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Sheaves and D-modules on Lorentzian manifolds

Benoît Jubin and Pierre Schapira*†

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Abstract

We introduce a class of causal manifolds which contains the globally hyperbolic spacetimes and we prove global propagation theorems for sheaves on such manifolds. As an application, we solve globally the Cauchy problem for hyperfunction solutions of hyperbolic systems.

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Introduction

A causal manifold (M, γ) is a real smooth manifold M endowed with an everywhere nonempty open convex cone γ in its tangent bundle TM. The main examples of such manifolds are provided by Lorentzian spacetimes, that is, time-oriented Lorentzian manifolds. Lorentzian spacetimes and the properties of their causal preorders are important objects of study in the mathematics of general relativity. A natural problem in this field is to solve *globally* the Cauchy problem for the wave operator or for related operators, with initial data on a Cauchy hypersurface. This problem has been and still is the object of an intense activity in mathematical physics (see for example [Ger70, HE73, BF00, BGP07, MS08, BF11]). It was initiated in the pioneering work of Jean Leray [Ler53].

In **Section 1** of this paper, using the tools of the Whitney normal cone and related notions as in [KS90] (see the appendix), we introduce the notion of a γ-set in M. The family of γ-sets is stable by union and intersection, which allows us to define causal preorders on a causal manifold and in particular the cc-preorder, the finest closed causal preorder. We then define a *Cauchy time function* $q\colon M \to \mathbb{R}$ (see Definition 1.50) as a submersive causal map which is proper on the future and the past (for the cc-preorder) of any point. The cc-preorder on a causal manifold with a Cauchy time function is proper (causal diamonds are compact). By theorems of Geroch and Bernal–Sánchez ([Ger70, BS05]), globally hyperbolic Lorentzian spacetimes may be endowed with Cauchy times functions. However, the situation here is more general: the cone γ need not have a smooth boundary nor be quadratic, and a causal manifold with a Cauchy time function may have causal loops. A triple (M, γ, q) composed of a causal manifold endowed with a Cauchy time function is called here a G-causal manifold, by reference to Geroch.

In **Section 2**, we apply the microlocal theory of sheaves of [KS90] to causal manifolds and obtain propagation results for sheaves on G-causal manifolds. More precisely, denote by T^*M the cotangent bundle to M, by $\gamma^{\circ a} = -\gamma^\circ$ the opposite polar cone of γ, and by $T^*_M M$ the zero-section of T^*M. Choose a field $\mathbf{k}$ and consider an object F of the bounded derived category $\mathrm{D^b}(\mathbf{k}_M)$ of sheaves of $\mathbf{k}$-modules on M. Denote as usual by $\mathrm{SS}(F)$ its microsupport, a closed conic co-isotropic subset of T^*M. We prove here the following results (see Theorem 2.13).

(1) If $\mathrm{SS}(F) \cap \gamma^{\circ a} \subset T^*_M M$, then for any compact set K the restriction morphism

$$\mathrm{R}\Gamma(M; F) \longrightarrow \mathrm{R}\Gamma(M \setminus J^-_{\mathrm{cc}}(K); F) \tag{0.1}$$

is an isomorphism. In other words, any "section" of F on $M \setminus J^-_{\mathrm{cc}}(K)$ extends uniquely to M.

(2) If $\mathrm{SS}(F) \cap (\gamma^\circ \cup \gamma^{\circ a}) \subset T^*_M M$, then setting $L = q^{-1}(0)$, the restriction morphism

$$\mathrm{R}\Gamma(M; F) \longrightarrow \mathrm{R}\Gamma(L; F|_L) \tag{0.2}$$

is an isomorphism. In other words, any "section" of F defined in a neighborhood of a Cauchy hypersurface L extends uniquely to the whole of M. In fact, we prove a more general result when replacing the Cauchy time function $q\colon M \to \mathbb{R}$ with a submersive morphism $f\colon M \to N$, the manifold N being contractible (see Corollary 2.11).

Both isomorphisms (0.1) and (0.2) are easy consequences of Theorem 2.9 below which gives a bound to the microsupport of direct images in a non-proper situation.

In **Section 3**, we apply the preceding results to the case where M is real analytic and F is the complex of hyperfunction (or analytic) solutions of a hyperbolic system $\mathscr{M}$. In [Sch13] (based on [KS90]), it is explained how the microlocal theory of sheaves allows one to solve the Cauchy problem and to give domains of propagation for the hyperfunction solutions of a linear hyperbolic system. We translate these results here in the case of causal manifolds with Cauchy time functions and show that the Cauchy problem may be solved *globally*. Note that the notion of hyperbolicity used here relies only on the characteristic variety of the system. In case of a linear equation $Pu = v$, it corresponds to "weakly hyperbolic" in the old terminology. Indeed, the framework of hyperfunctions is much more flexible than that of distributions as far as one wants to solve the Cauchy problem or to study analytic propagation. We end this section with several detailed examples: (1) in the case of a product of $\mathbb{R}$ (the "time") with a compact manifold, we give general sufficient conditions on a differential operator for the Cauchy problem to be globally well-posed; (2) we extend these results to the case of a product of $\mathbb{R}$ with a complete Riemannian manifold; (3) we give a propagation result for operators on a product of $\mathbb{C}$ (the "complex time") with a compact manifold; (4) in the general case of a globally hyperbolic Lorentzian spacetime, we solve globally the Cauchy problem for operators of wave type.

Erratum 0.1. *The statement* [DS98, Prop. 4.4 (ii)] *is not correct* (*see Example 2.16*). *Therefore,* [Sch13, Prop. 6.6] *as well as its corollaries are not correct. However, and that is what we shall show in this paper, most of the applications to causal manifolds are correct when assuming the spacetime globally hyperbolic.*

1 Causal manifolds

1.1 Notation and background

The statements of this subsection are all elementary and well-known. We recall them to fix some notations.

Unless otherwise specified, a manifold means a real C^∞-manifold and a morphism of manifolds $f\colon M \to N$ is a map of class C^∞.

Let M be a manifold. For any subset $A \subset M$, we denote by $\overline{A}$ its closure, by $\mathrm{Int}(A)$ its interior and we set $\partial A = \overline{A} \setminus \mathrm{Int}(A)$.

Cones in vector bundles

Recall that in a real finite-dimensional vector space $\mathbb{V}$ a cone is *proper* if its convex hull does not contain any nonzero linear subspace. The dual (see (1.1) below) of a cone is a convex closed cone. The dual of a cone with nonempty interior is proper.

Let $p\colon E \to M$ be a real (finite-dimensional) vector bundle over M. As usual, one denotes by $a\colon E \to E$ the antipodal map, $(x;\xi) \mapsto (x;-\xi)$. A subset γ of E is *conic* (or is a cone) if it is invariant by the action of $\mathbb{R}_{>0}$, that is, $\gamma_x \subset E_x$ is a cone for each

$x \in M$. Here, γ_x is the restriction of γ to the fibre E_x. If γ is closed and conic, then its projection by p on M coincides with its intersection with the zero-section of E and we identify the zero-section of E with M.

Let γ be a cone in E. We denote by γ° the polar (or dual) cone in E^* and by γ^a the image of γ by the antipodal map a:

$$\gamma^\circ = \{(x;\xi) \in E^*; \langle \xi, v\rangle \geq 0 \text{ for all } v \in \gamma_x\}, \tag{1.1}$$

$$\gamma^a = a(\gamma). \tag{1.2}$$

For a cone γ in a vector bundle E, one has to distinguish between its closure $\overline{\gamma}$ and its pointwise closure $\mathrm{cl}_{\mathrm{pw}}(\gamma)$. Similarly, one has to distinguish between its interior $\mathrm{Int}(\gamma)$ and its pointwise interior $\mathrm{Int}_{\mathrm{pw}}(\gamma)$. One has

$$\mathrm{Int}(\gamma) \subset \mathrm{Int}_{\mathrm{pw}}(\gamma) \subset \gamma \subset \mathrm{cl}_{\mathrm{pw}}(\gamma) \subset \overline{\gamma}$$

and all inclusions may be strict.

Example 1.1. Consider two nonempty open convex cones $\gamma_0^1 \subsetneq \gamma_0^2$ in a vector space $\mathbb{V}$ and two nonempty open subsets $U \subsetneq V \subset \mathbb{V}$ and define

$$\gamma = V \times \gamma_0^1 \cup U \times \gamma_0^2.$$

Then, for $x \in V \cap \partial U$, one has $\overline{\gamma_x} = \overline{\gamma_0^1}$ and $\overline{\gamma}_x = \overline{\gamma_0^2}$. Therefore, $\gamma^{\circ\circ} = \mathrm{cl}_{\mathrm{pw}}(\gamma)$ is not closed.

Lemma 1.2. *Let $p\colon E \to M$ be a real finite-dimensional vector bundle over M and let γ be an open convex cone. Then γ° is a closed convex cone, $\gamma = \mathrm{Int}(\mathrm{cl}_{\mathrm{pw}}(\gamma)) = \mathrm{Int}(\gamma^{\circ\circ})$.*

Proof. (i) First, γ° being a polar cone, it is convex. We shall prove that it is closed. Let $(x;\xi) \in \overline{\gamma^\circ}$. If $\gamma_x = \varnothing$, then $(x;\xi) \in \gamma^\circ$. Now assume $\gamma_x \neq \varnothing$. We choose a local chart U in a neighborhood of x so that $E|_U = U \times \mathbb{V}$. There exists a sequence $(x_n, \xi_n)_n \xrightarrow{n} (x,\xi)$ with $(x_n,\xi_n) \in \gamma^\circ$. Let $v \in \gamma_x$. Since γ is open, there exists an open cone $\theta \subset \mathbb{V}$ such that $(x,v) \in U \times \theta \subset \gamma$. Then for all n large enough, $x_n \in U$ and $\langle \xi_n, v\rangle \geq 0$. Therefore, $\langle \xi, v\rangle \geq 0$ and $(x;\xi) \in \gamma^\circ$.
(ii) Since $\gamma \subset \mathrm{cl}_{\mathrm{pw}}(\gamma)$ and γ is open, $\gamma \subset \mathrm{Int}(\mathrm{cl}_{\mathrm{pw}}(\gamma))$. On the other hand, $\mathrm{Int}(\mathrm{cl}_{\mathrm{pw}}(\gamma)) \subset \mathrm{Int}_{\mathrm{pw}}(\mathrm{cl}_{\mathrm{pw}}(\gamma)) = \gamma$ (recall that in a vector space, an open convex set is equal to the interior of its closure). Therefore, $\gamma = \mathrm{Int}(\mathrm{cl}_{\mathrm{pw}}(\gamma))$.
(iii) The set $U = \{x \in M; \gamma_x \neq \varnothing\}$ is open, and $\gamma^{\circ\circ}|_U = \mathrm{cl}_{\mathrm{pw}}(\gamma)|_U$ while $\gamma^{\circ\circ}|_{M\setminus U} = M \setminus U$ and $\mathrm{cl}_{\mathrm{pw}}(\gamma)|_{M\setminus U} = \varnothing$. Therefore, $\mathrm{Int}(\gamma^{\circ\circ}) = \mathrm{Int}(\mathrm{cl}_{\mathrm{pw}}(\gamma))$. Q.E.D.

Cotangent bundles

Let M be a manifold. We denote by $\tau\colon TM \to M$ and by $\pi\colon T^*M \to M$ its tangent and cotangent bundle, respectively.

For a submanifold N of M, we denote by $T_N M = (N \times_M TM)/TN$ the normal bundle of N in M and by $T_N^* M$ its dual, the conormal bundle of N in M. In particular, $T_M^* M$ is the zero-section.

For two manifolds M and N we denote by q_1 and q_2 the first and second projections defined on $M \times N$. We denote by Δ_M, or simply Δ, the diagonal of $M \times M$.

Let M_i $(i = 1, 2, 3)$ be manifolds. For short, we write $M_{ij} := M_i \times M_j$ $(1 \leq i, j \leq 3)$ and $M_{123} = M_1 \times M_2 \times M_3$. We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. For $A_1 \subset M_{12}$ and $A_2 \subset M_{23}$, one sets

$$A_1 \underset{2}{\circ} A_2 = q_{13}(q_{12}^{-1} A_1 \cap q_{23}^{-1} A_2).$$

Similarly, we denote by p_i the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by p_{ij} the projection $T^*M_{123} \to T^*M_{ij}$. We also need to introduce the map p_{ij^a}, the composition of p_{ij} and the antipodal map on T^*M_j and similarly with p_{i^aj}.

Let $\Lambda_1 \subset T^*M_{12}$ and $\Lambda_2 \subset T^*M_{23}$. We set

$$\Lambda_1 \overset{a}{\underset{2}{\circ}} \Lambda_2 := p_{13}(p_{12^a}^{-1} \Lambda_1 \cap p_{23}^{-1} \Lambda_2) = p_{13}(p_{12}^{-1} \Lambda_1 \cap p_{2^a3}^{-1} \Lambda_2). \tag{1.3}$$

To a morphism $f \colon M \to N$ of manifolds one associates the maps

$$\begin{tikzcd} TM \arrow[r, "f'"] \arrow[dr, "\tau"'] & M \times_N TN \arrow[r, "f_\tau"] \arrow[d, "\tau"] & TN \arrow[d, "\tau"] \\ & M \arrow[r, "f"] & N, \end{tikzcd} \qquad \begin{tikzcd} T^*M \arrow[dr, "\pi"'] & M \times_N T^*N \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*N \arrow[d, "\pi"] \\ & M \arrow[r, "f"] & N. \end{tikzcd} \tag{1.4}$$

We set

$$Tf := f_\tau \circ f' \colon TM \to TN, \tag{1.5}$$

and call Tf the tangent map to f.

We denote by Γ_f the graph of f. Then, after identifying $T\Gamma_f$ with its image in $T(M \times N)$, we have $T\Gamma_f = \Gamma_{Tf}$. We set

$$\Lambda_f := T^*_{\Gamma_f}(M \times N) = (T\Gamma_f)^\perp. \tag{1.6}$$

Then we have a commutative diagram in which p_1 and p_2 are induced by the projections $T^*(M \times N)$ to T^*M and T^*N and p_2^a is the composition of p_2 and the antipodal map of T^*N:

$$\begin{tikzcd} & \Lambda_f \arrow[dl, "p_1"'] \arrow[d, "\sim"] \arrow[dr, "p_2^a"] & \\ T^*M & M \times_N T^*N \arrow[l, "f_d"'] \arrow[r, "f_\pi"] & T^*N. \end{tikzcd} \tag{1.7}$$

We also set

$$T^*_M N := f_d^{-1} T^*_M M \simeq (T^*_M M \times T^*N) \cap \Lambda_f. \tag{1.8}$$

and call $T^*_M N$ the conormal bundle to M in N.

Quadratic forms

Let $\mathbb{V}$ be a real finite dimensional vector space, $\mathbb{V}^{\mathbb{C}}$ its complexification. Let Q be a quadratic form on $\mathbb{V}$. We keep the same notation Q to denote the quadratic form defined on $\mathbb{V}^{\mathbb{C}}$. We set $Q_{>0} := \{v \in \mathbb{V}; Q(v) > 0\}$ and similarly with $Q_{\geq 0}$. We denote by $\langle \cdot, \cdot \rangle_Q$ the bilinear form associated with Q.

Lemma 1.3. *Assume that Q has exactly one positive eigenvalue on $\mathbb{V}$.*

(a) *If $u, v \in \mathbb{V}$ and $Q(u) > 0$, then $Q(u + \sqrt{-1}v) \neq 0$.*

(b) *One has $\overline{Q_{>0}} = Q_{\geq 0}$.*

(c) *Let γ be a connected component of $Q_{>0}$. Then $\overline{\gamma} = \{v \in \mathbb{V}; \langle v, w \rangle_Q \geq 0$ for all $w \in \gamma\}$.*

The proof is left as an exercise.

