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► **To cite this version:**

Raoul Dzonou, Manuel Monteiro Marques, Laetitia Paoli. A convergence result for a vibro-impact problem with a general inertia operator. *Nonlinear Dynamics*, 2009, 2, pp.355-384. 10.1007/s11071-009-9484-1 . hal-01552482

**HAL Id: hal-01552482**

**<https://hal.science/hal-01552482>**

Submitted on 3 Jul 2017

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# A convergence result for a vibro-impact problem with a general inertia operator

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**Abstract** We consider a mechanical system with a finite number of degrees of freedom and non-trivial inertia matrix, submitted to a single perfect unilateral constraint. We assume that the local impact law consists in the transmission of the tangential component of the velocity and the reflexion of the normal component which is multiplied by the restitution coefficient  $e \in [0, 1]$ . Then, starting from the measure-differential formulation of the problem given by J.J. Moreau, we propose a velocity-based time-stepping method, reminiscent of the catching-up algorithm for sweeping processes and we prove that the numerical solutions converge to a solution of the problem.

**Keywords** Vibro-impact dynamics · Measure differential inclusion · Restitution coefficient · Sweeping process

## 1 Formulation of the problem

We consider a mechanical system with a finite number  $d$  of degrees of freedom. The configurations of the system are represented by  $q = (q_1, \dots, q_d) \in E := \mathbb{R}^d$ , where  $q \in E := \mathbb{R}^d$  denotes the representative point of the system in generalized coordinates. A motion is denoted by  $q : I \rightarrow E := \mathbb{R}^d$ , where  $I = [0, \tau]$ ,  $\tau > 0$  is a time interval. We assume that the system is subjected to an unilateral constraint: the trajectory must stay in a domain  $L$  geometrically expressed by

$$q(t) \in L = \{q \in E; g(q) \leq 0\} \quad \forall t \in I,$$

where  $g$  is a function of class  $C_{\text{loc}}^{1, \frac{1}{2}}(E, \mathbb{R})$  such that  $\nabla g$  does not vanish in a neighbourhood of

$$\{q \in E; g(q) = 0\}.$$

The system satisfied by the problem when impact is added, is

$$M(q)\ddot{q} = R + f(\cdot, q, p), \quad p = M(q)\dot{q}. \quad (1)$$

Here  $R$  describes the reaction of the constraint. For each  $q$ ,  $M(q)$  denotes the inertia operator which is a symmetric and positive definite (s.d.p)  $d \times d$  matrix.

Let us denote by  $\langle \cdot, \cdot \rangle_q$  and  $\langle \cdot, \cdot \rangle_q^*$  the inner products given by

$$\langle u, v \rangle_q = {}^t u M(q) v$$

and

$$\langle u, v \rangle_q^* = {}^t u M^{-1}(q) v.$$

The corresponding norms are

$$|u|_q = \sqrt{{}^t u M(q) u} \quad (\text{kinetic local norm})$$

and

$$|u|_q^* = \sqrt{{}^t u M^{-1}(q) u} \quad (\text{momentum local norm})$$

for all  $(u, v, q) \in E^3$ . Since discontinuous velocities may occur when  $g(q(t)) = 0$ , an appropriate functional framework is to look for a motion  $q$  such that  $\dot{q}$  has right and left limits  $\dot{q}(t+0)$  and  $\dot{q}(t-0)$ . Then it is clear that

$$\nabla g(q(t)) \cdot \dot{q}(t+0) \leq 0,$$

$$\nabla g(q(t)) \cdot \dot{q}(t-0) \geq 0.$$

For all  $q \in E$ , we define the set of right admissible velocities  $V(q)$ :

$$V(q) = \begin{cases} \{u \in E; \nabla g(q) \cdot u \leq 0\} & \text{if } g(q) \geq 0, \\ E & \text{otherwise,} \end{cases}$$

where  $v \cdot w$  denotes the Euclidian inner product of the vectors  $v$  and  $w$  in  $E$ . We define the exterior normal cone to  $L$  at point  $q$  by

$$N_L(q) = \begin{cases} \{\lambda \nabla g(q); \lambda \geq 0\} & \text{if } g(q) \geq 0, \\ \{0_E\} & \text{otherwise.} \end{cases}$$

We assume that the unilateral constraint is frictionless, hence the reaction belongs to the opposite of the normal cone to  $L$  at  $q$ ; see [9]. We can introduce the Stieltjes measure  $\dot{q} = d\dot{q}$  and we obtain the following measure differential inclusion (MDI):

$$f(t, q, p) dt - M(q) \ddot{q} = -dr \in N_L(q), \quad (2)$$

where  $dt$  is the Lebesgue's measure and  $dr$  is the measure which describes the reaction of the constraints.

When the velocity is discontinuous, i.e., if  $\dot{q}(t+0) \neq \dot{q}(t-0)$ , then by integrating (2) in a neighbourhood of  $\{t\}$ , we obtain

$$M(q)(\dot{q}(t+0) - \dot{q}(t-0)) = -dr(\{t\}) \in N_L(q(t)).$$

It follows that

$$\begin{cases} \dot{q}(t+0) \in V(q(t)), \\ \dot{q}(t+0) - \dot{q}(t-0) \in N_L^*(q(t)) \\ \quad = M^{-1}(q(t)) N_L(q(t)). \end{cases}$$

These relations do not define uniquely  $\dot{q}(t+0)$  and we have to complement the description of the dynamics with a constitutive impact law. We suppose that the local impact law consists in the transmission of the tangential component of the velocity and the reflexion of the normal component which is multiplied by a restitution coefficient  $e \in [0, 1]$ , i.e.,

$$\begin{aligned} \dot{q}(t+0) &= \text{proj}_{q(t)}(\dot{q}(t-0), V(q(t))) \\ &\quad - e \text{proj}_{q(t)}(\dot{q}(t-0), N_L^*(q(t))) \\ &= \dot{q}(t-0) - (1+e) \\ &\quad \times \text{proj}_{q(t)}(\dot{q}(t-0), N_L^*(q(t))), \end{aligned} \quad (3)$$

where  $\text{proj}_q(u, N_L^*(q))$  (resp.  $\text{proj}_q(u, V(q))$ ) denotes the projection operator on  $N_L^*(q)$  (resp. on  $V(q)$ ) relatively to the kinetic metric at  $q$ . We should observe that this impact law leads to non-increasing kinetic energy at impacts.

In [9] J.J. Moreau introduced a synthetic velocity-based formulation of the problem, given by the following measure differential inclusion:

$$f(t, q, p) - M(q) \ddot{q} \in N_{V(q)} \left( \frac{\dot{q}^+ + e\dot{q}^-}{1+e} \right) \quad (4)$$

with

$$N_{V(q)}(\dot{q}) = \begin{cases} \{y \in E; \langle y, x - \dot{q} \rangle \leq 0 \quad \forall x \in V(q)\} \\ \quad \text{if } \dot{q} \in V(q), \\ \emptyset \quad \text{otherwise.} \end{cases}$$

More precisely

**Definitio 1** A function  $q : I \rightarrow E$  is a solution of (2)–(3) with initial conditions  $(q_0, u_0) \in L \times V(q_0)$  iff:

- $q(t) \in L \quad \forall t \in I$
- $\dot{q} \in \text{bvem}(I; E)$  i.e.  $\dot{q} \in BV(I; E)$  and

$$\dot{q}(t) = \frac{\dot{q}(t+0) + e\dot{q}(t-0)}{1+e} \quad \forall t \in \text{Int}(I)$$

- $(q, \dot{q})$  satisfy the initial data in the following sense:

$$q(0) = q_0, \quad (5)$$

$$\dot{q}(0+0) = u_0 \quad (6)$$

- $(q, \dot{q})$  satisfy (4) in the following sense: for any positive measure  $d\mu$  over  $I$  with respect to which the Lebesgue measure  $dt$  and the Stieltjes measure  $d\dot{q}$  possess densities, respectively  $t'_\mu = \frac{dt}{d\mu} \in \mathcal{L}^1(I, d\mu; \mathbb{R})$  and  $\dot{q}'_\mu = \frac{d\dot{q}}{d\mu} \in \mathcal{L}^1(I, d\mu; E)$  we have

$$f(t, q, p)t'_\mu(t) - M(q)\dot{q}'_\mu(t) \in N_{V(q)}(\dot{q}(t))$$

$$d\mu \text{ a.e on } I. \quad (7)$$

Notice that (7) does not depend on the ‘base’ measure  $d\mu$  and that, at an instant of contact  $t$  such that  $\dot{q}(t+0) \neq \dot{q}(t-0)$ , (7) is equivalent to (3) (see [9]).

Let us give the assumptions under which we shall investigate the existence of a solution.

**H-1:**  $f : [0, T] \times E \times E \rightarrow E$  is a continuous function, which is locally Lipschitz continuous with respect to its second and third arguments.

We will denote by

$$C_{f,W} = \sup\{|f(t, q, M(q)v)|, (t, q, v) \in W\}$$

and  $L_{f,W}$  the associated Lipschitz constant on any compact set  $W \subset [0, T] \times E \times E$ .

**H-2:**  $g : E \rightarrow \mathbb{R}$  is a  $C_{\text{loc}}^{1,1/2}$  function, with nonzero gradient in the neighbourhood of its zero level set

$$S := \{q \in E; g(q) = 0\}.$$

**H-3:**  $M$  is a mapping of class  $C^1$  from  $E$  to the set of symmetric positive definite (s.d.p)  $d * d$  matrices  $\mathcal{F}$ .

It follows that mappings  $q \mapsto (M(q))^{1/2}$  and  $q \mapsto (M(q))^{-1}$  are locally Lipschitz continuous from  $E$  to the set of s.d.p  $d * d$  matrices  $\mathcal{F} \subset \mathcal{M}_d$  (see [11] for instance). Therefore, for any compact  $V$  of  $E$ , there exist  $\alpha_V > 0$  and  $\beta_V > 0$  such that

$$\alpha_V |u|^2 \leq |u|_q^2 \leq \beta_V |u|^2 \quad \forall u \in E, \forall q \in V, \quad (8)$$

an inequality which will be useful in the sequel.

First we show that there exists an interval  $[0, \tau^*] \subset [0, T]$ ,  $\tau^* > 0$  on which the Cauchy problem admits a solution (local existence). Moreover, energy estimates allow us to prove the existence of a ‘global’ solution, i.e., a solution defined on a time interval depending only on the data. Let us outline the main steps of the paper.

In Sect. 2 we introduce the numerical scheme (implicit and explicit versions) and we prove some local uniform estimates for the discrete velocities by using a fixed point argument. Next in Sect. 3, we establish that the total variation of the approximate velocities is uniformly bounded and by using Helly and Ascoli theorems, we extract converging subsequences. Finally, we study the properties of the limit, more precisely we check that the differential inclusion and the impact law are satisfied, which concludes the proof of the local existence result.

Then, in Sect. 4, we reason as in [13] to show that the convergence result can be extended to a fixed interval  $[0, \tau]$  depending only on the data, which leads to the existence of a ‘global’ solution.

Our results follow in the same spirit and they directly extend [5], where a constant and trivial mass matrix is considered, and [2] which tackle only the purely inelastic case  $e = 0$ , while in [6] and [7] both simplifying assumptions were made. Other approaches may be found in [14], in [1], in [10], in [12] and in [13]. The present results were announced and outlined in [3].

## 2 Numerical scheme

We shall define the numerical scheme on  $[0, \tilde{\tau}]$  with  $\tilde{\tau} \in (0, T]$  satisfying some requirements (see (10) below at the end of the proof of Lemma 1). More precisely, let  $n$  be any positive integer,  $h = \frac{\tilde{\tau}}{n}$  and  $t_{n,i} = ih$ ,  $0 \leq i \leq n$ . We define two finite sequences  $(q_{n,i})_{0 \leq i \leq n}$  and  $(u_{n,i})_{0 \leq i \leq n}$  of  $E$  by

$$q_{n,0} = q_0$$

$$u_{n,0} = -eu_0 + (1+e)\text{proj}_{q_0}(u_0, V(q_0)) = u_0$$

and for all  $0 \leq i \leq n-1$

$$q_{n,i+1} = q_{n,i} + hu_{n,i}$$

$$u_{n,i+1} = -eu_{n,i} + (1+e)\text{proj}_{q_{n,i+1}}\left(u_{n,i} + \frac{h}{1+e} \times M^{-1}(q_{n,i+1})f_{n,i+1}, V_{n,i+1}\right), \quad (9)$$

where  $f_{n,i+1}$  is an approximate value of  $f(t, q, p)$ ,  $V_{n,i+1} = V(q_{n,i+1})$  and the projection onto  $V_{n,i+1}$  is taken with respect to the kinetic metric at  $q_{n,i+1}$ .

The approximate value of the force can be updated explicitly i.e.,

$$f_{n,i+1} = f(t_{n,i+1}, q_{n,i+1}, M(q_{n,i+1})u_{n,i}),$$

$$0 \leq i \leq n-1.$$

But, in order to avoid instabilities that could appear with an explicit algorithm, we may choose to define an implicit updating given by

$$f_{n,i+1} = f(t_{n,i+1}, q_{n,i+1}, M(q_{n,i+1})u_{n,i+1}),$$

$$0 \leq i \leq n-1.$$

When no constraint is active at  $t_{n,i+1}$  then  $V(q_{n,i+1}) = E$  and (9) becomes

$$u_{n,i+1} = u_{n,i} + hM^{-1}(q_{n,i+1})f_{n,i+1}$$

which is an Euler's discretization of the ordinary differential equation

$$M(q)\ddot{q} = f(t, q, p).$$

We should notice that (9) can be rewritten as

$$f_{n,i+1} - M(q_{n,i+1}) \left( \frac{u_{n,i+1} - u_{n,i}}{h} \right)$$

$$\in N_{V_{n,i+1}} \left( \frac{u_{n,i+1} + eu_{n,i}}{1+e} \right)$$

which is a quite natural discretization of the measure differential inclusion (4).

In the implicit approach, the construction of the sequence  $(u_{n,i})_{0 \leq i \leq n}$  requires at each time step the computation of a fixed point which existence will be justified below. Since this fixed point argument will give some uniform estimates for the discrete velocities, we apply it also to the explicit approach although it is not necessary to define  $(u_{n,i})_{0 \leq i \leq n}$  in this case.

Let  $\tilde{\tau} \in (0, T]$  and  $\tilde{n}$  be the integer part of  $\frac{\tilde{\tau}}{h}$ ; we consider first the mapping  $G$  defined by

$$G : \begin{cases} (E \times E)^{\tilde{n}+1} \longrightarrow E^{\tilde{n}} \\ \{(\tilde{q}_{n,i}, \tilde{u}_{n,i})\}_{0 \leq i \leq \tilde{n}} \longmapsto \{f_{n,i}\}_{1 \leq i \leq \tilde{n}} \end{cases}$$

with

$$f_{n,i+1} = f(t_{n,i+1}, \tilde{q}_{n,i+1}, M(\tilde{q}_{n,i+1})\tilde{u}_{n,j(i)})$$

$$\text{for all } i \in \{0, \dots, \tilde{n}-1\}$$

and  $j(i) = i$  if we consider an explicit version of the scheme,  $j(i) = i+1$  if we use an implicit version.

