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**On the relation of Fuzzy subsets, Postean and Boolean lattices .**

**The  $\lambda$ -rainbow lattices. Transfinite Fuzzy subsets.**

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**§ 0. Introduction.** In this paper it is attempted to relate the definitions of the fuzzy subset lattices with the other classical definitions of Postean and Boolean lattices in the context of Universal algebra. In particular it is obtained a kind of equivalence (mutual isomorphic representations) of the fuzzy subsets, Postean and Boolean lattices. (corollaries 28,29,30.).This unification is worked out with

the axiomatic introduction of the  $\lambda$ -rainbow lattices (§1). In order to obtain these results, it is used the extended Stone's representation theorem (theorem 1). The way that this theorem is formulated makes use of the extended Birkhoff-Stone theorem (corollary 21). It is given new and more enlightening and simpler proofs for both theorems. They are discussed partial abelian semigroups, abelian groups, rings and modules that are also lattices. It is Introduced the concept of compatibility of the order relation and the algebraic structure. The previous equivalence theorems have important consequences in a kind of equivalence, up-to-lattices of the Fuzzy, 3-valued and 2-valued Logic. It is possible obtain the fuzzy logic as a general Logic (according to [J. Mesenquer 1987]). By introducing an ordering it is possible to investigate the relation of a general Logic, the Fuzzy, the 3-valued and the 2-valued Logic.

In the paragraph §2 is introduced the transfinite real, complex, quaternion Fuzzy subset lattices. They can be considered as a kind of Arithmetisation of the abstract  $\lambda$ -Fuzzy lattices. They have special good properties that can be understood only by a familiarity with the transfinite real numbers.

### **§1 The $\lambda$ -rainbow (or fibber) lattices and the $\lambda$ -Fuzzy subsets lattices.**

As it is known the category of Boolean lattices has been extensively studied, mainly because of its applications in Logic. The attempts to give alternative formulations of Logic led to other categories of lattices, like the Postean lattices (see e.g.[Carvallo M.1968],[Lukasiewicz,J1920],[Post E.L. 1921]), or the

lattices of Fuzzy subsets (see e.g., [Zadeth, L.A 1965]). A closer study of the Boolean, Postean Fuzzy -subsets, modular- complemented lattices leads naturally to a broad category of lattices which we define axiomatically in this paragraph and we call rainbow or fibber lattices. In conceiving the idea of rainbow lattices I was partly influenced at least by the terminology, of the Kantian ideas of transcendental analytic and synthetic logic and of Aristotle's term of "colors of the word". As we shall see transfinite numbers appear also.

**Remark:** If a lattice has minimum and maximum elements we denote them with 0,1 respectively.

We are now in a position to define the rainbow (or fibber) lattices: Let an non-empty order type which we denote by  $\lambda$ . Let a lattice, which we denote by L such that:

**The  $\lambda$ -rainbow (or fibber) axioms:**

$A_0$ : The order type  $\lambda$  is a sublattice of L and the L has 0 and 1.

$A_1$ : If  $\alpha \in \lambda$   $\alpha \neq 0$  and for a  $x \in L$  it holds  $x \wedge a = 0$  then  $x = 0$ .

$A_2$ : If  $\alpha \in \lambda$ ,  $\alpha \neq 1$  and for a  $x \in L$  holds  $x \vee a = 1$  then  $x = 1$ .

**The spectral axioms:**

For every  $x \in L$  there is a  $\lambda$ -family of elements of L  $\{x_\alpha \mid \alpha \in \lambda\}$  which, is called the  $\lambda$ -spectrum of the x, such that:

$B_0$ :  $x = \bigvee_{\alpha \in \lambda} (\alpha \wedge x_\alpha)$  The previous representation is called the  $\lambda$ -spectral expansion of  $x$ .

$B_1$ : If  $\alpha_1 \neq \alpha_2$  for two  $\lambda$ -indices of the spectrum of  $x$ , then  $x_{\alpha_1} \wedge x_{\alpha_2} = 0$

$B_2$ : For the  $\lambda$ -spectrum of  $x$  holds  $\bigvee_{\alpha \in \lambda} x_\alpha = 1$ .