Now let Q be a quadratic form on a manifold M and let $\langle \cdot\,, \cdot \rangle_Q$ be the associated bilinear form on TM. If Q is nondegenerate, it induces an isomorphism $\sharp\colon TM \to T^*M, v \mapsto \langle v, \cdot \rangle_Q$, with inverse denoted here by $\flat$. Therefore, there is an induced quadratic form, denoted by $Q^\sharp_x$, on each $T^*_x M$. As usual, we shall write $v^\sharp = \sharp(v)$ for $v \in TM$. We set $Q_{>0} := \{v \in TM; Q(v) > 0\}$ and similarly for $Q_{\geq 0}$.

Lemma 1.4. *Let M be a connected manifold and Q a continuous quadratic form on M with exactly one positive eigenvalue. Then*

(a) *$Q_{>0}$ has at most two connected components and $\mathrm{cl}_{\mathrm{pw}}(Q_{>0}) = \overline{Q_{>0}} = Q_{\geq 0}$.*

(b) *Suppose that $Q_{>0}$ has two connected components and let γ be one of them. Then*

  (i) *γ is an open convex proper cone and $\gamma_x \neq \varnothing$ for any $x \in M$,*

  (ii) *One has $\overline{\gamma} = \mathrm{cl}_{\mathrm{pw}}(\gamma)$ and $\gamma = \mathrm{Int}(\overline{\gamma})$.*

  (iii) *Suppose furthermore that Q is nondegenerate. Then $\gamma^\circ = \overline{\gamma}^\sharp$ and $\mathrm{Int}(\gamma^\circ) = \gamma^\sharp$.*

The proof is left as an exercise.

Preorders

Consider a preorder $\preceq$ on a manifold M and its graph $\Delta_\preceq \subset M \times M$. Then

$$\begin{aligned} &\Delta \subset \Delta_\preceq\,, \\ &\Delta_\preceq \circ \Delta_\preceq = \Delta_\preceq\,. \end{aligned}$$

In the sequel, we shall often identify $\preceq$ and $\Delta_\preceq$, that is, we shall call $\Delta_\preceq$ "a preorder". We denote by $\preceq^a$ the opposite preorder.

For a subset $A \subset M$, one sets

$$\begin{cases} J^-_\preceq(A) = q_1(q_2^{-1}(A) \cap \Delta_\preceq) = \{x \in M;\ \text{there exists } y \in A \text{ with } x \preceq y\}, \\ J^+_\preceq(A) = q_2(q_1^{-1}(A) \cap \Delta_\preceq) = \{x \in M;\ \text{there exists } y \in A \text{ with } y \preceq x\}. \end{cases} \tag{1.9}$$

For $x \in M$, we write $J^+_{\preceq}(x)$ and $J^-_{\preceq}(x)$ instead of $J^+_{\preceq}(\{x\})$ and $J^-_{\preceq}(\{x\})$ respectively. One calls $J^-_{\preceq}(A)$ (resp. $J^+_{\preceq}(A)$) the past (resp. future) of A for the preorder $\preceq$.

The next results are obvious:

- $J^-_{\preceq}(A) = \bigcup_{x\in A} J^-_{\preceq}(x)$, and similarly with $J^+_{\preceq}(A)$,
- $A \subset J^-_{\preceq}(A)$, $J^-_{\preceq}(J^-_{\preceq}(A)) = J^-_{\preceq}(A)$ and similarly with $J^+_{\preceq}(A)$,
- $\Delta_{\preceq^a} = v(\Delta_{\preceq})$ where $v\colon M \times M \to M \times M$ is the map $v(x, y) = (y, x)$,
- $J^-_{\preceq}(A) = J^+_{\preceq^a}(A)$,
- $A = J^+_{\preceq}(A) \Leftrightarrow M \setminus A = J^-_{\preceq}(M \setminus A)$. (Indeed, assume $A = J^+_{\preceq}(A)$ and let $x \notin A$, $y \in J^-_{\gamma}(x)$. Then $x \in J^+_{\preceq}(y)$ which shows that $y \notin A$.)

Definition 1.5. Let $\preceq$ be a preorder on M.

(a) The preorder is *closed* if $\Delta_{\preceq}$ is closed in $M \times M$.

(b) The preorder is *proper* if q_{13} is proper on $\Delta_{\preceq} \times_M \Delta_{\preceq}$.

In other words, a preorder $\preceq$ is proper if for any two compact subsets A and B of M, the so-called *causal diamond* $J^+_{\preceq}(A) \cap J^-_{\preceq}(B)$ is compact.

Proposition 1.6. *Let $\preceq$ be a preorder on M.*

(i) *If $\preceq$ is closed and A is a compact subset of M, then $J^-_{\preceq}(A)$ and $J^+_{\preceq}(A)$ are closed.*

(ii) *If $\preceq$ is proper, then it is closed.*

Proof. (i) One has $J^+_{\preceq}(A) = q_2(\Delta_{\preceq} \cap q_1^{-1}(A)) = q_2(\Delta_{\preceq} \cap (A \times M))$. The projection q_2 is proper on $A \times M$, hence closed, and $\Delta_{\preceq} \cap (A \times M)$ is closed, therefore $J^+_{\preceq}(A)$ is closed. The proof for $J^-_{\preceq}(A)$ is similar.
(ii) If the map $q_{13}\colon \Delta_{\preceq} \times_M \Delta_{\preceq} \to M \times M$ is proper, then it is closed. Therefore, its image $\Delta_{\preceq}$ is closed. Q.E.D.

1.2 Causal manifolds

In the literature, one often encounters time-orientable Lorentzian manifolds, to which one can associate a cone in TM (see Definition 1.17). Here, we only assume to be given a nowhere empty open convex cone $\gamma \subset TM$.

Recall that for a morphism of manifolds $f\colon M \to N$, the tangent map Tf is defined in (1.5).

Definition 1.7. (a) A *causal manifold* (M, γ) is a nonempty connected manifold M equipped with an open convex cone $\gamma \subset TM$ such that $\gamma_x \neq \varnothing$ for all $x \in M$.

(b) A *morphism of causal manifolds* $f\colon (M, \gamma_M) \to (N, \gamma_N)$ is a morphism of manifolds such that $Tf(\overline{\gamma_M}) \subset \overline{\gamma_N}$.

(c) A morphism of causal manifolds f is *strict* if $Tf(\gamma_M) \subset \gamma_N$.

The composition of causal morphisms (resp. strictly causal) morphisms is a causal (resp. strictly causal) morphism, so that causal manifolds and their causal (resp. strictly causal) morphisms form a category.

For U a open subset of M, we denote by γ_U the cone $U \times_M \gamma$ of TU. Then (U, γ_U) is a causal manifold and the embedding $U \hookrightarrow M$ induces a morphism of causal manifolds $(U, \gamma_U) \to (M, \gamma)$.

Notation 1.8. We will often denote by I an open interval of $\mathbb{R}$, which we will implicitly assume to contain $[0, 1]$. We denote by t a coordinate on I, by $(t; v)$ the coordinates on TI and by $(t; \tau)$ the associated coordinates on T^*I. To I we associate the causal manifold $(I, I \times \mathbb{R}_{>0})$ that we simply denote by $(I, +)$.

Proposition 1.9. *If $f\colon (M, \gamma_M) \to (N, \gamma_N)$ is a causal submersion and if $\mathrm{Int}\,(\overline{\gamma_N}) = \gamma_N$, then f is strictly causal. In particular, if $q\colon (M, \gamma) \to (I, +)$ is a submersive causal morphism, then q is strictly causal.*

Proof. Since f is submersive, then Tf is open, so $Tf(\gamma_M) \subset \mathrm{Int}\,(\overline{\gamma_N}) = \gamma_N$. Q.E.D.

Notation 1.10. In this paper, for a causal manifold (M, γ) we set

$$\lambda := \gamma^\circ \tag{1.10}$$

and similarly with λ_M, λ_N, etc. Note that, by Lemma 1.2, λ is a closed convex proper cone of T^*M and $\gamma = \mathrm{Int}(\lambda^\circ)$. Moreover, $\lambda \supset T^*_M M$.

Remark 1.11. One has $\overline{(\gamma_x)} = \lambda_x^\circ \subset (\overline{\gamma})_x$ but as seen in Example 1.1, the inclusion may be strict.

Proposition 1.12. *Let (M, γ_M) and (N, γ_N) be two causal manifolds and let $f\colon M \to N$ be a morphism of manifolds. Then f is a morphism of causal manifolds if and only if $\Lambda_f \overset{a}{\circ} \lambda_N \subset \lambda_M$.*

Proof. Notice first that $Tf(\overline{\gamma_M}) \subset \overline{\gamma_N}$ if and only if $\overline{\gamma_M} \circ \Gamma_{Tf} \subset \overline{\gamma_N}$. At each $(x, y) \in \Gamma_f$, the vector subspace $(\Lambda_f)_{(x,y)} \subset T^*_{(x,y)}(M \times N)$ is the orthogonal to the vector subspace $(\Gamma_{Tf})_{(x,y)} \subset T_{(x,y)}(M \times N)$. Hence, setting $E_1 = T_xM$, $E_2 = T_yN$, $\Gamma = (\Gamma_{Tf})_{(x,y)}$, we are reduced to prove that for two real finite-dimensional vector spaces E_1 and E_2, two closed convex cones $\gamma_1 \subset E_1$, $\gamma_2 \subset E_2$ and a linear graph $\Gamma \subset E_1 \times E_2$ one has

$$\gamma_1 \circ \Gamma \subset \gamma_2 \Leftrightarrow \Gamma^\perp \overset{a}{\circ} \gamma_2^\circ \subset \gamma_1^\circ.$$

By hypothesis, Γ is the graph of a linear map $u\colon E_1 \to E_2$. Therefore $\Gamma^\perp$ is the graph of the opposite transposed map $-{}^t u\colon E_2^* \to E_1^*$ and the result is clear since

$$u(\gamma_1) \subset \gamma_2^{\circ\circ} \Leftrightarrow {}^t u(\gamma_2^\circ) \subset \gamma_1^\circ$$

and $\gamma_2 = \gamma_2^{\circ\circ}$ (since γ_2 is closed and convex). Q.E.D.

Definition 1.13. (i) A *constant cone* contained in γ is a triple (φ, U, θ) where $\varphi\colon U \to \mathbb{R}^d$ is a chart and $\theta \subset \mathbb{R}^d$ is an open convex cone, such that in this chart, $U \times \theta \subset \gamma$ (that is, $\varphi(U) \times \theta \subset T\varphi(\gamma|_U)$). A constant cone (φ, U, θ) will often be denoted simply by $U \times \theta$.
(ii) A *basis of constant cones* contained in γ is a family of constant cones whose union is γ.

Although they are obvious, we state the two next lemmas which will be of frequent use.

Lemma 1.14. *Let (M, λ) be a causal manifold. Then there exists a basis of constant cones contained in γ.*

Proof. It follows immediately from the fact that $\gamma \subset TM$ is open. Q.E.D.

Lemma 1.15. *Let (M, λ) be a causal manifold and let $A \subset M$. Then $\gamma \subset N(A)$ if and only if there exists a basis of constant cones contained in γ such that for $U \times \theta$ belonging to this basis, $U \cap (U \cap A + \theta) \subset A$.*

Proof. This is a reformulation of (A.2). Q.E.D.

Example 1.16. For the classical notions of Lorentzian manifold, spacetime and globally hyperbolic spacetime, references are made to [BGP07, BEE96, HE73, MS08].

A *Lorentzian manifold* (M, g) is a *connected* C^∞-manifold M with a C^∞ nondegenerate bilinear form g on M of signature $(+, -, \dots, -)$. Let

$$g_{>0} = \{(x; v) \in TM; g_x(v, v) > 0\}.$$

By Lemma 1.4, $g_{>0}$ has at most two connected components. The Lorentzian manifold (M, g) is *time-orientable* if the cone $g_{>0}$ has itself two connected components. It is *time-oriented* if furthermore one connected component has been chosen.

Definition 1.17. A *Lorentzian spacetime* is a connected time-oriented Lorentzian manifold.

Let (M, g) be a Lorentzian spacetime. We denote by (M, γ_g), or (M, γ) if there is no risk of confusion, the associated causal manifold.

1.3 γ-sets and γ-topology

The definition of the normal cone $N(A)$ to a subset A as well as of the cone $D(A)$ and their main properties are recalled in the appendix. Recall that $N(A)$ is an open convex cone.

Definition 1.18. Let (M, γ) be a causal manifold. A subset $A \subset M$ is a *γ-set* if $\gamma \subset N(A)$.

Applying Lemma 1.15, we get:

$$A \text{ is a } \gamma\text{-set} \Leftrightarrow \begin{cases} \text{there exists a basis of constant cones } U \times \theta \text{ con-} \\ \text{tained in } \gamma \text{ such that } U \cap (U \cap A + \theta) \subset A. \end{cases} \tag{1.11}$$

Proposition 1.19. *Let (M, γ) be a causal manifold.*

(i) *A set A is a γ-set if and only if $\gamma_x \subset N_x(A)$ for all $x \in \partial A$.*

(ii) *A subset A of M is a γ-set if and only if $M \setminus A$ is a γ^a-set.*

(iii) *Let (M_i, γ_i) $(i = 1, 2)$ be causal manifolds. Assume that A_i is a γ_i-set for $i = 1, 2$. Then $A_1 \times A_2$ is a $(\gamma_1 \times \gamma_2)$-set.*

(iv) *Let $\gamma_1 \subset \gamma_2$ be two open convex cones in TM. If a set A is a γ_2-set, then it is a γ_1-set.*

Proof. (i) This follows from Proposition A.3 (iv).
(ii) This follows from Propositon A.3 (ii).
(iii) This follows immediately from Proposition A.3 (x).
(iv) Indeed, $\gamma_2 \subset N(A)$ implies $\gamma_1 \subset N(A)$. Q.E.D.

Remark 1.20. The converse to Proposition 1.21 (iii) is true (see the remark following Proposition A.3) but we do not need it.

Proposition 1.21. *Let (M, γ) be a causal manifold.*

(i) *The family of γ-sets is closed under arbitrary unions and intersections.*

(ii) *The family of γ-sets is closed under taking closure and interior.*

(iii) *If A is a γ-set, then $\overline{\mathrm{Int} A} = \overline{A}$ and $\mathrm{Int}\overline{A} = \mathrm{Int} A$.*

(iv) *If A is a γ-set and $\mathrm{Int} A \subset B \subset \overline{A}$, then B is a γ-set.*

Proof. We shall use Notation (1.10).
(i) Let $\{A_i\}_{i \in I}$ be a family of γ-sets, let $A = \bigcup_{i \in I} A_i$ and let $(x, v) \in D(A)$. We shall prove that $(x, v) \notin \gamma$.

We choose a chart at x. There is a sequence $\{(x_n, y_n, c_n)\}_n$ in $A \times (M \setminus A) \times \mathbb{R}_{>0}$ such that $x_n, y_n \xrightarrow{n} x$ and $v_n = c_n(y_n - x_n) \xrightarrow{n} v$. Choose a function $\rho \colon \mathbb{N} \to I$ such that $x_n \in A_{\rho(n)}$. Define

$$\begin{aligned} t_n &= \sup\{t \in [0, 1] \mid [x_n, x_n + t(y_n - x_n)] \subset A_{\rho(n)}\}, \\ z_n &= x_n + t_n(y_n - x_n), \\ x_{m,n} &= \begin{cases} x_n + (t_n - 1/m)(y_n - x_n) \text{ (for } m > 1/t_n) & \text{if } t_n \neq 0, \\ x_n & \text{if } t_n = 0. \end{cases} \end{aligned}$$

Then $x_{m,n} \in A_{\rho(n)}$.