Let

$$q_{n,1} = q_0 + hu_0$$

and denote by  $(V_i^0)_{0 \leq i \leq \tilde{n}}$  the sets defined by

$$\begin{cases} V_0^0 = V(q_0) & i = 0, \\ V_1^0 = V(q_{n,1}) & i = 1, \\ V_i^0 = V_1^0 & \text{if } 2 \leq i \leq \tilde{n}. \end{cases}$$

We consider the mapping  $F^0$  given by

$$F^0 : \begin{cases} E^{\tilde{n}} \longrightarrow (E \times E)^{\tilde{n}+1} \\ \{f_{n,i}\}_{1 \leq i \leq \tilde{n}} \longmapsto \{(\tilde{q}_{n,i}, \tilde{u}_{n,i})\}_{0 \leq i \leq \tilde{n}} \end{cases}$$

with

$$\begin{cases} \tilde{q}_{n,0} = q_0, \\ \tilde{u}_{n,0} = u_0, \end{cases}$$

and for all  $i \in \{0, \dots, \tilde{n}-1\}$

$$\begin{cases} \tilde{q}_{n,i+1} = \tilde{q}_{n,i} + h\tilde{u}_{n,i}, \\ \tilde{u}_{n,i+1} = -e\tilde{u}_{n,i} + (1+e)\text{proj}_{\tilde{q}_{n,i+1}} \left( \tilde{u}_{n,i} + \frac{h}{1+e} \right. \\ \left. \times M^{-1}(\tilde{q}_{n,i+1})f_{n,i+1}, V_{i+1}^0 \right). \end{cases}$$

Then  $F^0 \in C^0(E^{\tilde{n}}, (E \times E)^{\tilde{n}+1})$  and  $G \in C^0((E \times E)^{\tilde{n}+1}, E^{\tilde{n}})$ . It follows that the mapping  $F^0 \circ G \in C^0((E \times E)^{\tilde{n}+1}, (E \times E)^{\tilde{n}+1})$  and it leaves globally invariant the compact convex set defined in the following lemma:

**Lemma 1** *Let  $R > |u_0|_{q_0}$ , and let  $\alpha_0, \beta_0$  be the real numbers defined by (8) on  $V = \bar{B}(q_0, R)$ . If we denote*

$$C = \left( \bar{B}(q_0, R) \times \bar{B}\left(0, \frac{R}{\sqrt{\alpha_0}}\right) \right)^{\tilde{n}+1}$$

*then  $F^0 \circ G(C) \subset C$ , if  $\tilde{\tau}$  is small enough.*

*Proof* Let us define

$$W = [0, T] \times \bar{B}(q_0, R) \times \bar{B}\left(0, \frac{R}{\sqrt{\alpha_0}}\right),$$

and recall that

$$C_{f,W} = \sup\{|f(t, q, M(q)v)|; (t, q, v) \in W\}.$$

Let  $\{(\tilde{q}_{n,i}, \tilde{u}_{n,i})\}_{0 \leq i \leq \tilde{n}} \in C$ . Then we have  $(t_{n,i+1}, \tilde{q}_{n,i+1}, \tilde{u}_{n,j(i)}) \in W$ , for all  $i \in \{0, \dots, \tilde{n} - 1\}$ , and it follows that

$$|f_{n,i+1}| \leq C_{f,W} \quad \forall 0 \leq i \leq \tilde{n} - 1.$$

It remains to prove that  $F^0(\{f_{n,i}\}_{1 \leq i \leq \tilde{n}}) = (\{\tilde{q}_{n,i}, \tilde{u}_{n,i}\})_{0 \leq i \leq \tilde{n}} \in C$  that is

$$\tilde{q}_{n,i} \in \bar{B}(q_0, R) \quad \text{and} \quad |\tilde{u}_{n,i}| \leq \frac{R}{\sqrt{\alpha_0}} \quad \forall 0 \leq i \leq \tilde{n}.$$

We establish this result by induction on  $i \in \{0, \dots, \tilde{n}\}$ . If  $i = 0$  we have

$$(\tilde{q}_{n,0}, \tilde{u}_{n,0}) = (q_0, u_0) \in \bar{B}(q_0, R) \times \bar{B}\left(0, \frac{R}{\sqrt{\alpha_0}}\right).$$

Assume that  $(\tilde{q}_{n,i}, \tilde{u}_{n,i}) \in \bar{B}(q_0, R) \times \bar{B}\left(0, \frac{R}{\sqrt{\alpha_0}}\right)$  for all  $i \in \{0, \dots, j\}$  with  $j \in \{0, \dots, \tilde{n} - 1\}$ . Then we have

$$|\tilde{q}_{n,j+1} - q_0| = h \left| \sum_{i=0}^j \tilde{u}_{n,i} \right| \leq \frac{\tilde{\tau} R}{\sqrt{\alpha_0}}.$$

Let us assume from now on that

$$\frac{\tilde{\tau} R}{\sqrt{\alpha_0}} \leq R.$$

In order to simplify the notation, let

$$\tilde{x}_{n,i} = \tilde{u}_{n,i} + \frac{h}{1+e} M^{-1}(\tilde{q}_{n,i+1}) f_{n,i+1} \quad 0 \leq i \leq \tilde{n} - 1$$

and denote by  $L_{V,1}$  (resp.  $L_{V,2}$ ) the Lipschitz constant associated to the mapping  $M$  (resp.  $M^{1/2}$ ) on the compact set  $V$ .

Let us also denote  $P_{n,i}$  (resp.  $N_{n,i}$ ) the projection operator with respect to the kinetic metric at  $\tilde{q}_{n,i}$  onto  $V_i^0$  (resp. onto the conjugate convex cone to  $V_i^0$  with respect to the kinetic metric at  $\tilde{q}_{n,i}$ ). We first prove that

$$|P_{n,i+1}(\tilde{x}_{n,i}) - eN_{n,i+1}(\tilde{x}_{n,i})|_{\tilde{q}_{n,i+1}} \leq |\tilde{x}_{n,i}|_{\tilde{q}_{n,i+1}}.$$

Indeed, using  $e \in [0, 1]$ :

$$\begin{aligned} & |P_{n,i+1}(\tilde{x}_{n,i}) - eN_{n,i+1}(\tilde{x}_{n,i})|_{\tilde{q}_{n,i+1}}^2 \\ &= |P_{n,i+1}(\tilde{x}_{n,i})|_{\tilde{q}_{n,i+1}}^2 \\ &\quad - 2e(P_{n,i+1}(\tilde{x}_{n,i}), N_{n,i+1}(\tilde{x}_{n,i}))_{\tilde{q}_{n,i+1}} \end{aligned}$$

$$\begin{aligned} & + e^2 |N_{n,i+1}(\tilde{x}_{n,i})|_{\tilde{q}_{n,i+1}}^2 \\ &= |P_{n,i+1}(\tilde{x}_{n,i})|_{\tilde{q}_{n,i+1}}^2 + e^2 |N_{n,i+1}(\tilde{x}_{n,i})|_{\tilde{q}_{n,i+1}}^2 \\ &\leq |P_{n,i+1}(\tilde{x}_{n,i}) + N_{n,i+1}(\tilde{x}_{n,i})|_{\tilde{q}_{n,i+1}}^2 = |\tilde{x}_{n,i}|_{\tilde{q}_{n,i+1}}^2. \end{aligned}$$

Moreover

$$\begin{aligned} \tilde{u}_{n,i+1} &= -e\tilde{u}_{n,i} + (1+e)P_{n,i+1}(\tilde{x}_{n,i}) \\ &= P_{n,i+1}(\tilde{x}_{n,i}) - eN_{n,i+1}(\tilde{x}_{n,i}) \\ &\quad + \frac{eh}{1+e} M^{-1}(\tilde{q}_{n,i+1}) f_{n,i+1} \end{aligned}$$

and we deduce that

$$|\tilde{u}_{n,i+1}|_{\tilde{q}_{n,i+1}} \leq |\tilde{u}_{n,i}|_{\tilde{q}_{n,i+1}} + h |M^{-1/2}(\tilde{q}_{n,i+1}) f_{n,i+1}|.$$

Thus

$$\begin{aligned} |\tilde{u}_{n,i+1}|_{\tilde{q}_{n,i+1}} &\leq |\tilde{u}_{n,i}|_{\tilde{q}_{n,i}} + \|(M(\tilde{q}_{n,i+1}))^{1/2} \\ &\quad - (M(\tilde{q}_{n,i}))^{1/2}\| |\tilde{u}_{n,i}| \\ &\quad + h \|(M(\tilde{q}_{n,i+1}))^{-1/2}\| |f_{n,i+1}|. \end{aligned}$$

By definition of  $\alpha_0$  and  $L_{V,2}$  we obtain

$$\begin{aligned} |\tilde{u}_{n,i+1}|_{\tilde{q}_{n,i+1}} &\leq |\tilde{u}_{n,i}|_{\tilde{q}_{n,i}} + h L_{V,2} |\tilde{u}_{n,i}|^2 + h \frac{C_{f,W}}{\sqrt{\alpha_0}} \\ &\quad \forall 0 \leq i \leq j. \end{aligned}$$

Let  $\{v_{n,i}\}_{0 \leq i \leq n}$  be defined by

$$\begin{cases} v_{n,0} = |u_0|_{q_0}, \\ \frac{v_{n,i+1} - v_{n,i}}{h} = \frac{L_{V,2}}{\alpha_0} v_{n,i}^2 + \frac{C_{f,W}}{\sqrt{\alpha_0}} \quad \forall 0 \leq i \leq n-1, \end{cases}$$

and

$$\begin{aligned} v(t) &= v_{n,i} + \frac{1}{h}(t - t_{n,i})(v_{n,i+1} - v_{n,i}) \\ &\quad \text{if } t \in [t_{n,i}, t_{n,i+1}], \quad 0 \leq i \leq n-1. \end{aligned}$$

Then  $\{v_{n,i}\}_{0 \leq i \leq n}$  is given by an Euler scheme associated to the following Cauchy problem:

$$\begin{cases} \frac{d\theta}{dt} = k_1 \theta^2 + k_2, \\ \theta(0) = |u_0|_{q_0} \quad \text{with } k_1 = \frac{L_{V,2}}{\alpha_0}, \quad k_2 = \frac{C_{f,W}}{\sqrt{\alpha_0}}. \end{cases}$$

Consequently, by comparison,

$$|\tilde{u}_{n,i}|_{\tilde{q}_{n,i}} \leq v_{n,i} = v(ih) \leq \theta(ih) \quad \forall 0 \leq i \leq j+1,$$

with

$$\theta(t) = \sqrt{\frac{k_2}{k_1}} \tan(\sqrt{k_2 k_1} t + C),$$

$$C = \text{Arctan}\left(\sqrt{\frac{k_1}{k_2}} |u_0|_{q_0}\right).$$

It follows that

$$|\tilde{u}_{n,i}| \leq \frac{1}{\sqrt{\alpha_0}} |\tilde{u}_{n,i}|_{\tilde{q}_{n,i}} \leq \frac{\theta(ih)}{\sqrt{\alpha_0}} \quad \forall 0 \leq i \leq j+1.$$

So, if we choose  $\tilde{\tau} \in (0, T]$  such that

$$\tilde{\tau} \leq \sqrt{\alpha_0} \quad \text{and} \quad \theta(\tilde{\tau}) \leq R \quad (10)$$

we obtain  $F^0 \circ G(C) \subset C$ .  $\square$

Using Brouwer's fixed point theorem, we infer that there exists  $\{\tilde{q}_i^0, \tilde{u}_i^0\}_{0 \leq i \leq \tilde{n}} \in C$  such that

$$F^0 \circ G(\{\tilde{q}_i^0, \tilde{u}_i^0\}_{0 \leq i \leq \tilde{n}}) = \{\tilde{q}_i^0, \tilde{u}_i^0\}_{0 \leq i \leq \tilde{n}}.$$

By definition of  $F^0$  and  $G$  we have

$$\begin{cases} \tilde{q}_{n,0}^0 = q_0 = q_{n,0}, \\ \tilde{u}_{n,0}^0 = u_0 = u_{n,0}, \end{cases}$$

and

$$\begin{aligned} \tilde{q}_{n,1}^0 &= \tilde{q}_{n,0}^0 + h\tilde{u}_{n,0}^0 = q_{n,0} + hu_{n,0} = q_{n,1}, \\ \tilde{u}_{n,1}^0 &= -e\tilde{u}_{n,0}^0 + (1+e)\text{proj}_{\tilde{q}_{n,1}^0} \left( \tilde{u}_{n,0}^0 + \frac{h}{1+e} \right. \\ &\quad \left. \times M^{-1}(\tilde{q}_{n,1}^0) f_{n,1}^0, V_1^0 \right), \\ f_{n,1}^0 &= f(t_{n,1}, \tilde{q}_{n,1}^0, M(\tilde{q}_{n,1}^0) \tilde{u}_{n,j(0)}^0). \end{aligned}$$

We deduce that

$$\begin{aligned} \tilde{u}_{n,1}^0 &= -eu_{n,0} + (1+e) \\ &\quad \times \text{proj}_{\tilde{q}_{n,1}^0} \left( u_{n,0} + \frac{h}{1+e} M^{-1}(\tilde{q}_{n,1}^0) \right) \\ &\quad \times f(t_{n,1}, \tilde{q}_{n,1}^0, M(\tilde{q}_{n,1}^0) \tilde{u}_{n,j(0)}^0, V(\tilde{q}_{n,1}^0)). \end{aligned}$$

So  $(\tilde{q}_{n,1}^0, \tilde{u}_{n,1}^0)$  satisfies (9) for  $i=0$  and we get

$$\tilde{q}_{n,1}^0 = q_{n,1}, \quad \tilde{u}_{n,1}^0 = u_{n,1}$$

with  $(q_{n,1}, u_{n,1}) \in \bar{B}(q_0, R) \times \bar{B}(0, \frac{R}{\sqrt{\alpha_0}})$ . (As seen below, although we are applying Brouwer's fixed point theorem, the updated velocity is still uniquely defined.)

Now we consider the position  $q_{n,2}$ , indeed:

$$\tilde{q}_{n,2}^0 = \tilde{q}_{n,1}^0 + h\tilde{u}_{n,1}^0 = q_{n,1} + hu_{n,1} = q_{n,2}.$$

We define the sets

$$\begin{cases} V_i^1 = V(q_{n,i}) & 0 \leq i \leq 2, \\ V_i^1 = V_2^1 & 3 \leq i \leq \tilde{n}. \end{cases}$$

As in the case  $k=0$  we define the mapping  $F^1$  by

$$F^1 : \begin{cases} E^{\tilde{n}} \longrightarrow (E \times E)^{\tilde{n}+1} \\ \{f_{n,i}\}_{1 \leq i \leq \tilde{n}} \longmapsto \{(\tilde{q}_{n,i}, \tilde{u}_{n,i})\}_{0 \leq i \leq \tilde{n}} \end{cases}$$

with

$$\begin{cases} \tilde{q}_{n,0} = q_0, \\ \tilde{u}_{n,0} = u_0, \end{cases}$$

and

$$\begin{cases} \tilde{q}_{n,i+1} = \tilde{q}_{n,i} + h\tilde{u}_{n,i}, \\ \tilde{u}_{n,i+1} = -e\tilde{u}_{n,i} + (1+e)\text{proj}_{\tilde{q}_{n,i+1}} \left( \tilde{u}_{n,i} + \frac{h}{1+e} \right. \\ \quad \left. \times M^{-1}(\tilde{q}_{n,i+1}) f_{n,i+1}, V_{i+1}^1 \right). \end{cases}$$

With the same arguments as previously, we show that  $F^k \circ G(C) \subset C$  for  $k=1$ , with the same value of  $\tilde{\tau}$  as in the previous case  $k=0$ , and thus  $F^1 \circ G$  admits a fixed point on  $C$  which we denote  $\{\tilde{q}_i^1, \tilde{u}_i^1\}_{0 \leq i \leq \tilde{n}}$ . We have

$$\tilde{q}_{n,0}^1 = q_0 = q_{n,0}, \quad \tilde{u}_{n,0}^1 = u_0 = u_{n,0},$$

and

$$\tilde{q}_{n,1}^1 = q_0 + hu_0 = q_{n,1}.$$

Moreover  $\tilde{u}_{n,1}^1 \in \bar{B}(0, \frac{R}{\sqrt{\alpha_0}})$  and it satisfies

$$\begin{aligned} \tilde{u}_{n,1}^1 &= -eu_{n,0} + (1+e)\text{proj}_{q_{n,1}} \left( u_{n,0} + \frac{h}{1+e} \right. \\ &\quad \left. \times M^{-1}(q_{n,1}) f_{n,1}^1, V(q_{n,1}) \right). \end{aligned}$$

So

$$\begin{aligned} |\tilde{u}_{n,1}^1 - u_{n,1}|_{q_{n,1}} &\leq h \left| f(t_1, q_{n,1}, M(q_{n,1}) \tilde{u}_{n,j(0)}^1) \right. \\ &\quad \left. - f(t_1, q_{n,1}, M(q_{n,1}) u_{n,j(0)}) \right|_{q_{n,1}}. \end{aligned}$$