**Definition 1.** A Lattice that satisfies the axioms  $A_0, A_1, A_2, B_0, B_1, B_2$  as above, is called a  $\lambda$ -rainbow (or  $\lambda$ -fibber) lattice.

**Remark 2:** If the  $\lambda$  is the order type of an ordinal number  $\alpha$ , we write simply a-rainbow lattice. If  $\lambda=3=\{-1 < 0 < +1\}$  and the  $L$  is furthermore distributive we get the definition of a Postean lattice (see e.g.[Carvallo ,M.1968],[Lukasiewicz,J.1920],[Post, E.L. 1921]). In other words a Postean lattice is defined as a distributive 3-rainbow lattice. We shall prove that the Fuzzy-subsets lattices are  $\lambda$ -rainbow lattices .We shall describe in the first a way to construct  $\lambda$ -rainbow lattices from  $\lambda$ -order types.

Remark. Let us suppose that the order type is at the same time a commutative partial semigroup such that the order and the operations are compatible: In other words if we denote by  $i$  the identity mapping and for two elements  $a, b$  the sums  $i(a)+i(b), i(a \vee b)+i(a \wedge b)$  exist then it holds that  $i(a \vee b)+i(a \wedge b)=i(a)+i(b)$ . In other word, again the identity mapping is a (partial semigroup valued) lattice valuation .Then a  $\lambda$ -rainbow lattice on such a  $\lambda$  is called rainbow lattice with algebraic

fibber. If the partial semigroup  $\lambda$ , is a category, commutative group, ring, field, module, or vector space, then we put this term before the term algebraic.

**Lemma 3.** The (direct) product  $\prod_x \lambda$  in the category of lattices of the order type  $\lambda$  with extremes 0,1, X-times where X is an arbitrary set, is a distributive  $\lambda$ -rainbow lattice.

Proof: The order type  $\lambda$  is contained as a sublattice in the  $\prod_x \lambda$ , as the element with all the coordinates equal.

Thus the axioms  $A_0, A_1, A_2$  are satisfied .We define as the  $\lambda$ -spectrum of an element  $x$  of  $\prod_x \lambda$  the elements  $\{x_\alpha | \alpha \in \lambda\}$  such that the  $x_\alpha$  is the element with all the coordinates equal to 0 except at the places where the coordinates of  $x$  are equal to  $a$  where it is equal, to 1. It is direct that the axioms  $B_0, B_1, B_2$  are satisfied. The order type  $\lambda$  is a distributive lattice. Obviously the same holds for the (direct) product  $\prod_x \lambda$  Q.E.D.

**Remark 4.** The  $\lambda$ -rainbow lattices  $\prod_x \lambda$  with extremes 0,1,

(in other words as in lemma 3 ) are the  $\lambda$ -**Fuzzy subsets lattices**

**Corollary 5.** The  $\lambda$ -Fuzzy-subsets lattices are distributive,  $\lambda$ -rainbow lattices.

Proof: A  $\lambda$ -Fuzzy-subset lattice on a set X is exactly the (direct), product  $\prod_x \lambda$  in the category of lattices ;Where the  $\lambda$  is, an order type .Thus we apply the previous Lemma Q.E.D.

**Lemma 6.** Let two order types  $\lambda_1, \lambda_2$  such that  $\lambda_1 \leq \lambda_2$  (the  $\lambda$  is isomorphic to a suborder type of  $\lambda_2$ ). The lattice  $\prod_x \lambda_1$  is isomorphic to a *sublattice* of  $\prod_x \lambda_2$ .

*Proof* : Direct from the definitions and that  $\lambda_1 \leq \lambda_2$ .  
Q.E.D.