For all $n \in \mathbb{N}$, since t_n is a supremum, there exists a sequence $\{\delta_{m,n}\}_m$ in $\mathbb{R}_{\geq 0}$ with $\delta_{m,n} \xrightarrow{m} 0$ such that

$$y_{m,n} = x_n + (t_n + \delta_{m,n})(y_n - x_n) \notin A_{\rho(n)}.$$

We also have $x_{m,n}, y_{m,n} \xrightarrow{m} z_n$.

If $t_n = 0$, then $\delta_{m,n} > 0$, so we can define

$$c_{m,n} = \begin{cases} c_n/(1/m + \delta_{m,n}) & \text{if } t_n \neq 0, \\ c_n/\delta_{m,n} & \text{if } t_n = 0. \end{cases}$$

Then $c_{m,n} > 0$ and $c_{m,n}(y_{m,n} - x_{m,n}) = v_n$. This proves that $(z_n, v_n) \in D(A_{\rho(n)}) \subset TM \setminus \gamma$. Since γ is open and $(z_n, v_n) \xrightarrow{n} (x, v)$, this implies that $(x, v) \notin \gamma$. Therefore A is a γ-set.

The case of an intersection is deduced from (i) by Proposition 1.19 (ii).

(ii) follows immediately from Proposition A.3 (vi).

(iii) follows immediately from Proposition A.3 (vii).

(iv) The hypothesis and (iii) imply that $\mathrm{Int}B = \mathrm{Int}A$ and $\overline{B} = \overline{A}$. It then follows from Proposition A.3 (viii) that $N(A) = N(B)$ and B is a γ-set. Q.E.D.

γ-sets in vector spaces

In vector spaces endowed with constant cones, γ-sets are easy to characterize.

Let $\mathbb{V}$ be a real finite-dimensional vector space, let Ω be a nonempty convex open subset of $\mathbb{V}$ and let γ_0 be an open convex cone in $\mathbb{V}$. Set $\gamma = \Omega \times \gamma_0 \subset T\Omega$, so that (Ω, γ) is a causal manifold.

Proposition 1.22. *A subset A of $\Omega \subset \mathbb{V}$ is a γ-set if and only if*

$$\Omega \cap (A + \gamma_0) \subset A. \tag{1.12}$$

Proof. (i) Suppose that A satisfies (1.12). Then, for each open subset $W \subset \Omega$, we get

$$W \cap (A \cap W + \gamma_0) \subset A.$$

It follows from (1.11) that A is a γ-set.

(ii) Conversely, let $A \subset \Omega$ be a γ-set. Let $x \in A$ and let $v \in \gamma_0$ with $x + v \in \Omega$. We shall prove that $x + v \in A$. Define $t_\infty = \sup\{t \in [0,1]; [x, x+tv] \subset A\}$ and let $x_\infty = x + t_\infty v$. Then $x_\infty \in \Omega$ and $(x_\infty, v) \in \gamma \subset N(A)$. Therefore, there is a neighborhood $W \subset \Omega$ of x_∞ such that $W \cap (A \cap W + \mathbb{R}_{>0} v) \subset A$. Since W is a neighborhood of x_∞, there exists $\eta > 0$ such that $[x_\infty, x_\infty + \eta v] \subset A$, which is a contradiction unless $t_\infty = 1$. Q.E.D.

γ-topology

Proposition 1.21 allows us to generalize [KS90, Def. 3.5.1].

Definition 1.23. Let (M, γ) be a causal manifold. The *γ-topology* on M is the topology for which the open sets are the open sets of M which are γ-sets.

A subset $A \subset M$ is called γ-open if it is open for the γ-topology. In other words, if it is open in the usual topology and is a γ-set.

Remark 1.24. We shall not use the term γ-closed since a set which is closed for the γ-topology is not in general a γ-set, but is a γ^a-set.

As in [KS90, Def. 3.5.1], define the γ_0-topology on Ω by saying that an open set U of $\mathbb{V}$ is γ_0-open if

$$U = \Omega \cap (U + \gamma_0). \tag{1.13}$$

Applying Proposition 1.22, we get that the γ-topology and the γ_0-topology on Ω coincide.

1.4 The chronological preorder

Definition 1.25. For $A \subset M$, we denote by $I^+_\gamma(A)$ the intersection of all the γ-sets which contain A and call it the *chronological future* of A. We set $I^+_\gamma(x) = I^+_\gamma(\{x\})$.

Note that a set A is a γ-set if and only if $I^+_\gamma(A) = A$.

Lemma 1.26. *The relation $y \in I^+_\gamma(x)$ is a preorder.*

Proof. Let $y \in I^+_\gamma(x)$ and $z \in I^+_\gamma(y)$. Then $I^+_\gamma(x)$ is a γ-set which contains y and $I^+_\gamma(y)$ is the smallest γ-set which contains y. Therefore, $I^+_\gamma(y) \subset I^+_\gamma(x)$ and $z \in I^+_\gamma(x)$. Q.E.D.

Notation 1.27. We denote by $\preceq_\gamma$ the preorder given by $x \preceq_\gamma y$ if $y \in I^+_\gamma(x)$ and we denote by Δ_γ the graph of this preorder. Hence, using the notations (1.9), $I^+_\gamma(x) = J^+_{\preceq_\gamma}(x)$ and $\Delta_\gamma = \Delta_{\preceq_\gamma}$. We call $\preceq_\gamma$ the *chronological preorder*.

Remark 1.28. Recall Notation 1.8. On $(I, +)$ the chronological preorder $\preceq_\gamma$ is the usual order $\leq$.

Proposition 1.29. *Let $A \subset M$ be a closed subset. Then $I^+_\gamma(A) \setminus A$ is open.*

Proof. One has $\mathrm{Int}(I^+_\gamma(A)) \subset \mathrm{Int}(I^+_\gamma(A)) \cup A \subset I^+_\gamma(A)$. Applying Proposition 1.21 (iv), we get that $\mathrm{Int}(I^+_\gamma(A)) \cup A$ is a γ-set. Since it contains A, it contains $I^+_\gamma(A)$. Therefore $\mathrm{Int}(I^+_\gamma(A)) \cup A = I^+_\gamma(A)$ and $I^+_\gamma(A) \setminus A = \mathrm{Int}(I^+_\gamma(A)) \setminus A$ is open. Q.E.D.

Lemma 1.30. *Let (M, γ) be a causal manifold and consider a constant cone $U \times \theta$ contained in γ. Then, for $y, z \in U$ with $z - y \in \theta$, we have $z \in I^+_\gamma(y)$.*

Proof. Set $\gamma_1 = U \times \theta$. Then $z \in I^+_{\gamma_1}(y)$ by Proposition 1.22 and $I^+_{\gamma_1}(y) \subset U \cap I^+_\gamma(y)$ by Proposition 1.19 (iv) (applied with $M = U$). Q.E.D.

1.5 Causal paths

Notation 1.31. If a function $c\colon I \to M$ is left (resp. right) differentiable, we denote its left (resp. right) derivative by c'_l (resp. c'_r).

Definition 1.32. A path $c\colon I \to M$ is a piecewise smooth map. A path c is *causal* if $c'_l(t), c'_r(t) \in (\overline{\gamma})_{c(t)}$ for any $t \in I$ and it is *strictly causal* if $c'_l(t), c'_r(t) \in \gamma_{c(t)}$ for any $t \in I$.

Hence, a smooth path c is causal (resp. strictly causal) if and only if it defines a morphism (resp. strict morphism) of causal manifolds $c\colon (I,+) \to (M,\gamma)$.

Note that if c_1 and c_2 are two causal (resp. strictly causal) paths on I with $c_1(1) = c_2(0)$, the concatenation $c = c_1 \cup c_2$ (defined by glueing the two paths as usual) is causal (resp. strictly causal).

Lemma 1.33. *Let $f\colon (M,\gamma_M) \to (N,\gamma_N)$ be a morphism (resp. a strict morphism) of causal manifolds and let $c\colon I \to M$ be a causal path (resp. a strictly causal path). Then $f \circ c\colon I \to N$ is a causal path (resp. a strictly causal path).*

Proof. This is a direct consequence of the chain rule. Q.E.D.

Lemma 1.34. *Let $c\colon I \to M$ be a strictly causal path. Then for $t_1 \le t_2$ with $t_1, t_2 \in I$ we have $c(t_1) \preceq_\gamma c(t_2)$.*

Proof. It is enough to prove that for any $t_0 \in I$, there exists $\alpha > 0$ such that $c(t_0) \preceq_\gamma c(t)$ for $t \in (t_0, t_0+\alpha)$ and similarly $c(t) \preceq_\gamma c(t_0)$ for $t \in (t_0-\alpha, t_0)$. We may assume $t_0 = 0$. There exists a constant cone $U \times \theta$ contained in γ and containing $(c(0), c_r'(0)$. There exists $\alpha > 0$ such that $c(t) - c(0) \in \theta$ for $t \in (0,\alpha)$. By Lemma 1.30, this implies $c(0) \preceq_\gamma c(t)$ for $t \in (0,\alpha)$. The other case is similar, using $c_l'(0)$. Q.E.D.

Lemma 1.35. *Let $A \subset M$. One has $y \in I_\gamma^+(A)$ if and only if $y \in A$ or there exists a strictly causal path $c\colon I \to M$ such that $c(0) \in A$, $c(1) = y$.*

Proof. Let B be the union of A with the set of points that can be reached from A by a strictly causal path. We shall prove that $I_\gamma^+(A) = B$.

(i) To prove that $B \supset I_\gamma^+(A)$, it is enough to check that B is a γ-set. Choose a constant cone $U \times \theta$ contained in γ with U convex. By (1.11), it is enough to prove that $U \cap (B \cap U + \theta) \subset B$. Let $y' \in B \cap U$ and $v' \in \theta$ with $y' + v' \in U$. Since U is convex, $c\colon I \to U, t \mapsto y' + tv'$ is a strictly causal path for I a small enough neighborhood of $[0,1]$. Since $y' \in B$, there exists a strictly causal path $\tilde{c}$ with $\tilde{c}(0) \in A$ and $\tilde{c}(1) = y'$. Therefore, concatenating $\tilde{c}$ and c proves that $y' + v' \in B$.

(ii) Let us prove that $B \subset I_\gamma^+(A)$. Let $y \in B, y \notin A$. There exist $x \in A$ and a strictly causal curve c going from x to y. Then $y \in I_\gamma^+(x)$ by Lemma 1.34. Hence, $y \in I_\gamma^+(A)$. Q.E.D.

Remark 1.36. Using Lemma 1.35, we obtain an alternate proof that the family of γ-sets is closed under unions and intersections. Indeed, the proof of Lemma 1.35 says that for any $A \subset M$, there is a smallest γ-set containing A, and it is the union of A and the set of points that can be reached from A by a strictly causal path. Taking this as the definition of $I_\gamma^+(A)$, for any set $A \subset M$, one has $I_\gamma^+(A) = A$ if and only if A is a γ-set. Now, let $(A_i)_{i\in I}$ be a family of γ-sets and let $A = \bigcap_i A_i$. Then $A = \bigcap_i I_\gamma^+(A_i) = \bigcap_i \bigcup_{x\in A_i} I_\gamma^+(x) = \bigcup_{x\in\bigcap_i A_i} I_\gamma^+(x) = \bigcup_{x\in\bigcap_i A_i} I_\gamma^+(x) = I_\gamma^+(A)$, so A is a γ-set. The proof for unions is similar.

Example 1.37. Let $M = \mathbb{R}^2$ with coordinates (x_1, x_2) and let $(x_1, x_2; v_1, v_2)$ denote the coordinates on TM. Consider the cones

$$\theta_1 = \{(v_1, v_2); v_2 > |v_1|\}, \quad \theta_2 = \{(v_1, v_2); v_2 > \frac{1}{2}|v_1|\}.$$

Let

$$Z = \{(x_1, x_2); x_2 = |x_1|\},$$
$$\gamma = (M \setminus Z) \times \theta_2 \cup Z \times \theta_1.$$

Note that $\mathrm{Int}(\overline{\gamma}) = M \times \theta_2$. One has $I^+_\gamma(0) = \{0\} \cup \{(x_1, x_2); x_2 > |x_1|\}$ and this set is strictly contained in $I^+_{\mathrm{Int}(\overline{\gamma})}(0) = \{0\} \cup \{(x_1, x_2); x_2 > \frac{1}{2}|x_1|\}$. In other words, γ and $\mathrm{Int}(\overline{\gamma})$ define different chronological preorders.

1.6 Causal preorders

Definition 1.38. A preorder $\preceq$ is *causal* if $\Delta_\preceq$ is a $(\gamma^a \times \gamma)$-set.

Lemma 1.39. *Let $\preceq$ be a preorder on M. Assume that for any $x \in M$, $J^+_\preceq(x)$ is a γ-set. Then, for any $y \in M$, $J^-_\preceq(y)$ is a γ^a-set.*

Proof. For any x, $J^+_\preceq(x)$ is a γ-set containing x. Therefore, $I^+_\gamma(x) \subset J^+_\preceq(x)$ and $x \preceq_\gamma y$ implies $x \preceq y$. Let $y \in M$ and let $(x, v) \in \gamma$. By Lemma 1.14, there is a constant cone $U \times \theta \subset \gamma$ containing (x, v). We shall prove that $U \cap (U \cap J^-_\preceq(y) - \theta) \subset J^-_\preceq(y)$. Let $x' \in U$ and $v' \in \theta$ be such that $x' - v' \in U$ and $x' \preceq y$. We have $x' - (x' - v') = v' \in \theta$ so $x' - v' \preceq_\gamma x'$ by Lemma 1.30. Hence, $x' - v' \preceq x'$. By hypothesis, $x' \preceq y$ so by transitivity, $x' - v' \preceq y$, that is, $x' - v' \in J^-_\preceq(y)$. Therefore, $(x, -v) \in N(J^-_\preceq(y))$. Q.E.D.

Lemma 1.40. *Let $\preceq$ be a preorder on M. Assume that for any $x \in M$, $J^+_\preceq(x)$ is a γ-set. Then the preorder $\preceq$ is causal.*

Proof. By the hypothesis, Lemma 1.39 and Proposition 1.19 (iii), $J^-_\preceq(x) \times J^+_\preceq(x)$ is a $(\gamma^a \times \gamma)$-set. Then the result follows from Proposition 1.21 (i) and the equality

$$\Delta_\preceq = \bigcup_{x \in M} J^-_\preceq(x) \times J^+_\preceq(x).$$

Q.E.D.

Lemma 1.41. *One has $\Delta_\gamma = I^+_{\gamma^a \times \gamma}(\Delta)$.*

Proof. (i) Lemma 1.40 implies that Δ_γ is a $(\gamma^a \times \gamma)$-set which proves the inclusion "$\supset$". (ii) For the reverse inclusion, let $(x, y) \in \Delta_\gamma$. By Lemma 1.35, there exists a strictly causal path $c\colon I \to M$ with $c(0) = x$ and $c(1) = y$. The path $\widetilde{c} = (c_1, c_2)\colon I \to M \times M$ defined by $c_1(t) = c(1 - t)$ and $c_2(t) = c(t)$ is a strictly causal path (for the causal structure on $M \times M$ given by $\gamma^a \times \gamma$) with $\widetilde{c}(1/2) = (c(1/2), c(1/2)) \in \Delta$ and $\widetilde{c}(1) = (x, y)$. Therefore, again by Lemma 1.35, $(x, y) \in I^+_{\gamma^a \times \gamma}(\Delta)$. Q.E.D.

Remark 1.42. Since Δ is closed, Lemmas 1.29 and 1.41 imply that $\Delta_\gamma \setminus \Delta$ is open.

Theorem 1.43. *Let (M, γ) be a causal manifold and $\preceq$ be a preorder on M. Then the following assertions are equivalent:*

(i) *The preorder* $\preceq$ *is causal.*

(ii) *For any* $x \in M$, $J^+_{\preceq}(x)$ *is a* γ*-set.*

(iii) *For any* $y \in M$, $J^-_{\preceq}(y)$ *is a* γ^a*-set.*

(iv) *For any* $x \in M$, $I^+_\gamma(x) \subset J^+_{\preceq}(x)$.

