the unitary gauge or consider the covariant action but only for a restricted range of solutions, as discussed in [21].

## 5.2 Khronometric theories

In khronometric theories [21], the scalar field Lagrangian is invariant under the field redefinition  $\phi \rightarrow \tilde{\phi}(\phi)$ . At lowest order in derivatives, the action can be written in terms of the unit vector field  $u_\mu \equiv \partial_\mu \phi / \sqrt{-X}$  as

$$S = \frac{M_*^2}{2} \int d^4x \sqrt{-g} \left( {}^{(4)}R + \mathcal{K}^{\mu\nu}{}_{\rho\sigma} \nabla_\mu u^\rho \nabla_\nu u^\sigma \right), \quad (5.5)$$

with

$$\mathcal{K}^{\mu\nu}{}_{\rho\sigma} \equiv c_1 g^{\mu\nu} g_{\rho\sigma} + c_2 \delta_\rho^\mu \delta_\sigma^\nu + c_3 \delta_\sigma^\mu \delta_\rho^\nu + c_4 u^\mu u^\nu g_{\rho\sigma}, \quad (5.6)$$

where  $M_*$  is a constant with mass dimension and the  $c_i$  are dimensionless constants. Without loss of generality, we can set  $c_1 = 0$  in the action [64]. Moreover, by using the Gauss-Codazzi relation

and that  $\nabla_\mu u_\nu = K_{\mu\nu} - u_\mu u^\rho \nabla_\rho u^\nu$ , the above Lagrangian can be rewritten as

$$S_{\text{unitary}} = \frac{M_*^2}{2} \int d^3x dt \sqrt{h} N \left[ R + (c_2 - 1)K^2 + (c_3 + 1)K_{ij}K^{ij} + c_4 \frac{\partial_i N \partial^i N}{N^2} \right]. \quad (5.7)$$

In the following we will assume that  $c_3 > -1$ , which ensures that

$$M^2 = M_*^2(1 + c_3) > 0, \quad (5.8)$$

i.e. that gravitons have a strictly positive kinetic term. Expanding at second order and comparing with eq. (1.2), one finds

$$\alpha_{\text{T}} = \alpha_{\text{H}} = -\frac{c_3}{1 + c_3}, \quad \alpha_{\text{L}} = -\frac{3}{2} \frac{c_2 + c_3}{1 + c_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 = \frac{c_4}{1 + c_3}. \quad (5.9)$$

To compute the action for the propagating scalar degree of freedom, we can use action (4.18) with  $\alpha_{\text{H}} = \alpha_{\text{T}}$  and  $\beta_1 = \beta_2 = 0$ . Using the Hamiltonian constraint to replace  $\delta N$ , and performing an integration by parts, the quadratic action for  $\zeta$  can be written in Fourier space as

$$S^{\text{quad}} = \int \frac{d^3k}{(2\pi)^3} dt a^3 \frac{M^2}{2} \frac{k_H^2}{6(1 + \alpha_{\text{L}}) + \alpha_{\text{L}}\beta_3 k_H^2} \left\{ 6(1 + \alpha_{\text{L}})\beta_3 \dot{\zeta}_{\mathbf{k}} \dot{\zeta}_{-\mathbf{k}} \right. \\ \left. + \frac{2a^2 H^2 (1 + \alpha_{\text{T}})}{6(1 + \alpha_{\text{L}}) + \alpha_{\text{L}}\beta_3 k_H^2} (c_{\zeta 2} + c_{\zeta 4} k_H^2 + c_{\zeta 6} k_H^4) \zeta_{\mathbf{k}} \zeta_{-\mathbf{k}} \right\}, \quad (5.10)$$

where

$$c_{\zeta 2} = 36(1 + \alpha_{\text{L}})^2 \dot{H}, \quad c_{\zeta 4} = -6\alpha_{\text{L}}(1 + \alpha_{\text{L}}) \left[ \beta_3 \left( 1 + \frac{\dot{H}}{H^2} \right) + 2(1 + \alpha_{\text{T}}) \right], \quad (5.11) \\ c_{\zeta 6} = \alpha_{\text{L}}^2 \beta_3 [\beta_3 - 2(1 + \alpha_{\text{T}})],$$

and  $k_H \equiv k/(aH)$ . This action can be also derived from (D.11) in App. D.

If we now restrict our discussion to degenerate khronometric theories, the degeneracy conditions (2.15) and (2.16) impose constraints on the parameters of actions (5.10) and (5.5). We find, respectively,

$$\alpha_{\text{L}} = 0, \quad \beta_3 = 0, \quad \Leftrightarrow \quad c_2 + c_3 = 0, \quad c_4 = 0, \quad (\text{C}_{\text{I}}), \quad (5.12)$$

for conditions (2.15), and

$$\alpha_{\text{L}} = -1, \quad \beta_3 = 2(1 + \alpha_{\text{T}}), \quad \Leftrightarrow \quad 3c_2 = 2 - c_3, \quad c_4 = 2, \quad (\text{C}_{\text{II}}), \quad (5.13)$$

for conditions (2.16). These two families of degenerate khronometric theories were already identified in [29], together with two other families that lead to  $M^2 = 0$  and are thus not relevant here.

As shown in [29], the first family (5.12) is conformally-disformally related to general relativity, which means that there is no dynamical scalar degree of freedom in the absence of matter. The second family (5.13) is conformally-disformally related to the theory (in units where  $M_*^2/2 = 1$ )

$$f_2 = 1, \quad a_1 = -\frac{2}{X}, \quad a_2 = a_3 = 0, \quad a_4 = \frac{6}{X^2}, \quad a_5 = -\frac{4}{X^3}. \quad (5.14)$$

Substituting into the unitary gauge expression (2.11), one finds

$$\mathcal{L} = R + 3K_{ij}K^{ij} - K^2 + 2 \frac{\partial_i N \partial^i N}{N^2}, \quad (5.15)$$

and the trace part of  $K_{ij}$  automatically cancels, which means that there is no propagating scalar mode. Note that theories of this family possess the peculiar property that  $\beta_1 = \beta_2 = 0$  while  $\beta_3 \neq 0$ . This is not in contradiction with our discussion in Sec. 2.4 as  $\alpha_{\text{L}} = -1$  here.

## 6 Conclusions

In this work, we have studied the effective description of quadratic and cubic Higher-Order Scalar-Tensor theories, focusing in particular on the degenerate ones, i.e. DHOST theories. We considered the quadratic action of linear perturbations about a homogeneous and isotropic background, written in the unitary gauge where the scalar field is spatially uniform. In general, the quadratic action contains time and spatial derivatives of the lapse perturbation, which requires the introduction of three new (time-dependent) parameters which we denote  $\beta_1$ ,  $\beta_2$  and  $\beta_3$ . We also need another parameter,  $\alpha_L$ , in front of the trace of the extrinsic curvature (squared), in order to cover all DHOST theories.

The presence of time derivatives of the lapse is not in contradiction with the property that DHOST theories contain a single scalar degree of freedom, because of the existence of degeneracy conditions that ensure that the effective parameters  $\beta_1$ ,  $\beta_2$  and  $\beta_3$  in the quadratic action are not arbitrary but instead must be linked via three relations. One of these relations, (2.17), can easily be inferred from the requirement that the unitary gauge action is manifestly degenerate, but the other two relations cannot be immediately guessed within the unitary gauge, because they come from the degeneracy imposed at the level of the covariant action. However, as we have shown, they can be deduced from the requirement that the dispersion relation is linear.

