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Inference for asymptotically independent samples of extremes

Armelle Guillou⁽¹⁾, Simone A. Padoan⁽²⁾ and Stefano Rizzelli⁽²⁾

⁽¹⁾ *Institut Recherche Mathématique Avancée, UMR 7501
Université de Strasbourg et CNRS
7 rue René Descartes
67084 Strasbourg cedex
France*

e-mail: armelle.guillou@math.unistra.fr

⁽²⁾ *Department of Decision Sciences,
Bocconi University of Milan
via Roentgen 1
20136 Milano, Italy*

*e-mail: simone.padoan@unibocconi.it
stefano.rizzelli@phd.unibocconi.it*

Abstract: An important topic of the multivariate extreme-value theory is to develop probabilistic models and statistical methods to describe and measure the strength of dependence among extreme observations. The theory is well established for data whose dependence structure is compatible with that of asymptotically dependence models. On the contrary, in many applications data do not comply with asymptotic dependence models and in such cases there are less guidelines available. This is especially true when considering the componentwise maxima approach. In this paper we contribute to extending this part. We propose a statistical test based on the classical Pickands dependence function to verify whether asymptotic dependence or independence holds. Then, we present a new Pickands dependence function to describe the extremal dependence under asymptotic independence. We propose an estimator of the latter and we study its main asymptotic properties and its performance is illustrated by a simulation study.

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1. Introduction

Multivariate extreme-value theory provides the mathematical foundation for performing real data analysis of rare events. To characterize the joint upper tail of a multivariate distribution, two different approaches can be used: either by considering the componentwise maxima, or all the observations above a high threshold. A description of these methodologies can be found for instance in [Coles \(2001, Ch. 8\)](#), [Beirlant et al. \(2004, Ch. 8-9\)](#), [de Haan and Ferreira \(2006, Ch. 6\)](#) and [Resnick \(2007, Ch. 6\)](#), among others. Unfortunately the flexibility of the dependence structures provided by the classical theory of multivariate extreme-values may not be sufficient for statistical modelling (see e.g. [Ledford and Tawn, 1996, 1997](#)). To solve this issue, different coefficients of tail dependence or probabilistic models have been introduced. They allow to govern/describe the strength of the extremal dependence. In this paper, we are particularly interested in the notion of asymptotic independence which is common in real data analysis. This concept can be defined as follows.

Let \mathbf{Y} be a multivariate random vector of dimension d , with distribution function F and marginals F_j , $1 \leq j \leq d$. We say that F is in the max-domain of attraction of a multivariate extreme-value distribution G , if there exist sequences of constants $\mathbf{a}_n > \mathbf{0}$ and $\mathbf{b}_n \in \mathcal{R}^d$ such that

$$\lim_{n \rightarrow \infty} F^n(\mathbf{a}_n \mathbf{y} + \mathbf{b}_n) = G(\mathbf{y}),$$

for all $\mathbf{y} \in \mathcal{R}^d$. Under this condition, a particular case arises when G is equal to the product of its marginal distributions. In this setting, we say that \mathbf{Y} satisfies the property of asymptotic independence (or tail independence)

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which is equivalent to say that the elements of \mathbf{Y} are asymptotically independent in the upper tail, i.e.

$$\lim_{u \rightarrow 1} \Pr(F_j(Y_j) > u | F_i(Y_i) > u) = 0$$

for all $1 \leq i \neq j \leq d$. On the contrary the elements of \mathbf{Y} are said asymptotically dependent. The classical theory expects asymptotic dependence and independence as the only two possible scenarios, conceiving extremes as independent in the second case. Many efforts have been made to characterize a residual tail dependence in the data (if there is any) by offering new dependence coefficients or probabilistic and statistical models under asymptotic independence, see [Ledford and Tawn \(1996\)](#), [Coles \(2001, Ch 8.4\)](#), [Resnick \(2002\)](#), [Maulik and Resnick \(2004\)](#), [Ramos and Ledford \(2009, 2011\)](#), [Wadsworth and Tawn \(2013\)](#) and [Wadsworth et al. \(2017\)](#), to name a few. If we restrict our framework to the dimension $d = 2$, several statistical tests for checking asymptotic independence or tail independence have been proposed, among them, [Ledford and Tawn \(1996\)](#), [Draisma et al. \(2004\)](#), [Hüsler and Li \(2009\)](#) and [Falk et al. \(2010, Ch. 6.5\)](#) and the references therein. However, the extension to dimensions higher than 2 are still in its infancy. Recent proposals are based on the k th largest order statistics of the sample. Although these approaches are simple to implement, the performance of the resulting tests depends strongly on the choice of k , see e.g. [Kiriliouk et al. \(2016\)](#).

In this paper, we propose in [Section 2](#) an alternative approach to test asymptotic independence for an arbitrary dimension $d \geq 2$, based on the componentwise maxima. We illustrate the performance of our proposal up to dimension $d = 4$. Then, using again the componentwise maxima approach and in particular the framework proposed by [Ramos and Ledford \(2011\)](#), we introduce, in [Section 3](#), a new dependence function similar to the well-known Pickands dependence function which allows us to measure the residual dependence under asymptotic independence. Finally, we estimate this new dependence function and we establish the main asymptotic properties of the estimator. By means of a simulation study, its good performance is highlighted. All the proofs are postponed to the appendix.

Throughout the paper, the following notations are used. For any arbitrary dimension d and $f : \mathcal{X} \subset \mathcal{R}^d \rightarrow \mathcal{R}$, we set $\|f\|_\infty = \sup_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x})$. We denote by $\ell^\infty(\mathcal{X})$ the space of all bounded real-valued functions on \mathcal{X} . The symbol “ \rightsquigarrow ” stands for convergence in distribution of random vectors, but also for weak convergence of bounded real-valued functions in $\ell^\infty(\mathcal{X})$, the difference will be clear from the context.

2. A test for asymptotic independence

A d -dimensional random vector $\mathbf{X} = (X_1, \dots, X_d)$ follows the law of a multivariate extreme-value distribution if the one-dimensional marginal distributions, $G_j(x) = \Pr(X_j \leq x)$ for all $x \in \mathcal{R}$, $j = 1, \dots, d$, are Generalized Extreme-Value (GEV) distributions, and the joint distribution takes the form

$$G(\mathbf{x}) = C(G_1(x_1), \dots, G_d(x_d)), \quad \mathbf{x} \in \mathcal{R}^d,$$

where C is an extreme-value copula, i.e.,

$$C(\mathbf{u}) = \exp\left(-V\left((-\log u_1)^{-1}, \dots, (-\log u_d)^{-1}\right)\right), \quad \mathbf{u} \in (0, 1]^d,$$

with $V : (0, \infty)^d \rightarrow [0, \infty)$ (see [de Haan and Ferreira, 2006](#), Ch. 1, 6, for details). Consider the map $L : [0, \infty)^d \mapsto [0, \infty)$, defined by $L(\mathbf{z}) := V(1/\mathbf{z})$ with $\mathbf{z} = 1/\mathbf{y}$ for $\mathbf{y} \in (0, \infty)^d$. The function L is known as the stable tail dependence. As it is a homogeneous function of order one, i.e. $L(a\mathbf{z}) = aL(\mathbf{z})$ for all $a > 0$, we have

$$L(\mathbf{z}) = (z_1 + \dots + z_d)A(\mathbf{t}), \quad \mathbf{z} \in [0, \infty)^d,$$

with $t_j = z_j/(z_1 + \dots + z_d)$ for $j = 2, \dots, d$, $t_1 = 1 - t_2 - \dots - t_d$, and A is the restriction of L into the d -dimensional unit simplex,

$$\mathcal{S}_d := \{(v_1, \dots, v_d) \in [0, 1]^d : v_1 + \dots + v_d = 1\}.$$

The function A is well-known as the Pickands dependence function (see [Pickands, 1981](#)), and is often used to quantify the dependence among the elements of \mathbf{X} . Indeed, A satisfies the constraint $1/d \leq \max(t_1, \dots, t_d) \leq$

$A(\mathbf{t}) \leq 1$ for all $\mathbf{t} \in \mathcal{S}_d$, with lower and upper bounds corresponding to the complete dependence and independence cases, respectively (see Falk et al., 2010, Ch. 4, for details). Thus, estimating this Pickands dependence function is crucial for analysing multivariate extremes, and it has been an extensively discussed topic in the literature, see Klüppelberg and May (2006), Zhang et al. (2008), Gudendorf and Segers (2011), Bücher et al. (2011), Berghaus et al. (2013) or Vettori et al. (2017), among others.

2.1. A slightly modified version of the Pickands dependence estimator proposed by Marcon et al. (2017)

This estimator is based on the madogram concept, a notion borrowed from geostatistics in order to capture the spatial structure. Starting from independent and identically distributed (i.i.d.) copies X_1, \dots, X_n , of X , our modified estimator is defined as

$$\widehat{A}_n(\mathbf{t}) := \frac{\widehat{v}_n(\mathbf{t}) + c(\mathbf{t})}{1 - \widehat{v}_n(\mathbf{t}) - c(\mathbf{t})} \quad (2.1)$$

where

$$\begin{aligned} \widehat{v}_n(\mathbf{t}) &:= \frac{1}{n} \sum_{i=1}^n \left(\bigvee_{j=1}^d \{G_{n,j}^{(1)}(X_{i,j})\}^{1/t_j} - \frac{1}{d} \sum_{j=1}^d \{G_{n,j}^{(1)}(X_{i,j})\}^{1/t_j} \right) \\ c(\mathbf{t}) &:= \frac{1}{d} \sum_{j=1}^d \frac{t_j}{1 + t_j} \end{aligned} \quad (2.2)$$

with

$$G_{n,j}^{(a)}(X_{i,j}) := G_{n,j}(X_{i,j}) \left(\frac{1+a}{a} \frac{1}{n} \sum_{k=1}^n G_{n,j}^{1/a}(X_{k,j}) \right)^{-a}, \quad j = 1, \dots, d, \text{ for } a > 0,$$

and the empirical distribution functions denoted by

$$G_{n,j}(x) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_{i,j} \leq x\}}, \quad j = 1, \dots, d.$$

By convention, here $u^{1/0} = 0$ for $0 < u < 1$. Compared to the proposal in Marcon et al. (2017), our slightly modified version based on the use of $G_{n,j}^{(1)}$ instead of $G_{n,j}$ in (2.2), ensures that the new Pickands estimator \widehat{A}_n now satisfies $\widehat{A}_n(\mathbf{e}_j) = 1$ for all $j = 1, \dots, d$, where $\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the j th canonical unit vector (see Appendix A.1). This is a necessary condition that a function needs to satisfy in order to be a valid Pickands dependence function (see e.g. Marcon et al., 2017). Although as established in Appendix A.1, our modified estimator shares the same asymptotic properties as the estimator discussed in Marcon et al. (2017), our modification greatly improves the latter for finite samples.

2.2. Construction of our statistical test

Using our estimator for A , we want now to construct a statistical test to check asymptotic independence in dimensions higher than or equal to two. To this aim, we consider the following system of hypotheses

$$\begin{cases} H_0 : & A(\mathbf{t}) = 1, \forall \mathbf{t} \in \mathcal{S}_d \\ H_1 : & A(\mathbf{t}) < 1, \text{ for some } \mathbf{t} \in \mathcal{S}_d. \end{cases}$$

Note that H_0 means that all the components of X are asymptotically independent, whereas under H_1 some elements of X are asymptotically dependent.