(v) *One has* $\Delta_\gamma \subset \Delta_{\preceq}$.

Proof. (ii)⇒(i). It is proved in Lemma 1.40.

(ii)⇔(iii). By Lemma 1.39.

(i)⇒(v). By Lemma 1.41, Δ_γ is the smallest $(\gamma^a \times \gamma)$-set containing the diagonal. By hypothesis, $\Delta_{\preceq}$ is a $\gamma^a \times \gamma$-set and contains the diagonal. The result follows.

(iv)⇔(v). Obvious.

(iv)⇒(ii). We shall apply (1.11). Let $U \times \theta$ be a constant cone contained in γ and let us prove that $U \cap (U \cap J^+_{\preceq}(x) + \theta) \subset J^+_{\preceq}(x)$. Let $y' \in U \cap J^+_{\preceq}(x)$ and let $v' \in \theta$ be such that $y' + v' \in U$. By Lemma 1.30, $y' \preceq_\gamma y' + v'$. Since $\Delta_\gamma \subset \Delta_{\preceq}$, we obtain $y' \preceq y' + v'$. Since $x \preceq y'$, we get $y' + v' \in J^+_{\preceq}(x)$. Q.E.D.

Graphs of transitive relations, closed sets, and γ-sets in a causal manifold, are all closed under intersections. This justifies Item (a) of the following definition.

Definition 1.44. (a) One denotes by $\preceq_{\rm cc}$ the finest closed causal preorder and by $\Delta_{\rm cc}$ its graph, that is, $\Delta_{\rm cc}$ is the intersection of all graphs of closed causal preorders. One calls it the canonical closed causal preorder, cc-preorder for short. One denotes by $J^+_{\rm cc}(A)$ and $J^-_{\rm cc}(A)$ the future and past sets of A for the cc-preorder.

(b) One denotes by $\preceq_{\rm ps}$ the preorder given by $x \preceq_{\rm ps} y$ if there exists a causal path $c\colon I \to M$ with $c(0) = x$ and $c(1) = y$ and calls it the piecewise smooth preorder, ps-preorder for short. One denotes by $\Delta_{\rm ps}$ its graph and one denotes by $J^+_{\rm ps}(A)$ and $J^-_{\rm ps}(A)$ the future and past sets of A for the ps-preorder.

(c) If there is a risk of confusion, we denote by Δ^M_γ the chronological preorder on M, and similarly for $\Delta^M_{\rm ps}$ and $\Delta^M_{\rm cc}$.

The cc-preorder was introduced first for Lorentzian spacetimes in [SW96]. One has

$$\Delta_\gamma \subset \Delta_{\rm cc} \text{ and } \Delta_\gamma \subset \Delta_{\rm ps}. \tag{1.14}$$

The first inclusion is obvious by construction and the second one follows from Lemma 1.35. Applying Theorem 1.43 (v)⇒(i), one gets:

Corollary 1.45. *The preorder* $\preceq_{\rm ps}$ *is causal.*

Proposition 1.46. *Let* $f\colon (M, \gamma_M) \to (N, \gamma_N)$ *be a causal morphism.*

(a) *The function* f *is increasing as a function from* $(M, \preceq_{\rm ps})$ *to* $(N, \preceq_{\rm ps})$.

(b) *If f is strictly causal, then it is increasing as a function from $(M, \preceq_\gamma)$ to $(N, \preceq_\gamma)$.*

(c) *If either $\Delta^N_{\mathrm{ps}} \subset \Delta^N_{\mathrm{cc}}$ or if f is strictly causal, then f is increasing as a function from $(M, \preceq_{\mathrm{cc}})$ to $(N, \preceq_{\mathrm{cc}})$.*

Proof. (a) follows from Lemma 1.34 and the definition of $\preceq_{\mathrm{ps}}$.
(b) follows from Lemmas 1.34 and 1.33.
(c) Let $A = \{(x, y) \in M^2; f(x) \preceq_{\mathrm{cc}} f(y)\}$. We shall prove that $A \supset \Delta_{\mathrm{cc}}$. For that purpose, using Theorem 1.43, it is enough to check that A is the graph of a closed preorder and $A \supset \Delta_\gamma$.
(c)-(i) A is clearly the graph of a preorder and since $A = (f^{-1} \times f^{-1})(\Delta_{\mathrm{cc}})$, it is closed.
(c)-(ii) Let $x \preceq_\gamma y$. If f is strictly causal then $f(x) \preceq_{\mathrm{cc}} f(y)$ by (b) and (1.14). If $\Delta^N_{\mathrm{ps}} \subset \Delta^N_{\mathrm{cc}}$, then $f(x) \preceq_{\mathrm{cc}} f(y)$ by (a) and (1.14). Hence $\Delta_\gamma \subset A$. Q.E.D.

Proposition 1.47. *Let (M, g) be a Lorentzian spacetime and let (M, γ) be the associated causal manifold.*

(a) *One has*

$$\Delta_\gamma \subset \Delta_{\mathrm{ps}} \subset \overline{\Delta_\gamma} \subset \Delta_{\mathrm{cc}}. \tag{1.15}$$

(b) *The preoreder Δ_{ps} is a proper order if and only if the preorder Δ_{cc} is a proper order and in this case, one has $\overline{\Delta_\gamma} = \Delta_{\mathrm{ps}} = \Delta_{\mathrm{cc}}$.*

One shall be aware that the inclusion $\overline{\Delta_\gamma} \subset \Delta_{\mathrm{cc}}$ may be strict since the closure of a transitive relation need not be transitive, even in Lorentzian spacetimes.

Proof. (a) The only inclusion left to prove is $\Delta_{\mathrm{ps}} \subset \overline{\Delta_\gamma}$. It is classical, see for instance [MS08, Prop. 2.17].
(b) If Δ_{ps} is proper, then it is closed, so equal to $\preceq_{\mathrm{cc}}$ which is therefore a proper preorder. For the converse and the second claim, see [MS08, Remark. 2.20]. Q.E.D.

We now extend the classical definition of global hyperbolicity of Lorentzian spacetimes to general causal manifolds as follows:

Definition 1.48. A causal manifold (M, γ) is *globally hyperbolic* if Δ_{cc} is a proper order.

Example 1.49. Let $M = \mathbb{R}^2 \setminus \{(1, 0)\}$ and $\gamma = M \times (\mathbb{R}_{>0})^2$. Then (M, γ) is a causal manifold. One easily checks that

$$\begin{aligned}
I^+_\gamma((0,0)) &= \{(0,0)\} \cup (\mathbb{R}_{>0})^2, \\
J^+_{\mathrm{ps}}((0,0)) &= (\mathbb{R}_{\geq 0})^2 \setminus \big([1, +\infty) \times \{0\}\big), \\
J^+_{\mathrm{cc}}((0,0)) &= \overline{I^+_\gamma((0,0))} = (\mathbb{R}_{\geq 0})^2 \setminus \{(1,0)\}.
\end{aligned}$$

In particular, $J^+_{\mathrm{ps}}((0,0))$ is neither closed nor open.

1.7 Cauchy time functions and G-causal manifolds

The terminology G-causal below is not inspired by gravitation but by the name of Geroch.

Definition 1.50. (a) A *Cauchy time function* on a causal manifold (M, γ) is a submersive causal morphism $q\colon (M, \gamma) \to (\mathbb{R}, +)$ which is proper on the sets $J^+_{\mathrm{cc}}(K)$ and $J^-_{\mathrm{cc}}(K)$ for any compact set $K \subset M$.
(b) A *G-causal manifold* (M, γ, q) is the data of a causal manifold (M, γ) together with a Cauchy time function q.

Proposition 1.51. *A Cauchy time function on a causal manifold (M, γ) is strictly causal and is increasing as a function from $(M, \preceq_{\mathrm{cc}})$ to $(\mathbb{R}, \leq)$. It is strictly increasing on strictly causal paths.*

Proof. Both results follow from Propositions 1.9 and 1.46, since $\preceq_{\mathrm{cc}}^{(\mathbb{R},+)}$ is $\leq$. Q.E.D.

The above proposition implies that a causal manifold with a Cauchy time function cannot have strictly causal loops. However, it may have causal loops, as Example 3.13 shows.

In the definition of a Cauchy time function, it is enough to assume properness on the future and past of each point.

Proposition 1.52. *Let $q\colon (M, \gamma) \to (\mathbb{R}, +)$ be a submersive causal morphism which is proper on the sets $J^+_{\mathrm{cc}}(x)$ and $J^-_{\mathrm{cc}}(x)$ for any $x \in M$. Then q is a Cauchy time function on (M, γ).*

Proof. The proof is classical. Let $K \subset M$ be a compact set and let $a, b \in \mathbb{R}, a \leq b$. We shall prove that $q^{-1}([a, b]) \cap J^-_{\mathrm{cc}}(K)$ is compact, the case of $J^+_{\mathrm{cc}}(K)$ being similar.

For any $x \in K$, let $y_x \in I^+_\gamma(x) \setminus \{x\}$. Then $x \in I^-_\gamma(y_x) \setminus \{y_x\}$, and this set being open, there is a compact neighborhood of x, say V_x, included in $I^-_\gamma(y_x) \setminus \{y_x\}$. Since $q^{-1}([a, b]) \cap J^-_{\mathrm{cc}}(V_x)$ is closed (by Proposition 1.6 (i)) and contained in the compact set $q^{-1}([a, b]]) \cap J^-_{\mathrm{cc}}(y_x)$, it is compact.

We cover K with finitely many V_x's, say the family $\{V_{x_i}\}_{i\in I}$. Then $q^{-1}([a, b]) \cap J^-_{\mathrm{cc}}(K)$ is closed and contained in the compact set $\bigcup_i (q^{-1}([a, b]) \cap J^-_{\mathrm{cc}}(V_{x_i}))$. Therefore, it is compact. Q.E.D.

Proposition 1.53. *If a causal manifold admits a Cauchy time function, then its cc-preorder is proper.*

Proof. Let q be a Cauchy time function on a causal manifold (M, γ). Since q is increasing, one has $q(J^-_{\mathrm{cc}}(x)) \subset (-\infty, q(x)]$ and $q(J^+_{\mathrm{cc}}(x)) \subset [q(x), +\infty)$ for any $x \in M$. Let K and L be compact subsets of M. There exist real numbers $a \leq b$ such that $J^+_{\mathrm{cc}}(K) \cap J^-_{\mathrm{cc}}(L) \subset q^{-1}([a, b])$. Since q is proper on $J^+_{\mathrm{cc}}(K)$, the set $q^{-1}([a, b]) \cap J^+_{\mathrm{cc}}(K)$ is compact. Recalling that $J^-_{\mathrm{cc}}(L)$ is closed, the set $J^+_{\mathrm{cc}}(K) \cap J^-_{\mathrm{cc}}(L) = q^{-1}([a, b]) \cap J^+_{\mathrm{cc}}(K) \cap J^-_{\mathrm{cc}}(L)$ is compact. Q.E.D.

Proposition 1.54. *Let q be a Cauchy time function on (M, γ) and let $x \in M$. Then $I^+_\gamma(x)$ is not relatively compact and $q(I^+_\gamma(x)) = q(J^+_{\mathrm{cc}}(x)) = [q(x), +\infty)$. In particular, G-causal manifolds cannot be compact and Cauchy time functions are surjective.*

Proof. One has $q(I^+_\gamma(x)) \subset q(J^+_{cc}(x)) \subset [q(x), +\infty)$ by Proposition 1.51. Since $I^+_\gamma(x)$ is connected, the set $q(I^+_\gamma(x))$ is an interval. By properness of q on $J^+_{cc}(x)$, hence on $\overline{I^+_\gamma(x)}$, this interval is bounded if and only if $I^+_\gamma(x)$ is relatively compact. Suppose that this is the case. Then $q\left(\overline{I^+_\gamma(x)}\right) = [q(x), t_\infty]$ for some $t_\infty \in [q(x), +\infty)$. Let $y \in \overline{I^+_\gamma(x)}$ with $q(y) = t_\infty$, fix a chart U at y, and let $v \in \gamma_y$. The set $\overline{I^+_\gamma(x)}$, being the closure of a γ-set, is also a γ set. Therefore, for $\varepsilon > 0$ small enough, the path $c\colon (-\varepsilon, \varepsilon) \to U, t \mapsto y + tv$ satisfies $c\big([0, \varepsilon)\big) \subset \overline{I^+_\gamma(x)}$ and this shows that there exists $\eta > 0$ such that $[q(x), t_\infty + \eta) \subset q\left(\overline{I^+_\gamma(x)}\right)$, which contradicts our assumption. Q.E.D.

Theorem 1.55. *If a Lorentzian spacetime is globally hyperbolic, then it admits a Cauchy time function.*

Proof. See [MS08, Thm. 3.75] and [Ger70, Prop. 8] (which suffices in view of Proposition 1.52). See also [FS11] for a more general version. Q.E.D.

2 Sheaves on causal manifolds

2.1 Microsupport

We recall here a few basic results on the microlocal theory of sheaves and refer to [KS90].

For simplicity, we denote by $\mathbf{k}$ a field, although all results would remain true when $\mathbf{k}$ is a commutative unital ring of finite global dimension.

Let $\mathrm{D^b}(\mathbf{k}_M)$ denote the bounded derived category of sheaves of $\mathbf{k}$-modules on M. For $F \in \mathrm{D^b}(\mathbf{k}_M)$, its microsupport, or singular support, denoted by $\mathrm{SS}(F)$, is a closed conic co-isotropic subset of T^*M whose intersection with $T^*_M M$ is $\mathrm{supp}(F)$, the support of F (see [KS90, Def. 5.1.2]). Roughly speaking, the microsupport describes the set of codirections of non propagation.

For a locally closed subset A of M, one denotes by $\mathbf{k}_A$ the constant sheaf on A with stalk $\mathbf{k}$ extended by 0 on $X \setminus A$. For $F \in \mathrm{D^b}(\mathbf{k}_M)$, one sets $F_A := F \otimes \mathbf{k}_A$. Recall that if Z is closed in M, then $\mathrm{R}\Gamma(M; F_Z) \simeq \mathrm{R}\Gamma(Z; F|_Z)$.

We shall also make use of the dualizing complex on M denoted by ω_M. Recall that ω_M is isomorphic to the orientation sheaf shifted by the dimension of M. It is an invertible sheaf and for a morphism $f\colon M \to N$, one has $f^!\omega_N \simeq \omega_M$ and $f^!\mathbf{k}_N \simeq \omega_{M/N} = \omega_M \otimes f^{-1}\omega_N^{\otimes -1}$.

We first recall a few results of constant use:

Proposition 2.1 (see [KS90, Prop. 5.3.8])**.** *Let $Z, U \subset M$. Assume that Z is closed and U is open. Then $\mathrm{SS}(\mathbf{k}_Z) \subset N(Z)^\circ$ and $\mathrm{SS}(\mathbf{k}_U) \subset N(U)^{\circ a}$.*

Theorem 2.2 ([KS90, Prop. 5.4.14])**.** *Let $F_1, F_2 \in \mathrm{D^b}(\mathbf{k}_M)$.*

(i) *Assume that $\mathrm{SS}(F_1) \cap \mathrm{SS}(F_2)^a \subset T^*_M M$. Then $\mathrm{SS}(F_1 \otimes F_2) \subset \mathrm{SS}(F_1) + \mathrm{SS}(F_2)$.*

(ii) *Assume that $\mathrm{SS}(F_1) \cap \mathrm{SS}(F_2) \subset T^*_M M$. Then $\mathrm{SS}(\mathrm{R}\mathscr{H}om\,(F_1, F_2)) \subset \mathrm{SS}(F_1)^a + \mathrm{SS}(F_2)$.*

Consider a morphism of manifolds $f\colon M \to N$ and recall the maps f_d and f_π in (1.4).