**Lemma 7.** A lattice is complemented if and only if it is a 2-rainbow lattice.

**Remark 8.** A lattice is called complemented if it has extremes, 0,1 and for any element  $x$  there is an element  $x'$  such that,  $xvx'=1$  and  $x \wedge x'=0$ . The  $x'$  is called the complement of  $x$ . (See also [v Neumann J.1960] part 1 ch 1 axiom v pp 2 or the paper with the same title in his collected works ). It is not postulated uniqueness of the complement of an element .

*Proof* : The order type  $\lambda$  is the two element set  $\{0,1\}$  thus the, order type of 2. Thus the axioms  $A_0, A_1, A_2$  are satisfied .As the  $\{0,1\}$  spectrum of any element  $x$  we define the two element set  $\{x_0, x_1\}$  with  $x_0=x'$  and  $x_1=x$ . By  $x'$  we denote the complement of  $x$ . It holds by the definition of the complement that  $(0 \wedge x_0) \vee (1 \wedge x_1) = (0 \wedge x') \vee (1 \wedge x) = 0 \vee x = x$ ; and  $x_0 \wedge x_1 = x' \wedge x = 0, x_0 \vee x_1 = x' \vee x = 1$ .

Thus the axioms  $B_0, B_1, B_2$  are satisfied. Conversely in every 2-rainbow lattice the  $x_0$  can be taken as a complement of  $x=x_1$ . Q.E.D.

**Corollary 9.** A lattice is a Boolean lattice if and only if it is, a distributive 2-rainbow lattice.

Proof: Direct from the previous lemma and that every Boolean, lattice is defined as a complemented (distributive) lattice. Q.E.D.

## §2 .The transfinite real, complex, quaternion Fuzzy subsets and lattices.

**Remark 10.** In the next we introduce the transfinite real, complex,, quaternion Fuzzy subsets .We shall examine at first the elementary cases of the real, complex and quaternion fuzzy subsets. In the definition of the ordinary  $[0,1]$ -fuzzy subsets (see e.g.[Zadeth,,L.A.1965]) what matters is the interval  $[0,1]$  as an order type and not what numbers are the extremes of it. Let us consider the 2-point compactification of the linearly ordered field of the real numbers  $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ , where the  $-\infty$ ,  $+\infty$  are the minimum and maximum of  $\bar{\mathbb{R}}$ . As it is known the  $\bar{\mathbb{R}}$  is order isomorphic order type to the order type  $[0,1]$ . Therefore the  $[0,1]$ -Fuzzy subsets can be considered as  $\bar{\mathbb{R}}$ -Fuzzy subsets. The latter form is some how more convenient as we can make use of the algebraic structure of the field  $\bar{\mathbb{R}}$ . Furthermore negative values of the index or characteristic function of a fuzzy subset can appear.

It is very natural to substitute the linearly ordered (commutative) field  $\bar{\mathbb{R}}$  with the field of the complex numbers  $\mathbb{C}$  or the (non-commutative) field of the quaternion numbers  $\mathbb{H}$ . (See also [v.Weizacker C.F.-Sheibe,E. Sussmann G. 1958]. Actually we have to use the compactifications of them  $\bar{\mathbb{C}}, \bar{\mathbb{H}} = \mathbb{C} \cup \{\infty\}, \bar{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ . Thus the elements of the products in the category of sets  $\prod_x \bar{\mathbb{R}}, \prod_x \bar{\mathbb{C}}, \prod_x \bar{\mathbb{H}}$ , where the  $x$  is an arbitrary set, are the real ,complex and