Assuming that the extreme-value copula C has continuous partial derivatives over the sets $\{\mathbf{u} \in [0, 1]^d : 0 < u_i < 1\}$, by Theorem 2.4 in Marcon et al. (2017) and according to Appendix A.1, we have under H_0

$$\sqrt{n} \left(\widehat{A}_n(\mathbf{t}) - 1 \right)_{\mathbf{t} \in \mathcal{S}_d} \rightsquigarrow -4 \left(\int_0^1 \mathbb{A}(v^{t_1}, \dots, v^{t_d}) dv \right)_{\mathbf{t} \in \mathcal{S}_d}, \quad \text{as } n \rightarrow \infty, \quad (2.3)$$

where \mathbb{A} is a centered Gaussian process on $[0, 1]^d$ with continuous sample paths and covariance function equal to

$$\text{Cov}(\mathbb{A}(\mathbf{u}), \mathbb{A}(\mathbf{v})) = \prod_{j=1}^d u_j \wedge v_j - \sum_{j=1}^d \left(u_j \wedge v_j \prod_{i \neq j} u_i v_i \right) + (d-1) \prod_{j=1}^d u_j v_j.$$

As a consequence, by the continuous mapping theorem (see e.g. [van der Vaart, 2000](#), Ch. 2.1), it follows that

$$\widehat{S}_n := \sup_{\mathbf{t} \in \mathcal{S}_d} \sqrt{n} \left| \widehat{A}_n(\mathbf{t}) - 1 \right| \rightsquigarrow S := \sup_{\mathbf{t} \in \mathcal{S}_d} 4 \left| \int_0^1 \mathbb{A}(v^{t_1}, \dots, v^{t_d}) dv \right|, \quad \text{as } n \rightarrow \infty.$$

This convergence can be used as the cornerstone of our test. Denoting by $Q_S(\alpha)$, $\alpha \in (0, 1)$, the $(1 - \alpha)$ -quantile function for the distribution of the random variable S , H_0 can be rejected with a $(1 - \alpha)\%$ confidence level whenever \widehat{s}_n , the observed value of \widehat{S}_n , exceeds $Q_S(\alpha)$. Unfortunately, there is no closed form for the function $Q_S(\alpha)$, however an approximation can still be computed with a Monte Carlo simulation as follows.

Note that for any $u, v \in [0, 1]$ and $\mathbf{t}, \mathbf{w} \in \mathcal{S}_d$, the covariance function of the Gaussian process \mathbb{A} in (2.3), evaluated at the indexes $u^{\mathbf{t}}, v^{\mathbf{w}} \in [0, 1]^d$, is equal to

$$\text{Cov}(\mathbb{A}(v^{t_1}, \dots, v^{t_d}), \mathbb{A}(u^{w_1}, \dots, u^{w_d})) = \prod_{j=1}^d (v^{t_j} \wedge u^{w_j}) - \sum_{j=1}^d (v^{t_j} \wedge u^{w_j}) v^{1-t_j} u^{1-w_j} + (d-1)uv. \quad (2.4)$$

Thus, for any fixed $\alpha \in (0, 1)$, an approximation of the quantile $Q_S(\alpha)$ can be obtained by adhering to the following four steps:

1. Divide the unit interval $(0, 1)$ and the simplex \mathcal{S}_d in p and m equally spaced points, where p and m are positive integers. Let v_1, \dots, v_m and $\mathbf{t}_1, \dots, \mathbf{t}_p$ be the two partitions of $(0, 1)$ and \mathcal{S}_d , respectively. The sequences v_1, \dots, v_m and $\mathbf{t}_1, \dots, \mathbf{t}_p$ form a finite sequence of positions $v_r^{k,1}, \dots, v_r^{k,d} \in [0, 1]^d$, with $r = 1, \dots, m$ and $k = 1, \dots, p$, on which the process \mathbb{A} is simulated.
2. Sample n^* realizations

$$x_i(v_1^{t_{1,1}}, \dots, v_1^{t_{1,d}}), \dots, x_i(v_m^{t_{p,1}}, \dots, v_m^{t_{p,d}}), \quad i = 1, \dots, n^*,$$

of a zero-mean Gaussian process at $v_r^{k,1}, \dots, v_r^{k,d}$, for $r = 1, \dots, m$ and $k = 1, \dots, p$, with a $(mp \times mp)$ variance-covariance matrix defined through the covariance function in (2.4).

3. Simulate samples that approximately follow the distribution of the random variable S , the integral and the sup in S being approximated by a sum and the max for sufficiently large values of m and p . This leads to the realizations

$$\widetilde{s}_i = \max_{1 \leq k \leq p} 4 \left| \frac{1}{m} \sum_{r=1}^m x_i(v_r^{k,1}, \dots, v_r^{k,d}) \right|, \quad i = 1, \dots, n^*.$$

4. An approximation of the quantile $Q_S(\alpha)$, denoted by $\widetilde{Q}_S(\alpha)$, can then be obtained by computing the sample quantile of the realizations $\widetilde{s}_1, \dots, \widetilde{s}_{n^*}$ for sufficiently large n^* .

2.3. Numerical results

We illustrate the performance of our statistical test through a simulation study. Precisely, we estimate some values of the significance level α and the power $1 - \beta$ of the test by computing the empirical proportion of simulated samples under the null hypothesis and the alternative hypothesis that rejected the null hypothesis, respectively. For simplicity we focus on the significance levels $\alpha = 0.05$ and 0.01 .

In order to perform the simulation study, the first step consists of computing the approximated quantile $\widetilde{Q}_S(\alpha)$, for a given α , following our algorithm. In particular, the goodness of the approximation relies on the values of the indexes m , p and n^* . Clearly, the larger their values are, the more accurate the approximation is. We set $n^* = 500\,000$. We consider increasing values of m and p and for each combination we compute \widetilde{Q}_S . We stop the search of a

better value for these indexes when the value of $\widetilde{Q}_S(\alpha)$ does not increase anymore, up to the second decimal. The calculation of \widetilde{Q}_S requires a considerable computational effort, therefore we derive its values only for a dimension $d = 2, 3, 4$ of the vector \mathbf{X} .

Then, in a second step, we compute the rejection rates. To this aim, we consider two experiments:

First experiment: We focus on the multivariate logistic extreme-value model introduced by Tawn (1990), with dependence parameter $\psi \in (0, 1]$, $\psi = 1$ corresponding to independent components of \mathbf{X} , whereas complete dependence can be reached when $\psi \rightarrow 0$. We consider 20 equally spaced values of ψ in $(0, 1]$. For each of them, we simulate n independent observations from a logistic extreme-value distribution with unit Fréchet margins. Then we estimate the Pickands dependence function by (2.1) and we compute \widehat{s}_n . We repeat this task 5000 times and we compute the proportion of times that $\widehat{s}_n > \widetilde{Q}_S(\alpha)$. This experiment is repeated for different values of the samples sizes n and different dimension d of \mathbf{X} . Table 1 reports the estimated values of the significance levels α .

TABLE 1

Estimated significance levels α . From left to right: the dimension of \mathbf{X} , the true significance level, the approximate $(1 - \alpha)$ -quantile and the empirical proportion of simulated samples under H_0 that rejected the null hypothesis.

d	α	$\widetilde{Q}_S(\alpha)$	n			
			25	50	100	200
2	0.05	0.960	0.0380	0.0460	0.0512	0.0524
	0.01	1.204	0.0060	0.0082	0.0102	0.0102
3	0.05	1.300	0.0364	0.0452	0.0508	0.0574
	0.01	1.540	0.0056	0.0068	0.0084	0.0092
4	0.05	1.480	0.0398	0.0454	0.0548	0.0576
	0.01	1.740	0.0064	0.0082	0.0096	0.0126

We see that accurate estimates of α are already obtained with the sample size $n = 50$, indicating a good performance of our statistical test. Figure 1 displays the estimated powers of the test. In the top and bottom rows the results obtained with $\alpha = 0.05$ and $\alpha = 0.01$ are reported, respectively. The panels from left to right illustrate the results for the dimensions 2, 3 and 4. Once again, the test shows a good performance already with the sample size $n = 50$. Indeed in the case $d = 2$ we see that the power of the test reaches 1 with mild dependence levels, i.e. $\psi = 0.5$. This figure also outlines that the power of the test improves with the dimensions of \mathbf{X} and that, as expected, for any fixed dimension $d = 2, 3, 4$, it also improves with the sample size.

Second experiment: We consider the inverted multivariate logistic extreme-value model (see e.g. Ledford and Tawn, 1997; Wadsworth et al., 2017), with dependence parameter $\psi \in (0, 1]$, $\psi = 1$ corresponding to exact independence of the components of \mathbf{X} , whereas asymptotic dependence is reached as $\psi \rightarrow 0$. This time, we consider 10 equally spaced values of ψ in $(0, 1]$. For each of them, we simulate 366 values from an inverted logistic distribution with exponential margins. Then, we compute the normalized componentwise maxima and we repeat this procedure in order to obtain n normalized maxima from which we estimate the Pickands dependence function and we calculate \widehat{s}_n . We repeat this task 5000 times and we compute the proportion of times that $\widehat{s}_n > \widetilde{Q}_S(0.05)$. This procedure has been done for different values of d and n and the results are summarized in Table 2.

With $d = 2$, the rejection rates are close to 0.05 whenever ψ is larger than 0.5. Otherwise, the rejection rate is greater than 0.05 and it reaches 1 when ψ approaches 0. In these cases, it can be observed that the normalized maxima show quite a strong dependence, which indeed seems that of an asymptotic dependence model rather than asymptotic independence. The strength of the dependence is reduced whenever the normalized maxima are computed on sequences larger than 366, resulting in improvements in the performances of our test. The test performance deteriorates as the dimension of \mathbf{X} increases. This behavior is expected, because with our test the null hypothesis is rejected whenever a pair of variables turns out to be asymptotically dependent. In conclusion this study highlights the good performance of our statistical test not only for exactly independent multivariate data but also for asymptotically independent data.