Definition 2.3. Let $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_N)$. One says that f is non-characteristic for G if f_d is proper on $f_\pi^{-1}\,\mathrm{SS}(G)$.

Since $f_\pi^{-1}\,\mathrm{SS}(G)$ is conic in $M \times_N T^*N$, this is equivalent to the condition

$$f_d^{-1} T^*_M M \cap f_\pi^{-1}\,\mathrm{SS}(G) \subset M \times_N T^*_N N. \tag{2.1}$$

Theorem 2.4 ([KS90, Prop. 5.4.4, 5.4.5])**.** *Consider a morphism of manifolds $f\colon M \to N$.*

(i) *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ and assume that f is proper on $\mathrm{supp}(F)$. Then $\mathrm{SS}(\mathrm{R}f_*F) \subset f_\pi f_d^{-1}(\mathrm{SS}(F))$. If f is a closed embedding then this inclusion is an equality.*

(ii) *Let $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_N)$ and assume that f non-characteristic for G. Then $\mathrm{SS}(f^{-1}G) \subset f_d f_\pi^{-1}(\mathrm{SS}(G))$. If f is a submersion then this inclusion is an equality.*

Consider a morphism of manifolds $f\colon M \to N$ and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. When f is proper on $\mathrm{supp}(F)$, Theorem 2.4 gives a bound to the microsupport of $\mathrm{R}f_*F$. However, we shall have to consider a non proper situation. The next lemma already appeared in [GS14] (with a slightly different formulation) but we give here another and more elementary proof. It is a variation on [KS90, Exe. V.7].

Lemma 2.5. *Let $f\colon M \to N$ be a morphism of manifolds and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. Assume to be given an increasing family $\{Z_n\}_n$ $(n \in \mathbb{N})$ of closed subsets of M such that $Z_n \subset \mathrm{Int}(Z_{n+1})$ for all n and $M = \bigcup_n Z_n$. Then*

$$\mathrm{SS}(\mathrm{R}f_*F) \subset \overline{\bigcup_n \mathrm{SS}\big(\mathrm{R}f_*(F_{Z_n})\big)}. \tag{2.2}$$

Proof. Let $p \in T^*N$ with $p \notin \overline{\bigcup_n \mathrm{SS}\big(\mathrm{R}f_*(F_{Z_n})\big)}$.
(i) We may assume $p \notin T^*_N N$, otherwise the result is obvious.
(ii) We may assume that N is open in some vector space $\mathbb{V}$ and there exist an open neighborhood V of $\pi(p)$ in N, an open cone λ_0 in $\mathbb{V}^*$ with $p \in V \times \lambda_0$ such that

$$(V \times \lambda_0) \cap \mathrm{SS}\big(\mathrm{R}f_*(F_{Z_n})\big) = \varnothing \text{ for all } n.$$

Let γ_1 be a closed convex proper cone contained in $(\lambda_0)^{\circ a}$ and let $\Omega_0 \subset \Omega_1$ be two γ_1-open subsets of $\mathbb{V}$ with

$$\begin{aligned}&\Omega_1 \setminus \Omega_0 \subset N,\\ &\text{for any } x \in \Omega_1, (x+\gamma_1) \setminus \Omega_0 \text{ is compact.}\end{aligned}$$

Applying [KS90, Prop. 5.2.1], we obtain the isomorphism

$$\mathrm{R}\Gamma(\Omega_1 \cap N; \mathrm{R}f_*(F_{Z_n})) \xrightarrow{\sim} \mathrm{R}\Gamma(\Omega_0 \cap N; \mathrm{R}f_*(F_{Z_n})). \tag{2.3}$$

or equivalently, the isomorphism

$$\mathrm{R}\Gamma\big(f^{-1}(\Omega_1 \cap N) \cap Z_n; F\big) \xrightarrow{\sim} \mathrm{R}\Gamma\big(f^{-1}(\Omega_0 \cap N) \cap Z_n; F\big). \tag{2.4}$$

Consider a distinguished triangle

$$\mathrm{R}\Gamma_{f^{-1}(\Omega_1 \cap N)} F \to \mathrm{R}\Gamma_{f^{-1}(\Omega_0 \cap N)} F \to G \xrightarrow{+1}$$

Then $\mathrm{R}\Gamma(Z_n; G) \simeq 0$ and by the Grothendieck Mittag-Leffler theorem (see for example [KS90, Prop. 2.7.1 (iii)]), we get $\mathrm{R}\Gamma(X; G) \simeq 0$, whence the isomorphism

$$\mathrm{R}\Gamma(\Omega_1 \cap N; \mathrm{R}f_* F) \xrightarrow{\sim} \mathrm{R}\Gamma(\Omega_0 \cap N; \mathrm{R}f_* F). \tag{2.5}$$

It follows from the definition of the microsupport that $p \notin \mathrm{SS}(\mathrm{R}f_* F)$. Q.E.D.

The next result is well-known from the specialists, but, to our knowledge, is not in the literature. Consider the Cartesian square of real manifolds

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ N \arrow[r, "i"'] & M. \end{tikzcd} \tag{2.6}$$

Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. One has a natural isomorphism [KS90, Prop. 3.1.9]:

$$i^! \mathrm{R}f_* F \simeq \mathrm{R}g_* j^! F. \tag{2.7}$$

Lemma 2.6 (Non-characteristic base change formula)**.** *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. Assume that j is non-characteristic for F, i is non-characteristic for $\mathrm{R}f_* F$ and f is submersive. Then the isomorphism (2.7) induces the isomorphism $i^{-1}\mathrm{R}f_* F \simeq \mathrm{R}g_* j^{-1} F$.*

Proof. Using [KS90, Prop. 5.4.13] and the hypotheses that i and j are non-characteristic, we get the isomorphisms

$$\begin{aligned} i^! \mathrm{R}f_* F &\simeq i^{-1}\mathrm{R}f_* F \otimes i^! \mathbf{k}_M, \\ \mathrm{R}g_* j^! F &\simeq \mathrm{R}g_*(j^{-1} F \otimes j^! \mathbf{k}_X). \end{aligned}$$

On the other hand, we have the isomorphisms

$$\begin{aligned} \mathrm{R}g_*(j^{-1} F \otimes j^! \mathbf{k}_X) &\simeq \mathrm{R}g_*(j^{-1} F \otimes g^{-1} i^! \mathbf{k}_M) \\ &\simeq \mathrm{R}g_* j^{-1} F \otimes i^! \mathbf{k}_M. \end{aligned}$$

The isomorphism $j^! \mathbf{k}_X \simeq g^{-1}(i^! \mathbf{k}_M)$ follows from the hypothesis that f is submersive. The last isomorphism follows since locally on N, $i^! \mathbf{k}_M$ is free of finite rank (up to a shift). Using (2.6), we get the the isomorphism

$$i^{-1}\mathrm{R}f_* F \otimes i^! \mathbf{k}_M \simeq \mathrm{R}g_* j^{-1} F \otimes i^! \mathbf{k}_M.$$

The result follows since $i^! \mathbf{k}_M$ is invertible. Q.E.D.

2.2 Propagation and Cauchy problem

Recall Notation 1.10 in which we set $\lambda = \gamma^\circ$.

Proposition 2.7. *Let (M, γ) be a causal manifold. Let $Z, U \subset M$. Assume that U is open and is a γ-set and that Z is closed and is a γ^a-set. Then $\mathrm{SS}(\mathbf{k}_U) \subset \lambda^a$ and $\mathrm{SS}(\mathbf{k}_Z) \subset \lambda^a$.*

Proof. By hypothesis, $\gamma \subset N(U)$. Hence, $\mathrm{SS}(\mathbf{k}_U) \subset N(U)^{\circ a} \subset \lambda^a$ by Proposition 2.1. The same proof applies to Z. Q.E.D.

Corollary 2.8. *Let (M, γ) be a causal manifold and $\preceq$ be a closed causal preorder on M. Let $Z, U \subset M$. Assume that U is open and $U = J^+_{\preceq}(U)$ and assume that Z is closed and $Z = J^-_{\preceq}(Z)$. Then $\mathrm{SS}(\mathbf{k}_U) \subset \lambda^a$ and $\mathrm{SS}(\mathbf{k}_Z) \subset \lambda^a$.*

Proof. Apply Proposition 2.7 and Theorem 1.43, (i)⇒(ii). Q.E.D.

Recall that a morphism of manifolds $f\colon M \to N$ gives rise to the maps

$$T^*M \xleftarrow{f_d} M \times_N T^*N \xrightarrow{f_\pi} T^*N.$$

Theorem 2.9. *Let $f\colon (M, \gamma_M) \to (N, \gamma_N)$ be a morphism of causal manifolds, let $\preceq$ be a closed causal preorder on M and let $F \in \mathrm{D^b}(\mathbf{k}_M)$. Assume that*

(a) *$f\colon M \to N$ is submersive,*

(b) *for any compact $K \subset M$, the map f is proper on the closed set $J^-_{\preceq}(K)$,*

(c) *$\mathrm{SS}(F) \cap \lambda_M \subset T^*_M M$.*

Then

$$\mathrm{SS}(\mathrm{R}f_* F) \cap \mathrm{Int}(\lambda_N) = \varnothing. \tag{2.8}$$

Proof. (i) Let K be a compact subset of M and let $Z = J^-_{\preceq}(K)$. Then Z is closed by Proposition 1.6, $Z = J^-_{\preceq}(Z)$ and $\mathrm{SS}(\mathbf{k}_Z) \subset \lambda^a_M$ by Corollary 2.8. Since the cone λ_M is closed convex and proper, we obtain by applying Proposition 2.2 (i)

$$\mathrm{SS}(F_Z) \cap \lambda_M \subset T^*_M M. \tag{2.9}$$

Since f is submersive (hence f_d is injective), we deduce from (2.9)

$$f_d^{-1}(\mathrm{SS}(F_Z)) \cap f_d^{-1}\lambda_M \subset M \times_N T^*_N N$$

and using the fact that f is causal

$$f_d^{-1}(\mathrm{SS}(F_Z)) \cap f_\pi^{-1}\lambda_N \subset M \times_N T^*_N N.$$

Since f is proper on $\mathrm{supp}(F_Z)$, we get:

$$\mathrm{SS}(\mathrm{R}f_* F_Z) \cap \lambda_N \subset f_\pi f_d^{-1}(\mathrm{SS}(F_Z)) \cap \lambda_N \subset T^*_N N. \tag{2.10}$$

(ii) Let W be an open relatively compact subset of N and set $V = f^{-1}W$, $f_V = f|_V$. Since f is proper on the sets $J^-_{\preceq}(K)$ (K compact in M), we may construct by induction an exhaustive sequence $\{K_n\}_n \in \mathbb{N}$ of compact subsets of M such that

$$J^-_{\preceq}(K_n) \cap V \subset \mathrm{Int}(J^-_{\preceq}(K_{n+1})) \cap V.$$

Set $Z_n = J^-_{\preceq}(K_n)$ and $F_n = F_{Z_n}$. By (2.10)

$$\mathrm{SS}(\mathrm{R}f_* F_n) \cap T^*W \cap \mathrm{Int}(\lambda_N) = \varnothing.$$

Applying Lemma 2.5 to the map $f_V \colon V \to W$, we get:

$$\mathrm{SS}(\mathrm{R}f_* F) \cap T^*W \cap \mathrm{Int}(\lambda_N) = \varnothing. \tag{2.11}$$

Since this result holds for any W open relatively compact, the proof is complete. Q.E.D.

Corollary 2.10. *Let I be a finite set and let $\{\gamma_{M,i}\}_{i\in I}$ and $\{\gamma_{N,i}\}_{i\in I}$ be two families of open convex cones in TM and TN respectively. Let $f \colon M \to N$ be a morphism of manifolds which defines morphisms of causal manifolds $f \colon (M, \gamma_{M,i}) \to (N, \gamma_{N,i})$. For each $i \in I$ let $\preceq_i$ be a closed preorder on M, causal for γ_i. Let $F \in \mathrm{D^b}(\mathbf{k}_M)$. Assume*

(a) *f is submersive,*

(b) *for any compact $K \subset M$ and all $i \in I$, the map f is proper on $J^-_{\preceq_i}(K)$,*

(c) *$\mathrm{SS}(F) \cap \lambda_{M,i} \subset T^*_M M$ for all $i \in I$,*

(d) *$\bigcup_i \mathrm{Int}(\lambda_{N,i}) = T^*N \setminus T^*_N N$.*

Then

$$\mathrm{SS}(\mathrm{R}f_* F) \subset T^*_N N. \tag{2.12}$$

In other words, $\mathrm{R}f_* F$ is a local system (in the derived sense). Moreover, f is surjective.

Proof. The inclusion (2.12) follows from Theorem 2.9. Since M is nonempty, $\mathrm{R}f_* \mathbf{k}_M$ is a non-zero local system on N. Since N is connected, the result follows. Q.E.D.

Corollary 2.11. *We make the same hypotheses as in Corollary 2.10 and we assume moreover that N is contractible. For $a \in N$, set $M_a = f^{-1}(\{a\})$. Then the restriction morphism*

$$\mathrm{R}\Gamma(M; F) \to \mathrm{R}\Gamma(M_a; F|_{M_a}) \tag{2.13}$$

is an isomorphism.

Proof. It follows from Corollary 2.10 that $\mathrm{SS}(\mathrm{R}f_*F) \subset T^*_N N$. In other words, all cohomology objects of $\mathrm{R}f_*F$ are local systems on N. This last manifold being contractible, we get:

$$\mathrm{R}\Gamma(N; \mathrm{R}f_*F) \simeq (\mathrm{R}f_*F)_a. \tag{2.14}$$

Consider the Cartesian square

$$\begin{array}{ccc} M_a & \xrightarrow{\iota} & M \\ {\scriptstyle f_a}\downarrow & \square & \downarrow{\scriptstyle f} \\ \{a\} & \xrightarrow[j]{} & N. \end{array}$$

It follows from the hypotheses that ι is non-characteristic for F and it follows from Corollary 2.10 that j is non-characteristic for $\mathrm{R}f_*F$. Applying Lemma 2.6, we get

$$j^{-1}\mathrm{R}f_*F \simeq \mathrm{R}f_{a*}\iota^{-1}F. \tag{2.15}$$

By using (2.14), we get

$$\mathrm{R}\Gamma(M; F) \simeq \mathrm{R}\Gamma(N; \mathrm{R}f_*F) \simeq (\mathrm{R}f_*F)_a \simeq j^{-1}\mathrm{R}f_*F \simeq \mathrm{R}f_{a*}\iota^{-1}F \simeq \mathrm{R}\Gamma(M_a; F|_{M_a}).$$

Q.E.D.

2.3 Sheaves on G-causal manifolds

We shall particularize the results of Subsection 2.2 to the case where $N = \mathbb{R}$, the interesting case in practice.

In the sequel, we denote by t a coordinate on $\mathbb{R}$ and by $(t; \tau)$ the associated coordinates on $T^*\mathbb{R}$. We shall write for short $\{\tau \geq 0\}$ instead of $\{(t; \tau) \in T^*\mathbb{R}; \tau \geq 0\}$ and similarly with $\tau \leq 0$.

Proposition 2.12. *Let (M, γ, q) be a G-causal manifold and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. Assume that $\mathrm{SS}(F) \cap \lambda \subset T^*_M M$. Then*

$$\mathrm{SS}(\mathrm{R}q_*F) \subset \{\tau \leq 0\}. \tag{2.16}$$

Proof. Apply Theorem 2.9 with $N = \mathbb{R}$, $f = q$, $\gamma_N = \{\tau \geq 0\}$. Q.E.D.