In the explicit case,  $\tilde{u}_{n,j(0)}^1 = \tilde{u}_{n,0}^1 = u_0$  so  $\tilde{u}_{n,1}^1 = u_{n,1}$ . Otherwise, since  $u_{n,1} \in \bar{B}(0, \frac{R}{\sqrt{\alpha_0}})$  we apply the Lipschitz property of  $f$  on  $W$  and we get

$$\begin{aligned} |\tilde{u}_{n,1}^1 - u_{n,1}|_{q_{n,1}} &\leq hL_{f,W}\sqrt{\beta_0}\|M(q_{n,1})\| |\tilde{u}_{n,1}^1 - u_{n,1}| \\ &\leq hL_{f,W}\sqrt{\frac{\beta_0}{\alpha_0}}\beta_0 |\tilde{u}_{n,1}^1 - u_{n,1}|_{q_{n,1}}. \end{aligned}$$

Let us assume from now on that

$$0 < h \leq h^* < \frac{\sqrt{\alpha_0}}{L_{f,W}(\beta_0)^{3/2}}. \quad (11)$$

Then  $\tilde{u}_{n,1}^1 = u_{n,1}$  and thus

$$\begin{aligned} \tilde{q}_{n,2}^1 &= \tilde{q}_{n,1}^1 + h\tilde{u}_{n,1}^1 = q_{n,1} + hu_{n,1} = q_{n,2}, \\ \tilde{u}_{n,2}^1 &= -e\tilde{u}_{n,1}^1 + (1+e) \\ &\quad \times \text{proj}_{\tilde{q}_{n,2}^1} \left( \tilde{u}_{n,1}^1 + \frac{h}{1+e} M^{-1}(\tilde{q}_{n,2}^1) f_{n,2}^1, V_2^1 \right), \\ f_{n,2}^1 &= f(t_{n,2}, \tilde{q}_{n,2}^1, M(\tilde{q}_{n,2}^1) \tilde{u}_{n,j(1)}^1). \end{aligned}$$

We deduce that

$$\begin{aligned} \tilde{u}_{n,2}^1 &= -eu_{n,1} + (1+e) \\ &\quad \times \text{proj}_{\tilde{q}_{n,2}^1} \left( u_{n,1} + \frac{h}{1+e} M^{-1}(\tilde{q}_{n,2}^1) \right) \\ &\quad \times f(t_{n,2}, \tilde{q}_{n,2}^1, M(\tilde{q}_{n,2}^1) \tilde{u}_{n,j(1)}^1, V(\tilde{q}_{n,2}^1)). \end{aligned}$$

Hence  $(\tilde{q}_{n,i}^1, \tilde{u}_{n,i}^1)$  satisfies (9) for  $i = 1, 2$  and we get

$$\tilde{q}_{n,i}^1 = q_{n,i}, \quad \tilde{u}_{n,i}^1 = u_{n,i}, \quad i = 1, 2$$

with  $(q_{n,i}, u_{n,i}) \in \bar{B}(q_0, R) \times \bar{B}(0, \frac{R}{\sqrt{\alpha_0}})$  for  $i = 1, 2$ .

So we define all the approximate positions and velocities  $\{q_{n,i}, u_{n,i}\}_{0 \leq i \leq \tilde{n}}$  by iterating this fixed point procedure. More precisely, for  $k \in \{2, \dots, \tilde{n} - 1\}$  we define recursively the sets

$$\begin{cases} V_i^k = V(q_{n,i}) & 0 \leq i \leq k+1, \\ V_i^k = V_{k+1}^k & k+2 \leq i \leq \tilde{n} \end{cases}$$

and the mapping  $F^k$  by

$$F^k : \begin{cases} E^{\tilde{n}} \longrightarrow (E \times E)^{\tilde{n}+1}, \\ \{f_{n,i}\}_{1 \leq i \leq \tilde{n}} \longmapsto \{(\tilde{q}_{n,i}^1, \tilde{u}_{n,i}^1)\}_{0 \leq i \leq \tilde{n}} \end{cases}$$

with

$$\begin{cases} \tilde{q}_{n,0} = q_0, \\ \tilde{u}_{n,0} = u_0, \end{cases}$$

and

$$\begin{cases} \tilde{q}_{n,i+1} = \tilde{q}_{n,i} + h\tilde{u}_{n,i}, \\ \tilde{u}_{n,i+1} = -e\tilde{u}_{n,i} + (1+e)\text{proj}_{\tilde{q}_{n,i+1}} \left( \tilde{u}_{n,i} + \frac{h}{1+e} \right. \\ \quad \left. \times M^{-1}(\tilde{q}_{n,i+1}) f_{n,i+1}, V_{i+1}^k \right). \end{cases}$$

By using finite induction, we see that, for all  $h \in (0, h^*]$ , the system (9) possesses a solution which is a fixed point of  $F^{\tilde{n}-1} \circ G$  in  $C$ .

In order to obtain an estimate of the discrete accelerations, we will apply the following lemma due to J.J. Moreau [8]:

**Lemma 2** (J.J. Moreau) *Let  $C$  be a closed convex set of a Hilbert space  $H$ ,  $C$  containing a ball  $B(a, r)$ . Then*

$$\begin{aligned} \forall x \in H : |x - \text{proj}(x, C)| \\ \leq \frac{1}{2r} (|x - a|^2 - |\text{proj}(x, C) - a|^2). \end{aligned}$$

We choose  $C = V(q_{n,j})$  and we also use the following result.

**Lemma 3** *For all  $\tilde{R} > 0$ , there exist  $\rho > 0$ ,  $\delta > 0$  and  $\alpha > 0$  such that, for all  $q \in \bar{B}(q_0, \tilde{R})$ , there exists  $a_q \in E$  such that*

$$|a_q| \leq \alpha, \quad \bar{B}(a_q, \rho) \subset V(\tilde{q}) \quad \forall \tilde{q} \in \bar{B}(q, \delta).$$

*Proof* The lower semi-continuity of the mapping  $q \mapsto V(q)$  implies that for all  $q \in E$  there exist  $a_q \in E$ ,  $r_q > 0$  and  $\delta_q > 0$  such that

$$\bar{B}(a_q, r_q) \subset V(\tilde{q}) \quad \forall \tilde{q} \in \bar{B}(q, \delta_q)$$

and the result follows by a standard compactness argument (the bound on the  $a_q$  is easy to obtain, as the sets  $V(q)$  are cones).  $\square$

We take  $\tau^* \in (0, T]$  satisfying (10) and such that  $\frac{R}{\sqrt{\alpha_0}} \tau^* \leq \delta_0$ , i.e.,

$$\tau^* \leq \sqrt{\alpha_0}, \quad \theta(\tau^*) \leq R \quad \text{and} \quad \frac{R}{\sqrt{\alpha_0}} \tau^* \leq \delta_0, \quad (12)$$

where  $\delta_0$  is a constant obtained from the previous lemma for  $\tilde{R} = R$ .

We define the sequence of approximate solutions on  $[0, \tau^*]$  by considering piecewise constant velocities and then integrating over the time-interval:

$$u_n(t) = u_{n,i} \quad \text{if } t \in [t_{n,i}, t_{n,i+1}] \cap [0, \tau^*]$$

and

$$q_n(t) = q_0 + \int_0^t u_n(s) ds \quad \forall t \in [0, \tau^*]$$

for all  $n \geq n^* = \lceil T/h^* \rceil + 1$  (i.e. for all  $h \in (0, h^*)$ ).

### 3 Local convergence result

#### 3.1 Convergence result

We already know that the sequences  $(u_n)_{n \geq n^*}$  and  $(q_n)_{n \geq n^*}$  are uniformly bounded on  $[0, \tau^*]$ . Moreover the sequence  $(q_n)_{n \geq n^*}$  is equicontinuous on  $[0, \tau^*]$ , more precisely the functions  $(q_n)_{n \geq n^*}$  are  $\frac{R}{\sqrt{\alpha_0}}$ -Lipschitz continuous on  $[0, \tau^*]$ . From Ascoli-Arzela's theorem, it follows that, possibly by extracting a subsequence still denoted  $(q_n)_{n \geq n^*}$ , there exists  $q \in C^0([0, \tau^*]; E)$  such that

$$q_n \rightarrow q \quad \text{strongly in } C^0([0, \tau^*]; E).$$

Moreover  $q$  is  $\frac{R}{\sqrt{\alpha_0}}$ -Lipschitz continuous on  $[0, \tau^*]$ .

Next we prove that the limit function  $q$  satisfies the constraints i.e.,  $g(q(t)) \leq 0$  for all  $t \in [0, \tau^*]$ . Indeed, it is a direct consequence of the following lemma.

**Lemma 4** *Let  $n \geq n^* = \lceil T/h^* \rceil + 1$ . We have*

$$\begin{aligned} g(q_n(t)) &\leq hK \frac{R}{\sqrt{\alpha_0}} + \frac{eh^2}{1+e} j \frac{C_{f,W}}{\alpha_0} K \\ &\quad + t_{n,j} \varphi \left( h \frac{R}{\sqrt{\alpha_0}} \right) \frac{R}{\sqrt{\alpha_0}} \\ &\quad \forall t \in [0, t_{n,j}] \cap [0, \tau^*], \end{aligned}$$

where  $K = \sup\{|\nabla g(q)|; q \in \bar{B}(q_0, R)\}$  and  $\varphi$  is the continuity modulus of  $\nabla g$  on  $V = \bar{B}(q_0, R)$ .

*Proof* We will prove this result by induction on  $j$ . For  $j = 0$  we have  $t_{n,j} = 0$  and  $g(q_n(t)) = g(q_0) \leq 0$ . Assume now that (13) holds for some  $j \in \{0, \dots, \tilde{n}\}$ . For

all  $t \in [t_{n,j}, t_{n,j+1}] \cap [0, \tau^*]$  we have

$$\begin{aligned} g(q_n(t)) &= g(q_{n,j}) + \int_{t_{n,j}}^t \nabla g(q_n(s)) \cdot u_n(s) ds \\ &\leq g(q_{n,j}) + \int_{t_{n,j}}^t \nabla g(q_{n,j}) \cdot u_{n,j} ds \\ &\quad + \int_{t_{n,j}}^t |\nabla g(q_n(s)) - \nabla g(q_{n,j})| |u_{n,j}| ds. \end{aligned}$$

We consider two cases.

• If  $g(q_{n,j}) > 0$ , then the constraint is active at  $t_{n,j}$  and  $j \geq 1$ . We rewrite the last inequality as

$$\begin{aligned} g(q_n(t)) &\leq g(q_{n,j}) + \int_{t_{n,j}}^t \nabla g(q_{n,j}) \cdot \frac{u_{n,j} + eu_{n,j-1}}{1+e} ds \\ &\quad + \frac{e}{1+e} \int_{t_{n,j}}^t \nabla g(q_{n,j}) \cdot (u_{n,j} - u_{n,j-1}) \\ &\quad + \int_{t_{n,j}}^t |\nabla g(q_n(s)) - \nabla g(q_{n,j})| |u_{n,j}| ds. \end{aligned}$$

By definition of the scheme we have

$$\begin{aligned} &\frac{1}{1+e} (u_{n,j} + eu_{n,j-1}) \\ &= \text{proj}_{q_{n,j}} \left( u_{n,j-1} + \frac{h}{1+e} M_{n,j}^{-1} f_{n,j}, V(q_{n,j}) \right), \end{aligned}$$

where  $M_{n,j} = M(q_{n,j})$ . It follows that

$$\begin{aligned} g(q_n(t)) &\leq g(q_{n,j}) + \frac{e}{1+e} \int_{t_{n,j}}^t \nabla g(q_{n,j}) \cdot (u_{n,j} - u_{n,j-1}) \\ &\quad + \int_{t_{n,j}}^t |\nabla g(q_n(s)) - \nabla g(q_{n,j})| |u_{n,j}| ds \\ &\leq g(q_{n,j}) \\ &\quad - e \int_{t_{n,j}}^t \nabla g(q_{n,j}) \cdot \left( u_{n,j-1} + \frac{h}{1+e} M_{n,j}^{-1} f_{n,j} \right. \\ &\quad \left. - \text{proj}_{q_{n,j}} \left( u_{n,j-1} + \frac{h}{1+e} M_{n,j}^{-1} f_{n,j} \right) \right) \\ &\quad + e \int_{t_{n,j}}^t \nabla g(q_{n,j}) \cdot \frac{h}{1+e} M_{n,j}^{-1} f_{n,j} \\ &\quad + \int_{t_{n,j}}^t |\nabla g(q_n(s)) - \nabla g(q_{n,j})| |u_{n,j}| ds \end{aligned}$$

$$\begin{aligned} &\leq g(q_{n,j}) + \frac{eh}{1+e} \int_{t_{n,j}}^t |\nabla g(q_{n,j}) M_{n,j}^{-1} f_{n,j}| ds \\ &\quad + \int_{t_{n,j}}^t |\nabla g(q_n(s)) - \nabla g(q_{n,j})| |u_{n,j}| ds \end{aligned}$$

since  $-M_{n,j}^{-1} \nabla g(q_{n,j}) \in V(q_{n,j})$ .

Then, recalling that  $(q_{n,j}, u_{n,j}) \in \bar{B}(q_0, R) \times \bar{B}(0, \frac{R}{\sqrt{\alpha_0}})$ , we get

$$\begin{aligned} g(q_n(t)) &\leq g(q_{n,j}) + \frac{eh^2}{1+e} K |M_{n,j}^{-1} f_{n,j}| \\ &\quad + h\varphi\left(h \frac{R}{\sqrt{\alpha_0}}\right) \frac{R}{\sqrt{\alpha_0}} \end{aligned}$$

and the induction hypothesis yields

$$\begin{aligned} g(q_n(t)) &\leq hK \frac{R}{\sqrt{\alpha_0}} + (j+1) \frac{eh^2}{1+e} \frac{C_{f,W}}{\alpha_0} K \\ &\quad + t_{n,j+1} \varphi\left(h \frac{R}{\sqrt{\alpha_0}}\right) \frac{R}{\sqrt{\alpha_0}}. \end{aligned}$$

• If  $g(q_{n,j}) \leq 0$  then

$$\begin{aligned} g(q_n(t)) &\leq \int_{t_{n,j}}^t \nabla g(q_{n,j}) \cdot u_{n,j} ds \\ &\quad + \int_{t_{n,j}}^t |\nabla g(q_n(s)) - \nabla g(q_{n,j})| |u_{n,j}| ds \\ &\leq hK \frac{R}{\sqrt{\alpha_0}} + h\varphi\left(h \frac{R}{\sqrt{\alpha_0}}\right) \frac{R}{\sqrt{\alpha_0}} \\ &\leq hK \frac{R}{\sqrt{\alpha_0}} + \frac{eh^2(j+1)}{1+e} \frac{C_{f,W}}{\alpha_0} K \\ &\quad + t_{n,j+1} \varphi\left(h \frac{R}{\sqrt{\alpha_0}}\right) \frac{R}{\sqrt{\alpha_0}}. \end{aligned}$$

□

Now we observe that, since  $\nabla g$  does not vanish in a neighbourhood of  $S := \{q \in E; g(q) = 0\}$ , there exists  $r > 0$  such that, if  $q \in \bar{B}(q_0, R)$  with  $|g(q)| \leq r$ , then  $\nabla g(q) \neq 0$ .