3. Asymptotic independence for componentwise maxima

Being able to test asymptotic independence versus asymptotic dependence is obviously important, but the models obtained via the classical multivariate extreme-value theory work in general well only under asymptotic depen-

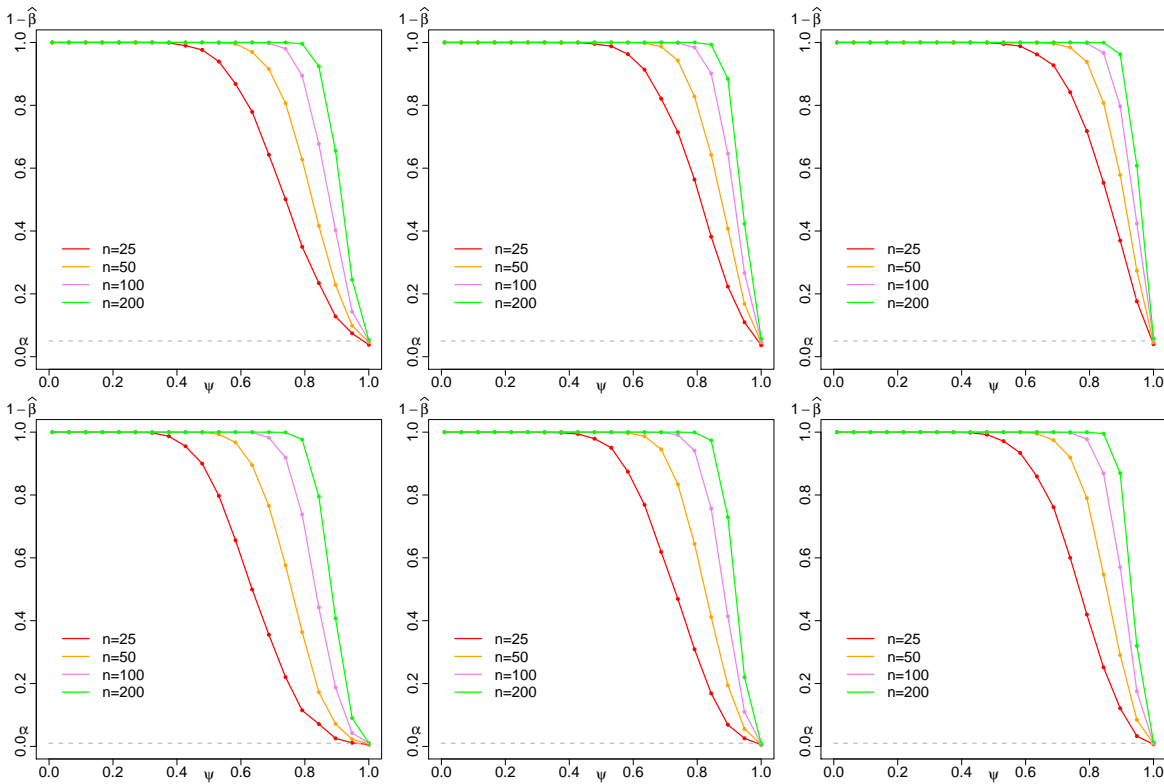


FIG 1. Estimated power functions $1 - \beta$. Points report the empirical proportion of simulated samples under H_1 that rejected the null hypothesis as a function of ψ . From left to right, the dimension of X : 2, 3 and 4 and from the top to the bottom: $\alpha = 0.05$ and $\alpha = 0.01$.

TABLE 2

Estimated significance levels α . From left to right: the dimension of X , the sample size and the empirical proportion of simulated samples under H_0 that rejected the null hypothesis for different values of ψ .

d	n	ψ									
		1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
2	25	0.0398	0.0406	0.0390	0.0458	0.0518	0.0540	0.0898	0.2356	0.6046	0.9846
	50	0.0470	0.0428	0.0448	0.0492	0.0580	0.0610	0.1686	0.4350	0.8924	1.0000
	100	0.0518	0.0520	0.0512	0.0520	0.0534	0.1200	0.2760	0.7160	0.9948	1.0000
	200	0.0492	0.0528	0.0528	0.0552	0.0530	0.1726	0.5038	0.9404	1.0000	1.0000
3	25	0.0316	0.0310	0.0368	0.0416	0.0512	0.0602	0.1674	0.4234	0.8604	0.9990
	50	0.0426	0.0440	0.0526	0.0498	0.0524	0.1298	0.3180	0.7344	0.9910	1.0000
	100	0.0514	0.0500	0.0560	0.0536	0.0640	0.1208	0.5450	0.9484	1.0000	1.0000
	200	0.0548	0.0554	0.0574	0.0652	0.0752	0.1332	0.6332	0.8188	0.9984	1.0000
4	25	0.0580	0.0518	0.0524	0.0654	0.1068	0.1648	0.3490	0.6842	0.8696	1.0000
	50	0.0518	0.0554	0.0548	0.0784	0.1120	0.1888	0.3482	0.6454	0.9238	1.0000
	100	0.0568	0.0575	0.0577	0.0657	0.1560	0.2900	0.5895	0.9975	0.9996	1.0000
	200	0.0532	0.0536	0.0584	0.0932	0.1866	0.3666	0.7316	0.9896	1.0000	1.0000

dence. Since asymptotic independence often arises in applications, it is thus crucial to develop some general models that accommodate both situations. In this section, we consider the framework of Ramos and Ledford (2009) (see also Ledford and Tawn, 1997). More precisely, if Y is a d -dimensional random vector with common unit Fréchet margins, i.e. $\Pr(Y \leq y) = e^{-1/y}$ for every $y > 0$, this theory relies on the joint survival function of Y which is assumed to be multivariate regularly varying with index $-1/\eta$, where $\eta \in (0, 1]$, i.e. $\Pr(Y > \mathbf{y}) = \tau(\mathbf{y})(y_1 \cdots y_d)^{-1/d\eta}$

with τ a slowly varying function satisfying

$$\lim_{r \rightarrow \infty} \frac{\tau(r y_1, \dots, r y_d)}{\tau(r, \dots, r)} = g(\mathbf{y})$$

for all $\mathbf{y} \in (0, \infty]^d$. The function g here is homogeneous of order 0, i.e. such that $g(a y_1, \dots, a y_d) = g(y_1, \dots, y_d)$ for any $a > 0$. This framework implies that the joint survival function can be rephrased for all $\mathbf{y} > \mathbf{1}$, the vector of ones, as

$$\Pr(\mathbf{Y} > \mathbf{y}) = \lim_{r \rightarrow \infty} \frac{\Pr(\mathbf{Y} > r\mathbf{y})}{\Pr(\mathbf{Y} > r\mathbf{1})} = \eta \int_{\mathcal{S}_d} \bigwedge_{j=1}^d \left(\frac{w_j}{y_j} \right)^{1/\eta} dH_\eta(\mathbf{w})$$

where H_η is a non-negative measure satisfying the condition

$$\eta \int_{\mathcal{S}_d} \bigwedge_{j=1}^d w_j^{1/\eta} dH_\eta(\mathbf{w}) = 1.$$

This measure H_η is a particular case of the hidden angular measure introduced by [Resnick \(2002\)](#) (see also [Maulik and Resnick, 2004](#)) when $\eta < 1$ and it is a rescaled version of the classical angular measure when $\eta = 1$, see [Ramos and Ledford \(2009\)](#) for details. According to [Ramos and Ledford \(2011\)](#) we assume that H_η is a finite measure. We recall that η is the so-called coefficient of tail dependence, which measures the level of dependence within the asymptotic independence framework. Specifically, $\eta = 1$ corresponds to the case of asymptotic dependence, whereas $\eta < 1$ corresponds to the case of asymptotic independence. More precisely, when the coefficient falls in the following sets: $1/2 < \eta < 1$, $\eta = 1/2$ or $0 < \eta < 1/2$, then we say that among the variables there is a positive association, independence or negative association, respectively, within asymptotic independence (see e.g. [Ledford and Tawn, 1996](#)).

3.1. A η -Pickands dependence function

Consider now, n i.i.d. copies $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ of \mathbf{Y} and for a small $\varepsilon > 0$, define $\mathbf{M}_{n,\varepsilon} = (M_{n,1,\varepsilon}, \dots, M_{n,d,\varepsilon})$ as the vector of componentwise maxima, precisely

$$M_{n,j,\varepsilon} = \bigvee_{i=1, \dots, n} \{Y_{i,j} | Y_{i,j} > \varepsilon\}, \quad j = 1, \dots, d.$$

Let b_n be a sequence of normalizing constants defined by the equation $n\Pr(\mathbf{Y} > b_n) = 1$. Then, differently from the classical theory (e.g. [de Haan and Ferreira, 2006](#), Ch. 6), here the limiting distribution for the normalized vector of componentwise maxima $\mathbf{M}_{n,\varepsilon}$ is obtained as

$$G_\eta(\mathbf{y}) := \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \Pr(\mathbf{M}_{n,b_n\varepsilon} \leq b_n\mathbf{y}), \quad \mathbf{y} \in (0, \infty]^d,$$

see [Ramos and Ledford \(2011\)](#) for details. When a limiting distribution exists with nondegenerate margins, then G_η is called a multivariate η -extreme-value distribution. Specifically, a d -dimensional random vector \mathbf{Z} follows the law of a multivariate η -extreme-value distribution, if the one-dimensional marginal distributions are $G_{\eta,j}(y) = \exp(-\sigma_{\eta,j}y^{-1/\eta})$, for all $y > 0$, $j = 1, \dots, d$, and the joint distribution takes the form

$$G_\eta(\mathbf{y}) = C_\eta(G_{\eta,1}(y_1), \dots, G_{\eta,d}(y_d)), \quad \mathbf{y} \in (0, \infty]^d, \tag{3.1}$$

where C_η is an η -extreme-value copula, i.e.

$$C_\eta(\mathbf{u}) = \exp \left\{ -V_\eta \left(\left(\frac{\sigma_{\eta,1}}{-\log u_1} \right)^\eta, \dots, \left(\frac{\sigma_{\eta,d}}{-\log u_d} \right)^\eta \right) \right\} \quad \mathbf{u} \in (0, 1]^d$$

with $V_\eta : (0, \infty]^d \rightarrow [0, \infty)$ a homogeneous function of order $-1/\eta$ and

$$\sigma_{\eta,j} := V_\eta(\infty, \dots, \infty, 1, \infty, \dots, \infty) = \eta \int_{\mathcal{S}_d} w_j^{1/\eta} dH_\eta(\mathbf{w}). \tag{3.2}$$

Introduce now $L_\eta(\mathbf{z}) := V_\eta((\boldsymbol{\sigma}_\eta/\mathbf{z})^\eta)$, for all $\mathbf{z} = \boldsymbol{\sigma}_\eta/\mathbf{y}^{1/\eta}$. This function is called the η -stable tail dependence function and using the homogeneity property, it can be rewritten as

$$L_\eta(\mathbf{z}) = (z_1 + \cdots + z_d)A_\eta(\mathbf{t}), \quad \mathbf{z} \in [0, \infty)^d,$$

where $t_j = z_j/(z_1 + \cdots + z_d)$ for $j = 2, \dots, d$, $t_1 = 1 - t_2 - \cdots - t_d$. Here, the function A_η is called the η -Pickands dependence function and it satisfies the following properties.

Proposition 3.1. *The η -Pickands dependence function A_η satisfies:*

1. For all $\eta \in (0, 1]$, $A_\eta(\mathbf{e}_j) = 1$, $j = 1, \dots, d$;
2. $A_1(\mathbf{t}) = A(\mathbf{t})$, for all $\mathbf{t} \in \mathcal{S}_d$;
3. For every $\eta \in (0, 1]$ and $\mathbf{t} \in \mathcal{S}_d$,

$$1/d \leq \max(t_1, \dots, t_d) \leq A_\eta(\mathbf{t}) \leq 1.$$

4. $A_\eta(\mathbf{t})$ is convex, i.e. $A_\eta(a\mathbf{t}_1 + (1-a)\mathbf{t}_2) \leq aA_\eta(\mathbf{t}_1) + (1-a)A_\eta(\mathbf{t}_2)$, for all $a \in [0, 1]$ and $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{S}_d$.

Similarly to the classical literature, a η -madogram function can be defined as the expected distance between the maximum and the mean of the variables $G_{\eta,1}^{1/\eta t_1}(Z_1), \dots, G_{\eta,d}^{1/\eta t_d}(Z_d)$, that is,

$$v_\eta(\mathbf{t}) = \mathbb{E} \left[\left| \bigvee_{j=1}^d \{G_{\eta,j}^{1/\eta t_j}(Z_j)\} - \frac{1}{d} \sum_{j=1}^d G_{\eta,j}^{1/\eta t_j}(Z_j) \right| \right].$$

This function can also be linked to the η -Pickands dependence function as follows.