Theorem 2.13. *Let (M, γ, q) be a G-causal manifold and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$.*

(i) *Assume that $\mathrm{SS}(F) \cap \lambda^a \subset T^*_M M$ and let B be a closed subset satisfying $B = J^-_{\preceq}(B)$ and $B \subset q^{-1}((-\infty, a])$ for some $a \in \mathbb{R}$. Then*

$$\mathrm{R}\Gamma_B(M; F) \simeq 0. \tag{2.17}$$

(ii) *Assume that $\mathrm{SS}(F) \cap (\lambda \cup \lambda^a) \subset T^*_M M$. Then, setting $M_0 = q^{-1}(0)$, the natural restriction morphism below is an isomorphism:*

$$\mathrm{R}\Gamma(M; F) \xrightarrow{\sim} \mathrm{R}\Gamma(M_0; F|_{M_0}). \tag{2.18}$$

Proof. (i) By Corollary 2.8, $\mathrm{SS}(\mathbf{k}_B) \subset \lambda^a$. Applying Proposition 2.2 (ii), we get

$$\mathrm{SS}(\mathrm{R}\Gamma_B(F)) \subset \mathrm{SS}(F) + \lambda.$$

Therefore, $\mathrm{SS}(\mathrm{R}\Gamma_B F) \cap \lambda^a \subset T^*_M M$ and we may apply Proposition 2.12 to this sheaf (with λ^a instead of λ). We obtain

$$\mathrm{SS}(\mathrm{R}q_*\mathrm{R}\Gamma_B F) \subset \{\tau \geq 0\}.$$

Since $\mathrm{supp}(\mathrm{R}\Gamma_B F) \subset (-\infty, a]$ for some $a \in \mathbb{R}$, this implies

$$\mathrm{R}\Gamma_B(M; F) \simeq \mathrm{R}\Gamma(\mathbb{R}; \mathrm{R}q_*\mathrm{R}\Gamma_B F) \simeq 0.$$

(ii) Apply Corollary 2.11. Q.E.D.

Remark 2.14. In the paper [DS98], the notion of a λ-propagator is introduced. Essentially, on a causal manifold (M, γ), an object $K \in \mathrm{D^b}(\mathbf{k}_{M\times M})$ is a λ-propagator if it satisfies:

$$\begin{cases} \text{(i) the identity morphism } \mathbf{k}_M \to \mathbf{k}_M \text{ factors through } \mathrm{R}q_{2!}K, \\ \text{(ii) } \mathrm{SS}(K) \subset T^*M \times \lambda^a, \\ \text{(iii) } \mathrm{SS}(K) \cap (T^*M \times T^*_M M) \subset T^*_{M\times M}(M \times M). \end{cases} \tag{2.19}$$

Denote by $\preceq$ any closed causal preorder on M. Then one can prove along the lines of loc. cit. that

$$\begin{cases} \text{Assume that there exists a } \lambda\text{-propagator } K. \text{ Let } B \subset M \text{ be a closed} \\ \text{subset such that } B \text{ is } \gamma\text{-proper, } B \neq M \text{ and } B = J^-_{\preceq}(B). \text{ Then for any} \\ F \in \mathrm{D^b}(\mathbf{k}_M) \text{ satisfying } \mathrm{SS}(F) \cap \lambda^a \subset T^*_M M, \text{ one has } \mathrm{R}\Gamma_B(M; F) \simeq 0. \end{cases} \tag{2.20}$$

In other words, one recovers the conclusions of Theorem 2.13 (indeed, one can also treat the Cauchy problem) when assuming the existence of a propagator instead of that of a Cauchy time function.

However, it seems difficult to construct λ-propagators, contrarily to what is written in [DS98, Prop. 4.4 (ii)] (see below).

Erratum 2.15. *In* [DS98, Prop. 4.4 (ii)]*, it is asserted that under mild conditions on the preorder, the constant sheaf* (*or a variant of this sheaf*) *on the graph of the causal preorder is a propagator. However, the proof is not complete and indeed, the result is not correct without extra hypotheses, as seen in Example 2.16 below. Note that the results of* [Sch13, § 6]*, being built on this wrong statement, they should be replaced with those of this paper.*

Example 2.16. Let $M = \mathbb{R}^2$ be the plane with coordinates (x, t). Define the open convex cones

$$\gamma_0^- = \{(v, w) \in \mathbb{R}^2; w > \max(0, -v)\}$$
$$\gamma_0^+ = \{(v, w) \in \mathbb{R}^2; w > |v|\}$$

and define the open convex cone $\gamma \subset TM \simeq M \times \mathbb{R}^2$ by

$$\gamma = (\mathbb{R} \times \mathbb{R}_{<0}) \times \gamma_0^- \cup M \times \gamma_0^+$$

so that (M, γ) is a causal manifold. The cc-preorder is given by

$$J_{\mathrm{cc}}^+((x,t)) = (x,t) + \overline{\gamma_0^+} \text{ if } t > 0, \text{ and } (x,t) + \overline{\gamma_0^-} \text{ if } t \leq 0.$$

In particular, (M, γ) is easily seen to be globally hyperbolic. One also checks that if $0 < \alpha < 1$, then $(x,t) \mapsto t + \alpha x$ is a Cauchy time function on (M, γ).

We shall prove that $((0,-1),(0,0)) \in (\mathrm{SS}(\mathbf{k}_{\Delta_{\mathrm{cc}}}))_{((0,0),(1,0))}$, which implies that $\mathbf{k}_{\Delta_{\mathrm{cc}}}$ is not a propagator (it does not satisfy (2.19) (iii)). Define the open balls $U_1 = B((0,0);1/4)$ and $U_2 = B((1,0);1/4)$. Then $U_1 \times U_2$ is a neighborhood of $((0,0),(1,0)) \in M^2$ and one has

$$\Delta_{\mathrm{cc}} \cap (U_1 \times U_2) = \{((x_1,t_1),(x_2,t_2)) \in U_1 \times U_2; t_1 \leq \min(0,t_2)\}.$$

By [KS90, Prop. 5.3.1], this implies that

$$(\mathrm{SS}(\mathbf{k}_{\Delta_{\mathrm{cc}}}))_{((0,0),(1,0))} = \{((\xi_1,\tau_1),(\xi_2,\tau_2)) \in \mathbb{R}^2 \times \mathbb{R}^2; 0 \leq \tau_2 \leq -\tau_1\}$$

which contains $((0,-1),(0,0))$.

Note that one can obtain a similar counter-example with a continuous cone, namely, on $M = \mathbb{R}^2$, consider

$$\gamma = (\mathbb{R} \times \mathbb{R}_{<0}) \times \gamma_0^- \cup (\mathbb{R} \times \mathbb{R}_{\geq 0}) \times \{(v,w) \in \mathbb{R}^2; w > \max(-v, \sqrt{t}|v|)\},$$

but one cannot choose a Lipschitz continuous cone.

3 Hyperbolic systems on causal manifolds

3.1 The Cauchy problem for $\mathscr{D}$-modules

Let $(X, \mathscr{O}_X)$ be a complex manifold and denote as usual by $\mathscr{D}_X$ the sheaf of rings of holomorphic (finite order) differential operators on X. References for $\mathscr{D}$-module theory are made to to [Kas03].

Let $\mathscr{M}$ be a left coherent $\mathscr{D}_X$-module. By [KS90, Thm. 11.3.3], the characteristic variety of $\mathscr{M}$ is equal to the microsupport of the complex of its holomorphic solutions:

$$\mathrm{char}(\mathscr{M}) = \mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)).$$

Let Y be a complex submanifold of the complex manifold X. One says that Y is non-characteristic for $\mathscr{M}$ if

$$\mathrm{char}(\mathscr{M}) \cap T_Y^*X \subset T_X^*X.$$

With this hypothesis, the induced system $\mathscr{M}_Y$ by $\mathscr{M}$ on Y is a coherent $\mathscr{D}_Y$-module and one has the Cauchy–Kowalesky–Kashiwara theorem [Kas70]:

Theorem 3.1. *Assume Y is non-characteristic for $\mathscr{M}$. Then $\mathscr{M}_Y$ is a coherent $\mathscr{D}_Y$-module and the morphism*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y). \tag{3.1}$$

is an isomorphism.

3.2 Review on hyperbolic systems

We shall very briefly recall how to apply the preceding results to hyperbolic systems, using the results of [KS90]. For a detailed exposition, we refer to [Sch13].

Now let M be a real analytic manifold, say of dimension n, and let X be a complexification of M. Consider a vector bundle $\tau\colon E \to M$. It gives rise to a morphism of vector bundles over M, $\tau'\colon TE \to E \times_M TM$ which by duality gives the map $\tau_d\colon E \times_M T^*M \to T^*E$. By restricting to the zero-section of E, we get the map:

$$T^*M \hookrightarrow T^*E.$$

Applying this construction to the bundle $T^*_M X$ above M, and using the Hamiltonian isomorphism we get the maps

$$T^*M \hookrightarrow T^*T^*_M X \simeq T_{T^*_M X}T^*X. \tag{3.2}$$

Definition 3.2. Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module.

(a) We set

$$\mathrm{hypchar}_M(\mathscr{M}) = T^*M \cap C_{T^*_M X}(\mathrm{char}(\mathscr{M})) \tag{3.3}$$

and call $\mathrm{hypchar}_M(\mathscr{M})$ the *hyperbolic characteristic variety* of $\mathscr{M}$ along M.

(b) A vector $\theta \in T^*M$ such that $\theta \notin \mathrm{hypchar}_M(\mathscr{M})$ is called *hyperbolic* with respect to $\mathscr{M}$.

(c) A submanifold N of M is called *hyperbolic* for $\mathscr{M}$ if

$$T^*_N M \cap \mathrm{hypchar}_M(\mathscr{M}) \subset T^*_M M, \tag{3.4}$$

that is, any nonzero vector of $T^*_N M$ is hyperbolic for $\mathscr{M}$.

(d) For a differential operator P, we set $\mathrm{hypchar}(P) = \mathrm{hypchar}_M(\mathscr{D}_X/\mathscr{D}_X \cdot P)$.

By construction, $\mathrm{hypchar}_M(\mathscr{M})$ is a closed conic subset of T^*M, invariant by the antipodal map $(x;\xi) \mapsto (x;-\xi)$.

Example 3.3. Assume we have a local coordinate system $(x + \sqrt{-1}y)$ on X with $M = \{y = 0\}$ as above and let $(x + \sqrt{-1}y; \xi + \sqrt{-1}\eta)$ be the coordinates on T^*X so that $T^*_M X = \{y = \xi = 0\}$. Let $(x_0;\theta_0) \in T^*M$ with $\theta_0 \neq 0$. Let P be a differential operator with principal symbol $\sigma(P)$. Applying the definition of the normal cone, we find that $(x_0;\theta_0)$ is hyperbolic for P if and only if

$$\begin{cases}\text{there exist an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ and an open conic}\\ \text{neighborhood } \gamma \text{ of } \theta_0 \in \mathbb{R}^n \text{ such that } \sigma(P)(x;\theta + \sqrt{-1}\eta) \neq 0 \text{ for}\\ \text{all } \eta \in \mathbb{R}^n,\ x \in U \text{ and } \theta \in \gamma.\end{cases} \tag{3.5}$$

As noticed by M. Kashiwara (see [BS73, § 2.]), it follows from the local Bochner's tube theorem that Condition (3.5) can be simplified: $(x_0;\theta_0)$ is hyperbolic for P if and only if

$$\begin{cases}\text{there exist an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ such that}\\ \sigma(P)(x;\theta_0 + \sqrt{-1}\eta) \neq 0 \text{ for all } \eta \in \mathbb{R}^n, \text{ and } x \in U.\end{cases} \tag{3.6}$$

Hence, one recovers the classical notion of a (weakly) hyperbolic operator (see [Ler53]).

Now, consider the sheaves

$$\mathscr{A}_M = \mathscr{O}_X|_M, \quad \mathscr{B}_M = H^n_M(\mathscr{O}_X) \otimes \mathrm{or}_M \simeq \mathrm{R}\Gamma_M(\mathscr{O}_X) \otimes \mathrm{or}_M\,[n]. \tag{3.7}$$

Here, or_M is the orientation sheaf on M and $n = \dim M$. The sheaf $\mathscr{A}_M$ is the sheaf of (complex valued) real analytic functions on M and the sheaf $\mathscr{B}_M$ is the sheaf of Sato's hyperfunctions on M. Recall that the sheaf $\mathscr{B}_M$ is flabby and the sheaf $\mathscr{A}_M$ is Γ-acyclic, that is, $\mathrm{R}\Gamma(U;\mathscr{A}_M)$ is concentrated in degree 0 for each open set $U \subset M$. Applying [KS90, Cor. 6.4.4], we get:

Theorem 3.4 (see KS90)**.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then*

$$\begin{aligned}&\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M)) \subset \mathrm{hypchar}_M(\mathscr{M}),\\ &\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{A}_M)) \subset \mathrm{hypchar}_M(\mathscr{M}).\end{aligned}$$

In other words, hyperfunction (as well as real analytic) solutions of the system $\mathscr{M}$ propagate in the hyperbolic directions.

Now we consider the following situation: $N \hookrightarrow M$ is a real analytic smooth closed submanifold of M of codimension d and $Y \hookrightarrow X$ is a complexification of N in X.

The next result was announced without proof in [Sch13]. For sake of completeness, we give a proof here.

Lemma 3.5. *Let M, X, N, Y be as above and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Assume that N is hyperbolic for $\mathscr{M}$. Then Y is non-characteristic for $\mathscr{M}$ in a neighborhood of N.*

Proof. (i) Consider the exact sequence of vector bundles over N:

$$0 \to T^*_M X \times_M N \times_Y T^*_Y X \to N \times_Y T^*_Y X \to T^*_N M \to 0$$

It defines the inclusion $T^*_N M \subset C_{T^*_M X}(T^*_Y X)$ hence the inclusion $T^*_N M \subset C_{T^*_M X}(T^*_Y X) \cap T^*M$. Using local coordinates one checks the equality:

$$C_{T^*_M X}(T^*_Y X) \cap T^*M = T^*_N M. \tag{3.8}$$

(ii) Now let $(x;\theta) \in \mathrm{char}(\mathscr{M}) \cap T^*_Y X \cap (M \times_X T^*X)$ and denote by $L \subset T^*_x X$ the cone $(x;\mathbb{C}^\times \cdot \theta)$. Using (3.3), (3.4) and (3.8), we get

$$C_{T^*_M X}(L) \cap T^*M \subset \mathrm{hypchar}(\mathscr{M}) \cap T^*_N M \subset T^*_M M. \tag{3.9}$$

Choose a local coordinate system (x, y) on X so that $M = \{y = 0\}$ and let $(x, y; \xi, \eta)$ denote the associated coordinates. Identifying $T_{T^*_M X} T^*X$ with T^*X, we get

$$\begin{aligned}&T^*M = \{(x, 0; \xi, 0)\}, \quad T^*_M X = \{(x, 0; 0, \eta)\},\\ &C_{T^*_M X}(L) = T^*_M X + L.\end{aligned}$$

Then (3.9) implies $\theta = 0$. Q.E.D.

The following result is easily deduced from Theorem 3.4. See [Sch13] for details.

Theorem 3.6. *Let M be a real analytic manifold, X a complexification of M, $\mathscr{M}$ a coherent $\mathscr{D}_X$-module. Let $N \hookrightarrow M$ be a real analytic smooth closed submanifold of M and $Y \hookrightarrow X$ is a complexification of N in X. We assume*

$$T^*_N M \cap \mathrm{hypchar}_M(\mathscr{M}) \subset T^*_M M, \tag{3.10}$$

that is, N is hyperbolic for $\mathscr{M}$. Then Y is non-characteristic for $\mathscr{M}$ in a neighborhood of N and the isomorphism (3.1) induces the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)|_N \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N). \tag{3.11}$$

In other words, the Cauchy problem in a neighborhood of N for hyperfunctions on M is well-posed for hyperbolic systems.