From the previous lemma we infer that there exists  $n_0 \geq n^*$  such that, for all  $n \geq n_0$ , we have  $g(q_n(t)) \leq r$  for all  $t \in [0, \tau^*]$ . We define

$$\tilde{S} := \{q \in \bar{B}(q_0, R); |g(q)| \leq r\},$$

and let

$$0 < \underline{m}_0 = \inf_{q \in \tilde{S}} \|\nabla g(q)\| \leq \bar{m}_0 = \sup_{q \in \tilde{S}} \|\nabla g(q)\|. \quad (14)$$

Next we focus on the total variation of the approximate velocities  $(u_n)_{n \geq n_0}$ . We start with the following remark: for all  $q \in E$ , for all  $x \in N_L^*(q)$ , we have  $-x \in V(q)$ . Indeed, we have

$$N_L^*(q) = M^{-1}(q) N_L(q).$$

Hence, if  $g(q) \geq 0$ , for  $x \in N_L^*(q)$  there exists  $\lambda \geq 0$  such that  $x = \lambda M^{-1}(q) \nabla g(q)$  and  $-x \in V(q)$  since

$$\begin{aligned} -x \cdot \nabla g(q) &= -\lambda^t \nabla g(q) M^{-1}(q) \nabla g(q) \\ &= -\lambda (|\nabla g(q)|_q^*)^2 \leq 0. \end{aligned}$$

If  $g(q) < 0$ , the previous result remains true since  $V(q) = E$  et  $N_L^*(q) = \{0_E\}$ .

For the sake of simplicity, we will omit from now on the index  $n$  and we denote by  $P_j$  (resp.  $N_j$ ) the projection onto  $V(q_j)$  (resp. onto  $N_L^*(q_j)$ ) with respect to the kinetic metric at  $q_j$ , for all  $j \in \{0, \dots, \tilde{n}\}$  and for all  $n \geq n_0$ . Similarly, let  $M_j = M(q_j)$  and  $|\cdot|_j = |\cdot|_{q_j}$  for all  $j \in \{0, \dots, \tilde{n}\}$  and for all  $n \geq n_0$ . Next we prove:

**Lemma 5** *There exists a constant  $C_{0,1} > 0$  such that*

$$\begin{aligned} |u_{j+1} - u_j|_{j+1}^2 &\leq h C_{0,1} \\ \forall j \in \{0, \dots, \tilde{n} - 1\}, \forall n &\geq n_0. \end{aligned} \quad (15)$$

*Proof* Let  $n \geq n_0$ . By the definition of the scheme, we have

$$u_{j+1} = -e u_j + (1+e) P_{j+1}(x_j)$$

with

$$x_j = u_j + \frac{h}{1+e} M_{j+1}^{-1} f_{j+1} \quad \forall j \in \{0, \dots, \tilde{n} - 1\}.$$

So

$$\begin{aligned} u_{j+1} - u_j &= -(1+e)x_j + (1+e)P_{j+1}(x_j) \\ &\quad + hM_{j+1}^{-1}f_{j+1} \\ &= -(1+e)N_{j+1}(x_j) + hM_{j+1}^{-1}f_{j+1}. \end{aligned}$$

If  $g(q_{j+1}) < 0$  then  $N_L^*(q_{j+1}) = \{0_E\}$  and  $N_{j+1}(x_j) = 0$ . Thus,

$$|u_{j+1} - u_j|_{j+1} = h |M_{j+1}^{-1} f_{j+1}|_{j+1} \leq \frac{h}{\sqrt{\alpha_0}} C_{f,w}.$$

Otherwise, if  $g(q_{j+1}) \geq 0$ , we have  $q_{j+1} \in \tilde{S}$  and  $N_{j+1}(x) = (x, \nabla g(q_{j+1})^+ H(q_{j+1}))$  with

$$H : \begin{cases} \tilde{S} \rightarrow E \\ q \mapsto \frac{1}{{}^t G(q) M^{-1}(q) G(q)} G(q), \end{cases}$$

where  $G(q) = \nabla g(q)$  for all  $q \in E$ .

Moreover, since we assume that  $n \geq n_0$ , we have  $g(q_j) \leq r$  and

$$|g(q_{j+1}) - g(q_j)| \leq \sup_{y \in V} \|\nabla g(y)\| \frac{R}{\sqrt{\alpha_0}} h.$$

Without loss of generality, we may assume that

$$\sup_{y \in V} \|\nabla g(y)\| \frac{R}{\sqrt{\alpha_0}} h \leq r \quad \forall n \geq n_0.$$

Thus  $q_j \in \tilde{S}$ . Let us introduce the following notation:

$$\tilde{N}_j(x) = (x \cdot \nabla g(q_j))^+ H(q_j) \quad \forall x \in E.$$

We rewrite  $u_{j+1} - u_j$  as

$$\begin{aligned} u_{j+1} - u_j &= -(1+e)(N_{j+1}(x_j) - N_{j+1}(u_j)) \\ &\quad - (1+e)(N_{j+1}(u_j) - \tilde{N}_j(u_j)) + \lambda_j \end{aligned}$$

with

$$\lambda_j = -(1+e)\tilde{N}_j(u_j) + h M_{j+1}^{-1} f_{j+1}.$$

Hence

$$|u_{j+1} - u_j|_{j+1} \leq A_{j+1} + B_{j+1} + C_{j+1}$$

with

$$A_{j+1} = (1+e) |N_{j+1}(x_j) - N_{j+1}(u_j)|_{j+1}$$

$$B_{j+1} = (1+e) |N_{j+1}(u_j) - \tilde{N}_j(u_j)|_{j+1}$$

$$C_{j+1} = (1+e) |\tilde{N}_j(u_j)|_{j+1} + h |M_{j+1}^{-1} f_{j+1}|_{j+1}.$$

The contraction property of the projection operator yields

$$A_{j+1} \leq (1+e) \left| \frac{h}{1+e} M_{j+1}^{-1} f_{j+1} \right|_{j+1} \leq \frac{h}{\sqrt{\alpha_0}} C_{f,w}.$$

Next, in order to estimate  $B_{j+1}$ , we prove the following lemma.

**Lemma 6** *There exists  $C_1 > 0$ , independent of  $n$  and  $j$ , such that*

$$|N_{j+1}(u_j) - \tilde{N}_j(u_j)| \leq C_1 h^{1/2}. \quad (16)$$

*Proof* Since  $g \in C_{\text{loc}}^{1,1/2}(E, \mathbb{R})$ , there exists  $L'_0 > 0$  such that

$$|\nabla g(q') - \nabla g(q)| \leq L'_0 |q' - q|^{1/2} \quad \forall (q, q') \in \tilde{S}^2.$$

Moreover, since  $M^{-1/2}$  is locally Lipschitz continuous and  $\tilde{S}$  is compact, we deduce that  $H \in C^{0,1/2}(\tilde{S}; \mathbb{R})$ , i.e., there exists  $L''_0 > 0$  such that

$$|H(q') - H(q)| \leq L''_0 |q' - q|^{1/2} \quad \forall (q, q') \in \tilde{S}^2.$$

We denote by  $L_{V,3}$  the Lipschitz constant associated to  $q \mapsto M^{-1}(q)$  on the compact set  $V = \bar{B}(q_0, R)$  and let  $G_{j+k} := \nabla g(q_{j+k})$  and

$$a_{j+k} := \frac{G_{j+k}}{{}^t G_{j+k} M_{j+k}^{-1} G_{j+k}}$$

with  $k = 0, 1$ . We have

$$\begin{aligned} N_{j+1}(u_j) - \tilde{N}_j(u_j) &= (u_j \cdot G_{j+1})^+ M_{j+1}^{-1} a_{j+1} - (u_j \cdot G_j)^+ M_j^{-1} a_j \\ &= ((u_j \cdot G_{j+1})^+ - (u_j \cdot G_j)^+) M_{j+1}^{-1} a_{j+1} \\ &\quad + (u_j \cdot G_j)^+ ((M_{j+1}^{-1} - M_j^{-1}) a_{j+1} \\ &\quad + M_j^{-1} (a_{j+1} - a_j)). \end{aligned}$$

It follows that

$$\begin{aligned} |N_{j+1}(u_j) - \tilde{N}_j(u_j)| &\leq |(u_j \cdot G_{j+1})^+ - (u_j \cdot G_j)^+| |M_{j+1}^{-1} a_{j+1}| \\ &\quad + |(u_j \cdot G_j)^+| (\|M_{j+1}^{-1} - M_j^{-1}\| |a_{j+1}| \\ &\quad + \|M_j^{-1}\| |a_{j+1} - a_j|) \\ &\leq |u_j| |G_{j+1} - G_j| \|M_{j+1}^{-1}\| |a_{j+1}| \\ &\quad + |u_j| |G_j| (\|M_{j+1}^{-1} - M_j^{-1}\| |a_{j+1}| \\ &\quad + \|M_j^{-1}\| |a_{j+1} - a_j|). \end{aligned}$$

Then we recall that  $u_j \in \bar{B}(0, \frac{R}{\sqrt{\alpha_0}})$  and

$$\frac{|u|^2}{\beta_0} \leq |u|_q^{*2} \leq \frac{|u|^2}{\alpha_0} \quad \forall q \in \bar{B}(q_0, R), \quad \forall u \in E.$$

Thus

$$|a_{j+1}| = \frac{|G_{j+1}|}{(|G_{j+1}|_j^*)^2} \leq \beta_0 \frac{1}{|G_{j+1}|} \leq \frac{\beta_0}{\underline{m}_0},$$

and we conclude the proof by taking

$$C_1 = \frac{R}{\sqrt{\alpha_0}} \left( L'_0 \sqrt{\frac{R}{\alpha_0}} \frac{\beta_0}{\alpha_0 \underline{m}_0} + L_{V,3} \sqrt{T} \frac{R \bar{m}_0 \beta_0}{\sqrt{\alpha_0 \underline{m}_0}} + \frac{L''_0}{\alpha_0} \bar{m}_0 \sqrt{\frac{R}{\alpha_0}} \right).$$

□

We infer that

$$B_{j+1} \leq (1+e) \sqrt{\beta_0} C_1 h^{1/2}.$$

Finally, we obtain the following estimate for the terms  $C_{j+1}$  above:

$$C_{j+1} \leq h C_{f,W} \left( \frac{1}{\sqrt{\alpha_0}} + \frac{\sqrt{\beta_0}}{\alpha_0} \right). \quad (17)$$

Indeed,  $|M_{j+1}^{-1} f_{j+1}|_{j+1} \leq \frac{C_{f,W}}{\sqrt{\alpha_0}}$  and  $\tilde{N}_j(u_j) = 0$  if  $j = 0$ . Moreover, if  $j \geq 1$ , then, by the definition of the scheme, we have

$$\begin{aligned} u_j &= -e u_{j-1} + (1+e) P_j(x_{j-1}) \\ &= P_j(x_{j-1}) - e N_j(x_{j-1}) + \frac{eh}{1+e} M^{-1}(q_j) f_j. \end{aligned}$$

Thus

$$\begin{aligned} u_j - \frac{eh}{1+e} M^{-1}(q_j) f_j \\ = P_j(x_{j-1}) - e N_j(x_{j-1}) \in V(q_j) \end{aligned}$$

and we have

$$\begin{aligned} |\tilde{N}_j(u_j)| &= \left| \tilde{N}_j \left( u_j - \frac{eh}{1+e} M^{-1}(q_j) f_j \right. \right. \\ &\quad \left. \left. + \frac{eh}{1+e} M^{-1}(q_j) f_j \right) \right| \end{aligned}$$

$$\begin{aligned} &= \left( \left( u_j - \frac{eh}{1+e} M^{-1}(q_j) f_j \right. \right. \\ &\quad \left. \left. + \frac{eh}{1+e} M^{-1}(q_j) f_j \right) \cdot \nabla g(q_j) \right)^+ \\ &\quad \times \frac{|M^{-1}(q_j) \nabla g(q_j)|}{|\nabla g(q_j) M^{-1}(q_j) \nabla g(q_j)|} \\ &\leq \left( \left( u_j - \frac{eh}{1+e} M_j^{-1} f_j \right) \cdot \nabla g(q_j) \right)^+ \\ &\quad + \left( \frac{eh}{1+e} M_j^{-1} f_j \cdot \nabla g(q_j) \right)^+ \\ &\quad \times \frac{|M^{-1}(q_j) \nabla g(q_j)|_j}{\sqrt{\alpha_0} |\nabla g(q_j) M_j^{-1} \nabla g(q_j)|}. \end{aligned}$$

Since  $u_j - \frac{eh}{1+e} M^{-1}(q_j) f_j \in V(q_j)$  we get

$$\left( \left( u_j - \frac{eh}{1+e} M^{-1}(q_j) f_j \right) \cdot \nabla g(q_j) \right)^+ = 0.$$

Hence

$$\begin{aligned} |\tilde{N}_j(u_j)| &\leq \frac{eh}{1+e} (M_j^{-1} f_j \cdot \nabla g(q_j))^+ \\ &\quad \times \frac{1}{\sqrt{\alpha_0} |M^{-1}(q_j) \nabla g(q_j)|_j} \\ &\leq \frac{eh}{1+e} |(M_j^{-1} f_j, M_j^{-1} \nabla g_j)_j| \\ &\quad \times \frac{1}{\sqrt{\alpha_0} |M^{-1}(q_j) \nabla g(q_j)|_j} \\ &\leq \frac{eh}{1+e} \frac{|M^{-1}(q_j) f_j|_j}{\sqrt{\alpha_0}} \leq \frac{eh}{1+e} \frac{C_{f,W}}{\alpha_0} \end{aligned}$$

which immediately yields (17).

From the previous estimates, we have in both cases ( $g(q_{j+1}) < 0$  or  $g(q_{j+1}) \geq 0$ ):

$$\begin{aligned} |u_{j+1} - u_j|_{j+1}^2 &\leq h \left( 2 \frac{T^{1/2} C_{f,W}}{\sqrt{\alpha_0}} + (1+e) \sqrt{\beta_0} C_1 \right. \\ &\quad \left. + T^{1/2} \frac{\sqrt{\beta_0}}{\alpha_0} C_{f,W} \right)^2. \end{aligned}$$

□

With these preliminary results, we can now prove:

**Proposition 1** *There exists  $A > 0$  such that*

$$\sum_{j=0}^{\lfloor \tau^*/h \rfloor - 1} |u_{j+1} - u_j| \leq A \quad \forall n \geq n_0.$$

*Proof* First we observe that

$$\sum_{j=0}^{\lfloor \tau^*/h \rfloor - 1} |u_{j+1} - u_j| \leq \frac{1}{\sqrt{\alpha_0}} \sum_{j=0}^{\lfloor \tau^*/h \rfloor - 1} |u_{j+1} - u_j|_{j+1} \quad \forall n \geq n_0. \quad (18)$$

By construction, we have

$$u_{j+1} - P_{j+1}(x_j) = \frac{e}{1+e}(u_{j+1} - u_j)$$

with

$$x_j = u_j + \frac{h}{1+e} M_{j+1}^{-1} f_{j+1}$$

and

$$\begin{aligned} |u_{j+1} - P_{j+1}(x_j)|_{j+1} &= e |P_{j+1}(x_j) - u_j|_{j+1} \\ &\leq e |x_j - P_{j+1}(x_j)|_{j+1} \\ &\quad + \frac{eh}{1+e} |M_{j+1}^{-1} f_{j+1}|_{j+1} \end{aligned}$$

for all  $j \in \{0, \dots, \tilde{n} - 1\}$ , for all  $n \geq 1$ . By Lemma 3, we know that there exist  $\rho > 0$  and  $a = a_{q_0} \in E$  such that

$$\bar{B}(a, \rho) \subset V(q) \quad \forall q \in \bar{B}(q_0, \delta_0)$$

thus

$$\bar{B}_q(a, \rho') \subset V(q) \quad \forall q \in \bar{B}(q_0, \delta_0)$$

with  $\rho' = \rho \sqrt{\alpha_{\bar{V}}}$ ,  $\bar{V} = \bar{B}(q_0, R + \delta_0)$ . Recalling (12), we can apply Moreau's lemma at each time step and we obtain

$$\begin{aligned} |x_j - P_{j+1}(x_j)|_{j+1} \\ \leq \frac{1}{2\rho'} (|x_j - a|_{j+1}^2 - |P_{j+1}(x_j) - a|_{j+1}^2). \end{aligned}$$

We develop the squares in the right-hand side of the inequality; we have

$$|x_j - a|_{j+1}^2$$

$$\begin{aligned} &= \left| u_j + \frac{h}{1+e} M_{j+1}^{-1} f_{j+1} - a \right|_{j+1}^2 \\ &= |u_j - a|_{j+1}^2 + 2 \left\langle u_j - a, \frac{h}{1+e} M_{j+1}^{-1} f_{j+1} \right\rangle_{j+1} \\ &\quad + \frac{h^2}{(1+e)^2} |M_{j+1}^{-1} f_{j+1}|_{j+1}^2. \end{aligned}$$