Proposition 3.2. *Any random vector \mathbf{Z} with a η -extreme-value distribution admits a η -Pickands dependence function A_η which satisfies*

$$A_\eta(\mathbf{t}) = \frac{1}{\eta} \frac{v_\eta(\mathbf{t}) + c_\eta(\mathbf{t})}{1 - v_\eta(\mathbf{t}) - c_\eta(\mathbf{t})} \quad (3.3)$$

for all $\mathbf{t} \in \mathcal{S}_d$, where

$$c_\eta(\mathbf{t}) = \frac{1}{d} \sum_{j=1}^d \frac{t_j}{t_j + 1/\eta}. \quad (3.4)$$

This η -Pickands dependence function can be used to represent the level of dependence among the elements of \mathbf{Z} , and thus in the next section, we estimate this function and derive the main asymptotic properties of the estimator.

3.2. An estimator of the η -Pickands dependence function

Let $\mathbf{Z}_1, \dots, \mathbf{Z}_n$ be i.i.d. copies of \mathbf{Z} and define

$$H_n(\mathbf{y}) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\mathbf{Z}_i \leq \mathbf{y}\}}, \quad \mathbf{y} \in (0, \infty]^d$$

and its associated empirical process

$$\mathbb{H}_n(\mathbf{y}) = \sqrt{n}(H_n(\mathbf{y}) - G_\eta(\mathbf{y})), \quad \mathbf{y} \in (0, \infty]^d.$$

In order to estimate the η -Pickands dependence function we first assume that we have at our disposal an estimator $\widehat{\eta}_n$ for η satisfying the condition:

Condition 1. *Let $\widehat{\eta}_n$ be an estimator of η satisfying:*

- (i) $\widehat{\eta}_n \rightarrow \eta$ a.s. as $n \rightarrow \infty$;

(ii) One of the following holds true

- (a) $\sqrt{n}(\widehat{\eta}_n - \eta) = n^{-1/2} \sum_{i=1}^n \rho(\mathbf{Z}_i) + o_p(1)$, where $\rho : (0, \infty)^d \mapsto \mathcal{R}$ is a measurable function such that $E\rho(\mathbf{Z}) = 0$ and $E\rho^2(\mathbf{Z}) < \infty$;
- (b) $\sqrt{n}(\widehat{\eta}_n - \eta) = \chi(\mathbb{H}_n) + o_p(1)$, where $\chi : \ell^\infty((0, \infty)^d) \mapsto \mathcal{R}$ is a bounded linear functional.

In the spirit of (2.1) in Section 2, we propose the following estimator for A_η :

$$\widehat{A}_{\widehat{\eta}_n, n}(\mathbf{t}) := \frac{1}{\widehat{\eta}_n} \frac{\widehat{v}_{\widehat{\eta}_n, n}(\mathbf{t}) + \widehat{c}_{\widehat{\eta}_n, n}(\mathbf{t})}{1 - \widehat{v}_{\widehat{\eta}_n, n}(\mathbf{t}) - \widehat{c}_{\widehat{\eta}_n, n}(\mathbf{t})}$$

where

$$\begin{aligned} \widehat{v}_{\widehat{\eta}_n, n}(\mathbf{t}) &:= \frac{1}{n} \sum_{i=1}^n \left(\bigvee_{j=1}^d \{H_{n,j}^{(\widehat{\eta}_n)}(Z_{i,j})\}^{1/\widehat{\eta}_n t_j} - \frac{1}{d} \sum_{j=1}^d \{H_{n,j}^{(\widehat{\eta}_n)}(Z_{i,j})\}^{1/\widehat{\eta}_n t_j} \right) \\ \widehat{c}_{\widehat{\eta}_n, n}(\mathbf{t}) &:= \frac{1}{n d} \sum_{i=1}^n \sum_{j=1}^d \{H_{n,j}^{(\widehat{\eta}_n)}(Z_{i,j})\}^{1/\widehat{\eta}_n t_j} \end{aligned} \quad (3.5)$$

with

$$H_{n,j}^{(a)}(Z_{i,j}) = H_{n,j}(Z_{i,j}) \left(\frac{1+a}{a} \frac{1}{n} \sum_{k=1}^n H_{n,j}^{1/a}(Z_{k,j}) \right)^{-a}, \quad j = 1, \dots, d, \quad \text{for } a > 0,$$

and the empirical distribution functions denoted by

$$H_{n,j}(x) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{Z_{i,j} \leq x\}}, \quad j = 1, \dots, d.$$

Note that (3.5) comes from the fact that c_η defined in Proposition 3.2 can be viewed as

$$c_\eta(\mathbf{t}) = E \left(\frac{1}{d} \sum_{j=1}^d \{G_{\eta,j}(Z_j)\}^{1/\eta t_j} \right)$$

and thus in (3.5) we use the empirical counterpart. Another option would have been to replace η by an estimator in (3.4).

We are now able to state our main result on the convergence of a rescaled version of $\widehat{A}_{\widehat{\eta}_n, n}$.

Theorem 3.1. *Under Condition 1(i), we have*

$$\|\widehat{A}_{\widehat{\eta}_n, n} - A_\eta\|_\infty \rightarrow 0 \text{ a.s. as } n \rightarrow \infty. \quad (3.6)$$

Under Conditions 1(ii), we have in $\ell^\infty(\mathcal{S}_d)$, as $n \rightarrow \infty$,

$$\sqrt{n}(\widehat{A}_{\widehat{\eta}_n, n}(t) - A_\eta(t))_{t \in \mathcal{S}_d} \rightsquigarrow \left(-\frac{(1 + \eta A_\eta(t))^2}{\eta} \int_0^1 \mathbb{A}_\eta(v^{\eta t_1}, \dots, v^{\eta t_d}) dv \right)_{t \in \mathcal{S}_d}, \quad (3.7)$$

where \mathbb{A}_η is a stochastic process defined as

$$\mathbb{A}_\eta(\mathbf{u}) := \mathbb{B}_\eta(\mathbf{u}) - \sum_{j=1}^d \dot{C}_{\eta,j}(\mathbf{u}) \mathbb{B}_\eta(1, \dots, 1, u_j, 1, \dots, 1), \quad \mathbf{u} \in [0, 1]^d,$$

with C_η an η -extreme-value copula such that its partial derivative $\dot{C}_{\eta,j}(\mathbf{u}) := \partial C_\eta / \partial u_j(\mathbf{u})$ exists and is continuous on $\{\mathbf{u} \in [0, 1]^d : 0 < u_j < 1\}$, for all $j = 1, \dots, d$, and \mathbb{B}_η a C_η -Brownian bridge, i.e. a zero-mean Gaussian process on $[0, 1]^d$ with continuous sample paths and covariance function equal to

$$\text{Cov}(\mathbb{B}_\eta(\mathbf{u}), \mathbb{B}_\eta(\mathbf{v})) = C_\eta(\mathbf{u} \wedge \mathbf{v}) - C_\eta(\mathbf{u})C_\eta(\mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in [0, 1]^d.$$

3.3. Examples of estimators satisfying Condition 1

Our η -Pickands dependence function requires an estimator of η which satisfies Condition 1. Below, two examples of such estimators are proposed.

Example 1. Let $Z^* = \max(Z_1, \dots, Z_d)$, where \mathbf{Z} follows the distribution (3.1). Then, for any $y > 0$, the distribution of Z^* is $G_\eta(y) := G_\eta(y, \dots, y)$. This distribution can be seen as a two-parameter Fréchet family of distributions. Let $\widehat{\eta}_n$ be the Maximum Likelihood (ML) estimator. By Propositions 3.1 and 3.3 in Bücher and Segers (2017), it follows that the ML estimator satisfies Conditions 1(i) and 1(ii)(a).

Example 2. Let $\widehat{\eta}_n$ be the Generalized Probability Weighted Moment (GPWM) estimator of η introduced by Guillou et al. (2014). The next theorem shows that the GPWM estimator admits a stochastic representation implying that Condition 1(ii)(b) is satisfied. The almost sure consistency of $\widehat{\eta}_n$ is a direct consequence.

Theorem 3.2. Let $\widehat{\eta}_n$ be the GPWM estimator. For a, b two integers and $Q_\eta(u) := G_\eta^{\leftarrow}(u)$, introduce the parameter

$$\mu_{a,b} := \int_0^1 Q_\eta(u) u^a (-\log u)^b du$$

and on $u \in (0, 1)$ the two functions

$$\begin{aligned} \gamma(u) &:= \mu_{1,2} u (-\log u) - \mu_{1,1} u (-\log u)^2 \\ \varphi(u) &:= \frac{1}{\eta V_\eta^\eta(1, \dots, 1)} u (-\log u)^{1+\eta}. \end{aligned}$$

Then,

$$\sqrt{n}(\widehat{\eta}_n - \eta) = -\frac{2}{\mu_{1,1}^2} \int_0^1 \mathbb{H}_n(Q_\eta(u), \dots, Q_\eta(u)) \frac{\gamma(u)}{\varphi(u)} du + o(1) \text{ a.s.} \quad (3.8)$$

Consequently, as $n \rightarrow \infty$

$$\begin{aligned} \widehat{\eta}_n &\rightarrow \eta \text{ a.s.} \\ \sqrt{n}(\widehat{\eta}_n - \eta) &\rightsquigarrow -\frac{2}{\mu_{1,1}^2} \int_0^1 \mathbb{H}(Q_\eta(u), \dots, Q_\eta(u)) \frac{\gamma(u)}{\varphi(u)} du \end{aligned}$$

where \mathbb{H} is a tight centered Gaussian process on $(0, \infty]^d$, with covariance function

$$\text{Cov}(\mathbb{H}(\mathbf{z}), \mathbb{H}(\mathbf{y})) = G_\eta(\mathbf{z} \wedge \mathbf{y}) - G_\eta(\mathbf{z})G_\eta(\mathbf{y}), \quad \mathbf{z}, \mathbf{y} \in (0, \infty]^d.$$

3.4. Simulation

The performance of our estimator $\widehat{A}_{\widehat{\eta}_n, n}$ is illustrated in a simulation study with two different experiments.

First experiment: We consider the bivariate η -asymmetric logistic model in equation (4.3) of Ramos and Ledford (2011), with dependence parameters $\psi \in (0, 1]$, $\varrho > 0$ and $\eta \in (0, 1]$. For simplicity we focus on $\varrho = 1$ and the interesting case given by $\alpha < \eta$. In this framework the η -asymmetric logistic model is the limiting distribution for normalized componentwise maxima obtained from a random vector with the joint tail probability in equation (3.3) of Ramos and Ledford (2009), which is indeed an asymptotic independence model. For every $\eta \in (0, 1]$, the strength of the dependence within asymptotic independence increases for decreasing values of the parameter ψ . We simulate n values from the η -asymmetric logistic model and we estimate the η -Pickands dependence function with $\widehat{A}_{\widehat{\eta}_n, n}$. We repeat these steps 1000 times and we compute a Monte Carlo approximation of the Mean Integrated Squared Error (MISE), i.e.,

$$\text{MISE}(\widehat{A}_{\widehat{\eta}_n, n}, A_\eta) = \mathbb{E} \left\{ \int_{S_d} (\widehat{A}_{\widehat{\eta}_n, n}(\mathbf{t}) - A_\eta(\mathbf{t}))^2 d\mathbf{t} \right\}.$$

TABLE 3

Estimates (standard deviation) of η and MISE for the η -Pickands dependence function, based on a bivariate η -asymmetric logistic dependence model with $\eta = 0.7$. The first line corresponds to the GPWM method, whereas the second line is the ML method.