Note that for $f \in \mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)$, its wavefront set does not intersect $T^*_Y X \cap T^*_M X$ and thus its restriction to $\mathscr{E}xt^j_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N)$ is well-defined. In particular, if $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X \cdot P$, this means that if f is a hyperfunction on M defined in a neighborhood of N solution of $Pf = 0$, then its wavefront set does not intersect $T^*_Y X \cap T^*_M X$ and therefore its restriction is well-defined (see [SKK73]).

3.3 Hyperbolic systems on causal manifolds

Theorem 3.7. *Let M and S be a real analytic manifolds, X a complexification of M, $\mathscr{M}$ a coherent $\mathscr{D}_X$-module. Let I be a finite set and let $\{\gamma_{M,i}\}_{i\in I}$ and $\{\gamma_{S,i}\}_{i\in I}$ be two families of open convex cones in TM and TS respectively. Let $f\colon M \to S$ be a morphism of manifolds which defines morphisms of causal manifolds $f\colon (M, \gamma_{M,i}) \to (S, \gamma_{S,i})$. For each $i \in I$ let $\preceq_i$ be a closed preorder on M, causal for γ_i. Assume*

(a) *f is submersive and surjective,*

(b) *S is contractible,*

(c) *for any compact $K \subset M$ and all $i \in I$, the map f is proper on $J^-_{\preceq_i}(K)$,*

(d) $\mathrm{hypchar}(\mathscr{M}) \cap \lambda_{M,i} \subset T^*_M M$ *for all $i \in I$,*

(e) $\bigcup_i \mathrm{Int}(\lambda_{S,i}) = T^*S \setminus T^*_S S$.

Let $a \in S$ and assume that $N := f^{-1}(\{a\})$ is real analytic. Let Y be a complexification of N in X. Then the isomorphism (3.1) induces the isomorphism

$$\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M) \xrightarrow{\sim} \mathrm{RHom}_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N). \tag{3.12}$$

Recall that on a topological space M, $\mathrm{RHom} \simeq \mathrm{R}\Gamma(M; \bullet) \circ \mathrm{R}\mathscr{H}om$.

Proof. Set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)$. Then $\mathrm{SS}(F) \subset \mathrm{hypchar}(\mathscr{M})$ by Theorem 3.4. Applying Corollary 2.11, we get $\mathrm{R}\Gamma(M; F) \simeq \mathrm{R}\Gamma(N; F|_N)$ and, applying Theorem 3.6, we get $\mathrm{R}\Gamma(N; F|_N) \simeq \mathrm{RHom}_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N)$. Q.E.D.

Corollary 3.8. *Let (M, γ, q) be a G-causal manifold and assume that M is real analytic. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module satisfying* $\mathrm{hypchar}(\mathscr{M}) \cap \lambda \subset T^*_M M$.

(a) *Let A be a closed subset satisfying either $A = J^+_{\mathrm{cc}}(A)$ and $A \subset q^{-1}([a, +\infty))$ or $A = J^-_{\mathrm{cc}}(A)$ and $A \subset q^{-1}((-\infty, a])$ for some $a \in \mathbb{R}$. Then $\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \Gamma_A \mathscr{B}_M) \simeq 0$. In particular, hyperfunction solutions of the system $\mathscr{M}$ defined on $M \setminus A$ extend uniquely to the whole of M as hyperfunction solutions of the system.*

(b) *Let $N = q^{-1}(0)$ and assume that N is real analytic. Let Y be a complexification of N in X. Then the restriction morphism $\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M) \to \mathrm{RHom}_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N)$ is an isomorphism. In other words, the Cauchy problem for hyperfunctions with initial data on N is globally well-posed.*

Note that if K is compact, then $A = J^+_{\mathrm{cc}}(K)$ or $A = J^-_{\mathrm{cc}}(K)$ will satisfy the condition in (a).

Proof. (a) Apply Theorem 2.13 (i).
(b) Apply Theorem 2.13 (ii) or Theorem 3.7. Q.E.D.

Remark 3.9. Theorem 3.7 and Corollary 3.8 remain true when replacing the sheaves of hyperfunctions by those of real analytic functions, that is, replacing $\mathscr{B}_M$ and $\mathscr{B}_N$ with $\mathscr{A}_M$ and $\mathscr{A}_N$, respectively.

3.4 Examples

In this subsection, all manifolds will be real (or complex) analytic and the differential operators we consider will have analytic coefficients.

We shall often assume that $M = N \times \mathbb{R}$. In this situation, for $(x, t) \in M$, the vector $(x, t; dt) = (x, t; 0, 1)$ is well-defined in T^*M.

3.4.1 Product with a compact Cauchy hypersurface

Let us translate Corollary 3.8 in a particular situation.

Proposition 3.10. *Let N be a real analytic compact manifold and let $M = N \times \mathbb{R}$. Let P be a differential operator on M of order m which is hyperbolic in the codirection $(x, t; dt)$ for all $(x, t) \in M$.*

(a) *The Cauchy problem*

$$\begin{cases} Pf = 0 \\ (f, \dots, \partial_t^{m-1} f)|_{t=0} = (h_0, \dots, h_{m-1}) \end{cases} \tag{3.13}$$

is globally well-posed for hyperfunctions and for analytic functions. In other words, for any $h = (h_0, \dots, h_{m-1})$ in $\mathscr{B}(N)^m$ (resp. $\mathscr{A}(N)^m$) there exists a unique $f \in \mathscr{B}(M)$ (resp. $f \in \mathscr{A}(N)$) solution of (3.13).

(b) *The operator P is a surjective endomorphism of $\Gamma(M; \mathscr{B}_M)$ and of $\Gamma(M; \mathscr{A}_M)$.*

(c) *Any hyperfunction (resp. analytic function) f solution of the equation $Pf = 0$ defined on $N \times (a,b)$ with $-\infty \le a < b \le +\infty$ extends uniquely as a hyperfunction (resp. analytic function) solution of this equation on M.*

Proof. (a)-(b) Consider the open convex cone $\gamma = TN \times \{(t;v) \in T\mathbb{R}; v > 0\}$ in TM. The projection on the second factor, $q\colon M \to \mathbb{R}, (x,t) \mapsto t$, is a Cauchy time function on (M,γ). One has $\gamma^\circ = T^*_N N \times \{(t;\tau) \in T^*\mathbb{R}; \tau \ge 0\}$. Since $(x,t;0,1)$ is a hyperbolic codirection for all $(x,t) \in M$, one has $\mathrm{hypchar}(P) \cap \gamma^\circ \subset T^*_M M$, so we can apply Corollary 3.8 (b).

Since $\mathrm{RHom}_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N) \simeq \mathrm{R}\Gamma(N; \mathscr{B}_N^m)$ is concentrated in degree 0, we get that $\mathrm{Ext}^1_{\mathscr{D}}(\mathscr{M}, \mathscr{B}_M) = 0$, that is, the operator $P\colon \mathscr{B}(M) \to \mathscr{B}(M)$ is surjective and its kernel is isomorphic to $\mathscr{B}(N)^m$, the isomorphism being given (for example) by $f \mapsto (f, \dots, \partial_t^{m-1} f)|_{t=0}$. Indeed, an isomorphism $\mathscr{M}_Y \simeq \mathscr{D}_X/t \cdot \mathscr{D}_X + \mathscr{D}_X \cdot P$ is described as follows: any $R(x,t;\partial_x,\partial_t) \in \mathscr{D}_X|_Y$ may be written uniquely as

$$R(x,t;\partial_x,\partial_t) = t \cdot Q + S \cdot P + \sum_{j=0}^{m-1} R_j(x,\partial_x) \cdot \partial_t^j$$

by the Späth–Weierstrass division theorem (see [Kas70, SKK73] for details).

(c) follows from Corollary 3.8 (a). Q.E.D.

Example 3.11. Let P be a differentoial operator of order 2 such that

$$\begin{cases} P = \partial_t^2 - R, \\ \sigma_2(R)|_{T^*_M X} \le 0, \\ \sigma_2(R) \text{ does not depend on } \tau. \end{cases} \tag{3.14}$$

Then P is hyperbolic in the codirections $(x,t;\pm dt)$ for all $(x,t) \in M$. Indeed, choose a local coordinate system $(z;\zeta)$ on T^*Y, $z = x + \sqrt{-1}y$, $\zeta = \xi + \sqrt{-1}\eta$. Denote by $(t + \sqrt{-1}t'; \tau + \sqrt{-1}\tau')$ the coordinates on $T^*\mathbb{C}$. Hence, $(x,t;\sqrt{-1}\eta, \sqrt{-1}\tau')$ is a local coordinate system on $T^*_M X$. Then, denoting by $\sigma(P)$ the principal symbol of P,

$$\begin{aligned} \sigma(P)(x,t;\sqrt{-1}\eta, \sqrt{-1}\tau' + \theta) &= (\theta + \sqrt{-1}\tau')^2 - \sigma_2(R)(x,t;\sqrt{-1}\eta) \\ &= \theta^2 - \tau'^2 + \sigma_2(R)(x,t;\eta) + 2\sqrt{-1}\theta\tau' \neq 0 \text{ for } \theta \neq 0. \end{aligned}$$

If $(g_t)_{t\in\mathbb{R}}$ is an analytic family of Riemannian metrics on N and $(\Delta_t)_{t\in\mathbb{R}}$ are the associated Laplace–Beltrami operators, then

$$P = \partial_t^2 - \Delta_t \tag{3.15}$$

is such an example.

Being hyperbolic in a given codirection depends only on the top-order part of the operator. If a differential operator is hyperbolic in a given codirection, then so are its powers. Therefore, all operators of the form $P^r + Q$ where Q is a differential operator on M of order at most $2r-1$ are also examples.

Remark 3.12. Similar results do not hold in general with the sheaves of distributions or of C^∞-functions. For example, it is well-known since Hadamard that the Cauchy problem is not well-posed in the space of C^∞-functions on $\mathbb{R}^2$ for the operator $\partial_t^2 - \partial_x$. However, if one assumes that the operator R in Example 3.13 is elliptic, then the operator $P = \partial_t^2 - R$ is hyperbolic in the classical sense and the Cauchy problem is well-posed in the spaces of C^∞-functions and of distributions on $N \times \mathbb{R}$.

Example 3.13. Let us particularize Example 3.11 to the case $N = \mathbb{S}^1$, hence $M = \mathbb{S}^1 \times \mathbb{R}$. We define $\gamma, \preceq, q$ as above and we denote denote by x a coordinate on $\mathbb{S}^1$ (hence, $x + 2\pi = x$). Then the path $[0, 2\pi] \ni s \mapsto (s, 0) \in M$ is a causal loop.

Consider the differential operator $P(x, t; \partial_x, \partial_t) = \partial_t^2 - \partial_x^2$. The Cauchy problem (3.13) (with $m = 2$) is globally well-posed in various spaces of functions or generalized functions. In fact, writing $f(x, t) = f_0(x + t) + f_1(x - t)$, we get

$$f_0(x) + f_1(x) = h_0(x), \quad f_0'(x) - f_1'(x) = h_1(x).$$

Therefore, $2f_0 = h_0 + \int h_1$ and $2f_1(x) = h_0 - \int h_1$. Note that $\int h_1$ is not necessarily periodic, but replacing $f(x, t)$ with $f(x, t) + (\int_0^{2\pi} h_1)t$, we may assume from the beginning that $\int_0^{2\pi} h_1 = 0$ so that $\int h_1$ is 2π-periodic. If one works in the space of real analytic functions or in the space of hyperfunctions, this result is in accordance with Corollary 3.8.

Example 3.14. We consider Example 3.13 and replace $\mathbb{R}$ with coordinate t with the circle $\mathbb{R}/a \cdot \mathbb{Z}$ for some $a > 0$. Hence, now M is a torus. Set $S = \mathbb{R}/a \cdot \mathbb{Z}$ and keep the notation of the previous example. Hence $q\colon M \to S$ is a submersive morphism of causal manifolds, but not a time function since $S \neq \mathbb{R}$. Moreover, the Cauchy problem (3.13) (with $m = 2$) is not globally well-posed, except for $a = 2\pi$. Note that Theorem 3.7 does not apply since S is not contractible.

3.4.2 Complex time

Example 3.15. In this example, we treat the case where the time is complex. For simplicity, we restrict ourselves to an elementary situation.

Let N be a real analytic compact manifold and let $M = N \times \mathbb{C}$. Let Y be a complexification of N. We denote by $w = t + \sqrt{-1}t'$ the complex coordinate on $\mathbb{C}$ and by $(w; \tau + \sqrt{-1}\tau')$ the coordinates on $T^*\mathbb{C}$. Let $M = N \times \mathbb{C}$ viewed as a real manifold. Consider the left ideal $\mathscr{I}$ and the left $\mathscr{D}_M$-module $\mathscr{M}$:

$$\mathscr{I} = \mathscr{D}_M \cdot P + \mathscr{D}_M \cdot \overline{\partial}_w, \quad \mathscr{M} = \mathscr{D}_M / \mathscr{I},$$

where $\overline{\partial}_w$ is the Cauchy–Riemann operator on $\mathbb{C}$ and

$$P = \partial_w^2 - R$$

where R is a differential operator on M (holomorphic in w) of order ≤ 2 whose symbol of order 2 depends neither on w nor on $\tau + \sqrt{-1}\tau'$ and satisfies

$$\sigma_2(R)|_{T_N^*Y} \leq 0.$$

Then any $(x,w;0,\theta+\sqrt{-1}\theta')\in T^*M$, $\theta\neq 0$ is hyperbolic for $\mathscr{M}$. Indeed, the system of equations

$$\begin{aligned}&\sigma(\overline{\partial}_w)\big(x,w;\sqrt{-1}\eta,(\theta+\sqrt{-1}\theta')+\sqrt{-1}(\tau+\sqrt{-1}\tau')\big)=0,\\&\sigma(P)\big(x,w;\sqrt{-1}\eta,(\theta+\sqrt{-1}\theta')+\sqrt{-1}(\tau+\sqrt{-1}\tau')\big)=0\end{aligned}$$

is equivalent to

$$(\theta+\sqrt{-1}\tau)^2-\sigma_2(R)(x;\sqrt{-1}\eta)=0$$

which has no solutions for $\theta\neq 0$. In other words,

$$\mathrm{hypchar}_M(\mathscr{M})\cap(T^*_N N\times T^*\mathbb{C})\subset T^*_N N\times(\mathbb{C}\times\sqrt{-1}\mathbb{R}). \tag{3.16}$$

Since the codirections $\sqrt{-1}\theta'$ are not hyperbolic for the system, we cannot apply Theorem 3.7 to solve the Cauchy problem with data on the submanifold $N\times\{0\}$ of M and, indeed, one easily sees that the Cauchy problem

$$\begin{cases}Pf=0,\\ \overline{\partial_w}f=0,\\ (f,\partial_w f)|_{w=0}=(h_0,h_1)\end{cases}$$

is not well posed. However, one has propagation results:

$$\begin{cases}\text{Let } \Omega_0 \text{ be an open subset of } \mathbb{C} \text{ whose intersection with any line } \mathbb{R}+\sqrt{-1}a,\\ a\in\mathbb{R}, \text{ is connected and let } \Omega_1=\Omega_0+\mathbb{R}_{\geq 0}\times\{0\}. \text{ Set } U_i=N\times\Omega_i\ (i=0,1).\\ \text{Then one has the restriction isomorphism}\\ \mathrm{R}\Gamma(U_1;\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M))\xrightarrow{\sim}\mathrm{R}\Gamma(U_0;\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M)).\end{cases} \tag{3.17}$$

To prove (3.17), one may proceed as follows. Let $q\colon N\times\mathbb{C}\to\mathbb{C}$ denote the projection and set $F=\mathrm{R}q_*\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M)$. Since q is proper, it follows from (3.16) and Theorem 3.4 that $\mathrm{SS}(F)\subset\mathbb{C}\times\sqrt{-1}\mathbb{R}\subset T^*\mathbb{C}$. Then the isomorphism $\mathrm{R}\Gamma(\Omega_1;F)\xrightarrow{\sim}\mathrm{R}\Gamma(\Omega_0;F)$ follows from [KS90, Prop. 5.2.1]. Indeed, with the notations of loc. cit., choose $\gamma=\mathbb{R}_{\leq 0}\times\{0\}$ and $U=\Omega_1$. Then, for any $x\in\Omega_1$, the set $(x+\gamma)\setminus\Omega_0$ is compact.