Remarkably, all cases can be summarized by only two different sets of degeneracy conditions at the level of the linear perturbations in the unitary gauge. The first set, (2.15), is characterized by the condition  $\alpha_L = 0$ , while  $\beta_1$  is left arbitrary and the other two coefficients  $\beta_2$  and  $\beta_3$  are constrained in terms of  $\beta_1$ ,  $\alpha_T$  and  $\alpha_H$ . By contrast, with the alternative set of degenerate conditions (2.16), the parameter  $\alpha_L$  remains arbitrary while the three coefficients  $\beta_1$ ,  $\beta_2$  and  $\beta_3$  are determined in terms of  $\alpha_L$ ,  $\alpha_T$  and  $\alpha_H$ . Among all quadratic and cubic DHOST theories, the subclass containing Horndeski, beyond-Horndeski and the theories conformally-disformally related to them stands out as satisfying (2.15) only. By contrast, all the other subclasses verify the conditions (2.16), including one that satisfies  $\alpha_L = 0$  as well, i.e. both  $\mathcal{C}_I$  and  $\mathcal{C}_{II}$  (see Table 1).

Let us turn to phenomenology. Our analysis shows that all DHOST subclasses, except the one containing Horndeski and beyond-Horndeski, suffer from the problem that the effective Newton's constant becomes infinite in the linear regime. This analysis, which is restricted to linear perturbations, thus seems to indicate that one cannot recover standard gravity in these theories, although a fully nonlinear treatment would be necessary for a definite conclusion. Even if they cannot account for gravity as we know it, such theories could still be interesting for other contexts, such as in the early Universe.

In cosmology, we derived the quadratic action governing the dynamics of the linear scalar mode. By imposing only the condition  $\mathcal{C}_U$ , defined in (2.17), we obtained an action with a nonlinear dispersion relation and we checked that the curvature perturbation  $\zeta$  is conserved on large scales. When the full degeneracy conditions are imposed, the dispersion relation simplifies and becomes linear.

We have also studied the impact of a fully general conformal-disformal transformation, with both functions depending on  $X$  and  $\phi$ , on the quadratic perturbative action, thus extending previous results on the transformation of the effective parameters. This expresses, at the level of linear perturbations, the underlying structure of transformations within DHOST theories, which enlarges those for Horndeski theories, where  $C$  and  $D$  are restricted to be independent of  $X$ , and those for beyond Horndeski, where  $D$  can depend on  $X$  but not  $C$ .

In the main body of this paper, we have implicitly assumed that matter is minimally coupled to the metric. In this case, two conformally-disformally related theories are not physically equivalent. It is also possible to relax the assumption of minimal coupling and attribute to each matter species

four parameters that characterize, at the linear level, the non-minimal coupling to the metric. These four parameters transform under a general conformal-disformal transformation, similarly to the parameters of the scalar-tensor action. In App. D, we derived the dispersion relation for the scalar mode in the presence of matter, which enables to verify that the dispersion relation remains the same in all frames. Moreover, the mixing between matter and scalar perturbations can be quantified with a frame-invariant parameter. If we consider  $N_S$  matter species, their coupling to the metric is described by  $4N_S$  parameters, which add to the 6 independent parameters for the gravitational sector (9 parameters minus three degeneracy constraints). Taking into account frame transformations, characterized by 4 parameters, we end up with  $2(2N_S + 1)$  physically independent parameters.

Note that one can use the disformal transformations to simplify the scalar-tensor action, although at the price of complicating the coupling between matter and the metric. For example, for DHOST theories in the first subclass, one can use a conformal-disformal transformation to rewrite the scalar-tensor sector as a Horndeski theory, i.e. with  $\alpha_L = 0$ ,  $\alpha_H = 0$  and  $\beta_1 = \beta_2 = \beta_3 = 0$ , while matter has a complicated coupling to the metric and scalar field. This choice between the Jordan frame (where matter is minimally coupled) and the Horndeski frame is analogous to the choice between the Jordan frame and the Einstein frame for ordinary scalar-tensor theories.

Finally, we have also discussed Lorentz-breaking theories inspired by Horava's gravity, as well as khronons. In particular, for the former, we have shown that the covariant formulations of these models lead to nondegenerate theories. Therefore, they contain an extra degree of freedom even if it is not visible at the linear level in the unitary gauge; see discussion in App. B.2.

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## A (3 + 1) decomposition of scalar-tensor Lagrangians in the unitary gauge

In the unitary gauge, specified by eq. (2.8), each of the elementary Lagrangians defined in eqs. (2.3) and (2.5) can be expressed in terms of the velocities  $V$  and  $K_{ij}$ . For the quadratic Lagrangians, eq. (2.3), one finds

$$\begin{aligned}
 L_1^{(2)} &= A^2 K_{ij} K^{ij} + V^2 - 2\partial_i A \partial^i A , \\
 L_2^{(2)} &= (AK + V)^2 , \\
 L_3^{(2)} &= -A^2 V (AK + V) , \\
 L_4^{(2)} &= A^2 (-V^2 + \partial_i A \partial^i A) , \\
 L_5^{(2)} &= A^4 V^2 .
 \end{aligned}
 \tag{A.1}$$

Moreover, the cubic Lagrangians in eq. (2.5) read

$$\begin{aligned}
L_1^{(3)} &= -(AK + V)^3, \\
L_2^{(3)} &= -(AK + V)(A^2 K_{ij} K^{ij} + V^2 - 2\partial_i A \partial^i A), \\
L_3^{(3)} &= -A^3 K_j^i K_{il} K^{lj} + 3AK^{ij} \partial_i A \partial_j A + 3V \partial_i A \partial^i A - V^3, \\
L_4^{(3)} &= A^2 V (AK + V)^2, \\
L_5^{(3)} &= -A^2 (-V^2 + \partial_i A \partial^i A) (AK + V), \\
L_6^{(3)} &= A^2 V (A^2 K_{ij} K^{ij} + V^2 - 2\partial_i A \partial^i A), \\
L_7^{(3)} &= -A^3 K^{ij} \partial_i A \partial_j A - 2A^2 V \partial_i A \partial^i A + A^2 V^3, \\
L_8^{(3)} &= A^4 V (-V^2 + \partial_i A \partial^i A), \\
L_9^{(3)} &= -A^4 V^2 (AK + V), \\
L_{10}^{(3)} &= A^6 V^3.
\end{aligned} \tag{A.2}$$

The terms that depend on the Ricci tensor can also be expressed in the unitary gauge. The simplest way to do this is to rewrite them in terms of the quartic and quintic Horndeski Lagrangians, respectively defined as

$$L_4^H = f_2 {}^{(4)}R - 2f_{2X} (\Box \phi^2 - \phi^{\mu\nu} \phi_{\mu\nu}), \tag{A.3}$$

$$L_5^H = f_3 {}^{(4)}G_{\mu\nu} \phi^{\mu\nu} + \frac{1}{3} f_{3X} (\Box \phi^3 - 3\Box \phi \phi_{\mu\nu} \phi^{\mu\nu} + 2\phi_{\mu\nu} \phi^{\mu\sigma} \phi_\sigma^\nu). \tag{A.4}$$

In our terminology, they correspond to a quadratic and a cubic Lagrangian, respectively. Indeed, they are of the form (2.1), with

$$a_1 = -a_2 = 2f_{2X}, \quad a_3 = a_4 = a_5 = 0, \tag{A.5}$$

and

$$3b_1 = -b_2 = \frac{3}{2} b_3 = f_{3X}, \quad b_i = 0 \quad (i = 4, \dots, 10). \tag{A.6}$$

Therefore, the full action (2.1) can be rewritten as

$$S[g, \phi] = \int d^4x \sqrt{-g} \left( L_4^H + \tilde{C}_{(2)}^{\mu\nu\rho\sigma} \phi_{\mu\nu} \phi_{\rho\sigma} + L_5^H + \tilde{C}_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\alpha\beta} \right), \tag{A.7}$$

where the tensors  $\tilde{C}_{(2)}^{\mu\nu\rho\sigma}$  and  $\tilde{C}_{(3)}^{\mu\nu\rho\sigma\alpha\beta}$  are defined with the new functions

$$\tilde{a}_1 = a_1 - 2f_{2X}, \quad \tilde{a}_2 = a_2 + 2f_{2X}, \tag{A.8}$$

$$\tilde{b}_1 = b_1 - \frac{1}{3} f_{3X}, \quad \tilde{b}_2 = b_2 + f_{3X}, \quad \tilde{b}_3 = b_3 - \frac{2}{3} f_{3X}, \tag{A.9}$$

while all the other functions remain unchanged.