ψ	$\widehat{\eta}_n$				MISE($\widehat{A}_{\widehat{\eta}_n, n}, A_\eta$)			
	n				n			
	25	50	100	200	25	50	100	200
0.1	0.661(0.115)	0.678(0.084)	0.690(0.062)	0.695(0.043)	0.0111	0.0037	0.0013	0.0005
	0.800(0.201)	0.763(0.128)	0.741(0.085)	0.728(0.055)	0.0110	0.0036	0.0013	0.0005
0.2	0.667(0.116)	0.679(0.084)	0.688(0.062)	0.692(0.044)	0.0480	0.0195	0.0088	0.0041
	0.807(0.204)	0.761(0.128)	0.740(0.088)	0.724(0.057)	0.0457	0.0187	0.0086	0.0040
0.3	0.665(0.116)	0.680(0.087)	0.692(0.064)	0.696(0.046)	0.1176	0.0542	0.0262	0.0133
	0.811(0.211)	0.768(0.130)	0.745(0.087)	0.730(0.059)	0.1133	0.0527	0.0256	0.0131
0.4	0.673(0.114)	0.687(0.088)	0.694(0.062)	0.697(0.045)	0.2177	0.1021	0.0523	0.0260
	0.810(0.204)	0.770(0.129)	0.744(0.084)	0.729(0.057)	0.2118	0.1000	0.0514	0.0257
0.5	0.670(0.113)	0.684(0.085)	0.692(0.062)	0.695(0.044)	0.3602	0.1795	0.0952	0.0481
	0.805(0.201)	0.766(0.129)	0.742(0.085)	0.728(0.057)	0.3531	0.1765	0.0940	0.0476
0.6	0.670(0.115)	0.685(0.085)	0.691(0.062)	0.696(0.046)	0.4566	0.2252	0.1192	0.0556
	0.822(0.206)	0.778(0.127)	0.751(0.089)	0.734(0.063)	0.4758	0.2275	0.1156	0.0585

TABLE 4

Estimates (standard deviation) of η and MISE for the η -Pickands dependence function, based on componentwise maxima with approximate bivariate η -asymmetric logistic model with $\eta = 0.7$. The first line corresponds to the GPWM method, whereas the second line is the ML method.

ψ	$\widehat{\eta}_n$				MISE($\widehat{A}_{\widehat{\eta}_n, n}, A_\eta$)			
	n				n			
	25	50	100	200	25	50	100	200
0.1	0.668(0.115)	0.684(0.089)	0.692(0.061)	0.695(0.044)	0.0108	0.0034	0.0013	0.0005
	0.800(0.204)	0.764(0.128)	0.742(0.086)	0.730(0.060)	0.0106	0.0033	0.0013	0.0005
0.2	0.664(0.116)	0.681(0.086)	0.687(0.061)	0.693(0.045)	0.0456	0.0187	0.0088	0.0040
	0.810(0.213)	0.765(0.133)	0.743(0.091)	0.728(0.064)	0.0442	0.0183	0.0079	0.0039
0.3	0.670(0.120)	0.686(0.089)	0.696(0.063)	0.698(0.045)	0.1088	0.0563	0.0257	0.0119
	0.804(0.194)	0.766(0.119)	0.744(0.078)	0.732(0.055)	0.1080	0.0546	0.0255	0.0117
0.4	0.684(0.119)	0.699(0.091)	0.707(0.063)	0.711(0.045)	0.2265	0.1146	0.0593	0.0279
	0.829(0.209)	0.783(0.125)	0.759(0.082)	0.745(0.054)	0.2207	0.1129	0.0584	0.0276
0.5	0.711(0.119)	0.727(0.088)	0.734(0.063)	0.738(0.044)	0.3820	0.1970	0.1113	0.0657
	0.846(0.197)	0.806(0.122)	0.786(0.080)	0.773(0.049)	0.3819	0.1973	0.1112	0.0661
0.6	0.751(0.125)	0.766(0.094)	0.775(0.070)	0.781(0.050)	0.7105	0.4298	0.3018	0.2333
	0.898(0.201)	0.849(0.121)	0.831(0.082)	0.820(0.053)	0.7293	0.4387	0.3096	0.2381

This study is done for different values of the sample size n and different values of the dependence parameter ψ . The results are summarized in Table 3. For each value of ψ , between the second and the fifth column the mean of the estimates for η obtained with the GPWM (first row) and ML (second row) estimator are reported for increasing sample size. In parentheses is the standard deviation. Between the sixth and ninth columns the approximated MISE is reported. Accurate estimates are obtained with all the dependence levels. GPWM and ML estimators provide similar results, although those of the former seem slightly better. According to the MISE, the better performances are obtained with stronger dependence strengths. For every dependence level the accuracy of estimates increases with increasing sample size.

Second experiment: We show the performance of the estimator $\widehat{A}_{\widehat{\eta}_n, n}$ under a more realistic scenario. We simulate $n \times 366$ realizations of a bivariate random vector with the joint tail probability given in equation (3.3) of Ramos and Ledford (2009). To do this we use the algorithm described in Appendix B of Ramos and Ledford (2009). Precisely, we set $u = 10$ and $\lambda = 1 - \exp(-0.1) - 0.2$ which satisfies the required monotonicity condition. With the minimum between pairs of all observations we compute b_n as the $1 - 1/n$ empirical quantile. For each block of 366 observations we compute the componentwise maxima using only the pairs that are both greater than $\varepsilon b_n = 13.780$, i.e. the 0.93-quantile of a unit Fréchet distribution. We standardize the maxima by dividing them by b_n . With the n normalized maxima we estimate the η -Pickands dependence function by $\widehat{A}_{\widehat{\eta}_n, n}$. We repeat these steps 1000 times and we compute an approximation of the MISE. Table 4 collects the results. We see that although in this case the data are only approximately coming from a η -asymmetric logistic model the estimates of η and A_η are similar to

those obtained in Table 3, indicating a good performance of our estimator.

Appendix A: Proofs

A.1. Some properties of \widehat{A}_n

Note that,

$$\widehat{v}_n(\mathbf{e}_j) = \frac{1}{n} \sum_{i=1}^n \left(\frac{G_{n,j}(X_{i,j})}{2n^{-1} \sum_{k=1}^n G_{n,j}(X_{k,j})} - \frac{1}{d} \frac{G_{n,j}(X_{i,j})}{2n^{-1} \sum_{k=1}^n G_{n,j}(X_{k,j})} \right) = \frac{1}{2} - \frac{1}{2d}, \quad j = 1, \dots, d.$$

Therefore, $\widehat{A}_n(\mathbf{e}_j) = 1$ for all $j = 1, \dots, d$.

The distribution function of the i.i.d. random variables $X_{1,j}, \dots, X_{n,j}$, $j = 1, \dots, d$, being continuous, there is almost surely no ties and thus

$$G_{n,j}^{(1)}(X_{i,j}) = G_{n,j}(X_{i,j}) \left(\frac{2}{n} \sum_{k=1}^n G_{n,j}(X_{k,j}) \right)^{-1} = \frac{n}{n+1} G_{n,j}(X_{i,j}).$$

Then, with simple adjustments of the proof of Theorem 2.4 in [Marcon et al. \(2017\)](#), the weak convergence of \widehat{A}_n and its almost sure consistency follow. \square

A.2. Proof of Proposition 3.1

Our definition of L_η combining with (6.3) in [Ramos and Ledford \(2011\)](#) entails

$$A_\eta(\mathbf{t}) = \eta \int_{\mathcal{S}_d} \max \left(\frac{t_1 w_1^{1/\eta}}{\sigma_{\eta,1}}, \dots, \frac{t_d w_d^{1/\eta}}{\sigma_{\eta,d}} \right) dH_\eta(\mathbf{w}), \quad \mathbf{t} \in \mathcal{S}_d.$$

Then, Property 1 follows by the definition of $\sigma_{\eta,j}$ given in (3.2).

When $\eta = 1$, according to Section 3, we have

$$\lim_{n \rightarrow \infty} \frac{\Pr(\mathbf{Y} > n\mathbf{x})}{\Pr(\mathbf{Y} > n\mathbf{1})} = \int_{\mathcal{S}_d} \bigwedge_{j=1}^d \left(\frac{w_j}{x_j} \right) dH_1(\mathbf{w}).$$

Now, this limit can also be rephrased with the classical theory (see e.g. [de Haan and Ferreira, 2006](#), Ch. 6), where

$$\lim_{n \rightarrow \infty} \frac{\Pr(\mathbf{Y} > n\mathbf{x})}{\Pr(\mathbf{Y} > n\mathbf{1})} = \frac{d \int_{\mathcal{S}_d} \bigwedge_{j=1}^d \left(\frac{w_j}{x_j} \right) dH(\mathbf{w})}{R(1, \dots, 1)},$$

with H and R defined in pages 218 and 225 in [de Haan and Ferreira \(2006\)](#). Therefore, Property 2 follows from the relations

$$d^{-1} R(1, \dots, 1) dH_1(\mathbf{w}) = dH(\mathbf{w}), \quad \mathbf{w} \in \mathcal{S}_d$$

and $\sigma_{1,j} = 1/R(1, \dots, 1)$, $j = 1, \dots, d$.

For every $\mathbf{t} \in \mathcal{S}_d$ we have

$$\eta \int_{\mathcal{S}_d} \max \left(\frac{t_1 w_1^{1/\eta}}{\sigma_{\eta,1}}, \dots, \frac{t_d w_d^{1/\eta}}{\sigma_{\eta,d}} \right) dH_\eta(\mathbf{w}) \leq \eta \int_{\mathcal{S}_d} \sum_{j=1}^d \left(\frac{t_j w_j^{1/\eta}}{\sigma_{\eta,j}} \right) dH_\eta(\mathbf{w}) = 1,$$

from which the upper bound in Property 3 follows. To derive the lower bound, it is sufficient to remark that for every $\mathbf{t} \in \mathcal{S}_d$, we have

$$\begin{aligned} \eta \int_{\mathcal{S}_d} \max\left(\frac{t_1 w_1^{1/\eta}}{\sigma_{\eta,1}}, \dots, \frac{t_d w_d^{1/\eta}}{\sigma_{\eta,d}}\right) dH_\eta(\mathbf{w}) &\geq \bigvee_{1 \leq i < j \leq d} \left(\eta \int_{\mathcal{S}_d} \max\left(\frac{t_i w_i^{1/\eta}}{\sigma_{\eta,i}}, \frac{t_j w_j^{1/\eta}}{\sigma_{\eta,j}}\right) dH_\eta(\mathbf{w}) \right) \\ &= \bigvee_{1 \leq i < j \leq d} \left(t_i + t_j - \eta \int_{\mathcal{S}_d} \min\left(\frac{t_i w_i^{1/\eta}}{\sigma_{\eta,i}}, \frac{t_j w_j^{1/\eta}}{\sigma_{\eta,j}}\right) dH_\eta(\mathbf{w}) \right) \\ &\geq \bigvee_{1 \leq i < j \leq d} (t_i + t_j - \min(t_i, t_j)) = \bigvee_{1 \leq j \leq d} t_j. \end{aligned}$$

Finally, the convexity in Property 4 can be shown similar to the convexity of A . \square