quaternion Fuzzy subsets of  $x$ . The norm of the characteristic functions of them give real Fuzzy subsets. But the advantage of the complex and quaternion Fuzzy subsets is that besides their real norm they have respectively a phase  $e^{i\theta}$ ,  $e^{q\theta}$  (where  $i$  is the imaginary unit and the  $q$  is a unit vector of the Euclidean space  $\mathbb{R}^3$ ). The way to define the transfinite real, complex and quaternion Fuzzy subsets is plausible and formalises a non-Archimedean phased Fuzziness. It is to be understood that the next arguments do not have to depend on the [Conway J.H.1976],[Robinson A. 1966]. It can be proved that all the three different techniques of the transfinite real numbers, the surreal numbers and the ordinal real numbers give by inductive limit or union the same class of numbers known already as the class No. This class can be called the totally ordered, finitary Newton-Leibniz realm of numbers. Till the moment that this shall be understood we can be content with the transfinite real numbers in [Glazal A.1937]. Although logically not necessary, any additional references to the ordinal real numbers shall be in parenthesis and shall not affect the arguments. The author believes that the Ordinal real numbers shall be indispensable, in the future, for numerical and quantitative applications of mathematics to other sciences.

The transfinite real numbers were defined at first by A.Glazal in 1937(!) (see [Glazal A.1937]). His technique was what it is now known as formal power series fields. His main idea was to define by formal power series fields, Archimedean complete (although non-Archimedean) linearly ordered fields. The

concept of non-Archimedean order is known in the process management of operating systems in computer science and in operation research (Goal programming) as pre-emptive prioritisation.

(It can be proved that they are fundamental (Cauchy) complete (in the order topology) and Dedekind complete. I repeat that the reader can be content with their definition in [Glazal A.1937] which we repeat concisely. Let any order type which we denote by  $\lambda$ . Let the Lexicographic product of  $\lambda$ -copies of the field of the real numbers  $\mathbb{R}$ , which we denote by  $\mathbb{L}\prod_{\lambda}\mathbb{R}$ . Its elements are the elements of the Group-product  $\prod_{\lambda}\mathbb{R}$  which have zero all the components except on a well-ordered subset of  $\lambda$ . This subset is called as usually the support of the element. The group  $\mathbb{L}\prod_{\lambda}\mathbb{R}$  is a linearly ordered group in the lexicographic ordering. The formal power series field  $\mathbb{R}(\lambda) = \mathbb{R}(\mathbb{L}\prod_{\lambda}\mathbb{R})$  is defined as the transfinite real numbers of Archimedean base  $\lambda$ . Let us denote by  $\mathbb{R}(\mathbb{R}(\lambda))$  the real closure of the transfinite real numbers; it can be obtained by adjunction of the square roots of its positive elements. But the  $\mathbb{R}(\lambda)$  is a formal power series field, thus it contains the  $n$ -roots of its positive elements (see [v. Neumann] B.H. 1949] 4.91 Corollary pp 211). In other words it is Pythagorean complete. Thus  $\mathbb{R}(\mathbb{R}(\lambda)) = \mathbb{R}(\lambda)$ .

Consequently its extensions of degree 2 with the imaginary unit  $C(\lambda) = R(\lambda)$  (i) is also algebraically closed. The algebraically closed and formal power series field  $C(\lambda)$  we call transfinite complex numbers of Archimedean base  $\lambda$ . It holds obviously that  $C(\lambda) = C(\prod_{\lambda} R)$ . Similarly the field that is obtained by the quaternion extension of  $R(\lambda)$ , which is the field  $H(\lambda) = H(\prod_{\lambda} R)$  is called the transfinite quaternion numbers of Archimedean base  $\lambda$ . Let us consider the extensions of them  $\bar{R}(\lambda) = R(\lambda) \cup \{-\infty\} \cup \{+\infty\}$ ,  $\bar{C}(\lambda) = C(\lambda) \cup \{\infty\}$ ,  $\bar{H}(\lambda) = H(\lambda) \cup \{\infty\}$  where the  $-\infty, +\infty$  are the minimum and maximum of the order type  $\bar{R}(\lambda)$ .

**Definition 11.** The products  $\prod_X \bar{R}(\lambda), \prod_X \bar{C}(\lambda), \prod_X \bar{H}(\lambda)$  in the category of sets, where the  $X$  is an arbitrary set, are called the **transfinite, real, complex, quaternion Fuzzy subsets** of  $X$ .