In order to estimate  $|P_{j+1}(x_j) - a|_{j+1}^2$ , we rewrite it as follows:

$$\begin{aligned} |a - u_{j+1}|_{j+1}^2 \\ &= |a - P_{j+1}(x_j) - (u_{j+1} - P_{j+1}(x_j))|_{j+1}^2 \\ &= |P_{j+1}(x_j) - a|_{j+1}^2 - \frac{2e}{1+e} \\ &\quad \times \langle a - P_{j+1}(x_j), u_{j+1} - u_j \rangle_{j+1} \\ &\quad + \frac{e^2}{(1+e)^2} |u_{j+1} - u_j|_{j+1}^2. \end{aligned}$$

Hence

$$\begin{aligned} |P_{j+1}(x_j) - a|_{j+1}^2 \\ &= |u_{j+1} - a|_{j+1}^2 \\ &\quad + \frac{2e}{1+e} \langle a - P_{j+1}(x_j), u_{j+1} - u_j \rangle_{j+1} \\ &\quad - \frac{e^2}{(1+e)^2} |u_{j+1} - u_j|_{j+1}^2. \end{aligned}$$

From the previous results we deduce that

$$\begin{aligned} |u_{j+1} - u_j|_{j+1} \\ &\leq (1+e) |x_j - P_{j+1}(x_j)|_{j+1} \\ &\quad + h |M_{j+1}^{-1} f_{j+1}|_{j+1} \\ &\leq \frac{1+e}{2\rho'} [A_{j+1} + B_{j+1} + C_{j+1} + D_{j+1}] \\ &\quad + \frac{h}{\sqrt{\alpha_0}} C_{f,w} \end{aligned} \quad (19)$$

with

$$A_{j+1} = |u_j - a|_{j+1}^2 - |u_{j+1} - a|_{j+1}^2,$$

$$B_{j+1} = \frac{e^2}{(1+e)^2} |u_{j+1} - u_j|_{j+1}^2,$$

$$C_{j+1} = \frac{2e}{1+e} \left\langle a - P_{j+1}(x_j), u_j - u_{j+1} \right\rangle_{j+1},$$

$$D_{j+1} = 2 \left\langle u_j - a, \frac{h}{1+e} M_{j+1}^{-1} f_{j+1} \right\rangle_{j+1}$$

$$+ \frac{h^2}{(1+e)^2} |M_{j+1}^{-1} f_{j+1}|_{j+1}^2.$$

But

$$\sum_{j=0}^{\lfloor \tau^*/h \rfloor - 1} A_{j+1}$$

$$\leq |u_0 - a|_{q_0}^2 + L_{1,V} \tau^* \frac{R}{\sqrt{\alpha_0}} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right)^2. \quad (20)$$

Indeed, for all  $j \in \{0, \dots, \leq \tilde{n} - 1\}$ :

$$A_{j+1} = |u_j - a|_{j+1}^2 - |u_{j+1} - a|_{j+1}^2$$

$$= |u_j - a|_j^2 - |u_{j+1} - a|_{j+1}^2$$

$$+ {}^t(u_j - a)(M_{j+1} - M_j)(u_j - a)$$

$$\leq |u_j - a|_j^2 - |u_{j+1} - a|_{j+1}^2$$

$$+ hL_{V,1} \frac{R}{\sqrt{\alpha_0}} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right)^2.$$

By Lemma 5, we have

$$B_{j+1} = \frac{e^2}{(1+e)^2} |u_{j+1} - u_j|_{j+1}^2 \leq \frac{e^2}{(1+e)^2} C_{0,1} h,$$

$$\forall j \in \{0, \dots, \tilde{n} - 1\},$$

thus

$$\sum_{j=0}^{\lfloor \tau^*/h \rfloor - 1} B_{j+1} \leq \frac{e^2}{(1+e)^2} C_{0,1} \tau^*. \quad (21)$$

Next we obtain an estimate of  $C_{j+1}$  for all  $j \in \{0, \dots, \tilde{n} - 1\}$ : we have

$$C_{j+1} = \frac{2e}{1+e} \left\langle a - \frac{1}{1+e} (u_{j+1} + eu_j), u_j - u_{j+1} \right\rangle_{j+1}.$$

Since  $a \in V(q_{j+1})$  and

$$\frac{u_{j+1} + eu_j}{1+e} = P_{j+1}(x_j),$$

we have

$$\left\langle x_j - \frac{u_{j+1} + eu_j}{1+e}, a - \frac{1}{1+e} (u_{j+1} + eu_j) \right\rangle_{j+1} \leq 0,$$

and since

$$x_j - \frac{u_{j+1} + eu_j}{1+e} = \frac{u_j - u_{j+1}}{1+e} + \frac{h}{1+e} M_{j+1}^{-1} f_{j+1}$$

we obtain

$$\left\langle \frac{1}{1+e} (u_j - u_{j+1}), a - \frac{1}{1+e} (u_{j+1} + eu_j) \right\rangle_{j+1}$$

$$\leq \left\langle \frac{h}{1+e} M_{j+1}^{-1} f_{j+1}, \right.$$

$$\left. \frac{1}{1+e} (u_{j+1} - a + e(u_j - a)) \right\rangle_{j+1}.$$

It follows that

$$\left\langle \frac{1}{1+e} (u_j - u_{j+1}), a - \frac{1}{1+e} (u_{j+1} + eu_j) \right\rangle_{j+1}$$

$$\leq h \frac{C_{f,W}}{(1+e)} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right)$$

and thus

$$\sum_{j=1}^{\lfloor \tau^*/h \rfloor - 1} C_{j+1} \leq \frac{2e\tau^* C_{f,W}}{(1+e)} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right). \quad (22)$$

Finally, for all  $j \in \{0, \dots, \tilde{n} - 1\}$ :

$$D_{j+1} = \frac{2h}{1+e} f_{j+1} \cdot (u_j - a)$$

$$+ \left( \frac{h}{1+e} \right)^2 |M_{j+1}^{-1} f_{j+1}|_{j+1}^2,$$

thus

$$|D_{j+1}| \leq \frac{2h}{1+e} C_{f,W} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right) + \frac{h^2 C_{f,W}^2}{(1+e)^2 \alpha_0}$$

and

$$\sum_{j=0}^{\lfloor \tau^*/h \rfloor - 1} |D_{j+1}|$$

$$\leq \frac{2\tau^*}{1+e} C_{f,W} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right) + \frac{h\tau^* C_{f,W}^2}{(1+e)^2 \alpha_0}. \quad (23)$$

By combining (18)–(19) and (20)–(23) we may conclude.  $\square$

We infer that the sequence  $(u_n)_{n \geq n_0}$  is uniformly bounded and bounded in variation on  $[0, \tau^*]$ . By

Helly's theorem, possibly by extracting a subsequence still denoted  $(q_n, u_n)_{n \geq n_0}$ , there exists  $v \in BV(0, \tau^*; E)$  such that

$$u_n(t) \longrightarrow v(t) \quad \forall t \in I^* = [0, \tau^*].$$

Since we want to obtain bvem velocities (see Definition 1), we define

$$u(t) = \frac{1}{1+e} (v^+(t) + e v^-(t)), \quad \forall t \in [0, \tau^*].$$

Of course we make the usual conventions for the values of  $v$  at  $\tau^*$  and 0, i.e.,  $v^-(0) := v(0)$  and  $v^+(\tau^*) := v(\tau^*)$ . Let us observe that  $v(0) = \lim_{n \rightarrow +\infty} u_n(0) = u_0$  and that  $u^\pm(t) = v^\pm(t)$  for all  $t \in (0, \tau^*)$ . So  $u \in bvem(I^*; E)$ . Moreover, since  $(q_n)_{n \geq n_0}$  strongly converges to  $q$  in  $C^0([0, \tau^*]; E)$  and

$$\begin{aligned} q_n(t) &= q_0 + \int_0^t u_n(s) ds \\ \forall t \in I^* &= [0, \tau^*], \quad \forall n \geq n_0 \end{aligned}$$

we obtain in the limit

$$q(t) = q_0 + \int_0^t u(s) ds \quad \forall t \in I^*$$

and so  $q$  admits a derivative (in the classical sense) almost everywhere in  $[0, \tau^*]$  and  $\dot{q}(t) = u(t)$  for almost every  $t \in [0, \tau^*]$ .

### 3.2 Variational inequality

In the line of [2], we now prove the following proposition:

**Proposition 2** (Variational inequality) *Let  $0 \leq s \leq t \leq \tau^*$  and assume that  $z \in V(y)$  for all  $y$  in a neighbourhood  $\omega$  of  $q([s, t])$ . Then*

$$\begin{aligned} & \int_s^t \left[ f(\xi, q, p) \cdot (z - v) + \left( \frac{dM}{dq} v \right) v \cdot \left( z - \frac{1}{2} v \right) \right] d\xi \\ & \leq (M(q(t))v(t) - M(q(s))v(s)) \cdot z \\ & \quad - \frac{1}{2} (|v(t)|_{q(t)}^2 - |v(s)|_{q(s)}^2). \end{aligned} \quad (24)$$

*Proof* From the uniform convergence of the sequence  $(q_n)_{n \geq n^*}$  to  $q$  we infer that there exists an integer  $n_1$  such that  $q_n([s, t]) \subset \omega$  for all  $n \geq n_1$ . For all  $n \geq n_1$  we denote by  $(t_{n,i})_{i=j}^k$  the discretization nodes that

belong to  $(s, t]$  i.e.  $t_{n,j-1} \leq s < t_{n,j} \leq \dots \leq t_{n,k} \leq t < t_{n,k+1}$ . Since  $q_n([s, t]) \subset \omega$ , we deduce that  $z \in V(q_{n,i+1})$  for all  $j-1 \leq i \leq k-1$ . We recall that

$$\begin{aligned} & \frac{1}{1+e} (u_{n,i+1} + e u_{n,i}) \\ & = P_{n,i+1} \left( u_{n,i} + \frac{h}{1+e} M_{n,i+1}^{-1} f_{n,i+1} \right). \end{aligned}$$

By definition of the projection operator we have

$$\begin{aligned} & \langle h M_{n,i+1}^{-1} f_{n,i+1} - u_{n,i+1} + u_{n,i}, (1+e)z \\ & \quad - (u_{n,i+1} + e u_{n,i}) \rangle_{n,i+1} \leq 0. \end{aligned}$$

Thus

$$\begin{aligned} & \langle h M_{n,i+1}^{-1} f_{n,i+1}, (1+e)z \rangle_{n,i+1} \\ & \quad - \langle h M_{n,i+1}^{-1} f_{n,i+1}, u_{n,i+1} + e u_{n,i} \rangle_{n,i+1} \\ & \leq (1+e) \langle u_{n,i+1} - u_{n,i}, z \rangle_{n,i+1} \\ & \quad - \langle u_{n,i+1} - u_{n,i}, u_{n,i+1} + e u_{n,i} \rangle_{n,i+1}. \end{aligned}$$

We develop each member of the right-hand side:

$$\begin{aligned} & \langle u_{n,i+1} - u_{n,i}, z \rangle_{n,i+1} \\ & = (M_{n,i+1} u_{n,i+1} - M_{n,i+1} u_{n,i}) \cdot z \\ & = (M_{n,i+1} u_{n,i+1} - M_{n,i} u_{n,i}) \cdot z \\ & \quad - (M_{n,i+1} - M_{n,i}) u_{n,i} \cdot z \\ & = \langle u_{n,i+1} - u_{n,i}, u_{n,i+1} \rangle_{n,i+1} \\ & = -\langle u_{n,i+1}, u_{n,i+1} \rangle_{n,i+1} + \langle u_{n,i+1}, u_{n,i} \rangle_{n,i+1} \\ & = -|u_{n,i+1}|_{n,i+1}^2 - \frac{1}{2} [|u_{n,i+1} - u_{n,i}|_{n,i+1}^2 \\ & \quad - |u_{n,i}|_{n,i+1}^2 - |u_{n,i+1}|_{n,i+1}^2] \\ & = -\frac{1}{2} |u_{n,i+1}|_{n,i+1}^2 - \frac{1}{2} |u_{n,i+1} - u_{n,i}|_{n,i+1}^2 \\ & \quad + \frac{1}{2} |u_{n,i}|_{n,i+1}^2 \\ & = -\frac{1}{2} |u_{n,i+1}|_{n,i+1}^2 - \frac{1}{2} |u_{n,i+1} - u_{n,i}|_{n,i+1}^2 \\ & \quad + \frac{1}{2} |u_{n,i}|_{n,i}^2 + \frac{1}{2} (M_{n,i+1} - M_{n,i}) u_{n,i} \cdot u_{n,i}. \end{aligned}$$

Similarly

$$\begin{aligned} & -\langle u_{n,i+1} - u_{n,i}, e u_{n,i} \rangle_{n,i+1} \\ & = -e \langle u_{n,i+1}, u_{n,i} \rangle_{n,i+1} + e |u_{n,i}|_{n,i+1}^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{e}{2} [ |u_{n,i+1} - u_{n,i}|_{n,i+1}^2 - |u_{n,i+1}|_{n,i+1}^2 \\
&\quad - |u_{n,i}|_{n,i+1}^2 ] + e |u_{n,i}|_{n,i+1}^2 \\
&= \frac{e}{2} [ |u_{n,i+1} - u_{n,i}|_{n,i+1}^2 - |u_{n,i+1}|_{n,i+1}^2 \\
&\quad + |u_{n,i}|_{n,i+1}^2 ] \\
&= \frac{e}{2} [ |u_{n,i+1} - u_{n,i}|_{n,i+1}^2 - |u_{n,i+1}|_{n,i+1}^2 + |u_{n,i}|_{n,i}^2 \\
&\quad + (M_{n,i+1} - M_{n,i})u_{n,i} \cdot u_{n,i} ].
\end{aligned}$$

It follows that

$$\begin{aligned}
&-\langle u_{n,i+1} - u_{n,i}, u_{n,i+1} + eu_{n,i} \rangle_{n,i+1} \\
&= \frac{-(1+e)}{2} [ |u_{n,i+1}|_{n,i+1}^2 - |u_{n,i}|_{n,i}^2 ] + A_i
\end{aligned}$$

with

$$\begin{aligned}
A_i &= \frac{1+e}{2} (M_{n,i+1} - M_{n,i})u_{n,i} \cdot u_{n,i} \\
&\quad + \frac{e-1}{2} |u_{n,i+1} - u_{n,i}|_{n,i+1}^2.
\end{aligned}$$

As  $e \in [0, 1]$  we have

$$\begin{aligned}
(1+e)\langle u_{n,i+1} - u_{n,i}, z \rangle_{n,i+1} \\
&\quad - \langle u_{n,i+1} - u_{n,i}, u_{n,i+1} + eu_{n,i} \rangle_{n,i+1} \\
&\leq (1+e) \left[ (M_{n,i+1}u_{n,i+1} - M_{n,i}u_{n,i}) \cdot z \right. \\
&\quad \left. - \frac{1}{2} (|u_{n,i+1}|_{n,i+1}^2 - |u_{n,i}|_{n,i}^2) \right] \\
&\quad - (1+e)(M_{n,i+1} - M_{n,i})u_{n,i} \cdot \left( z - \frac{u_{n,i}}{2} \right).
\end{aligned}$$

Next we obtain

$$\begin{aligned}
&hf_{n,i+1} \cdot \left( z - \frac{u_{n,i+1} + eu_{n,i}}{1+e} \right) \\
&\quad + (M_{n,i+1} - M_{n,i})u_{n,i} \cdot \left( z - \frac{u_{n,i}}{2} \right) \\
&\leq (M_{n,i+1}u_{n,i+1} - M_{n,i}u_{n,i}) \cdot z \\
&\quad - \frac{1}{2} (|u_{n,i+1}|_{n,i+1}^2 - |u_{n,i}|_{n,i}^2)
\end{aligned}$$

which yields

$$\sum_{i=j-1}^{k-1} hf_{n,i+1} \cdot \left( z - \frac{u_{n,i+1} + eu_{n,i}}{1+e} \right)$$

$$\begin{aligned}
&\quad + \sum_{i=j-1}^{k-1} (M_{n,i+1} - M_{n,i})u_{n,i} \cdot \left( z - \frac{u_{n,i}}{2} \right) \\
&\leq (M_{n,k}u_{n,k} - M_{n,j-1}u_{n,j-1}) \cdot z \\
&\quad - \frac{1}{2} (|u_{n,k}|_{n,k}^2 - |u_{n,j-1}|_{n,j-1}^2). \tag{25}
\end{aligned}$$