A.3. Proof of Proposition 3.2

For all $\eta \in (0, 1]$ and $\mathbf{t} \in \mathcal{S}_d$, set

$$v_\eta(\mathbf{u}; \mathbf{t}) := \bigvee_{j=1}^d u_j^{1/\eta t_j} - \frac{1}{d} \sum_{j=1}^d u_j^{1/\eta t_j}, \quad \mathbf{u} \in [0, 1]^d.$$

By convention $u^{1/\eta t} = 0$ when $t = 0$ and $u \in [0, 1]$. By Lemma A.1 in [Marcon et al. \(2017\)](#) we have

$$\begin{aligned} v_\eta(\mathbf{t}) &= \int_{[0,1]^d} v_\eta(\mathbf{u}; \mathbf{t}) dC_\eta(\mathbf{u}) \tag{A.1} \\ &= \frac{1}{d} \sum_{j=1}^d \int_0^1 C_\eta(1, \dots, 1, v^{\eta t_j}, 1, \dots, 1) dv - \int_0^1 C_\eta(v^{\eta t_1}, \dots, v^{\eta t_d}) dv \\ &= \frac{1}{d} \sum_{j=1}^d \int_0^1 v^{\eta t_j} dv - \int_0^1 v^{\eta A_\eta(\mathbf{t})} dv \\ &= \frac{1}{d} \sum_{j=1}^d \frac{1}{1 + \eta t_j} - \frac{1}{1 + \eta A_\eta(\mathbf{t})}. \end{aligned}$$

The result (3.3) follows by solving the above equality for A_η . \square

A.4. Proof of Theorem 3.1

We start with some notation. Let $\widehat{C}_n := \sqrt{n}(\widehat{C}_n - C_\eta)$, where \widehat{C}_n is the empirical copula defined as

$$\widehat{C}_n(\mathbf{u}) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\widehat{U}_i \leq \mathbf{u}\}}, \quad \mathbf{u} \in [0, 1]^d,$$

with $\widehat{U}_i = (H_{n,1}(Z_{i,1}), \dots, H_{n,d}(Z_{i,d}))$. Define now, for all $\mathbf{t} \in \mathcal{S}_d$,

$$M(\cdot, \mathbf{t}) := 1 - \int_0^1 C_\eta(v^{\cdot t_1}, \dots, v^{\cdot t_d}) dv, \tag{A.2}$$

$$\widehat{M}_n(\cdot, \mathbf{t}) := 1 - \int_0^1 \widehat{C}_n(v^{\cdot t_1}, \dots, v^{\cdot t_d}) dv. \tag{A.3}$$

We will prove Theorem 3.1 with $H_{n,j}^{(\widehat{\eta}_n)}$ in $\widehat{v}_{\widehat{\eta}_n,n}$ and $\widehat{c}_{\widehat{\eta}_n,n}$ replaced by $H_{n,j}$. Indeed, this slight modification has no impact on the convergences (3.6) and (3.7) since

$$H_{n,j}^{(\widehat{\eta}_n)}(Z_{i,j}) = H_{n,j}(Z_{i,j}) \left(1 + \frac{1 + \widehat{\eta}_n}{\widehat{\eta}_n} O\left(\frac{1}{n}\right) \right)^{-\widehat{\eta}_n} =: H_{n,j}^{(\widehat{\eta}_n)}(Z_{i,j}) e_n,$$

and thus (A.2) and (A.3) can be slightly changed by replacing in the integrals v^{t_j} by $v^{t_j} e_n^{t_j}$, $j = 1, \dots, d$, without any impact. In view of this remark, we pursue the proof of Theorem 3.1 with $M(\cdot, \mathbf{t})$ and $\widehat{M}_n(\cdot, \mathbf{t})$ defined in (A.2) and (A.3) without taking care of the adjustment with e_n .

We start to prove (3.7). To this aim, note that from (A.1) we have

$$M(\eta, \mathbf{t}) = \frac{\eta A_\eta(\mathbf{t})}{1 + \eta A_\eta(\mathbf{t})}$$

and thus the following decomposition holds

$$\begin{aligned} \sqrt{n}(\widehat{A}_{\widehat{\eta}_n,n}(\mathbf{t}) - A_\eta(\mathbf{t})) &= \sqrt{n} \left(\frac{1}{\widehat{\eta}_n} \frac{\widehat{M}_n(\widehat{\eta}_n, \mathbf{t})}{1 - \widehat{M}_n(\widehat{\eta}_n, \mathbf{t})} - \frac{1}{\eta} \frac{M(\eta, \mathbf{t})}{1 - M(\eta, \mathbf{t})} \right) \\ &= \frac{\sqrt{n}}{\widehat{\eta}_n} \left(\frac{\widehat{M}_n(\widehat{\eta}_n, \mathbf{t})}{1 - \widehat{M}_n(\widehat{\eta}_n, \mathbf{t})} - \frac{M(\eta, \mathbf{t})}{1 - M(\eta, \mathbf{t})} \right) + \frac{M(\eta, \mathbf{t})}{1 - M(\eta, \mathbf{t})} \sqrt{n} \left(\frac{1}{\widehat{\eta}_n} - \frac{1}{\eta} \right) \\ &=: L_n(\mathbf{t}) + R_n(\mathbf{t}), \end{aligned}$$

for all $\mathbf{t} \in \mathcal{S}_d$. We derive a tractable expression for L_n by means of the following three results.

Lemma A.1. *We have the following decomposition*

$$\sqrt{n}(\widehat{M}_n(\widehat{\eta}_n, \mathbf{t}) - M(\eta, \mathbf{t})) = \sqrt{n}(\widehat{M}_n(\widehat{\eta}_n, \mathbf{t}) - M(\widehat{\eta}_n, \mathbf{t})) + \sqrt{n}(M(\widehat{\eta}_n, \mathbf{t}) - M(\eta, \mathbf{t})) + o_p(1).$$

Proof. The proof uses arguments from van der Vaart and Wellner (2007). Since

$$\begin{aligned} \sqrt{n}(\widehat{M}_n(\widehat{\eta}_n, \mathbf{t}) - M(\eta, \mathbf{t})) &= \left\{ \sqrt{n}(\widehat{M}_n(\widehat{\eta}_n, \mathbf{t}) - M(\widehat{\eta}_n, \mathbf{t})) - \sqrt{n}(\widehat{M}_n(\eta, \mathbf{t}) - M(\eta, \mathbf{t})) \right\} \\ &\quad + \sqrt{n}(\widehat{M}_n(\eta, \mathbf{t}) - M(\eta, \mathbf{t})) + \sqrt{n}(M(\widehat{\eta}_n, \mathbf{t}) - M(\eta, \mathbf{t})), \end{aligned}$$

it remains to show that

$$\| \sqrt{n}(\widehat{M}_n(\widehat{\eta}_n, \mathbf{t}) - M(\widehat{\eta}_n, \mathbf{t})) - \sqrt{n}(\widehat{M}_n(\eta, \mathbf{t}) - M(\eta, \mathbf{t})) \|_\infty = o_p(1). \quad (\text{A.4})$$

By Condition 1(ii) we have that $\sqrt{n}(\widehat{\eta}_n - \eta)$ is asymptotically tight. Thus, for every $\varepsilon > 0$, there exists a compact set $K \equiv K_\varepsilon \subseteq \mathcal{R}$ such that

$$\liminf_{n \rightarrow \infty} \Pr(\sqrt{n}(\widehat{\eta}_n - \eta) \in K) > 1 - \varepsilon.$$

Furthermore, by the compactness of K , there exist $\delta > 0$, $p := p(\delta) \in \mathbb{N}$ and $\{h_1, \dots, h_p\} \subseteq K$ such that $K \subseteq \cup_{1 \leq s \leq p} (h_s - \delta, h_s + \delta)$. Therefore,

$$\begin{aligned} \{ \sqrt{n}(\widehat{\eta}_n - \eta) \in K \} &\subseteq \left\{ \sqrt{n}(\widehat{\eta}_n - \eta) \subseteq \bigcup_{s=1}^p (h_s - \delta, h_s + \delta) \right\} \\ &= \bigcup_{s=1}^p \left\{ \widehat{\eta}_n \in \left(\eta + n^{-1/2}(h_s - \delta), \eta + n^{-1/2}(h_s + \delta) \right) \right\}. \end{aligned}$$

Consequently, it follows that, with probability at least $1 - \varepsilon$,

$$\begin{aligned}
& \|\sqrt{n}(\widehat{M}_n(\widehat{\eta}_n, \mathbf{t}) - M(\widehat{\eta}_n, \mathbf{t})) - \sqrt{n}(\widehat{M}_n(\eta, \mathbf{t}) - M(\eta, \mathbf{t}))\|_\infty \\
& \leq \sup_{\mathbf{t} \in \mathcal{S}_d} \max_{1 \leq s \leq p} \sup_{|h-h_s| < \delta} |\sqrt{n}(\widehat{M}_n(\eta_{n,h_s}, \mathbf{t}) - M(\eta_{n,h_s}, \mathbf{t})) - \sqrt{n}(\widehat{M}_n(\eta, \mathbf{t}) - M(\eta, \mathbf{t}))| \\
& \leq \sup_{\mathbf{t} \in \mathcal{S}_d} \max_{1 \leq s \leq p} |\sqrt{n}(\widehat{M}_n(\eta_{n,h_s}, \mathbf{t}) - M(\eta_{n,h_s}, \mathbf{t})) - \sqrt{n}(\widehat{M}_n(\eta, \mathbf{t}) - M(\eta, \mathbf{t}))| \\
& \quad + \sup_{\mathbf{t} \in \mathcal{S}_d} \max_{1 \leq s \leq p} \sup_{|h-h_s| < \delta} |\sqrt{n}(\widehat{M}_n(\eta_{n,h_s}, \mathbf{t}) - M(\eta_{n,h_s}, \mathbf{t})) - \sqrt{n}(\widehat{M}_n(\eta_{n,h}, \mathbf{t}) - M(\eta_{n,h}, \mathbf{t}))| \\
& =: I_{n,1} + I_{n,2}
\end{aligned}$$

where $\eta_{n,\bullet} := \eta + n^{-1/2}\bullet$. Showing (A.4) is thus equivalent to proving that both $I_{n,1}$ and $I_{n,2}$ tends to 0 in probability, as $n \rightarrow \infty$. Using (A.2) and (A.3) we obtain

$$\begin{aligned}
I_{n,1} & = \sup_{\mathbf{t} \in \mathcal{S}_d} \max_{1 \leq s \leq p} \left| \int_0^1 (\widehat{\mathbb{C}}_n(v^{\eta_{n,h_s} t_1}, \dots, v^{\eta_{n,h_s} t_d}) - \widehat{\mathbb{C}}_n(v^{\eta t_1}, \dots, v^{\eta t_d})) dv \right| \\
& \leq \sup_{\mathbf{t} \in \mathcal{S}_d} \max_{1 \leq s \leq p} \sup_{v \in (0,1)} \left| \widehat{\mathbb{C}}_n(v^{\eta_{n,h_s} t_1}, \dots, v^{\eta_{n,h_s} t_d}) - \widehat{\mathbb{C}}_n(v^{\eta t_1}, \dots, v^{\eta t_d}) \right|
\end{aligned}$$

and

$$\begin{aligned}
I_{n,2} & = \sup_{\mathbf{t} \in \mathcal{S}_d} \max_{1 \leq s \leq p} \sup_{|h-h_s| < \delta} \left| \int_0^1 (\widehat{\mathbb{C}}_n(v^{\eta_{n,h_s} t_1}, \dots, v^{\eta_{n,h_s} t_d}) - \widehat{\mathbb{C}}_n(v^{\eta_{n,h} t_1}, \dots, v^{\eta_{n,h} t_d})) dv \right| \\
& \leq \sup_{\mathbf{t} \in \mathcal{S}_d} \max_{1 \leq s \leq p} \sup_{|h-h_s| < \delta} \sup_{v \in (0,1)} \left| \widehat{\mathbb{C}}_n(v^{\eta_{n,h_s} t_1}, \dots, v^{\eta_{n,h_s} t_d}) - \widehat{\mathbb{C}}_n(v^{\eta_{n,h} t_1}, \dots, v^{\eta_{n,h} t_d}) \right|.
\end{aligned}$$