**Remark 12.** The norm of the transfinite real, complex, quaternion numbers of some Archimedean base  $\lambda$ , is defined as in the minimal case of the real, complex, and quaternion numbers. Only that the norm is not a positive real number but a transfinite positive real number. By taking the norm, the transfinite real, complex, quaternion Fuzzy subsets define  $\bar{R}(\lambda)$ -fuzzy subsets. Such (real) Fuzzy subsets we call **transfinite Fuzzy subsets**. They are special Fuzzy subsets, since the order types  $\bar{R}(\lambda)$  have special good properties; (fundamental completeness in the order topology, Dedekind completeness, they contain initial segments of the ordinal numbers e.t.c.).

The applications of the (real) transfinite Fuzzy subsets are as the applications of the general Fuzzy subsets and even better.

**Remark.** By the way that the transfinite real numbers are defined (over an arbitrary order type which is its Archimedean base) it is clear that any order type  $\lambda$  is order embeddable in some field of transfinite real numbers. Therefore after lemma 6 every  $\lambda$ -Fuzzy subsets lattice is isomorphic with the a sublattice of some transfinite real Fuzzy subsets lattice.

**Corollary (The transfinite Fuzzy subsets lattice reduction )**

Any  $\lambda$ -Fuzzy subsets lattice is isomorphic with a sublattice of some transfinite (real) Fuzzy subsets lattice.

Remark. It can be proved that any order type can be constructed by the ordinal numbers. This constitutes a genuine and better Arithmetisation of the  $\lambda$ -Fuzzy subsets).

**§3 Rings that are also lattices. Auxiliary rings.**

As it is known J.v.Neumann has studied the orthomodular 2-rainbow lattices that appear as lattices of projections in Hilbert spaces. He has also introduced and studied a special category of modular 2-rainbow lattices that he called continuous geometry (see [v.Neumann,J. 1960]). He proved that such lattices admit valuations in the real numbers. He proved that the modular 2-rainbow lattices have an up-to-isomorphism representation as lattices of ideals of rings. The ring that corresponds to every such lattice is called auxiliary ring (see [v.Neumann J.1960] part ii). This technique is not little laborious.

In the case of the Boolean lattices the auxiliary ring is again the Boolean algebra and the ring operations are  $x+y=(x\wedge y')\vee(x'\wedge y)$  and  $x.y=x\wedge y$ . Any element of the Boolean lattice corresponds to the principal ideal that it defines. The principal lattice-ideals and the principal ring-ideals coincide. The correspondence is a lattice isomorphism. The present representation theorem gives also a result of the same type. In other words an auxiliary ring appears also. Thus it can be also considered as a **J.v.Neumann- type representation**.

**Definition 13.** Let us denote a lattice by  $L$  and a ring by  $R$ . The ring  $R$  is called auxiliary to the lattice  $L$  if there is an isomorphism of  $L$  with a lattice of ideals of  $R$ .

Such an isomorphism is called a **J.v.Neumann-type representation**.

**Remark 14:** By the Stone's representation theorem we deduce that every Boolean lattice is isomorphic with a sublattice of the Power- set  $P(X)$  which is also a lattice of a set  $X$ . Considering the characteristic functions of the subsets of  $X$  we deduce that every Boolean lattice is isomorphic with a sublattice of the abelian  $Z_2$ -linear algebra  $\prod_x Z_2$ . The product is considered in the category of rings and simultaneously in the category of lattices. The Galois field  $Z_2$  is considered as a lattice with the ordering  $0 < 1$ .

It might be interesting to have a closer look to modules and rings that are also lattices (see also remark after remark 2).

Let a ring  $R$  and an lattice ordering on it. We do not suppose any compatibility of the ordering and the ring operations.

E.g. we may consider the rings  $Z_n$  with the linear ordering

$$0 < 1/\pi_1 < 1/\pi_2 < \dots < 1/2 < 1.$$

Or we may consider any linearly ordered ring e.g. the field real numbers.