For the ADM decomposition of the Horndeski Lagrangians we make use of the results of Ref. [4]. In particular,  $L_4^H$  and  $L_5^H$  can be rewritten, respectively, as

$$L_4^H = f_2 R - (f_2 + 2A^2 f_{2X}) \left( K^2 - K_j^i K_i^j \right) - 2A f_{2\phi} K, \tag{A.10}$$

and

$$L_5^H = -AF_3 \left( K^{ij} R_{ij} - \frac{1}{2} KR \right) - \frac{1}{3} A^3 f_{3X} \left( K^3 - 3KK_j^i K_i^j + 2K_j^i K_k^j K_i^k \right) - \frac{1}{2} A^2 (f_{3\phi} - F_{3\phi}) R - \frac{1}{2} A^2 f_{3\phi} \left( K^2 - K_j^i K_i^j \right), \quad f_{3X} \equiv F_{3X} + \frac{F_3}{2X}. \quad (\text{A.11})$$

In the unitary gauge, we can use the relations

$$A = \frac{\dot{\phi}}{N}, \quad V \equiv \frac{1}{N} \mathcal{D}_t A, \quad \partial_i A = -A \mathbf{a}_i, \quad \mathbf{a}_i \equiv \frac{\partial_i N}{N}. \quad (\text{A.12})$$

Combining all previous results, the ADM decomposition of the full Lagrangian in the unitary gauge leads to the following expression:

$$\begin{aligned} \mathcal{L} = & -A^3 (b_1 K^3 + b_2 K K_j^i K_i^j + b_3 K_j^i K_k^j K_i^k) \\ & + A^2 \left[ (-3b_1 + b_4 A^2 + f_{3X}) K^2 + (-b_2 + b_6 A^2 - f_{3X}) K_j^i K_i^j \right] V \\ & - A \left[ 3b_1 + b_2 - A^2 (2b_4 + b_5) + A^4 b_9 \right] K V^2 \\ & - \left[ b_1 + b_2 + b_3 - A^2 (b_4 + b_5 + b_6 + b_7) + A^4 (b_8 + b_9) - b_{10} A^6 \right] V^3 \\ & + \left( f_2 + a_1 A^2 + A^2 \frac{f_{3\phi}}{2} \right) K_{ij} K^{ij} - \left( f_2 - a_2 A^2 + A^2 \frac{f_{3\phi}}{2} \right) K^2 \\ & + A (4f_{2X} + 2a_2 - a_3 A^2) K V + \left[ a_1 + \alpha_2 - (a_3 + a_4) A^2 + a_5 A^4 \right] V^2 \\ & - 2f_{2\phi} A K - AF_3 \left( K^{ij} R_{ij} - \frac{1}{2} KR \right) \\ & + A^3 (3b_3 - b_7 A^2 - 2f_{3X}) K^{ij} \mathbf{a}_i \mathbf{a}_j + A^3 (2b_2 - b_5 A^2 + 2f_{3,X}) K \mathbf{a}_i \mathbf{a}^i \\ & + A^2 \left[ 2b_2 + 3b_3 - A^2 (b_5 + 2b_6 + 2b_7) + A^4 b_8 \right] V \mathbf{a}_i \mathbf{a}^i \\ & + \left[ f_2 - \frac{1}{2} A^2 (f_{3\phi} - F_{3\phi}) \right] R + \left[ 4f_{2X} A^2 - (2a_1 - a_4 A^2) A^2 \right] \mathbf{a}_i \mathbf{a}^i. \end{aligned} \quad (\text{A.13})$$

One can then expand this Lagrangian at quadratic order around a FLRW background and obtain an expression of the form (1.2). One finds that the effective coefficients of the quadratic action are given by

$$\begin{aligned} M^2 &= 2f_2 - X[2a_1 + f_{3\phi} - 6(b_2 + b_3)H\mu^2], \\ \alpha_T &= -1 + \frac{2f_2 + Xf_{3\phi}}{M^2}, \\ \alpha_H &= -1 + \frac{1}{M^2} \left[ 2f_2 - X(4f_{2X} + 2f_{3X}H\mu^2 + f_{3\phi}) \right], \\ \alpha_L &= \frac{3X}{M^2} \left[ a_1 + a_2 - (9b_1 + 5b_2 + 3b_3)H\mu^2 \right], \\ \beta_1 &= \frac{X}{2M^2} \left\{ 4f_{2X} + 2a_2 + Xa_3 + 2[2f_{3X} - 9b_1 - b_2 - X(3b_4 + b_6)]H\mu^2 \right\}, \\ \beta_2 &= -\frac{2X}{M^2} \left\{ a_1 + a_2 + X(a_3 + a_4) + X^2 a_5 - 3[3b_1 + b_2 + X(2b_4 + b_5) + X^2 b_9]H\mu^2 \right\}, \\ \beta_3 &= -\frac{2X}{M^2} \left\{ 4f_{2X} - 2a_1 - Xa_4 + [4f_{3X} + 6b_2 + 3b_3 + X(3b_5 + b_7)]H\mu^2 \right\}. \end{aligned} \quad (\text{A.14})$$

When the cubic terms are absent, these relations are equivalent to the expressions (2.14).

Using the terminology of [29] and [33], all the subclasses of DHOST theories are summarized in Table 1. For each subclass, we indicate the number of free functions and the degeneracy conditions verified by the effective parameters.

## B Degeneracy conditions

In this appendix, we concentrate on the full nonlinear action of quadratic DHOST theories. Their kinetic Lagrangian can be written in the form [26]

$$L_{\text{kin}} = \mathcal{A}(\phi, X, A)\dot{A}^2 + 2\mathcal{B}^{ij}(\phi, X, A)\dot{A}K_{ij} + \mathcal{K}^{ijkl}(\phi, X, A)K_{ij}K_{kl}, \quad (\text{B.1})$$

where  $\mathcal{A}(\phi, X, A)$  is a polynomial in the variable  $A$  and

$$\begin{aligned} \mathcal{B}^{ij}(\phi, X, A) &= \mathcal{B}_1 h^{ij} + \mathcal{B}_2 D^i \phi D^j \phi, \\ \mathcal{K}^{ijkl}(\phi, X, A) &= \kappa_1 h^{i(k} h^{j)l} + \kappa_2 h^{ij} h^{kl} + \hat{\mathcal{K}}^{ijkl}. \end{aligned} \quad (\text{B.2})$$

Here  $\mathcal{B}_1, \mathcal{B}_2, \kappa_1, \kappa_2$  are polynomials in  $A$ , and the tensor  $\hat{\mathcal{K}}^{ijkl}$  vanishes when  $\partial_i \phi = 0$ .

The Lagrangian is degenerate when the determinant of the kinetic matrix,

$$D(\phi, X, A^2) \equiv \mathcal{A} - \mathcal{K}_{ijkl}^{-1} \mathcal{B}^{ij} \mathcal{B}^{kl} = D_0(\phi, X) + A^2 D_1(\phi, X) + A^4 D_2(\phi, X), \quad (\text{B.3})$$

vanishes for any value of  $A$ . This gives the three independent relations

$$D_0(\phi, X) = 0, \quad D_1(\phi, X) = 0, \quad D_2(\phi, X) = 0, \quad (\text{B.4})$$

The expressions for  $D_0, D_1$  and  $D_2$  depend on the functions  $f_2$  and  $a_A$ , ( $A = 1, \dots, 5$ ) and on  $X$ , and can be found in eqs. (4.30)–(4.32) of [26].