Now, for every $v \in (0, 1)$ and small $\varepsilon > 0$, the map $\varphi : (0, 1) \rightarrow \ell^\infty([\eta - \varepsilon, \eta + \varepsilon]) : v \mapsto \varphi(v)$, defined by $(\varphi(v))(x) = v^x$, is continuously differentiable on $[\eta - \varepsilon, \eta + \varepsilon]$ for every $v \in (0, 1)$. The absolute value of its first derivative, i.e. $(\dot{\varphi}(v))(x) = v^x \log v$, is bounded above by $\xi_v = v^{\eta - \varepsilon} |\log v|$. Therefore, $(\varphi(v))(x)$ is a Lipschitz function and it satisfies the condition

$$|(\varphi(v))(x) - (\varphi(v))(y)| \leq \xi_v |x - y|, \quad \forall x, y \in [\eta - \varepsilon, \eta + \varepsilon].$$

Furthermore, there exists a positive constant ξ such that $\sup_{v \in (0,1)} \xi_v < \xi$, and thus for n sufficiently large ensuring that $\eta_{n,h_s}, \eta_{n,h_s} \in [\eta - \varepsilon, \eta + \varepsilon]$, we have:

$$\begin{aligned}
|v^{\eta_{n,h_s} t_j} - v^{\eta t_j}| & \leq \xi |\eta - \eta_{n,h_s}| = \xi n^{-1/2} |h_s| \rightarrow 0 \\
|v^{\eta_{n,h_s} t_j} - v^{\eta_{n,h} t_j}| & \leq \xi |\eta_{n,h_s} - \eta_{n,h}| = \xi n^{-1/2} |h_s - h| \leq \xi \delta n^{-1/2} \rightarrow 0,
\end{aligned}$$

as $n \rightarrow \infty$, for every $\mathbf{t} \in \mathcal{S}_d$, indexes $s \in \{1, \dots, p\}$, $j \in \{1, \dots, d\}$ and for every $|h - h_s| < \delta$. These results imply that

$$\sup_{\mathbf{t} \in \mathcal{S}_d} \max_{1 \leq s \leq p} \sup_{v \in (0,1)} \max_{1 \leq j \leq d} |v^{\eta_{n,h_s} t_j} - v^{\eta t_j}| \rightarrow 0, \quad n \rightarrow \infty \quad (\text{A.5})$$

and

$$\sup_{\mathbf{t} \in \mathcal{S}_d} \max_{1 \leq s \leq p} \sup_{|h-h_s| < \delta} \max_{1 \leq j \leq d} \sup_{v \in (0,1)} |v^{\eta_{n,h_s} t_j} - v^{\eta_{n,h} t_j}| \rightarrow 0, \quad n \rightarrow \infty. \quad (\text{A.6})$$

Since the first partial derivative of C_η exists and is continuous on $\{\mathbf{u} \in [0, 1]^d : 0 < u_j < 1\}$ for all $j = 1, \dots, d$, $\widehat{\mathbb{C}}_n \rightsquigarrow \mathbb{A}_\eta$ in $\ell^\infty([0, 1]^d)$ as $n \rightarrow \infty$ (see e.g. Fermanian et al., 2004; Segers, 2012). Therefore the sequence $\widehat{\mathbb{C}}_n$ is asymptotically uniformly equicontinuous in probability (see Theorem 1.5.7 in van der Vaart and Wellner, 1996). Combining this result with (A.5) and (A.6) entails that $I_{n,1}$ and $I_{n,2}$ tends to 0 in probability, as $n \rightarrow \infty$. Therefore (A.4) is established and thus Lemma A.1 follows. \square

Lemma A.2. *We have*

$$\sqrt{n}(M(\widehat{\eta}_n, \mathbf{t}) - M(\eta, \mathbf{t})) = \frac{A_\eta(\mathbf{t})}{(\eta A_\eta(\mathbf{t}) + 1)^2} \sqrt{n}(\widehat{\eta}_n - \eta) + o_p(1).$$

Proof. Let

$$\varphi : ((0, \infty), |\cdot|) \rightarrow (\ell^\infty(\mathcal{S}_d), \|\cdot\|_\infty) : a \mapsto M(a, \cdot)$$

be the map defined by

$$M(a, \cdot) = \frac{a A_\eta(\cdot)}{1 + a A_\eta(\cdot)}.$$

Its Hadamard derivative at $\eta \in (0, 1]$ is

$$h \mapsto (\dot{\varphi}_\eta(h)) = \frac{h A_\eta}{(\eta A_\eta + 1)^2}.$$

Indeed, for every $\epsilon_n \downarrow 0$ and $h_n \rightarrow h \in (0, \infty)$, as $n \rightarrow \infty$, such that $\eta + \epsilon_n h_n \in (0, \infty)$, we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{S}_d} \left| \frac{(\varphi(\eta + \epsilon_n h_n))(t) - (\varphi(\eta))(t)}{\epsilon_n} - (\dot{\varphi}_\eta(h))(t) \right| \\ &= \limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{S}_d} \left| \frac{1}{\epsilon_n} \left(\frac{(\eta + \epsilon_n h_n) A_\eta(t)}{(\eta + \epsilon_n h_n) A_\eta(t) + 1} - \frac{\eta A_\eta(t)}{\eta A_\eta(t) + 1} \right) - \frac{h A_\eta(t)}{(\eta A_\eta(t) + 1)^2} \right| \\ &= \limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{S}_d} \left| \frac{A_\eta(t)}{\eta A_\eta(t) + 1} \left| \frac{h_n}{(\eta + \epsilon_n h_n) A_\eta(t) + 1} - \frac{h}{\eta A_\eta(t) + 1} \right| \right| \\ &\leq \lim_{n \rightarrow \infty} d^2 \frac{|h_n - h| + |h h_n| \epsilon_n}{(d + \eta)(d + \eta + \epsilon_n h_n)} = 0. \end{aligned}$$

Lemma A.2 now follows from Theorem 20.8 in [van der Vaart \(2000\)](#) and under our Condition 1(ii). \square

Lemma A.3. *We have*

$$\sqrt{n} \left(\frac{\widehat{M}_n(\widehat{\eta}_n, \mathbf{t})}{1 - \widehat{M}_n(\widehat{\eta}_n, \mathbf{t})} - \frac{M(\eta, \mathbf{t})}{1 - M(\eta, \mathbf{t})} \right) = (1 + \eta A_\eta(\mathbf{t}))^2 \sqrt{n} (\widehat{M}_n(\widehat{\eta}_n, \mathbf{t}) - M(\eta, \mathbf{t})) + o_p(1).$$

Proof. The proof of this lemma is based on an application of the functional delta method after proving the Hadamard differentiability of the functional $\varphi(f) = f/(1 - f)$, with f in $\ell^\infty(\mathcal{S}_d)$, and the existence of the weak limit of $\sqrt{n}(M(\widehat{\eta}_n, \cdot) - M(\eta, \cdot))$ in $\ell^\infty(\mathcal{S}_d)$.

First, we start showing that the Hadamard derivative of φ at $M := M(\eta, \cdot)$ is

$$h \mapsto (\dot{\varphi}_M(h)) = \frac{h}{(1 - M)^2},$$

with h in $\ell^\infty(\mathcal{S}_d)$. Indeed, for every sequence $\epsilon_n \downarrow 0$ and $h_n \rightarrow h$ as $n \rightarrow \infty$, such that $M + \epsilon_n h_n$ in $\ell^\infty(\mathcal{S}_d)$, we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{S}_d} \left| \frac{(\varphi(M + \epsilon_n h_n))(t) - (\varphi(M))(t)}{\epsilon_n} - (\dot{\varphi}_M(h))(t) \right| \\ &= \limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{S}_d} \left| \frac{1}{\epsilon_n} \left(\frac{M(\eta, \mathbf{t}) + \epsilon_n h_n(\mathbf{t})}{1 - M(\eta, \mathbf{t}) - \epsilon_n h_n(\mathbf{t})} - \frac{M(\eta, \mathbf{t})}{1 - M(\eta, \mathbf{t})} \right) - \frac{h(\mathbf{t})}{(1 - M(\eta, \mathbf{t}))^2} \right| \\ &= \limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{S}_d} (1 + \eta A_\eta(\mathbf{t}))^2 \left| \frac{h_n(\mathbf{t}) - h(\mathbf{t}) + h(\mathbf{t}) \epsilon_n h_n(\mathbf{t}) (1 + \eta A_\eta(\mathbf{t}))}{1 - \epsilon_n h_n(\mathbf{t}) (1 + \eta A_\eta(\mathbf{t}))} \right| \\ &\leq \lim_{n \rightarrow \infty} (1 + \eta)^2 \frac{\|h_n - h\|_\infty + \epsilon_n \|h_n\|_\infty \|h\|_\infty (1 + \eta)}{1 - \epsilon_n \|h_n\|_\infty (1 + \eta)} = 0. \end{aligned}$$

Then, combining Lemmas A.1, A.2 with Proposition 3.1 in [Segers \(2012\)](#), we have under Condition 1(ii)(b) that

$$\sqrt{n} (\widehat{M}_n(\widehat{\eta}_n, \cdot) - M(\eta, \cdot)) =: T_{n,1}(\cdot) + T_{n,2}(\cdot) + o_p(1),$$

where for all $\mathbf{t} \in \mathcal{S}_d$, we have

$$T_{n,1}(\mathbf{t}) := - \int_0^1 \left(\mathbb{C}_n(v^{t_1\eta}, \dots, v^{t_d\eta}) - \sum_{j=1}^d \dot{\mathbb{C}}_{\eta,j}(v^{t_1\eta}, \dots, v^{t_d\eta}) \mathbb{C}_n(1, \dots, 1, v^{t_j\eta}, 1, \dots, 1) \right) dv$$

and

$$T_{n,2} := \frac{A_\eta}{(1 + \eta A_\eta)^2} \mathcal{X}(\mathbb{H}_n) + o_p(1).$$