The (direct) product  $\prod_X R$  over a set of indices  $X$  simultaneously

in the category of rings, and of lattices is a ring which is also a lattice. In the

previous examples the  $\prod_X Z_n$  are  $Z_n$ -modules and rings which are also

distributive  $n$ -rainbow lattices.

If  $n=p$  where  $p$  is a prime natural number then the distributive  $p$ -rainbow lattice

$\prod_X Z_p$  is also a  $Z_p$ -linear algebra.

In these commutative  $Z_p$ -linear algebras holds that  $x^p = x$  and  $px = 0$  (it is of, characteristic  $n$ ); which is a consequence of the Fermat theorem for the Galois fields  $Z_p$ .

The identity function has the property that  $x \vee y + x \wedge y = x + y$ .

Thus it is a Lattice valuation with values in the commutative semigroup  $\prod_X Z_n$ .

For  $n=2$  or  $3$  the relation of the lattice-operations and the ring operations are:

$$\begin{aligned} \lceil a.b \text{ if } a \neq b & \qquad \lceil a+b -a.b \text{ if } a \neq b \\ a \wedge b = \lfloor a \text{ if } a=b & \qquad a \vee b = \lfloor a \text{ if } a=b . \end{aligned}$$

The lattices  $\prod_x Z_p$   $p=2,3$  have as auxiliary rings again the  $\prod_x Z_p$ .

To every element corresponds the principal ring-ideal that it defines. The principal lattice-ideals and the principal ring-ideals, coincide.

#### §4 The inverse of the Birkhoff-Stone theorem in lattices

**Definition 15.** A set  $\Theta$ , that we denote by  $\Theta$ , of congruencies in a lattice,

that we denote by  $L$ , is said that it is separating iff for any two elements  $x, y \in L$  with  $x \neq y$  there is a congruence  $\partial \in \Theta$  such that  $x \not\equiv y \pmod{\partial}$ .

**Remark 16:** If the lattice  $L$  has minimum, that we denote by  $0$ , then as it is known to every congruence  $\partial$  corresponds an ideal that we denote by  $I_\partial$ , which is the equivalence class of  $0$ . Thus we have also the next definition :

**Definition 17** Let a lattice, be denoted by  $L$ , and a set of ideals of it, be denoted by  $\mathbf{I}$ . It is said that the  $\mathbf{I}$  is a separating set of ideals iff for any two  $x, y \in L$  with  $x \neq y$  there is an ideal  $i \in \mathbf{I}$  such that  $x \in i$  and  $y \notin i$ .

**Definition 18.** Let a lattice, be denoted by  $L$ . The set of its prime ideals we denote by  $\text{Spec}L$  and we call the spectrum of the lattice  $L$ .

**Definition 19.** A lattice, which is denoted by  $L$ , is said that it is with separating spectrum iff the set of ideals  $\text{Spec}L$  is separating in the  $L$ .

**Theorem 20.** (The extended Birkhoff-Stone theorem)

A lattice is with separating spectrum if and only if it is distributive .

Proof: see [Sirkoski,R 1969] Appendix 6 §13 first page of the paragraph

Q.E.D.

The Stone's representation theorem makes use of the fact that the spectrum of a Boolean lattice coincides with the set of its maximal ideals and that and the Birkhoff-Stone theorem which gives that any distributive lattice is with separating spectrum .

The Stone's representation theorem can be extended to any lattice with separating spectrum .

**Theorem 21.** (The extended Stone's isomorphic representation theorem ).

Let us denote by  $L$  a lattice with separating spectrum.

There is an isomorphism of  $L$  with a lattice of sets.

Proof: See [Birkhof,G 1967] ,or [Gratzer,G 1979] ,or[Sikorski,R 1969] or any comprehensive book about lattices. Q.E.D.



**Corollary 22.** (J.v.Neumann-type isomorphic representation )

Every distributive lattice has an auxiliary (abelian) ring.

**Proof:** The distributive 2-rainbow lattices  $\prod_x Z_2$  by the last remark of §3 have auxiliary ring the same set as a product abelian ring.