### B.1 Degeneracy conditions in terms of the effective parameters

For quadratic theories, one can express  $f_2$  and  $a_i$  in terms of  $\alpha_{\text{T}}, \alpha_{\text{H}}, \alpha_{\text{L}}, \beta_1, \beta_2$  and  $\beta_3$ :

$$\begin{aligned} f_2 &= \frac{M^2}{2} (1 + \alpha_{\text{T}}), & f_{2X} &= \frac{M^2}{4X} (\alpha_{\text{T}} - \alpha_{\text{H}}), \\ a_1 &= \frac{M^2}{2X} \alpha_{\text{T}}, & a_2 &= -\frac{M^2}{6X} (3\alpha_{\text{T}} - 2\alpha_{\text{L}}), \\ a_3 &= -\frac{M^2}{3X^2} (2\alpha_{\text{L}} - 3\alpha_{\text{H}} - 6\beta_2), & a_4 &= -\frac{M^2}{2X^2} (2\alpha_{\text{H}} - \beta_3), \\ a_5 &= -\frac{M^2}{6X^3} (-2\alpha_{\text{L}} + 3\beta_1 + 12\beta_2 + 3\beta_3). \end{aligned} \quad (\text{B.5})$$

Substituting these expressions into (B.4), one gets

$$\frac{X}{M^6} D_0 = \frac{\alpha_{\text{L}}}{3} [2(1 + \alpha_{\text{H}})^2 - (1 + \alpha_{\text{T}})\beta_3], \quad (\text{B.6})$$

$$\begin{aligned} \frac{X^2}{M^6} D_1 &= \frac{\alpha_{\text{L}}}{3} [4(1 + \alpha_{\text{H}})^2 - (1 + \alpha_{\text{T}})\beta_2 - (5 + 2\alpha_{\text{T}})\beta_3] \\ &\quad - [4(1 + \alpha_{\text{H}})\beta_1 + 2(1 + \alpha_{\text{T}})\beta_1^2 + \beta_3], \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} \frac{X^3}{M^6} D_2 &= \frac{\alpha_{\text{L}}}{3} [2(1 + \alpha_{\text{H}})^2 - (4 + \alpha_{\text{T}})\beta_2 - (4 + \alpha_{\text{T}})\beta_3] \\ &\quad - [\beta_2 + 4(1 + \alpha_{\text{H}})\beta_1 + 2(4 + \alpha_{\text{T}})\beta_1^2 + \beta_3]. \end{aligned} \quad (\text{B.8})$$

One can distinguish two cases, which yield the conditions  $\mathcal{C}_{\text{I}}$  and  $\mathcal{C}_{\text{II}}$ , given respectively in (2.15) and (2.16).

Subclass (see [33])	# free functions	Degeneracy	Remarks
${}^2\text{N-I/Ia}$	3	I	H, bH & conf-disf transf
${}^2\text{N-II/Ib}$	3	0	
${}^2\text{N-III/IIa}$	3	II	
${}^2\text{N-IV/IIb}$	3	0	
${}^2\text{M-I/IIIa}$	3	II	$\alpha_{\text{T}} = \alpha_{\text{H}} = -1$ $\alpha_{\text{L}} = \alpha_{\text{T}} = \alpha_{\text{H}} = -1$
${}^2\text{M-II/IIIb}$	3	II	
${}^2\text{M-III/IIIc}$	4	0	
${}^3\text{N-I}$	3	I	H, bH & conf-disf transf
${}^3\text{N-II}$	4	0	
${}^3\text{M-I}$	4	II	$\alpha_{\text{T}} = \alpha_{\text{H}} = -1$ $\alpha_{\text{T}} = \alpha_{\text{H}} = -1$
${}^3\text{M-II}$	3	II	
${}^3\text{M-III}$	1	0	
${}^3\text{M-IV}$	5	0	
${}^3\text{M-V}$	2	0	
${}^3\text{M-VI}$	6	0	
${}^3\text{M-VII}$	4	0	
${}^2\text{N-I/Ia} \ \& \ {}^3\text{N-I}$	4	I	H, bH & conf-disf transf
${}^2\text{N-II/Ib} \ \& \ {}^3\text{N-II}$	7	0	
${}^2\text{N-I/Ia} \ \& \ {}^3\text{M-I}$	5	II	
${}^2\text{N-I/Ia} \ \& \ {}^3\text{M-II}$	4	II	
${}^2\text{N-I/Ia} \ \& \ {}^3\text{M-III}$	3	I & II	
${}^2\text{N-I/Ia} \ \& \ {}^3\text{M-V}$	3	II	
${}^2\text{N-II/Ib} \ \& \ {}^3\text{M-VII}$	6	0	
${}^2\text{N-III/IIa} \ \& \ {}^3\text{M-I}$	6	II	
${}^2\text{N-III/IIa} \ \& \ {}^3\text{M-II}$	5	II	
${}^2\text{N-III/IIa} \ \& \ {}^3\text{M-III}$	4	II	
${}^2\text{N-III/IIa} \ \& \ {}^3\text{M-V}$	4	II	
${}^2\text{N-IV/IIb} \ \& \ {}^3\text{M-I}$	5	II	
${}^2\text{N-IV/IIb} \ \& \ {}^3\text{M-II}$	4	II	
${}^2\text{N-IV/IIb} \ \& \ {}^3\text{M-III}$	3	0	
${}^2\text{N-IV/IIb} \ \& \ {}^3\text{M-V}$	4	0	
${}^2\text{M-I/IIIa} \ \& \ {}^3\text{M-I}$	6	II	
${}^2\text{M-I/IIIa} \ \& \ {}^3\text{M-II}$	5	II	$\alpha_{\text{T}} = \alpha_{\text{H}} = -1$
${}^2\text{M-I/IIIa} \ \& \ {}^3\text{M-III}$	4	II	$\alpha_{\text{T}} = \alpha_{\text{H}} = -1$
${}^2\text{M-I/IIIa} \ \& \ {}^3\text{M-V}$	4	II	$\alpha_{\text{T}} = \alpha_{\text{H}} = -1$
${}^2\text{M-II/IIIb} \ \& \ {}^3\text{M-III}$	4	II	$\alpha_{\text{L}} = \alpha_{\text{T}} = \alpha_{\text{H}} = -1$
${}^2\text{M-II/IIIb} \ \& \ {}^3\text{M-IV}$	8	II	$\alpha_{\text{L}} = \alpha_{\text{T}} = \alpha_{\text{H}} = -1$
${}^2\text{M-II/IIIb} \ \& \ {}^3\text{M-VII}$	6	II	$\alpha_{\text{L}} = \alpha_{\text{T}} = \alpha_{\text{H}} = -1$
${}^2\text{M-III/IIIc} \ \& \ {}^3\text{M-V}$	6	0	
${}^2\text{M-III/IIIc} \ \& \ {}^3\text{M-VI}$	10	0	
${}^2\text{M-III/IIIc} \ \& \ {}^3\text{M-VII}$	7	0	

Table 1: Subclasses of DHOST theories, using the classification of Ref. [33]. Second column: number of free functions among  $f_2$ ,  $a_A$ ,  $f_3$  and  $b_A$ . In the degeneracy column, 0 stands for  $M^2 = 0$ , i.e. there are no tensor modes. If  $f_{3\phi} \neq 0$ , only  ${}^3\text{N-I}$  and  ${}^2\text{N-I/Ia} \ \& \ {}^3\text{N-I}$  are degenerate.

- $\alpha_L = 0$ : The condition (B.6) is automatically satisfied. From (B.7) and (B.8) we obtain

$$\beta_3 = -4\beta_1(1 + \alpha_H) - 2\beta_1^2(1 + \alpha_T), \quad \beta_2 = -6\beta_1^2. \quad (\text{B.9})$$

- $\alpha_L \neq 0$ : The three conditions can be solved for  $\beta_1$ ,  $\beta_2$  and  $\beta_3$ , obtaining

$$\begin{aligned} (1 + \alpha_T)\beta_1 &= -(1 + \alpha_L)(1 + \alpha_H), & (1 + \alpha_T)^2\beta_2 &= -6(1 + \alpha_L)(1 + \alpha_H)^2, \\ (1 + \alpha_T)\beta_3 &= 2(1 + \alpha_H)^2. \end{aligned} \quad (\text{B.10})$$

## B.2 Unitary gauge and extra degree of freedom

In the unitary gauge, we have  $\partial_i\phi = 0$  and therefore  $X = -A^2$  so that the vanishing of the determinant (B.3) is guaranteed by the single condition

$$D(\phi, X, -X) = 0. \quad (\text{B.11})$$

This condition is obviously satisfied if the three conditions of (B.4) are verified, but the converse is not true: there exist theories that look degenerate in the unitary gauge but are in fact not degenerate.