Of course, one may interchange the cones $\mathbb{R}_{\geq 0}\times\{0\}$ and $\mathbb{R}_{\leq 0}\times\{0\}$. For an open subset Ω of $\mathbb{C}$ whose intersection with any line $\mathbb{R}+\sqrt{-1}a$, $a\in\mathbb{R}$ is connected, denote by $\widetilde{\Omega}$ the open set $\Omega+\mathbb{R}\times\{0\}$ and for $U=N\times\Omega$ set $\widetilde{U}=N\times\widetilde{\Omega}$. One gets the isomorphism

$$\mathrm{R}\Gamma(\widetilde{U};\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M))\xrightarrow{\sim}\mathrm{R}\Gamma(U;\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M)).$$

3.4.3 Product with a Riemannian hypersurface

We shall need the following result.

Lemma 3.16 ([BEE96, Thm 3.66])**.** *Let (N, g) be a Riemannian manifold and let $f\colon \mathbb{R} \to \mathbb{R}_{>0}$ be a smooth function. Then the Lorentzian spacetime $(N \times \mathbb{R}, dt^2 - f(t)g)$ is globally hyperbolic if and only if g is complete, and in this case, the projection on the second factor $q\colon N \times \mathbb{R} \to \mathbb{R}$ is a Cauchy time function.*

Example 3.17. Let N be a real analytic manifold and set $M = N \times \mathbb{R}$ as in Example 3.11 but now, *we do not assume any more that N is compact.*

We still denote by t a coordinate on $\mathbb{R}$, by $(t; w)$ the coordinates on $T\mathbb{R}$ and by $(t; \tau)$ the coordinates on $T^*\mathbb{R}$ and we still consider a differential operator $P = \partial_t^2 - R$ as in (3.14). Since N is no more assumed to be compact, we need another hypothesis:

$$\text{there exist a smooth function } f\colon \mathbb{R} \to \mathbb{R}_{>0} \text{ and a smooth complete Riemannian metric } g \text{ on } N \text{ such that } \sigma_2(R)(x, t; \xi) \le f(t)|\xi|_{g_x}^2. \tag{3.18}$$

Note that this condition is automatically satisfied if N is compact. We want to solve the homogeneous Cauchy problem (3.13) for hyperfunctions (or analytic functions). If we chose γ as in the proof of Proposition 3.10, then the projection on the second factor would not be proper on the sets $J^+_{\preceq}(x)$ for any $x \in M$. We set

$$\gamma = \{(x, t; v, w) \in TM; w > 1/(2f(t))|v|_g\}. \tag{3.19}$$

One has $\gamma^\circ = \{(x, t; \xi, \tau) \in T^*M; \tau \ge 2f(t)|\xi|_g\}$. By Lemma 1.3, $\mathrm{hypchar}(P) \cap \gamma^\circ \subset T^*_M M$.

One checks that γ is the future cone of the Lorentzian spacetime $(M, dt^2 - (1/2f(t))g)$, which is globally hyperbolic by Lemma 3.16. Therefore, (M, γ, q) is a G-causal manifold and we can apply Corollary 3.8 which asserts that the Cauchy problem (3.13) for hyperfunctions (or analytic functions) is globally well-posed.

As in the compact case, if $(g_t)_{t\in\mathbb{R}}$ is an analytic family of *complete* Riemannian metrics on N and $(\Delta_t)_{t\in\mathbb{R}}$ are the associated Laplace–Beltrami operators, then the operator $P = \partial_t^2 - \Delta_t$ is such an example.

3.4.4 The wave operator on a globally hyperbolic Lorentzian spacetime

Definition 3.18. Let (M, g) be a real analytic Lorentzian manifold. The *wave operator* is defined by

$$\Box = -\operatorname{div}\operatorname{grad}. \tag{3.20}$$

A *wave-type operator* on (M, g) is a differential operator P whose symbol satisfies

$$\sigma(P)(x; \xi) = |\xi|_{g_x}^2 \tag{3.21}$$

for $(x; \xi) \in T^*M$.

A standard calculation shows that the wave operator on a Lorentzian manifold is a wave-type operator on that Lorentzian manifold.

Lemma 3.19. *Let (M, g) be a Lorentzian spacetime and let P be a wave-type operator, then* $\mathrm{hypchar}(P) \cap \mathrm{Int}(\gamma_g^\circ) \subset T^*_M M$.

Proof. This follows directly from Lemma 1.3. Q.E.D.

Theorem 3.20. *Let (M, g) be a real analytic globally hyperbolic Lorentzian spacetime and let P be a wave-type operator on M. Let $N \subset M$ be a real analytic Cauchy hypersurface and let v be an analytic vector field defined in a neighborhood of N and transversal to N. Then the Cauchy problem*

$$\begin{cases} Pf = 0 \\ (f|_N, v(f)|_N) = (h_0, h_1) \end{cases} \tag{3.22}$$

with $h_0, h_1 \in \mathscr{B}(N)$ (resp. $\mathscr{A}(N)$) has a unique global solution in $\mathscr{B}(M)$ (resp. $\mathscr{A}(M)$).

Furthermore, the operator P is a surjective endomorphism of $\mathscr{B}(M)$ and of $\mathscr{A}(M)$.

Proof. Since global hyperbolicity is a stable property, there exists a Lorentzian metric $\tilde{g}$ on M such that $(M, \tilde{g})$ is globally hyperbolic and $\overline{\gamma} \subset \tilde{\gamma} \cup \{0\}$, so $\tilde{\gamma}^\circ \subset \mathrm{Int}(\gamma^\circ) \cup \{0\}$. By Lemma 3.19, $\mathrm{hypchar}(P) \cap \tilde{\gamma}^\circ \subset \mathrm{hypchar}(P) \cap (\mathrm{Int}(\gamma^\circ) \cup T^*_M M) \subset T^*_M M$.

Let q be a Cauchy time function such that $q^{-1}(0) = N$. We apply Corollary 3.8 to the G-causal manifold $(M, \tilde{\gamma}, q)$ and the $\mathscr{D}_X$-module $\mathscr{D}_X / \mathscr{D}_X P$. Then the proof goes as for Proposition 3.10. Q.E.D.

One shall notice that in Theorem 3.20, there is no assumption that the initial data be compactly supported.

A Appendix: normal cones

References are made to [KS90].

Let A, B be two subsets of M. The *Whitney cone* $C(A, B)$ (see [KS90, Def. 4.1.1]) is a closed conic subset of TM. In a chart at $x_0 \in M$, it is described as follows.

$$v \in C_{x_0}(A, B) \subset T_{x_0}M \Leftrightarrow \begin{cases} \text{there exists a sequence } \{(x_n, y_n, \lambda_n)\}_n \subset A \times \\ B \times \mathbb{R}_{>0} \text{ such that} \\ x_n \overset{n}{\to} x_0,\ y_n \overset{n}{\to} x_0,\ \lambda_n(x_n - y_n) \overset{n}{\to} v. \end{cases} \tag{A.1}$$

For short, we define $C_x(A) = C_x(A, \{x\}) \subset T_x M$ and often identify it with $C(A, \{x\}) \subset TM$. This is a closed cone of $T_x M$, the set of limits when $y \in A$ goes to x of half-lines issued at x and passing through y. More generally, for N a smooth submanifold of M, the set $C(A, N)$ satisfies $N \times_M C(A, N) + TN = N \times_M C(A, N)$ and one denotes by $C_N(A)$ the image of $N \times_M C(A, N)$ in $T_N M = (N \times_M TM)/TN$.

Let us recall without proof some elementary properties of Whitney cones that we will need later.

Proposition A.1. *Let L, M, N be manifolds and let $g \colon L \to M$ and $f \colon M \to N$ be morphisms of manifolds. Let $A, A_1, A_2, B \subset M$. Then*

(i) *the Whitney cone $C(A, B) \subset TM$ is a closed cone,*

(ii) *$C(A, B) = -C(B, A)$,*

(iii) *if* $A_1 \subset A_2$, *then* $C(A_1, B) \subset C(A_2, B)$,

(iv) $C(A_1 \cup A_2, B) = C(A_1, B) \cup C(A_2, B)$,

(v) $C(\overline{A}, B) = C(A, B)$,

(vi) $C(A, B) \cap T_M M = \overline{A} \cap \overline{B}$,

(vii) $C(g^{-1}(A), g^{-1}(B)) \subset (Tg)^{-1} C(A, B)$,

(viii) $Tf(C(A, B)) \subset C(f(A), f(B))$.

Let A be a subset of M. Recall ([KS90, Def. 5.3.6]) that the *strict normal cone* of A is the set

$$N(A) = TM \setminus C(M \setminus A, A). \tag{A.2}$$

This is an open convex cone of TM (see below). In a chart at $x \in M$, one has:

$$(x, v) \in N(A) \Leftrightarrow \begin{cases} \text{there exists an open cone } \gamma_0 \text{ with } v \in \gamma_0 \\ \text{and an open neighborhood } U \text{ of } x \text{ such} \\ \text{that } U \cap (U \cap A + \gamma_0) \subset A. \end{cases} \tag{A.3}$$

As usual, for $x \in M$, one sets $N_x(A) = T_x M \cap N(A)$.

Proposition A.2. *Let* $A \subset M$ *and let* $x \in \overline{A}$. *Then* $\overline{N_x(A)} \subset C_x(A)$.

Proof. Let us choose a chart at x. Since $C_x(A)$ is closed, it is enough to check the inclusion $N_x(A) \subset C_x(A)$. Let $v \in N_x(A)$. There is a neighborhood U of x and a conic neighborhood γ_0 of v such that $U \cap (U \cap A + \gamma_0) \subset A$. Since $x \in \overline{A}$, there is a sequence $x_n \to x$ with $x_n \in A$. Let $c_n > 0$ be a sequence with $c_n \overset{n}{\to} +\infty$ and $c_n(x - x_n) \overset{n}{\to} 0$. Set $y_n = x_n + c_n^{-1} v$. Then for n large enough, $y_n \in A$ and

$$c_n(y_n - x) = c_n(y_n - x_n) + c_n(x_n - x) = v + c_n(x_n - x) \overset{n}{\to} v.$$

Q.E.D.

For short, we set

$$D(A) = C(M \setminus A, A) \tag{A.4}$$

and we call $D(A)$ the *cone of outgoing vectors of* A. Most of the following properties are direct consequences of the corresponding properties of the Whitney cone. We gather them in a proposition for later reference.

Proposition A.3. *Let* L, M, N *be manifolds and let* $g \colon L \to M$ *be a morphism of manifolds. Let* $A, A_1, A_2 \subset M$ *and* $B \subset N$. *Then*

(i) $N(A)$ *is an open convex cone,*

(ii) $N(M \setminus A) = N(A)^a$,

(iii) $N(\varnothing) = N(M) = TM$,

(iv) $N_x(A) = T_xM$ *if and only if* $x \notin \partial A$,

(v) $N(A_1) \cap N(A_2) \subset N(A_1 \cup A_2)$ *and* $N(A_1) \cap N(A_2) \subset N(A_1 \cap A_2)$,

(vi) $N(A) \subset N(\overline{A})$ *and* $N(A) \subset N(\mathrm{Int}A)$,

(vii) *if* $N_x(A) \neq \varnothing$ *for all* $x \in \partial A$, *then* $\overline{\mathrm{Int}A} = \overline{A}$ *and* $\mathrm{Int}\overline{A} = \mathrm{Int}A$,

(viii) *if* $\overline{A_1} = \overline{A_2}$ *and* $\mathrm{Int}A_1 = \mathrm{Int}A_2$, *then* $N(A_1) = N(A_2)$,

(ix) $(Tg)^{-1}(N(A)) \subset N(g^{-1}(A))$,

(x) $N(A) \times N(B) \subset N(A \times B)$.

Proof. (i) The set $N(A)$ is an open cone by its definition. Let us choose a chart at $x \in M$. Let $v, w \in T_xM$ with $v + w \in D_x(A)$. Then there is a sequence $\{(x_n, y_n, c_n)\}_n$ in $A \times (M \setminus A) \times \mathbb{R}_{>0}$ with $x_n \overset{n}{\to} x$, $y_n \overset{n}{\to} x$ and $c_n(y_n - x_n) \overset{n}{\to} v + w$. There are infinitely many $x_n + (1/c_n)v$ contained either in A or in $M \setminus A$. In the second case, $v \in D_x(A)$, and in the first case, $w \in D_x(A)$. This shows that if both v and w belong to $N_x(A)$, then so does $v + w$.

(ii)–(iv) are obvious.

(v) We have

$$\begin{aligned} D(A_1 \cup A_2) &= C(M \setminus (A_1 \cup A_2), A_1 \cup A_2) \\ &= C((M \setminus A_1) \cap (M \setminus A_2), A_1) \cup C((M \setminus A_1) \cap (M \setminus A_2), A_2) \\ &\subset C(M \setminus A_1, A_1) \cup C(M \setminus A_2, A_2) = D(A_1) \cup D(A_2). \end{aligned}$$

The second inclusion follows, using (ii).

(vi) We have

$$\begin{aligned} D(\overline{A}) &= C(\overline{A}, M \setminus \overline{A}) \\ &= C(A, M \setminus \overline{A}) \subset C(A, M \setminus A) = D(A). \end{aligned}$$

This proves the first inclusion. One deduces the second inclusion by using (ii).

(vii) Let $x \in \partial A$ and assume that $N_x(A) \neq \varnothing$. We shall prove that $x \in \overline{\mathrm{Int}A}$. We choose a chart at x. Let $v \in N_x(A)$. Then there exist a neighborhood V of x and a conic open neighborhood C of v such that $V \cap (x + C) \subset A$. Hence, there exists a sequence $\{t_n\}_n$, $t_n > 0$, $t_n \overset{n}{\to} 0$ such that $x + t_n v \in \mathrm{Int}A$. Therefore, $x \in \overline{\mathrm{Int}A}$. The other inclusion follows, using (ii).

(viii) By Proposition A.3 (v), using $\overline{M \setminus A} = M \setminus \mathrm{Int}A$, we get

$$D(A) = C(M \setminus A, A) = C(\overline{M \setminus A}, \overline{A}) = C(M \setminus \mathrm{Int}A, \overline{A}).$$

Hence, $D(A)$ depends only on $\overline{A}$ and $\mathrm{Int}A$.

(ix) By Proposition A.3 (vii), we have

$$\begin{aligned} D(f^{-1}(A)) &= C(M \setminus f^{-1}(A), f^{-1}(A)) \\ &= C(f^{-1}(L \setminus A), f^{-1}(A)) \\ &\subset Tf^{-1}(C(L \setminus A, A)) = Tf^{-1}(D(A)), \end{aligned}$$

and the result follows.

(x) Let $(v_1, v_2) \in T_{(x,y)}(M \times N)$. We choose local charts centered at x and y. Then $(x, y; v_1, v_2) \in N(A) \times N(B)$ if and only if there exist neighborhoods U of x, V of y and conic open neighborhoods γ_1 of v_1 and γ_2 of v_2, such that, setting $W = U \times V$

$$W \cap \big((W \cap A \times B) + \gamma_1 \times \gamma_2\big) \subset A \times B.$$

Therefore, $(v_1, v_2) \in N_{(x,y)}(A \times B)$. Q.E.D.

Remark A.4. One could improve Proposition A.3 (x) and show that for $(x, y) \in \overline{A \times B}$, $N_{(x,y)}(A \times B) = N_x(A) \times N_y(B)$ but we do not use this result.

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