Then we prove that

**Lemma 7** *We have the following convergence:*

$$\begin{aligned}
&\sum_{i=j-1}^{k-1} hf_{n,i+1} \cdot \left( z - \frac{1}{1+e} (u_{n,i+1} + eu_{n,i}) \right) \\
&\quad \xrightarrow{h \rightarrow 0} \int_s^t f(\xi, q(\xi), p(\xi)) \cdot (z - v(\xi)) d\xi.
\end{aligned}$$

*Proof* For all  $n \geq n_0$  let us define the function  $f_h$  by

$$f_h : \begin{cases} [0, \tau^*] \rightarrow E = \mathbb{R}^d \\ t \mapsto f_{n,i+1} \quad \text{if } t \in [t_{n,i}, t_{n,i+1}) \cap [0, \tau^*] \end{cases}$$

and let

$$A = \bigcup_{n \in \mathbb{N}^*} \left\{ t_{n,i} := ih; h = \frac{T}{n}, i \in \mathbb{N} \right\}.$$

The set  $A$  is countable hence it is a negligible set for Lebesgue's measure and the function  $f_h$  is a measurable step function. We will prove that

$$f_h \xrightarrow{h \rightarrow 0} f(\xi, q(\xi), M(q(\xi))v(\xi)) \quad \forall \xi \in [0, \tau^*] \setminus A.$$

Let  $\xi \in [0, \tau^*] \setminus A$  and  $h = \frac{T}{n}$  with  $n \geq n_0$ . There exists  $i \in \mathbb{N}$  such that  $\xi \in (t_{n,i}, t_{n,i+1})$  and

$$f_h(\xi) = f_{n,i+1} = f(t_{n,i+1}, q_{n,i+1}, M(q_{n,i+1})u_{n,j(i)}).$$

First we consider the explicit case, we recall that

$$u_{n,j(i)} = u_{n,i} = u_n(\xi).$$

We have

$$\begin{aligned}
&|f_h(\xi) - f(\xi, q(\xi), M(q(\xi))u_n(\xi))| \\
&= |f(t_{n,i+1}, q_{n,i+1}, M(q_{n,i+1})u_n(\xi)) \\
&\quad - f(\xi, q(\xi), M(q(\xi))u_n(\xi))| \\
&\leq |f(t_{n,i+1}, q_{n,i+1}, M(q_{n,i+1})u_n(\xi)) \\
&\quad - f(\xi, q_{n,i+1}, M(q_{n,i+1})u_n(\xi))|
\end{aligned}$$

$$\begin{aligned}
& + |f(\xi, q_{n,i+1}, M(q_{n,i+1})u_n(\xi)) \\
& - f(\xi, q(\xi), M(q(\xi))u_n(\xi))| \\
& \leq \omega_f(h) + L_{f,2}|q_{n,i+1} - q(\xi)| \\
& \quad + L_{f,3}\|M(q(\xi)) - M(q_{n,i+1})\|\|u_n(\xi)\|,
\end{aligned}$$

where  $\omega_f$  denotes the continuity modulus of  $f$  on  $W = [0, T] \times \bar{B}(q_0, R) \times \bar{B}(0, \frac{R}{\sqrt{\alpha_0}})$  and  $L_{f,2}$  and  $L_{f,3}$  are the Lipschitz constants of  $f$  with respect to its second and third arguments on  $W$ . But

$$\begin{aligned}
& |q(\xi) - q_{n,i+1}| \\
& = |q(\xi) - q_n(t_{n,i+1})| \\
& \leq |q(\xi) - q_n(\xi)| + |q_n(\xi) - q_n(t_{n,i+1})| \\
& \leq \|q - q_n\|_{C^0([0, \tau^*]; \mathbb{R}^d)} + \frac{R}{\sqrt{\alpha_0}}h.
\end{aligned}$$

Since  $(q_n, u_n)_{n \geq n_0}$  converges uniformly  $\times$  pointwise to  $(q, v)$  on  $[0, \tau^*]$  and  $M$  is continuous, we obtain

$$f_h(\xi) \rightarrow f(\xi, q(\xi), M(q(\xi))v(\xi)) \quad \forall \xi \in [0, \tau^*] \setminus A.$$

In the implicit case

$$u_{n,j(i)} = u_{n,i+1} = u_n(\xi + h).$$

We infer from Lemma 5 that

$$\begin{aligned}
& |u_n(\zeta + h) - v(\zeta)| \\
& \leq |u_n(\zeta + h) - u_n(\zeta)| + |u_n(\zeta) - v(\zeta)| \\
& \leq \left(\frac{C_{0,1}h}{\alpha_0}\right)^{1/2} + |u_n(\zeta) - v(\zeta)|
\end{aligned}$$

for all  $\zeta \in [0, \tau^* - h]$  and for all  $n \geq n_0$ . It follows that  $u_n(\zeta + h) \rightarrow v(\zeta)$  for all  $\zeta \in [0, \tau^*)$  and with similar computations as in the explicit case, we prove that

$$f_h(\xi) \rightarrow f(\xi, q(\xi), M(q(\xi))v(\xi)) \quad \forall \xi \in [0, \tau^*) \setminus A.$$

Since  $v(\xi) = u(\xi)$  for all  $\xi \in [0, \tau^*) \setminus F$  where  $F$  is a negligible set with respect to Lebesgue's measure ( $F$  is included in the set of discontinuity points of  $v$ , which is countable) we get finally in both cases

$$\begin{aligned}
& f_h(\xi) \rightarrow f(\xi, q(\xi), M(q(\xi))u(\xi)) \\
& \quad \forall \xi \in [0, \tau^*) \setminus (A \cup F).
\end{aligned}$$

Then, observing that

$$\begin{aligned}
& \sum_{i=j-1}^{k-1} h f_{n,i+1} \cdot \left(z - \frac{1}{1+e}(u_{n,i+1} + e u_{n,i})\right) \\
& = \int_{t_{n,j-1}}^{t_{n,k}} f_h(\xi) \cdot \left(z - \frac{1}{1+e}(u_n(\xi + h) + e u_n(\xi))\right) d\xi
\end{aligned}$$

we may conclude by means of Lebesgue's theorem.  $\square$

Next, recalling that the function  $u_n$  is constant on the subintervals  $[t_{n,i}, t_{n,i+1})$ ,  $i \in \{0, \dots, \tilde{n} - 1\}$ , we have

$$\begin{aligned}
& \int_{t_{n,i}}^{t_{n,i+1}} \frac{d}{dt} \left[ M(q_n(t))u_n(t) \cdot \left(z - \frac{1}{2}u_n(t)\right) \right] dt \\
& = (M_{n,i+1} - M_{n,i})u_{n,i} \cdot \left(z - \frac{1}{2}u_{n,i}\right),
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{i=j-1}^{k-1} (M_{n,i+1} - M_{n,i})u_{n,i} \cdot \left(z - \frac{1}{2}u_{n,i}\right) \\
& = \int_{t_{n,j-1}}^{t_{n,k}} \left( \frac{dM}{dq}(q_n(\xi))u_n(\xi) \right) u_n(\xi) \\
& \quad \cdot \left(z - \frac{1}{2}u_n(\xi)\right) d\xi.
\end{aligned}$$

Using Lebesgue's theorem once more, we obtain

$$\begin{aligned}
& \int_{t_{n,j-1}}^{t_{n,k}} \left( \frac{dM}{dq}(q_n(\xi))u_n(\xi) \right) u_n(\xi) \cdot \left(z - \frac{u_n(\xi)}{2}\right) d\xi \\
& \rightarrow \int_s^t \left( \frac{dM}{dq}(q(\xi))v(\xi) \right) v(\xi) \cdot \left(z - \frac{v(\xi)}{2}\right) d\xi.
\end{aligned}$$

Finally it remains to study the convergence of the right-hand side of (25). We have

$$\begin{aligned}
& (M_{n,k}u_{n,k} - M_{n,j-1}u_{n,j-1}) \cdot z \\
& \quad - \frac{1}{2}(|u_{n,k}|_{n,k}^2 - |u_{n,j-1}|_{n,j-1}^2) \\
& = (M(q_{n,k})u_n(t) - M(q_{n,j-1})u_n(s)) \cdot z \\
& \quad - \frac{1}{2}({}^t u_n(t)M(q_{n,k})u_n(t) \\
& \quad - {}^t u_n(s)M(q_{n,j-1})u_n(s)).
\end{aligned}$$

But

$$\begin{aligned} |q_{n,k} - q(t)| &= |q_n(t_{n,k}) - q(t)| \\ &\leq \|q - q_n\|_{C^0([0, \tau^*]; \mathbb{R}^d)} + \frac{R}{\sqrt{\alpha_0}} h \end{aligned}$$

and similarly

$$\begin{aligned} |q_{n,j-1} - q(s)| &= |q_n(t_{n,j-1}) - q(s)| \\ &\leq \|q - q_n\|_{C^0([0, \tau^*]; \mathbb{R}^d)} + \frac{R}{\sqrt{\alpha_0}} h. \end{aligned}$$

Since the mapping  $M$  is continuous and  $(u_n)_{n \geq n_0}$  converges pointwise to  $v$  on  $[0, \tau^*]$  we get

$$\begin{aligned} (M(q_{n,k})u_n(t) - M(q_{n,j-1})u_n(s)) \cdot z \\ \xrightarrow{n \rightarrow \infty} (M(q(t))u(t) - M(q(s))u(s)) \cdot z, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} ({}^t u_n(t) M(q_{n,k}) u_n(t) - {}^t u_n(s) M(q_{n,j-1}) u_n(s)) \\ \xrightarrow{n \rightarrow \infty} \frac{1}{2} (|v(t)|_{q(t)}^2 - |v(s)|_{q(s)}^2) \end{aligned}$$

which concludes the proof.  $\square$

### 3.3 Absolutely continuous motion

Let  $\mu$  be the measure defined by  $d\mu = |du| + dt$  and  $u'_\mu$  (resp.  $t'_\mu$ ) be the density of the measure  $du$  (resp.  $dt$ ) with respect to the non-negative measure  $d\mu$ . By Jeffery's theorem [4] we know that there exists a  $d\mu$ -negligible set  $N$  such that, for all  $t \in [0, \tau^*] \setminus N$ :

$$\begin{aligned} t'_\mu(t) &= \lim_{\varepsilon \rightarrow 0^+} \frac{dt([t, t + \varepsilon])}{d\mu([t, t + \varepsilon])}, \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{d\mu([t, t + \varepsilon])}, \\ u'_\mu(t) &= \lim_{\varepsilon \rightarrow 0^+} \frac{du([t, t + \varepsilon])}{d\mu([t, t + \varepsilon])}. \end{aligned}$$

Let  $N' = \{t \in I^*; v(t+0) = v(t-0) \neq v(t)\}$ ; it is a negligible set with respect to the measure  $d\mu$ . Now we can prove that

**Proposition 3** *Let  $t \in [0, \tau^*] \setminus N \cup N'$  and assume that  $u$  is continuous at  $t$ . Then*

$$\begin{aligned} f(t, q(t), p(t)) t'_\mu(t) - M(q(t)) u'_\mu(t) \\ \in N_{V(q(t))}(u(t)). \end{aligned}$$

*Proof* Let  $z$  be in the interior of  $V(q(t))$ . The lower semicontinuity of the map  $q \mapsto V(q)$  and the continuity of the map  $t \mapsto q(t)$  on  $L \cup \tilde{S}$  imply that  $z \in V(y)$  for any  $y$  in a neighbourhood of  $q([t, t + \varepsilon])$  for sufficiently small  $\varepsilon$ . Thanks to the variational inequality on the interval  $J_\varepsilon = [t, t + \varepsilon]$  we get

$$\begin{aligned} \int_t^{t+\varepsilon} \left[ f(\xi, q, p) \cdot (z - v) \right. \\ \left. + \left( \frac{dM}{dq} v \right) v \cdot \left( z - \frac{1}{2} v \right) \right] d\xi \\ \leq (M(q(t + \varepsilon))v(t + \varepsilon) - M(q(t))v(t)) \cdot z \\ - \left( \frac{1}{2} |v(t + \varepsilon)|_{q(t+\varepsilon)}^2 - \frac{1}{2} |v(t)|_{q(t)}^2 \right). \end{aligned} \quad (26)$$

Since  $u$  is continuous at  $t$  we have  $u(t+0) = u(t-0) = u(t)$ , thus  $u(t+0) = v(t+0) = u(t-0) = v(t-0) = u(t) = v(t)$  since  $t \notin N'$ . We choose  $(\varepsilon_i)_{i \in \mathbb{N}}$  such that  $\varepsilon_i \downarrow 0^+$  and  $u(t + \varepsilon_i) = v(t + \varepsilon_i) = u(t + \varepsilon_i + 0)$  for all  $i \in \mathbb{N}$  and we multiply (26) by  $\frac{1}{d\mu(J_{\varepsilon_i})} = \frac{\varepsilon_i}{d\mu(J_{\varepsilon_i}) \varepsilon_i}$ . When  $\varepsilon_i$  tends to zero the left-hand side converges to

$$\begin{aligned} t'_\mu(t) \left[ f(t, q(t), p(t)) \cdot (z - u(t)) \right. \\ \left. + \left( \frac{dM}{dq}(q(t))u(t) \right) u(t) \cdot \left( z - \frac{1}{2} u(t) \right) \right]. \end{aligned}$$

Next we consider the right-hand side, where the first term becomes

$$\begin{aligned} \frac{1}{d\mu(J_{\varepsilon_i})} [M(q(t + \varepsilon_i))u(t + \varepsilon_i) - M(q(t))u(t)] \cdot z \\ = \frac{M(q(t))du(J_{\varepsilon_i}) \cdot z}{d\mu(J_{\varepsilon_i})} \\ + \frac{1}{d\mu(J_{\varepsilon_i})} \left[ \int_t^{t+\varepsilon_i} \frac{dM}{dq}(q(s))u(s) ds \right] \\ \times u(t + \varepsilon_i) \cdot z \\ \xrightarrow{\varepsilon_i \rightarrow 0} M(q(t))u'_\mu(t) \cdot z \\ + t'_\mu(t) \left( \frac{dM}{dq}(q(t))u(t) \right) u(t) \cdot z. \end{aligned}$$

Now we rewrite the second term of the right-hand side as

$$\begin{aligned} |u(t + \varepsilon_i)|_{q(t+\varepsilon_i)}^2 - |u(t)|_{q(t)}^2 \\ = M(q(t + \varepsilon_i))u(t + \varepsilon_i) \cdot u(t + \varepsilon_i) \end{aligned}$$

$$\begin{aligned}
& -M(q(t))u(t) \cdot u(t) \\
& = M(q(t + \varepsilon_i))u(t + \varepsilon_i) \cdot du(J_{\varepsilon_i}) \\
& \quad + M(q(t))du(J_{\varepsilon_i}) \cdot u(t) \\
& \quad + \left( \int_t^{t+\varepsilon_i} \frac{dM}{dq}(q(s))u(s) ds \right) u(t + \varepsilon_i) \cdot u(t).
\end{aligned} \tag{27}$$

We divide by  $2d\mu(J_{\varepsilon_i})$  and, in the limit, we obtain

$$\begin{aligned}
& M(q(t))u'_\mu(t) \cdot u(t) \\
& \quad + t'_\mu(t) \left( \frac{dM}{dq}(q(t))u(t) \right) u(t) \cdot \frac{u(t)}{2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& [f(t, q(t), p(t))t'_\mu(t) \\
& \quad - M(q(t))u'_\mu(t)] \cdot (z - u(t)) \leq 0 \\
& \quad \forall z \in \text{Int}(V(q(t)))
\end{aligned}$$

and we may conclude by density of the interior in the cone  $V(q(t))$ .  $\square$

### 3.4 Impacts characterization

We are now interested in discontinuities of the velocity. First, we observe that, as in the inelastic case (see [2]), this can only happen when the left velocity is not admissible, since:

**Lemma 8** *If  $t_0 \in (0, \tau^*)$  is such that  $v^-(t_0) \in V(q(t_0))$ , then  $v^-(t_0) = v^+(t_0)$ .*

*Proof* Let  $z \in \text{Int}(V(q(t_0)))$ ; then  $z$  also belongs to  $V(y)$ , for all  $y$  in a neighbourhood of  $q([t_0 - \varepsilon, t_0 + \varepsilon])$ , if  $\varepsilon$  is small enough. We apply the variational inequality on  $[t_0 - \varepsilon, t_0 + \varepsilon]$ , and when  $\varepsilon$  tends to zero we get

$$\begin{aligned}
& M(q(t_0))(v^+(t_0) - v^-(t_0)) \cdot z \\
& \quad - \frac{1}{2}(|v^+(t_0)|_{q(t_0)}^2 - |v^-(t_0)|_{q(t_0)}^2) \geq 0.
\end{aligned}$$

By density, the same inequality holds for all  $z \in V(q(t_0))$ . By choosing  $z = v^-(t_0)$  we obtain

$$|v^+(t_0) - v^-(t_0)|_{q(t_0)}^2 \leq 0$$

which leads to  $v^+(t_0) = v^-(t_0)$ .  $\square$

When  $t = 0$  we have

$$\begin{aligned}
v^-(0) = v(0) & = \lim_{n \rightarrow +\infty} u_n(0) = u_0 \in V(q(0)) \\
& = V(q_0).
\end{aligned}$$

Thus, with the same arguments we obtain  $v^+(0) = v^-(0) = u_0$  and thus the initial conditions are satisfied.