An important consequence of the above discussion is that for a nondegenerate theory that satisfies (B.11), the extra scalar degree of freedom, which is known to be present because the theory is nondegenerate, does not show up in the linear perturbations about a homogeneous background. Indeed, in this case the kinetic Lagrangian quadratic in perturbations is given by

$$L_{\text{kin}}^{\text{quad}} = \bar{\mathcal{A}}\delta\dot{A}^2 + 2\bar{\mathcal{B}}^{ij}\delta\dot{A}\delta K_{ij} + \bar{\mathcal{K}}^{ijkl}\delta K_{ij}\delta K_{kl}, \quad (\text{B.12})$$

where all quantities are decomposed into a background component, denoted by a bar, and a perturbative component:  $\phi = \bar{\phi} + \delta\phi$ ,  $X = \bar{X} + \delta X$  and  $A = \bar{A} + \delta A$ . Note that  $\delta X$  and  $\delta A$  involve  $\delta\dot{\phi}$  and  $\partial_i\delta\phi$ . Since  $\bar{X} = -\bar{A}^2$ , we see that the degeneracy of the above quadratic Lagrangian is automatically guaranteed when the condition (B.11) is verified, even if the conditions (B.4) are not. The condition can be written as

$$\bar{\mathcal{A}} - \frac{3}{\bar{\kappa}_1 + 3\bar{\kappa}_2}\bar{\mathcal{B}}_1^2 = 0. \quad (\text{B.13})$$

In that case, it means that only one scalar degree of freedom shows up at the level of linear perturbations, independently of the gauge choice. However, the extra degree of freedom remains present and would show up at the nonlinear level or in an inhomogeneous background.

## C Disformal transformations

Here we study how action (1.2) transforms under a conformal and disformal redefinition of the metric, eq. (2.18). For simplicity, following [17] we work in the unitary gauge and set  $\phi = t$  (i.e.  $\mu = 1$  in eq. (2.12)), so that  $X = -1/N^2$ . In the unitary gauge, the disformal transformation (2.18) is of the form

$$\tilde{g}_{\mu\nu} = C(t, N)g_{\mu\nu} + D(t, N)\delta_\mu^0\delta_\nu^0. \quad (\text{C.1})$$

The ADM components of the metric  $\tilde{g}_{\mu\nu}$  are related to those of  $g_{\mu\nu}$  by

$$\tilde{N}^i = N^i, \quad \tilde{h}_{ij} = Ch_{ij}, \quad \tilde{N}^2 = CN^2 - D. \quad (\text{C.2})$$

Moreover, the other relevant geometrical quantities in the action transform as

$$\sqrt{-\tilde{g}} = \sqrt{-g} C^{3/2} \sqrt{C - D/N^2}, \quad (\text{C.3})$$

$$\tilde{R} = C^{-1} \left( R - 2D^i D_i \log C - \frac{1}{2} (\partial_i \log C)^2 \right), \quad (\text{C.4})$$

$$\tilde{K}_j^i = \frac{N}{\tilde{N}} \left[ K_j^i + \frac{1}{2NC} \left( \dot{C} + C_N (\dot{N} - N^i \partial_i N) \right) \delta_j^i \right]. \quad (\text{C.5})$$

The dimensionless time-dependent parameters defined in eq. (2.21) become

$$\alpha_C \equiv \frac{\dot{C}}{2HCN}, \quad \alpha_Y \equiv \frac{N}{2C} \frac{\partial C}{\partial N}, \quad \alpha_D \equiv \frac{D}{CN^2 - D}, \quad \alpha_X \equiv -\frac{1}{2CN} \frac{\partial D}{\partial N}, \quad (\text{C.6})$$

where all the quantities on the right-hand side are evaluated on the background. Using these definitions, from eq. (C.2) the homogeneous components of the metric and the lapse transform as

$$\tilde{a} = \sqrt{C} a, \quad \tilde{N}_0 = \sqrt{\frac{C}{1 + \alpha_D}} N_0. \quad (\text{C.7})$$

Moreover, the Hubble rate changes accordingly, i.e.,

$$\tilde{H} \equiv \frac{1}{\tilde{N}_0 \tilde{a}} \frac{d\tilde{a}}{dt} = H(1 + \alpha_C) \sqrt{\frac{1 + \alpha_D}{C}}. \quad (\text{C.8})$$

Furthermore, to compute how the action transforms we are interested in the following relations, derived from linearizing eqs. (C.2)–(C.5) and valid at linear order,

$$\begin{aligned} \delta N &= \Xi \frac{N}{\tilde{N}} \delta \tilde{N}, \\ \delta \sqrt{h} &= \frac{a^3}{\tilde{a}^3} \delta \sqrt{\tilde{h}} - 3a^3 \Xi \alpha_Y \frac{\delta \tilde{N}}{\tilde{N}}, \\ R &= C \tilde{R} + 4\alpha_Y \frac{\Xi}{a^2} \frac{\Delta \delta \tilde{N}}{\tilde{N}}, \\ \delta K_j^i &= \frac{\tilde{N}}{N} \delta \tilde{K}_j^i - \alpha_Y \frac{\Xi}{N} \frac{d}{dt} \left( \frac{\delta \tilde{N}}{\tilde{N}} \right) \delta_j^i + \delta_j^i H \Upsilon \frac{\delta \tilde{N}}{\tilde{N}}, \end{aligned} \quad (\text{C.9})$$

where

$$\Xi \equiv \frac{1}{(1 + \alpha_D)(1 + \alpha_X + \alpha_Y)}, \quad (\text{C.10})$$

and

$$\begin{aligned} \Upsilon \equiv & \Xi \left\{ \alpha_X + \alpha_Y + \alpha_D(1 + \alpha_X + \alpha_Y) + \alpha_C [(1 + \alpha_D)(1 + \alpha_X) + \alpha_Y(3 + \alpha_D)] \right. \\ & \left. - \frac{1}{2HC} \frac{\partial^2 C}{\partial t \partial N} - \alpha_Y \frac{\dot{\Xi}}{\Xi H} + \frac{\dot{N}_0}{HN_0} \left[ \alpha_X \alpha_Y + 3\alpha_Y^2 + \alpha_D \alpha_Y (1 + \alpha_X + \alpha_Y) - \frac{N_0^2}{2C} \frac{\partial^2 C}{\partial N^2} \right] \right\}. \end{aligned} \quad (\text{C.11})$$

The last ingredient is the transformation of the second-order perturbation of  $R$ , i.e.

$$\begin{aligned} \delta_2 R &= C \delta_2 \tilde{R} + 2C \alpha_Y \Xi \tilde{R} \frac{\delta \tilde{N}}{\tilde{N}} - 4\Xi \alpha_Y \frac{\delta \sqrt{\tilde{h}}}{\tilde{a}^3} \frac{\Delta \delta \tilde{N}}{a^2 \tilde{N}} + \left( \frac{2}{C} \frac{\partial^2 C}{\partial N^2} - 18\Xi^2 \alpha_Y^2 \right) \frac{(\partial_i \delta \tilde{N})^2}{a^2 \tilde{N}^2} \\ &+ \frac{\delta \tilde{N}}{a^2 \tilde{N}^2} \left( \frac{2}{C} \frac{\partial^2 C}{\partial N^2} - 8\alpha_Y^2 \right). \end{aligned} \quad (\text{C.12})$$