For any $\mathbf{u} \in [0, 1]^d$, $\mathbb{H}_n(\mathbf{u}) = \mathbb{H}_n(G_{\eta,1}^{\leftarrow}(u_1), \dots, G_{\eta,d}^{\leftarrow}(u_d))$, and so both terms are asymptotically equivalent to continuous functionals of the empirical process \mathbb{H}_n . Therefore, the weak convergence of $T_{n,1} + T_{n,2}$ follows from the continuous mapping theorem. A similar reasoning can be obtained if Condition 1(ii)(b) is replaced by Condition 1(ii)(a). In that case, we have the following decomposition

$$\sqrt{n}(\widehat{M}_n(\widehat{\eta}_n, \cdot) - M(\eta, \cdot)) =: T_{n,1} + \widetilde{T}_{n,2} + o_p(1),$$

where

$$T_{n,1}(\mathbf{t}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (W_{i,\mathbf{t}} - \mathbb{E}(W_{i,\mathbf{t}})), \quad \widetilde{T}_{n,2}(\mathbf{t}) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \widetilde{W}_{i,\mathbf{t}}, \quad \mathbf{t} \in \mathcal{S}_d$$

and

$$\begin{aligned} W_{i,\mathbf{t}} &= \bigvee_{j=1}^d G_\eta^{1/\eta t_j}(Z_{i,j}) + \sum_{j=1}^d \int_0^1 \dot{\mathbb{C}}_{\eta,j}(v^{t_1\eta}, \dots, v^{t_d\eta}) \mathbb{1}_{\{v > G_\eta^{1/\eta t_j}(Z_{i,j})\}} dv, \\ \widetilde{W}_{i,\mathbf{t}} &= \frac{A_\eta(\mathbf{t})}{(1 + \eta A_\eta(\mathbf{t}))^2} \rho(Z_i). \end{aligned}$$

Note that the new expression for $T_{n,1}$ is obtained by applying Fubini's theorem. The pair $(T_{n,1}, \widetilde{T}_{n,2})$ is asymptotically tight and so to show that its weak limit exists, it remains to prove that all its finite dimensional distributions converge. This can be done by applying the central limit theorem since, for all $k = 1, 2, \dots$, the i.i.d. random vectors

$$(W_{i,t_1}, \dots, W_{i,t_k}, \widetilde{W}_{i,t_1}, \dots, \widetilde{W}_{i,t_k}), \quad i = 1, \dots, n,$$

have finite second order moments under the assumptions of our Theorem 3.1 (see Nelsen, 2006, Theorem 2.2.7). This achieves the proof of Lemma A.3. \square

We come back now to the proof of Theorem 3.1. Combining the three previous lemmas with the definition of $M(\eta, \mathbf{t})$, we have

$$\begin{aligned} L_n + R_n &= \frac{(1 + \eta A_\eta(\mathbf{t}))^2}{\widehat{\eta}_n} \sqrt{n}(\widehat{M}_n(\eta, \mathbf{t}) - M(\eta, \mathbf{t})) + \frac{A_\eta(\mathbf{t})}{\widehat{\eta}_n} \sqrt{n}(\widehat{\eta}_n - \eta) + \eta A_\eta(\mathbf{t}) \sqrt{n} \left(\frac{1}{\widehat{\eta}_n} - \frac{1}{\eta} \right) + o_p(1) \\ &= \frac{(1 + \eta A_\eta(\mathbf{t}))^2}{\widehat{\eta}_n} \sqrt{n}(\widehat{M}_n(\eta, \mathbf{t}) - M(\eta, \mathbf{t})) + o_p(1) \\ &= - \frac{(1 + \eta A_\eta(\mathbf{t}))^2}{\eta} \int_0^1 \widehat{\mathbb{C}}_n(v^{\eta t_1}, \dots, v^{\eta t_d}) dv + o_p(1). \end{aligned}$$

As in the proof of Lemma A.1, using again the convergence $\widehat{\mathbb{C}}_n \rightsquigarrow \mathbb{A}_\eta$ in $\ell^\infty([0, 1]^d)$ as $n \rightarrow \infty$, (3.7) follows from the continuous mapping theorem and Slutsky's lemma.

It remains now to prove (3.6). Note that

$$\begin{aligned} \|\widehat{A}_{\widehat{\eta}_n, n} - A_\eta\|_\infty &= \sup_{\mathbf{t} \in \mathcal{S}_d} \left| \frac{1}{\widehat{\eta}_n} \frac{\widehat{M}_n(\widehat{\eta}_n, \mathbf{t})}{1 - \widehat{M}_n(\widehat{\eta}_n, \mathbf{t})} - \frac{1}{\eta} \frac{M(\eta, \mathbf{t})}{1 - M(\eta, \mathbf{t})} \right| \\ &= \sup_{\mathbf{t} \in \mathcal{S}_d} \left| \frac{1}{\widehat{\eta}_n \eta \{1 - \widehat{M}_n(\widehat{\eta}_n, \mathbf{t})\} \{1 - M(\eta, \mathbf{t})\}} \right| \times \sup_{\mathbf{t} \in \mathcal{S}_d} \left| \eta \{1 - M(\eta, \mathbf{t})\} \widehat{M}_n(\widehat{\eta}_n, \mathbf{t}) - \widehat{\eta}_n \{1 - \widehat{M}_n(\widehat{\eta}_n, \mathbf{t})\} M(\eta, \mathbf{t}) \right| \\ &=: T_{n,1} \times T_{n,2}. \end{aligned}$$

Since $\widehat{\eta}_n \rightarrow \eta$ a.s., for a small $\varepsilon > 0$ and large n , we have almost surely that

$$T_{n,1} \leq \frac{1 + 1/\eta}{\widehat{\eta}_n \int_0^1 \widehat{C}_n(v^{1+\varepsilon}, \dots, v^{1+\varepsilon}) dv} \rightarrow \frac{1 + 1/\eta}{\eta \int_0^1 C_\eta(v^{1+\varepsilon}, \dots, v^{1+\varepsilon}) dv} < \infty.$$

Now, using the Lipschitz property of order $k > 0$ of C_η , we have

$$\begin{aligned} T_{n,2} &\leq \|\eta\{1 - M(\eta, \mathbf{t})\} - \widehat{\eta}_n\{1 - \widehat{M}(\widehat{\eta}_n, \mathbf{t})\}\|_\infty + \|\{1 - \widehat{M}(\widehat{\eta}_n, \mathbf{t})\}\{1 - M(\eta, \mathbf{t})\}\|_\infty |\widehat{\eta}_n - \eta| \\ &\leq |\widehat{\eta}_n - \eta| \|1 - M(\eta, \mathbf{t})\|_\infty + \widehat{\eta}_n \|M(\eta, \mathbf{t}) - \widehat{M}_n(\widehat{\eta}_n, \mathbf{t})\|_\infty + |\widehat{\eta}_n - \eta| \\ &\leq 2|\widehat{\eta}_n - \eta| + \widehat{\eta}_n \|M(\widehat{\eta}_n, \mathbf{t}) - \widehat{M}_n(\widehat{\eta}_n, \mathbf{t})\|_\infty + \widehat{\eta}_n \|M(\eta, \mathbf{t}) - M(\widehat{\eta}_n, \mathbf{t})\|_\infty \\ &\leq 2|\widehat{\eta}_n - \eta| + \widehat{\eta}_n \|\widehat{C}_n - C_\eta\|_\infty + \widehat{\eta}_n k \int_0^1 \|v^{\widehat{\eta}_n t_1} - v^{\eta t_1}, \dots, v^{\widehat{\eta}_n t_d} - v^{\eta t_d}\|_\infty dv. \end{aligned}$$

Each term on the right-hand side of this inequality tend to 0 a.s. under our assumptions and according to similar arguments to those used in Lemma A.1 for the last term. Thus (3.6) is established and the proof of Theorem 3.1 is thus completed. \square

A.5. Proof of Theorem 3.2

According to Guillou et al. (2014), η can be rewritten as

$$\eta = 2 \left(1 - \frac{\mu_{1,2}}{\mu_{1,1}} \right).$$

A natural estimator can thus be obtained by replacing $Q_\eta(u)$ by the empirical version $G_n^{\leftarrow}(u)$ where $G_n(u) := G_n(u, \dots, u)$. This entails

$$\widehat{\eta}_n = 2 \left(1 - \frac{\widehat{\mu}_{1,2}}{\widehat{\mu}_{1,1}} \right),$$

where

$$\widehat{\mu}_{a,b} := \int_0^1 Q_n(u) u^a (-\log u)^b du.$$

Consequently, we can decompose the left-hand side of (3.8) as

$$\sqrt{n}(\widehat{\eta}_n - \eta) = 2 \sqrt{n} \left(\frac{\mu_{1,2}}{\mu_{1,1}} - \frac{\widehat{\mu}_{1,2}}{\widehat{\mu}_{1,1}} \right) = 2 \frac{\int_0^1 Q_n(u) \gamma(u) du}{n^{-1/2} \mu_{1,1} \int_0^1 Q_n(u) u (-\log u) du + \mu_{1,1}^2} =: 2 \frac{N_n}{D_n}$$

with

$$Q_n(u) := \sqrt{n}(Q_n(u) - Q_\eta(u)).$$

We start to study the numerator N_n . To this aim, we define the empirical and quantile processes:

$$\begin{aligned} \widetilde{\mathbb{H}}_n(u) &:= \sqrt{n}(\widetilde{G}_n(u) - u), \quad u \in (0, 1), \\ \widetilde{Q}_n(u) &:= \sqrt{n}(\widetilde{Q}_n(u) - u), \quad u \in (0, 1), \end{aligned}$$

where for i.i.d. copies U_1, \dots, U_n of $U = G_\eta(\max(Z_1, \dots, Z_d))$, we denote

$$\widetilde{G}_n(u) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}(U_i \leq u), \quad u \in (0, 1),$$

and as before $\widetilde{Q}_n := \widetilde{G}_n^{\leftarrow}$. Let $\dot{G}_\eta(y)$ and $\ddot{G}_\eta(y)$ be the first and second derivatives of $G_\eta(y)$ with respect to $y > 0$. The function defined in Theorem 3.2 is then equal to

$$\varphi(u) = \dot{G}_\eta(Q_\eta(u)), \quad u \in (0, 1).$$

We can easily check that G_η satisfies the conditions of Theorem 3 in Csörgő and Révész (1978), where

$$\sup_{u \in (0,1)} |\varphi(u) Q_n(u) - \widetilde{Q}_n(u)| = o(1) \text{ a.s.} \quad (\text{A.7})$$

and by Bahadur-Kiefer theorem (see e.g. Einmahl, 1996) we have

$$\sup_{u \in (0,1)} |\widetilde{Q}_n(u) + \widetilde{H}_n(u)| = o(1) \text{ a.s.} \quad (\text{A.8})$$

As by direct computations $\int_0^1 \left| \frac{\gamma(u)}{\varphi(u)} \right| du < \infty$, (A.7) and (A.8) entail

$$N_n = - \int_0^1 \widetilde{H}_n(u) \frac{\gamma(u)}{\varphi(u)} du + o(1) \text{ a.s.}$$

A similar reasoning implies that almost surely

$$D_n = -n^{-1/2} \mu_{1,1} \int_0^1 \widetilde{H}_n(u) \frac{u(-\log u)}{\varphi(u)} du + \mu_{1,1}^2 + o(1) = \mu_{1,1}^2 + o(1).$$

Assembling N_n and D_n , we deduce that

$$\sqrt{n}(\widehat{\eta}_n - \eta) = -\frac{2}{\mu_{1,1}^2} \int_0^1 \mathbb{H}_n(Q_\eta(u), \dots, Q_\eta(u)) \frac{\gamma(u)}{\varphi(u)} du + o(1) \text{ a.s.},$$

where we used the fact that $\widetilde{H}_n(u) = \mathbb{H}_n(Q_\eta(u), \dots, Q_\eta(u))$. Thus (3.8) is established. The other statements of the theorem are direct consequences. \square

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