Let us consider now  $t_0 \in (0, \tau^*)$  such that  $v^-(t_0) \notin V(q(t_0))$ . We infer immediately that  $q(t_0) \in \partial L$ . For the sake of simplicity, let us denote from now on  $v^+ = v^+(t_0)$ ,  $v^- = v^-(t_0)$  and  $M_0 = M(q(t_0))$ . Since  $u^\pm(t_0) = v^\pm(t_0)$ , we have to prove that

$$v^+ = -ev^- + (1+e)\text{proj}_{q(t_0)}(v^-, V(q(t_0))). \tag{28}$$

In the next lemma we establish that, if  $n$  is large enough, the approximate positions saturate the constraint at least once in a given neighbourhood of  $t_0$ .

**Lemma 9** *There exists  $\eta_0 > 0$  such that, for all  $\eta \in (0, \eta_0]$ , there exists  $n_\eta \geq n_0$  such that, for all  $n \geq n_\eta$ , there exists  $i_0 \in \{i, \dots, k\}$  such that  $V(q_{n,i_0}) \neq E$ , where  $t_{n,i}, \dots, t_{n,k}$  are the discretization nodes that belong to  $(t_0 - \eta, t_0 + \eta]$ .*

*Proof* Let  $\varepsilon = \frac{|v^+ - v^-|}{6} > 0$ . By definition of  $v^+$  and  $v^-$  there exists  $\eta_0 > 0$  such that  $[t_0 - \eta_0, t_0 + \eta_0] \subset [0, \tau^*]$  and

$$\begin{aligned}
|v(t_0 + \eta) - v(t_0 + 0)| & < \varepsilon, \\
|v(t_0 - \eta) - v(t_0 - 0)| & < \varepsilon, \\
\forall \eta & \in (0, \eta_0].
\end{aligned}$$

Without loss of generality, we may assume that

$$\eta_0 < \frac{\alpha_0 \varepsilon}{3C_{f,W}}.$$

Let  $\eta \in (0, \eta_0]$ . Since  $(u_n)_{n \geq n_0}$  converges pointwise to  $v$  on  $[0, \tau^*]$ , there exists  $n_\eta \geq n_0$  such that for all  $n \geq n_\eta$

$$\begin{aligned}
|u_n(t_0 + \eta) - v(t_0 + \eta)| & < \varepsilon, \\
|u_n(t_0 - \eta) - v(t_0 - \eta)| & < \varepsilon.
\end{aligned}$$

Without loss of generality, we may assume that  $h = \frac{T}{n} \leq \eta$  for all  $n \geq n_\eta$ . It follows that

$$2\varepsilon = \frac{|v^+ - v^-|}{3} < |u_n(t_0 + \eta) - u_n(t_0 - \eta)|$$

$$\forall n \geq n_\eta, \forall \eta \in (0, \eta_0].$$

If we assume that  $V(q_j) = E$  for all  $j \in \{i, \dots, k\}$ , then

$$\begin{aligned} & |u_n(t_0 + \eta) - u_n(t_0 - \eta)| \\ &= |u_{n,k} - u_{n,i-1}| \\ &\leq \sum_{j=i}^k |u_{n,j} - u_{n,j-1}| = \sum_{j=i}^k |hM_{n,j}^{-1}f_{n,j}| \\ &\leq \frac{3\eta C_{f,W}}{\alpha_0} < \varepsilon \end{aligned}$$

and we reach a contradiction.  $\square$

From now on, let us consider  $\eta \in (0, \eta_0]$  and  $n \geq n_\eta$ . In view of the previous result, we may define  $i_0$  as the first ‘‘discrete impact’’ on  $(t_0 - \eta, t_0 + \eta]$  i.e.,

$$t_{i_0} = \min\{t_j \in (t_0 - \eta, t_0 + \eta] ; V(q_{n,j}) \neq E\}.$$

First we estimate the distance between  $v^-$  and the last discrete velocity before the ‘‘discrete impact’’ at time  $t_{i_0}$ :

**Lemma 10** *We have*

$$\begin{aligned} & \left| u_{n,i_0-1} + \frac{h}{1+e} M_{n,i_0}^{-1} f_{n,i_0} - v^- \right| \\ & \leq \frac{4}{\alpha_0} \eta C_{f,W} + |u_n(t_0 - \eta) - v^-|. \end{aligned} \quad (29)$$

*Proof* Indeed

$$\begin{aligned} & \left| u_{n,i_0-1} + \frac{h}{1+e} M_{n,i_0}^{-1} f_{n,i_0} - v^- \right| \\ & \leq \frac{h}{\alpha_0} C_{f,W} + |u_{n,i_0-1} - u_n(t_0 - \eta)| \\ & \quad + |u_n(t_0 - \eta) - v^-|. \end{aligned}$$

But

$$\begin{aligned} & |u_n(t_0 - \eta) - u_{n,i_0-1}| \\ &= |u_{n,i-1} - u_{n,i_0-1}| \leq \sum_{j=i}^{i_0-1} |u_{n,j} - u_{n,j-1}| \\ &= \sum_{j=i}^{i_0-1} h |M_{n,j}^{-1} f_{n,j}| \leq \frac{3\eta}{\alpha_0} C_{f,W} \end{aligned}$$

since  $V(q_{n,j}) = E$  for all  $j \in \{i, \dots, i_0 - 1\}$ .  $\square$

Then, the continuity properties of the projection operator, the uniform Lipschitz continuity of  $(q_n)_{n \geq 1}$  and the uniform convergence of  $(q_n)_{n \geq n_0}$  to  $q$  on  $[0, \tau^*]$  yield

**Lemma 11** *Let  $\varepsilon > 0$ . Then, there exists  $\eta_\varepsilon \in (0, \eta_0]$  such that, for all  $\eta \in (0, \eta_\varepsilon]$ , there exists  $n_{\eta,\varepsilon} \geq n_\eta$  such that*

$$|\tilde{u} - u_{n,i_0}| < \varepsilon \quad \forall n \geq n_{\eta,\varepsilon},$$

where  $\tilde{u} = -ev^- + (1+e)\text{proj}_{q(t_0)}(v^-, V(q(t_0)))$ .

*Proof* We just need to observe that the projection operator is defined by

$$\begin{aligned} & \text{proj}_y(x, V(y)) \\ &= x - \frac{\max(0, x \cdot \nabla g(y))}{\nabla g(y) M^{-1}(y) \nabla g(y)} M^{-1}(y) \nabla g(y) \end{aligned}$$

for all  $(x, y) \in E^2$  if  $g(y) \geq 0$  and  $\nabla g(y) \neq 0$ . So, there exists  $\delta > 0$  such that

$$\forall x \in B(v^-, \delta), \quad \forall y \in \bar{B}(q(t_0), \delta)$$

$$g(y) \geq 0 \implies$$

$$|\text{proj}_y(x, V(y)) - \text{proj}_{q(t_0)}(v^-, V(q(t_0)))| \leq \frac{\varepsilon}{1+e}.$$

Then we consider  $\eta_\varepsilon \in (0, \eta_0]$  such that

$$\frac{R}{\sqrt{\alpha_0}} \eta_\varepsilon \leq \frac{\delta}{2}, \quad \frac{4}{\alpha_0} \eta_\varepsilon C_{f,W} \leq \frac{\varepsilon}{2(1+e)}$$

and, for all  $\eta \in (0, \eta_\varepsilon]$ , we define  $n_{\eta,\varepsilon} \geq n_\eta$  such that

$$\|q - q_n\|_{C^0([0, \tau^*]; E)} \leq \frac{\delta}{2} \quad \forall n \geq n_{\eta,\varepsilon},$$

$$|u_n(t_0 - \eta) - v^-| \leq \frac{\varepsilon}{2(1+e)}.$$

We conclude by applying the previous lemma.  $\square$

Next we estimate the variation of the discrete velocities after the first discrete impact.

**Lemma 12** *Let  $\varepsilon \in (0, 1]$ . There exists a positive constant  $C$  such that*

$$\sum_{j=i_0+1}^k |u_{n,j} - u_{n,j-1}|_{q_{n,j}} \leq C\varepsilon \quad (30)$$

for all  $n$  large enough.

*Proof* Let  $\varepsilon \in (0, 1]$ . Let  $a \in E$  be defined by

$$a = \tilde{u} - \varepsilon \frac{1}{\tau \nabla g(q(t_0)) M^{-1}(q(t_0)) \nabla(q(t_0))} \\ \times M^{-1}(q(t_0)) \nabla(q(t_0)).$$

Observing that  $\tilde{u} = -\text{proj}_{q(t_0)}(v^-, N_L^*(q(t_0))) + \text{proj}_{q(t_0)}(v^-, V(q(t_0))) \in V(q(t_0))$ , we get

$$B_{\tilde{q}}\left(a, \frac{\varepsilon}{2}\right) \subset \text{int}V(\tilde{q}), \quad |\tilde{u} - a|_{\tilde{q}} = \varepsilon.$$

Then, for all  $u \in \bar{B}_{q(t)}(a, \varepsilon/2)$  we have

$$\nabla g(q(t)) \cdot u \\ \leq -\varepsilon |\nabla g(q(t_0))|_{q(t_0)}^* + \frac{\varepsilon}{2} |\nabla g(q(t_0))|_{q(t_0)}^* \\ + \frac{\varepsilon}{2\sqrt{\alpha_0}} |\nabla g(q(t)) - \nabla g(q(t_0))|$$

and the continuity of the mappings  $t \mapsto \nabla g(q(t))$  and  $t \mapsto M^{-1}(q(t))$  implies that there exists  $\delta > 0$  such that

$$\bar{B}_q\left(a, \frac{\varepsilon}{2}\right) \subset V(q) \quad \forall q \in \bar{B}(\tilde{q}, \delta).$$

We apply the previous lemma to define  $\eta_\varepsilon$  and  $n_{\eta, \varepsilon}$  for all  $\eta \in (0, \eta_\varepsilon]$ .

Let  $\eta \in (0, \min(\varepsilon, \eta_\varepsilon)]$  and  $n \geq n_{\eta, \varepsilon}$ . We recall that  $t_{n, i_0+1}, \dots, t_{n, k}$  are the discretization nodes after  $t_{i_0}$  which belong to  $(t_0 - \eta, t_0 + \eta]$ .

Then for all  $j \in \{i_0 + 1, \dots, k\}$  we have  $|q_{n, j} - q(t_0)| < \delta$ . By using the definition of the numerical scheme

$$|u_{n, j} - u_{n, j-1}|_{q_{n, j}} \\ = (1+e) \left| -u_{n, j-1} \right. \\ \left. + P_{n, j} \left( u_{n, j-1} + \frac{h}{1+e} M_{n, j}^{-1} f_{n, j} \right) \right|_{q_{n, j}} \\ \leq (1+e) \left| -x_{n, j-1} + P_{n, j}(x_{n, j-1}) \right|_{q_{n, j}} \\ + h |M_{n, j}^{-1} f_{n, j}|_{q_{n, j}} \quad (31)$$

with

$$x_{n, j-1} = u_{n, j-1} + \frac{h}{1+e} M_{n, j}^{-1} f_{n, j}.$$

We apply Moreau's lemma to obtain

$$|x_{n, j-1} - P_{n, j}(x_{n, j-1})|_{q_{n, j}} \\ \leq \frac{1}{\varepsilon} (|x_{n, j-1} - a|_{q_{n, j}}^2 - |P_{n, j}(x_{n, j-1}) - a|_{q_{n, j}}^2)$$

and with the same computations as in Proposition 1 we get

$$|x_{n, j-1} - P_{n, j}(x_{n, j-1})|_{q_{n, j}} \leq \frac{1}{\varepsilon} [A_j + B_j + C_j + D_j]$$

with

$$A_j = |u_{n, j-1} - a|_{q_{n, j}}^2 - |u_{n, j} - a|_{q_{n, j}}^2, \\ B_j = \frac{e^2}{(1+e)^2} |u_{n, j} - u_{n, j-1}|_{q_{n, j}}^2, \\ C_j = \frac{2e}{1+e} \langle a - P_{n, j}(x_{n, j-1}), u_{n, j-1} - u_{n, j} \rangle_{q_{n, j}}, \\ D_j = 2 \left\langle u_{n, j-1} - a, \frac{h}{1+e} M_{n, j}^{-1} f_{n, j} \right\rangle_{q_{n, j}} \\ + \frac{h^2}{(1+e)^2} |M_{n, j}^{-1} f_{n, j}|_{q_{n, j}}^2.$$

Hence

$$\sum_{j=i_0+1}^k A_j \leq |u_{n, i_0} - a|_{q_{n, i_0}}^2 \\ + h(k - i_0) L_{V, 1} \frac{R}{\sqrt{\alpha_0}} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right)^2.$$

But

$$|u_{n, i_0} - a|_{q_{n, i_0}}^2 \\ \leq |u_{n, i_0} - a|_{q(t_0)}^2 + L_{V, 1} |q_{n, i_0} - q(t_0)| |u_{n, i_0} - a|^2 \\ \leq |u_{n, i_0} - a|_{q(t_0)}^2 \\ \times \left( 1 + \frac{L_{V, 1}}{\alpha_0} \left( \|q - q_n\|_{C^0(\{0, \tau^*\}; E)} + \frac{R}{\sqrt{\alpha_0}} \eta \right) \right)$$

and

$$|u_{n, i_0} - a|_{q(t_0)} \leq |u_{n, i_0} - \tilde{u}|_{q(t_0)} + |\tilde{u} - a|_{q(t_0)} \\ \leq \varepsilon \left( \frac{1}{\sqrt{\alpha_0}} + 1 \right).$$

Hence

$$\begin{aligned}
& \sum_{j=i_0+1}^k A_j \\
& \leq \varepsilon^2 \left( \frac{1}{\sqrt{\alpha_0}} + 1 \right)^2 \\
& \quad \times \left( 1 + \frac{L_{V,1}}{\alpha_0} \left( \|q - q_n\|_{C^0([0, \tau^*]; E)} + \frac{R}{\sqrt{\alpha_0}} \eta \right) \right) \\
& \quad + L_{V,1} \frac{R}{\sqrt{\alpha_0}} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right)^2 \eta.
\end{aligned}$$