Replacing all the geometrical terms in action (1.2) by their transformed quantities using eqs. (C.9) and (C.12) and making use of eqs. (C.7) and (C.8) for the homogeneous quantities, one obtains an action in terms of the metric  $\tilde{g}_{\mu\nu}$ . The last step is to make a time redefinition,

$$t \rightarrow \tilde{t} = \int \sqrt{\frac{C}{1 + \alpha_D}} dt, \quad (\text{C.13})$$

which sets the homogeneous “00” component of the metric to unity, i.e.  $\tilde{N}_0 = 1$ . The obtained action  $\tilde{S}$  has the same form as the original one, eq. (1.2), with the transformed  $\tilde{\alpha}_K$  and  $\tilde{\alpha}_B$  given by

$$\begin{aligned} \tilde{\alpha}_K &= \frac{\alpha_K \Xi^2 + 12 \Xi \Upsilon \alpha_B - 6 \Upsilon^2 (1 + \alpha_L)}{(1 + \alpha_C)^2} + 12 \Upsilon \beta_1 \frac{\dot{\Xi}}{H(1 + \alpha_C)^2} + \beta_2 \frac{\dot{\Xi}^2}{H^2(1 + \alpha_C)^2} \\ &\quad - \frac{1}{2 \tilde{M}^2 \tilde{a}^3 \tilde{H}^2} \frac{d}{dt} \left\{ \tilde{M}^2 \tilde{a}^3 \frac{12 \Xi H}{\tilde{N}_0} \left[ \Upsilon (\alpha_Y (1 + \alpha_L) + \beta_1) - \Xi \alpha_B \alpha_Y + \frac{\dot{\Xi}}{6H} (\beta_2 - 6 \alpha_Y \beta_1) \right] \right\}, \quad (\text{C.14}) \\ \tilde{\alpha}_B &= \frac{\alpha_B \Xi - (1 + \alpha_L) \Upsilon}{1 + \alpha_C} + \frac{\beta_1 \dot{\Xi}}{H(1 + \alpha_C)}. \end{aligned}$$

The remaining time-dependent functions  $\tilde{\alpha}_A$  and  $\tilde{\beta}_A$  are given by eq. (2.22).

## D Quadratic Lagrangian and dispersion relation in the presence of matter

We describe matter using a scalar field  $\sigma$  with a  $k$ -essence type action [65],

$$S_m = \int d^4x \sqrt{-g} P(Y), \quad Y \equiv \check{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma, \quad (\text{D.1})$$

where matter is coupled to a metric of the form

$$\check{g}_{\mu\nu} = C_m^{(\phi)}(\phi, X) g_{\mu\nu} + D_m^{(\phi)}(\phi, X) \partial_\mu \phi \partial_\nu \phi. \quad (\text{D.2})$$

In unitary gauge, this reads

$$\check{g}_{\mu\nu} = C_m(t, N) g_{\mu\nu} + D_m(t, N) \delta_\mu^0 \delta_\nu^0, \quad (\text{D.3})$$

with

$$C_m(t, N) = C_m^{(\phi)}(\phi(t), -\dot{\phi}(t)^2/N^2), \quad D_m(t, N) = \dot{\phi}^2(t) D_m^{(\phi)}(\phi(t), -\dot{\phi}(t)^2/N^2). \quad (\text{D.4})$$

Then, in analogy with equation (2.21), we introduce the parameters

$$\alpha_{C,m} \equiv \frac{\dot{C}_m}{2HC_m}, \quad \alpha_{Y,m} \equiv \frac{1}{2C_m} \frac{\partial C_m}{\partial N}, \quad \alpha_{X,m} \equiv -\frac{1}{2C_m} \frac{\partial D_m}{\partial N}, \quad \alpha_{D,m} \equiv \frac{D_m}{C_m - D_m}, \quad (\text{D.5})$$

where the right-hand sides are evaluated on the background. The first two parameters in the above equations,  $\alpha_{C,m}$  and  $\alpha_{D,m}$ , were introduced in Ref. [60], while the third in [47]. The total action is the sum of the gravitational action, eq. (1.2), and the matter action above. We will first discuss the case  $\alpha_L \neq 0$ ; the case  $\alpha_L = 0$  can be obtained by taking the smooth limit  $\alpha_L \rightarrow 0$ , as we will discuss below.

Variation of the total action with respect to  $\psi$  yields the scalar component of the momentum constraint, which reads

$$2\alpha_L \frac{\Delta\psi}{a^2} + 6H\delta N(1 + \alpha_B + \alpha_L) + 6\beta_1\delta\dot{N} - 6(1 + \alpha_L)\dot{\zeta} = 3(1 + \alpha_{D,m})c_m^2\dot{\sigma}_0 A_m \delta\sigma \quad (\text{D.6})$$

where  $\sigma_0 = \sigma_0(t)$  and  $\delta\sigma = \delta\sigma(t, \mathbf{x})$  respectively denote the homogeneous component of the scalar field and its perturbation. We also introduced the matter equation of state and sound speed squared including a lightcone-changing factor  $(1 + \alpha_{D,m})$  for later convenience,

$$w_m = \frac{P}{(1 + \alpha_{D,m})(2YP_Y - P)}, \quad c_m^2 = \frac{P_Y}{(1 + \alpha_{D,m})(P_Y + 2YP_{YY})}, \quad (\text{D.7})$$

as well as the combination

$$A_m \equiv -\frac{2C_m\sqrt{1 + \alpha_{D,m}}}{M^2}(P_Y + 2YP_{YY}). \quad (\text{D.8})$$

Substituting  $\psi$  into the total action using equation D.6 yields

$$\begin{aligned} S = \int d^3x dt a^3 \frac{M^2}{2} \left\{ \frac{1}{\alpha_L} \left[ 6(1 + \alpha_L)\dot{\zeta}^2 - 12\beta_1\delta\dot{N}\dot{\zeta} + (6\beta_1^2 + \alpha_L\beta_2)\delta\dot{N}^2 \right. \right. \\ + 12H \left( \beta_1(1 + \alpha_B)\delta\dot{N} - (1 + \alpha_B + \alpha_L)\dot{\zeta} \right) \delta N \\ + H^2 \left( 6\alpha_B^2 + 12\alpha_B + \alpha_L\alpha_K + 6(1 + \alpha_L) + \frac{\alpha_L\mathcal{M}_{22}^{(m)}}{H^2} \right) \delta N^2 \left. \right] \\ + \frac{1}{a^2} \left[ 4(1 + \alpha_H)\partial_i\zeta\partial_i\delta N + 2(1 + \alpha_T)(\partial_i\zeta)^2 + \beta_3(\partial_i\delta N)^2 \right] \\ + A_m \left[ \delta\dot{\sigma}^2 - \frac{c_m^2}{a^2}(\partial_i\delta\sigma)^2 + 6\dot{\sigma}_0c_m^2(1 + \alpha_{D,m}) \left( \zeta\delta\dot{\sigma} + \frac{1 + \alpha_L}{\alpha_L}\dot{\zeta}\delta\sigma \right) \right. \\ - 2\dot{\sigma}_0(1 + \alpha_{D,m}) \left( \delta N\delta\dot{\sigma}(1 + \alpha_{X,m} + \alpha_{Y,m}(1 - 3c_m^2)) + \frac{3c_m^2\beta_1}{\alpha_L}\delta\dot{N}\delta\sigma \right) \\ \left. - \frac{6H(1 + \alpha_B + \alpha_L)\dot{\sigma}_0c_m^2(1 + \alpha_{D,m})}{\alpha_L}\delta N\delta\sigma + 3\frac{c_m^4\dot{\sigma}_0^2(1 + \alpha_{D,m})^2}{2\alpha_L}A_m\delta\sigma^2 \right] \left. \right\}, \quad (\text{D.9}) \end{aligned}$$

where we have defined

$$\begin{aligned} \mathcal{M}_{22}^{(m)} &= 3\hat{\Omega}_m H^2 \alpha_{D,m}^{\text{eff}} + \dot{\sigma}_0^2(1 + \alpha_{D,m})^2 A_m \left[ (1 + \alpha_{X,m} + \alpha_{Y,m}) - 3c_m^2(\beta_1 + \alpha_{Y,m}) \right]^2, \\ \hat{\Omega}_m &= \frac{C_m^2\sqrt{1 + \alpha_{D,m}}(2YP_Y - P)}{3M^2H^2}, \\ \alpha_{D,m}^{\text{eff}} &\equiv \alpha_{D,m}(1 + \alpha_{X,m} + \alpha_{Y,m})^2 + 2(\alpha_{X,m} - \alpha_{Y,m}) + (\alpha_{X,m} + \alpha_{Y,m})^2 + \frac{1}{2C_m} \frac{\partial^2(D_m - C_m)}{\partial N^2} \\ &\quad + [1 + w_m(1 + \alpha_{D,m})] \left[ 6\beta_1(1 + \alpha_{X,m} + \alpha_{Y,m}) - 9c_m^2(\beta_1 + \alpha_{Y,m})^2 \right] \\ &\quad + 3w_m \left[ 2\alpha_{Y,m}(1 + \alpha_{D,m})(1 + \alpha_{X,m}) + \alpha_{Y,m}^2(3 + 2\alpha_{D,m}) \right. \\ &\quad \left. + 2\beta_1(1 + \alpha_{D,m})(1 + \alpha_{X,m} + \alpha_{Y,m}) + \frac{1}{2C_m} \frac{\partial^2 C_m}{\partial N^2} \right]. \quad (\text{D.10}) \end{aligned}$$

Let us now impose the unitary degeneracy condition, eq. (4.19). As discussed in Sec. 4, it is convenient to introduce  $\tilde{\zeta}$  defined in eq. (4.20). Replacing  $\zeta$  in terms of this variable in action (D.9) eliminates time derivatives of the lapse  $\delta N$ . In particular, the quadratic action takes the form

$$S = \int d^3x dt a^3 \frac{M^2}{2} \left[ \frac{6(1 + \alpha_L)}{\alpha_L} \dot{\tilde{\zeta}}^2 + A_m \dot{\delta\sigma}^2 + 2\dot{\mathbf{X}}^T \mathcal{V} \mathbf{X} + \frac{1}{a^2} \partial_i \mathbf{X}^T \mathcal{S} \partial_i \mathbf{X} + \mathbf{X}^T \mathcal{M} \mathbf{X} \right], \quad (\text{D.11})$$

where  $\mathbf{X}^T \equiv (\tilde{\zeta}, \delta N, \delta\sigma)$  and  $\mathcal{V}$  denotes the matrix of elements

$$\begin{aligned} \mathcal{V}_{12} &= -\frac{6}{\alpha_L} \left[ H(1 + \alpha_B + \alpha_L) - \dot{\beta}_1 + \frac{\beta_1 \dot{\alpha}_L}{1 + \alpha_L} \right], & \mathcal{V}_{13} &= \frac{3(1 + \alpha_L)}{\alpha_L} \dot{\sigma}_0 (1 + \alpha_{D,m}) c_m^2 A_m, \\ \mathcal{V}_{32} &= -\dot{\sigma}_0 (1 + \alpha_{D,m}) A_m \left[ 1 + \alpha_{X,m} + \alpha_{Y,m} - 3c_m^2 \left( \frac{\beta_1}{1 + \alpha_L} + \alpha_{Y,m} \right) \right], & & \\ \mathcal{V}_{11} &= \mathcal{V}_{21} = \mathcal{V}_{22} = \mathcal{V}_{23} = \mathcal{V}_{31} = \mathcal{V}_{33} = 0. \end{aligned} \quad (\text{D.12})$$

Moreover,  $\mathcal{M}$  and  $\mathcal{S}$  are symmetric matrices with elements given by

$$\begin{aligned} \mathcal{S}_{11} &= 2(1 + \alpha_T), & \mathcal{S}_{12} &= 2 \left[ 1 + \alpha_H + \frac{(1 + \alpha_T)\beta_1}{1 + \alpha_L} \right], & \mathcal{S}_{13} &= \mathcal{S}_{23} = 0, \\ \mathcal{S}_{22} &= 4 \frac{(1 + \alpha_H)\beta_1}{1 + \alpha_L} + 2 \frac{(1 + \alpha_T)\beta_1^2}{(1 + \alpha_L)^2} + \beta_3, & \mathcal{S}_{33} &= -c_m^2 A_m, \end{aligned} \quad (\text{D.13})$$

and

$$\begin{aligned} \mathcal{M}_{22} &= \frac{\alpha_L \mathcal{V}_{12}}{1 + \alpha_L} \left( \frac{\mathcal{V}_{12}}{6} - H \alpha_B \right) + H^2 \left[ \alpha_K - 6\alpha_B \left( 1 + \frac{3 + \alpha_M}{1 + \alpha_L} \beta_1 \right) \right] - 6 \frac{(\alpha_B H)}{1 + \alpha_L} \beta_1 + \mathcal{M}_{22}^{(m)}, \\ \mathcal{M}_{23} &= \frac{1}{2} \dot{\sigma}_0 (1 + \alpha_{D,m}) c_m^2 A_m \mathcal{V}_{12}, & \mathcal{M}_{33} &= \frac{3}{2\alpha_L} \dot{\sigma}_0^2 (1 + \alpha_{D,m})^2 c_m^4 A_m^2, \\ \mathcal{M}_{11} &= \mathcal{M}_{12} = \mathcal{M}_{13} = 0. \end{aligned} \quad (\text{D.14})$$

Variation of (D.11) with respect to  $\delta N$  gives the Hamiltonian constraint,

$$\mathcal{V}_{12} \dot{\tilde{\zeta}} + \mathcal{V}_{23} \dot{\delta\sigma} - \mathcal{S}_{21} \Delta \tilde{\zeta} - \mathcal{S}_{22} \Delta \delta N + \mathcal{M}_{22} \delta N + \mathcal{M}_{23} \delta\sigma = 0. \quad (\text{D.15})$$

This can be solved for  $\delta N$  and plugged back into action (D.11). After a spatial Fourier transform we get, denoting  $\hat{k} = k/a$ :

$$\begin{aligned} S &= \frac{1}{2(2\pi)^3} \int dt d^3k \frac{a^3 M^2}{\mathcal{M}_{22} + \hat{k}^2 \mathcal{S}_{22}} \left[ (c_{1,0} + c_{1,2} \hat{k}^2) \dot{\tilde{\zeta}}_{\mathbf{k}} \dot{\tilde{\zeta}}_{-\mathbf{k}} + \frac{c_{2,2} \hat{k}^2 + c_{2,4} \hat{k}^4 + c_{2,6} \hat{k}^6}{\mathcal{M}_{22} + \hat{k}^2 \mathcal{S}_{22}} \tilde{\zeta}_{\mathbf{k}} \tilde{\zeta}_{-\mathbf{k}} \right. \\ &+ (c_{3,0} + c_{3,2} \hat{k}^2) \dot{\delta\sigma}_{\mathbf{k}} \dot{\delta\sigma}_{-\mathbf{k}} + \frac{c_{4,0} + c_{4,2} \hat{k}^2 + c_{4,4} \hat{k}^4 + c_{4,6} \hat{k}^6}{\mathcal{M}_{22} + \hat{k}^2 \mathcal{S}_{22}} \delta\sigma_{\mathbf{k}} \delta\sigma_{-\mathbf{k}} \\ &\left. + 2c_{5,0} \dot{\tilde{\zeta}}_{\mathbf{k}} \dot{\delta\sigma}_{-\mathbf{k}} + 2c_{6,2} \hat{k}^2 \tilde{\zeta}_{\mathbf{k}} \dot{\delta\sigma}_{-\mathbf{k}} + 2(c_{7,0} + c_{7,2} \hat{k}^2) \delta\sigma_{\mathbf{k}} \dot{\tilde{\zeta}}_{-\mathbf{k}} + 2c_{8,2} \hat{k}^2 \tilde{\zeta}_{\mathbf{k}} \delta\sigma_{-\mathbf{k}} \right], \end{aligned} \quad (\text{D.16})$$