Moreover

$$\begin{aligned}
\sum_{j=i_0+1}^k B_j & \leq \frac{e^2}{(1+e)^2} C_{0,1} h(k-i_0) \leq \frac{e^2}{(1+e)^2} C_{0,1} \eta, \\
\sum_{j=i_0+1}^k C_j & \leq \frac{2e}{1+e} C_{f,W} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right) h(k-i_0) \\
& \leq \frac{2e}{1+e} C_{f,W} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right) \eta,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=i_0+1}^k |D_j| & \leq \frac{2h}{1+e} C_{f,W} \left( \frac{R}{\sqrt{\alpha_0}} + |a| \right) \eta \\
& \quad + \frac{h^2 C_{f,W}^2}{(1+e)^2 \alpha_0} \eta.
\end{aligned}$$

By combining these estimates with (31) we obtain the announced result.  $\square$

Finally, for all  $\varepsilon > 0$  and for all  $\eta \in (0, \min(\varepsilon, \eta_\varepsilon)]$  such that  $|v^+ - v(t_0 + \eta)| < \varepsilon$ , we have

$$\begin{aligned}
|v^+ - \tilde{u}| & \leq |v^+ - u_n(t_0 + \eta)| \\
& \quad + |u_n(t_0 + \eta) - u_{n,i_0}| + |u_{n,i_0} - \tilde{u}| \\
& \leq |v^+ - u_n(t_0 + \eta)| \\
& \quad + \frac{1}{\sqrt{\alpha_0}} \sum_{j=i_0+1}^k |u_{n,j} - u_{n,j-1}|_{q_{n,j}} \\
& \quad + |u_{n,i_0} - \tilde{u}| \\
& \leq |v^+ - u_n(t_0 + \eta)| + \left( \frac{C}{\sqrt{\alpha_0}} + 1 \right) \varepsilon
\end{aligned}$$

$$\leq \left( \frac{C}{\sqrt{\alpha_0}} + 2 \right) \varepsilon + |v(t_0 + \eta) - u_n(t_0 + \eta)|$$

for all  $n \geq n_{\eta, \varepsilon}$ . Passing to the limit as  $n$  tends to  $+\infty$ , then as  $\varepsilon$  tends to zero, we may conclude that

$$\begin{aligned}
u^+ & = v^+ = \tilde{u} \\
& = -eu^- + (1+e) \text{proj}_{q(t_0)}(v^-, V(q(t_0))).
\end{aligned}$$

#### 4 Global convergence result

In the previous sections we have already established a local convergence result for the proposed scheme. Now we will prove that the convergence holds on a time interval which depends only on the data. First we recall the following property of the solutions of the Cauchy problem.

**Proposition 4** (Energy estimate) *If  $R > |u_0|_{q_0}$ , there exists  $\tau(R) > 0$  such that, for any solution  $(q, u)$  of the Cauchy problem defined on  $[0, \tilde{\tau}]$ ,  $\tilde{\tau} \in (0, T]$  we have*

$$\begin{aligned}
|u(t)|_{q(t)} & \leq R, \quad |q(t) - q_0| \leq R \\
\forall t & \in [0, \min(\tilde{\tau}, \tau(R))].
\end{aligned}$$

*Proof* See Lemma 7.1 in [13].  $\square$

As a consequence, if we denote by  $(q_{n'}, u_{n'})_{n'}$  a subsequence of the approximate solutions converging to a solution  $(q, u)$  of the Cauchy problem on  $[0, \tilde{\tau}^*]$ , we have

$$\begin{aligned}
|u(t)|_{q(t)} & \leq R, \quad |q(t) - q_0| \leq R \\
\forall t & \in [0, \min(\tilde{\tau}^*, \tau(R))]
\end{aligned}$$

for all  $R > |u_0|_{q_0}$ . Next we infer a more precise estimate on the discrete velocities.

**Lemma 13** *Let  $R > |u_0|_{q_0}$ , and let  $(q_{n'}, u_{n'})_{n'}$  be a subsequence of the approximate solutions converging to a solution  $(q, u)$  of the Cauchy problem on  $[0, \tilde{\tau}^*]$ . Then,*

$$\begin{aligned}
\limsup_{h \rightarrow 0} \sup \{ |u_{n,i}|_{q_{n,i}}; 0 \leq t_{n,i} \leq \min(\tilde{\tau}^*, \tau(R)) \} \\
\leq \text{ess sup} \{ |u(t)|_{q(t)}; 0 \leq t \leq \min(\tilde{\tau}^*, \tau(R)) \}.
\end{aligned}$$

*Proof* We prove this result by contradiction. So, let us assume that there exists  $\varepsilon > 0$  such that

$$\limsup_{h \rightarrow 0} \lambda_h \geq \lambda + 2\varepsilon$$

with

$$\lambda_{h'} = \sup\{|u_{n,i}|_{q_{n,i}}; 0 \leq t_{n,i} \leq \min(\tilde{\tau}^*, \tau(R))\},$$

$$\forall h' = \frac{T}{n'}$$

and

$$\lambda = \text{ess sup}\{|u(t)|_{q(t)}; 0 \leq t \leq \min(\tilde{\tau}^*, \tau(R))\}.$$

So, there exists a subsequence, still denoted  $\lambda_{h'}$ , and there exists an integer  $i'$  not larger than  $n(h') = \lfloor \min(\tilde{\tau}^*, \tau(R))/h' \rfloor$  such that

$$\lambda_{h'} = |u_{n',i'}|_{q_{n',i'}} \geq \lambda + \varepsilon \quad \forall h' = \frac{T}{n'}.$$

We denote  $t_{i'} = i'h' \in [0, \tilde{\tau}^*]$ . Possibly extracting another subsequence, we may assume without loss of generality that  $(t_{i'})_{h' > 0}$  converges to a limit  $t^* \in [0, \tilde{\tau}^*]$ . With the estimates obtained in lemma 1, we have

$$|u_{n',i'}|_{q_{n',i'}} \leq |u_{n',j}|_{q_{n',j}} + (t_{n',i'} - t_{n',j})R_a,$$

$$\forall j \in \{0, \dots, i'\} \quad (32)$$

$$\text{with } R_a = \frac{L_{\tilde{v},z}R^2}{\alpha_{\tilde{v}}} + \frac{C_{f,\tilde{w}}}{\sqrt{\alpha_{\tilde{v}}}} \text{ with } \tilde{V} = \bar{B}(q_0, R), \tilde{W} = [0, T] \times \bar{B}(q_0, R) \times \bar{B}(0, \frac{R}{\sqrt{\alpha_{\tilde{v}}}}).$$

Thus

$$\lambda + \varepsilon \leq |u_{n',i'}|_{q_{n',i'}} \leq |u_{n',0}|_{q_{n',0}} + t_{n',i'}R_a$$

$$\forall h' > 0. \quad (33)$$

But

$$|u_{n',0}|_{q_{n',0}} = |u_0|_{q_0} = \lim_{t \rightarrow 0^+} |u(t)|_{q(t)} \leq \lambda$$

and passing to the limit as  $h'$  tends to zero in (33), we infer that  $t^* > 0$ .

Now (32) implies that

$$\lambda + \frac{\varepsilon}{2} \leq |u_{n',j}|_{q_{n',j}} \quad \forall t_{n',j} \in \left[ t_{n',i'} - \frac{\varepsilon}{4R_a}, t_{n',i'} \right].$$

Since  $(t_{n',i'})_{h' > 0}$  converges to  $t^* > 0$ , there exists  $\tilde{n}' \in \mathbb{N}^*$  such that  $[t^* - \frac{\varepsilon}{8R_a}, t^* - \frac{\varepsilon}{16R_a}] \subset [t_{n',i'} - \frac{\varepsilon}{4R_a}, t_{n',i'}]$

for all  $n' \geq \tilde{n}'$  and thus  $|u_{n',j}|_{q_{n',j}} \geq \lambda + \frac{\varepsilon}{2}$  if  $t_{n',j} \in [t^* - \frac{\varepsilon}{8R_a}, t^* - \frac{\varepsilon}{16R_a}]$ . Since  $|u_n(t)|_{q_n(t)}$  converges to  $|u(t)|_{q(t)}$  for almost every  $t$  in  $[0, \tilde{\tau}^*]$  we infer that

$$\text{ess sup}\left\{|u(t)|_{q(t)}; \max\left(0, t^* - \frac{\varepsilon}{8R_a}\right) < t < \min\left(t^* - \frac{\varepsilon}{16R_a}, \tau^*\right)\right\} \geq \lambda + \frac{\varepsilon}{2}$$

and we get a contradiction with the definition of  $\lambda$ .  $\square$

Then we obtain:

**Proposition 5** *Let  $R > |u_0|_{q_0}$  and  $\tau(R) > 0$  be such that, for any solution  $(q, u)$  of the Cauchy problem defined on  $[0, \tilde{\tau}]$ ,  $\tilde{\tau} \in (0, T]$  we have*

$$|u(t)|_{q(t)} \leq R, \quad |q(t) - q_0| \leq R$$

$$\forall t \in [0, \min(\tilde{\tau}, \tau(R))].$$

*Then, there exists a subsequence of the approximate solutions  $(q_n, u_n)_{n \geq 1}$  which converges to a solution  $(q, u)$  of the Cauchy problem on  $[0, \min(T, \tau(R))]$ .*

*Proof* We already know that there exists  $\tau^* \in (0, T]$  and a subsequence of the approximate solutions, still denoted  $(q_n, u_n)_{n \geq n_0}$ , which converges on  $[0, \tau^*]$  to a solution  $(q, u)$  of the Cauchy problem.

Let  $R > |u_0|_{q_0}$ . Of course, if  $\min(T, \tau(R)) \leq \tau^*$  the result is immediate. So let us assume now that  $\min(T, \tau(R)) > \tau^*$ . By proposition 4 we have

$$|u(t)|_{q(t)} \leq R, \quad |q(t) - q_0| \leq R$$

$$\forall t \in [0, \min(\tau^*, \tau(R))] = [0, \tau^*].$$

Applying Lemma 13, we get

$$\limsup_{h \rightarrow 0} \sup\{|u_{n,i}|_{q_{n,i}}; 0 \leq t_{n,i} \leq \tau^*\}$$

$$\leq \text{ess sup}\{|u(t)|_{q(t)}; 0 \leq t \leq \tau^*\} \leq R,$$

so there exists  $\tilde{h} > 0$  such that

$$\sup\{|u_{n,i}|_{q_{n,i}}; 0 \leq t_{n,i} \leq \tau^*\} \leq R + \frac{1}{2},$$

$$\forall h \in (0, \tilde{h}].$$

Since  $(q_n)_{n \geq n_0}$  converges uniformly to  $q$  on  $[0, \tau^*]$ , possibly by decreasing  $\tilde{h}$ , we also have  $|q_n(t) - q_0| \leq$

$R + \frac{1}{2}$  for all  $t \in [0, \tau^*]$ . Let now  $R_1 = R + 1$ ,  $V_1 = \bar{B}(q_0, R_1)$  and let  $\alpha_1$ ,  $\beta_1$  and  $\delta_1$  be as in (8) and in Lemma 3, with respect to  $V_1$ . Let  $h \in (0, \tilde{h}]$  and let  $t_{n,i_0} \in (\tau^* - 2\varepsilon, \tau^* - \varepsilon)$  with  $\frac{\tau^*}{4} > \varepsilon > 0$ . We have

$$|q_{n,i_0} - q_0| \leq R + \frac{1}{2}, \quad |u_{n,i_0}|_{q_{n,i_0}} \leq R + \frac{1}{2},$$

and we reproduce the construction proposed in Lemma 1 with  $V_1$  instead of  $V$  and  $\tilde{q}_0 = q_{n,i_0}$ ,  $\tilde{u}_0 = u_{n,i_0}$  instead of  $q_0$  and  $u_0$ . We obtain

$$\begin{aligned} |\tilde{q}_{n,i} - \tilde{q}_0| &= |q_{n,i_0+i} - q_{n,i_0}| \leq h \sum_{j=0}^{i-1} |u_{n,i_0+j}| \\ &\leq \frac{ihR_1}{\sqrt{\alpha_1}} \leq \frac{1}{2} \\ \text{if } ih &\leq \frac{\sqrt{\alpha_1}}{2R_1} \end{aligned}$$

and

$$\begin{aligned} \frac{|\tilde{u}_{n,i_0+i+1}|_{q_{n,i_0+i+1}} - |\tilde{u}_{n,i_0+i}|_{q_{n,i_0+i}}}{h} \\ \leq \frac{L_{\tilde{V},2}}{\alpha_1} |\tilde{u}_{n,i_0+i}|_{q_{n,i_0+i}}^2 + \frac{C_{f,W_1}}{\sqrt{\alpha_1}} \end{aligned}$$

with

$$W_1 = [0, T] \times \bar{B}(q_0, R_1) \times \bar{B}\left(0, \frac{R_1}{\sqrt{\alpha_1}}\right).$$

We can compare  $|u_{n,i_0+i}|_{q_{n,i_0+i}}$  to the solution  $\tilde{\theta}$  of the following Cauchy problem:

$$\begin{cases} \frac{d\tilde{\theta}}{dt} = k_1 \tilde{\theta}^2 + k_2, \\ \tilde{\theta}(0) = R + \frac{1}{2} \end{cases} \text{ with } k_1 = \frac{L_{\tilde{V},2}}{\alpha_1}, \quad k_2 = \frac{C_{f,W_1}}{\sqrt{\alpha_1}},$$

which is given by

$$\tilde{\theta}(t) = \sqrt{\frac{k_1}{k_2}} \tan(\sqrt{k_1 k_2} t + C)$$

$$\text{with } C = \text{Arctan}\left(\sqrt{\frac{k_1}{k_2}} \left(R + \frac{1}{2}\right)\right).$$

Then we have

$$\begin{aligned} |u_{n,i_0+i}| &\leq \frac{1}{\sqrt{\alpha_1}} |u_{n,i_0+i}|_{i_0+i} \leq \frac{\tilde{\theta}(ih)}{\sqrt{\alpha_1}} \\ \text{if } ih &\leq \frac{\sqrt{\alpha_1}}{2R_1}. \end{aligned}$$

So we choose  $\tilde{\tau}^* > 0$  such that

$$\begin{aligned} 0 < \tilde{\tau}^* &\leq \frac{\sqrt{\alpha_1}}{2R_1}, \quad \frac{R_1 \tilde{\tau}^*}{\sqrt{\alpha_1}} \leq \delta_1 \quad \text{and} \\ \tilde{\theta}(\tilde{\tau}^*) &\leq R_1. \end{aligned} \tag{34}$$

Then there exists  $\tilde{h}^* > 0$ , depending only on  $R_1$  and the data (see (11)), such that the fixed point technique of Lemma 1 allows us to construct  $(q_{n,i_0+i}, u_{n,i_0+i})$  for all  $ih \in [0, \tilde{\tau}^*]$  and for all  $h \in (0, \tilde{h}^*]$ , and

$$\begin{aligned} |u_{n,i_0+i}|_{q_{n,i_0+i}} &\leq R_1, \quad |q_{n,i_0+i} - q_0| \leq R_1, \\ |q_{n,i_0+i} - q_{n,i_0}| &\leq \min\left(\frac{1}{2}, \delta_1\right). \end{aligned}$$

Moreover, reproducing the same computations as in Proposition 1, with  $R_1$ ,  $W_1$  and  $\delta_1$  instead of  $R$ ,  $W$  and  $\delta$ , we obtain a uniform estimate for the discrete accelerations of the extended approximate solutions. It follows that we can extend the convergence property to  $[0, \min(\tau^* - 2\varepsilon + \tilde{\tau}^*, T)]$ . By choosing  $\varepsilon \in (0, \tilde{\tau}^*/4)$ , we obtain  $\tau^* - 2\varepsilon + \tilde{\tau}^* \geq \tau^* + \tilde{\tau}^*/2$  and, by means of a finite number of extensions, we finally obtain the convergence on  $[0, \min(\tau(R), T)]$ .  $\square$

**Acknowledgements** This work was partially financed by the European project SICONOS IST-2001 37172. Manuel Monteiro Marques was also supported by Fundação para a Ciência e a Tecnologia, Financiamento Base 2008—ISFL/1/2009.

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