where

$$\begin{aligned}
c_{1,0} &= \frac{6(1 + \alpha_L)}{\alpha_L} \mathcal{M}_{22} - \mathcal{V}_{12}^2, & c_{1,2} &= \frac{6(1 + \alpha_L)}{\alpha_L} \mathcal{S}_{22}, \\
c_{2,2} &= \mathcal{S}_{11} \mathcal{M}_{22}^2 + H \mathcal{S}_{12} \mathcal{M}_{22} \mathcal{V}_{12} (3 + \alpha_M) + \mathcal{M}_{22} \mathcal{V}_{12} \dot{\mathcal{S}}_{12} - \mathcal{V}_{12} \mathcal{S}_{12} \dot{\mathcal{M}}_{22} + \mathcal{S}_{12} \mathcal{M}_{22} \dot{\mathcal{V}}_{12}, \\
c_{2,4} &= -\mathcal{S}_{12}^2 \mathcal{M}_{22} + \mathcal{S}_{22} \left( 2\mathcal{S}_{12} \mathcal{M}_{22} + \mathcal{V}_{12} \dot{\mathcal{S}}_{12} \right) + \mathcal{S}_{12} \left( H \mathcal{S}_{22} \mathcal{V}_{12} (3 + \alpha_M) - \mathcal{V}_{12} \dot{\mathcal{S}}_{22} + \mathcal{S}_{22} \dot{\mathcal{V}}_{12} \right), \\
c_{2,6} &= \mathcal{S}_{22} \left( \mathcal{S}_{11} \mathcal{S}_{22} - \mathcal{S}_{12}^2 \right), & c_{3,0} &= A_m \mathcal{M}_{22} - \mathcal{V}_{32}^2, & c_{3,2} &= A_m \mathcal{S}_{22}, \\
c_{4,0} &= -\mathcal{M}_{22} \left[ \mathcal{M}_{23}^2 + (\mathcal{M}_{33} \mathcal{M}_{22} + \mathcal{V}_{32} \dot{\mathcal{M}}_{23}) \right] + \mathcal{M}_{23} \left[ H \mathcal{M}_{22} \mathcal{V}_{32} (3 + \alpha_M) - \mathcal{V}_{32} \dot{\mathcal{M}}_{22} + \mathcal{M}_{22} \dot{\mathcal{V}}_{32} \right], \\
c_{4,2} &= \mathcal{S}_{33} \mathcal{M}_{22}^2 - \mathcal{M}_{23} \mathcal{V}_{32} \dot{\mathcal{S}}_{22} + \mathcal{S}_{22} \left[ 2\mathcal{M}_{33} \mathcal{M}_{22} - \mathcal{M}_{23}^2 + \mathcal{V}_{32} \dot{\mathcal{M}}_{23} + \mathcal{M}_{23} (H \mathcal{V}_{32} (3 + \alpha_M) + \dot{\mathcal{V}}_{32}) \right], \\
c_{4,4} &= \mathcal{S}_{22} (2\mathcal{S}_{33} \mathcal{M}_{22} + \mathcal{S}_{22} \mathcal{M}_{33}), & c_{4,6} &= \mathcal{S}_{22}^2 \mathcal{S}_{33}, & c_{5,0} &= -\mathcal{V}_{12} \mathcal{V}_{32}, \\
c_{6,2} &= -\mathcal{V}_{32} \mathcal{S}_{12}, & c_{7,0} &= \mathcal{V}_{13} \mathcal{M}_{22} - \mathcal{V}_{12} \mathcal{M}_{23}, & c_{7,2} &= \mathcal{V}_{13} \mathcal{S}_{22}, & c_{8,2} &= -\mathcal{S}_{12} \mathcal{M}_{23}.
\end{aligned} \tag{D.17}$$

Let us focus on theories satisfying the degeneracy conditions  $\mathcal{C}_1$ . By imposing the remaining degeneracy conditions, the limit  $\alpha_L \rightarrow 0$  is finite and for the kinetic part of the action we find

$$S = \int d^3x dt a^3 \frac{M^2}{2} \left\{ A_{\tilde{\zeta}}^{(m)} \dot{\tilde{\zeta}}^2 + \frac{B_{\tilde{\zeta}}}{a^2} (\partial_i \tilde{\zeta})^2 + A_m \left[ \delta\dot{\sigma}^2 - \frac{c_m^2}{a^2} (\partial_i \delta\sigma)^2 \right] + A_{\tilde{\zeta}_m} \delta\sigma \dot{\tilde{\zeta}} + \frac{B_{\tilde{\zeta}_m}}{a^2} \partial_i \tilde{\zeta} \partial_i \delta\sigma \right\}, \tag{D.18}$$

where

$$A_{\tilde{\zeta}}^{(m)} = A_{\tilde{\zeta}} + \frac{\mathcal{M}_{22}^{(m)}}{[H(1 + \alpha_B) - \dot{\beta}_1]^2}, \tag{D.19}$$

$$A_{\tilde{\zeta}_m} = \frac{2\dot{\sigma}_0(1 + \alpha_{D,m}) A_m [1 + \alpha_{X,m} + \alpha_{Y,m} - 3c_m^2(\beta_1 + \alpha_{Y,m})]}{[H(1 + \alpha_B) - \dot{\beta}_1]^2}, \tag{D.20}$$

$$B_{\tilde{\zeta}_m} = -\frac{2\dot{\sigma}_0(1 + \alpha_{D,m}) c_m^2 A_m [1 + \alpha_H + \beta_1(1 + \alpha_T)]}{[H(1 + \alpha_B) - \dot{\beta}_1]^2}, \tag{D.21}$$

and  $A_{\tilde{\zeta}}$ ,  $B_{\tilde{\zeta}}$  are respectively defined in eqs. (4.9) and (4.10). Requiring that the time kinetic matrix is positive definite gives the no-ghost conditions

$$A_m \geq 0, \quad \alpha + 3\hat{\Omega}_m \alpha_{D,m}^{\text{eff}} \geq 0, \tag{D.22}$$

where  $\alpha$  was defined in equation (4.15). Requiring that the determinant of the kinetic matrix vanishes gives the dispersion relation

$$(\omega^2 - \hat{c}_s^2 k^2)(\omega^2 - c_m^2 k^2) = \lambda^2 \hat{c}_s^2 \omega^2 k^2, \tag{D.23}$$

where

$$\lambda^2 \equiv \frac{3[1 + w_m(1 + \alpha_{D,m})] \hat{\Omega}_m H^2}{\alpha + 3\hat{\Omega}_m \alpha_{D,m}^{\text{eff}}} [\alpha_H - \alpha_{X,m} - \alpha_{Y,m} + 3c_m^2(\alpha_{Y,m} + \beta_1) + \beta_1(1 + \alpha_T)]^2 \tag{D.24}$$

is a frame-independent parameter giving the amount of kinetic mixing between matter and the scalar field, which generalizes the one introduced in [47] (and to which it reduces in the limit  $\alpha_{Y,m} = \beta_1 = 0$ ). Moreover, in equation (D.24),  $\hat{c}_s^2$  is defined as

$$\hat{c}_s^2 \equiv \frac{\alpha}{\alpha + 3\hat{\Omega}_m \alpha_{D,m}^{\text{eff}}} c_s^2 - \frac{3[1 + w_m(1 + \alpha_{D,m})] \hat{\Omega}_m}{\alpha + 3\hat{\Omega}_m \alpha_{D,m}^{\text{eff}}} [1 + \alpha_H + \beta_1(1 + \alpha_T)]^2, \tag{D.25}$$

where  $c_s^2$  is the sound speed in the absence of matter, defined in eq. (4.16).

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