Q.E.D.

The next corollaries are results about the relation of the Boolean, Postean, and Fuzzy-subset lattices.

**Corollary 23.** (The  $Z_2$  reduction)

Every Postean and Fuzzy-subset lattice is isomorphic to a lattice of sets and has an auxiliary ring.

**Corollary 24.** Every distributive  $\lambda$ -rainbow lattice is isomorphic with a sublattice of a  $\lambda'$ -rainbow lattice  $\prod_x \lambda$ , for every order type  $\lambda'$  with extremes, in other words with a sublattice of a  $\lambda'$ -Fuzzy subsets lattice.

**Proof:** We apply the theorem 20 and the lemma 6 .The  $Z_2$  2-rainbow lattice is the minimal (or trivial ) rainbow lattice.

**Corollary 25.** (The  $\lambda$ -Fuzzy representation of distributive  $\lambda$ -rainbow lattices)

Every distributive  $\lambda$ -rainbow lattice is isomorphic with a sublattice of a  $\lambda$ -Fuzzy lattice.

**Corollary** Every distributive  $\lambda$ -rainbow lattice is isomorphic with a sublattice of a transfinite (real) Fuzzy subset lattice.

Proof: We apply the transfinite Fuzzy reduction of the Fuzzy subsets lattices of §2 and the previous lemma.

**Corollary 26.** (The  $Z_3$  reduction). Every distributive lattice and in particular every Fuzzy-subset lattice is isomorphic with a sublattice of a Postean lattice of type  $\prod_x Z_3$ .

**Corollary 27.** (The Fuzzy-reduction)

Every distributive lattice isomorphic with a sublattice of a Fuzzy-subset lattice

**Corollary 28.** (The  $Z_2 \leftrightarrow$  Fuzzy equivalence)

Every Boolean lattice is isomorphic with a sublattice of a Fuzzy-subset lattice and every Fuzzy-subset lattice is isomorphic with a sublattice of a Boolean lattice.

**Corollary 29.** (The  $Z_2 \leftrightarrow Z_3$  equivalence)

Every Boolean lattice is isomorphic with a sublattice of a Postean lattice and every Postean lattice is isomorphic with a sublattice of a Boolean lattice.

**Corollary 30.** (The  $Z_3 \leftrightarrow$  Fuzzy equivalence)

Every Postean lattice is isomorphic with a sublattice of a Fuzzy-subset lattice and every Fuzzy-subset lattice is isomorphic to a sublattice of a Postean lattice.

**Corollary 31.** (The equivalence up-to-lattices of the Fuzzy, 3-valued and 2-valued Logic.)

The Fuzzy, 3-valued, 2-valued Logics are up-to-lattices equivalent in the sense that there are mutual isomorphic representations of the logical (lattice) orders of them.

(See also [A.I.Appudia etc 1977],[J.Barwise,S Feferman 1985],[Carvallo ,M.1968],[Conway J.H.1976],[Mesenqeur J.1989])

**Remark 32.** We remind to you that by the extended Stone's representation theorem for any distributive lattice we get as a corollary that:

Every partially ordered set has a monomorphic image in a lattice of sets (relative to inclusion ).

Every partially ordered set has a monomorphic image in the full binary tree.

This is a very useful result that is not mentioned widely.

It can be also considered as an extension of the Stone's representation theorem, in two directions: a) to every (partially)ordered set b) the representation is not order-isomorphism but order-monomorphism (isotonous mapping ).

About its proof: As it is known, it holds a "linearisation" theorem for any (partially) ordered set :Every (partially) ordered set has an order-monomorphic image in an order type .The order types are distributive lattices. Thus by the extended Stone's representation theorem every (partially) ordered set has an order-monomorphic image in a lattice (relative to inclusion) of sets.

About the second assertion: The lattice of sets that was mentioned in the previous assertion is a Boolean lattice of the type  $\prod_x Z_2$ . The lattice direct-

product ordering of it has by the identity mapping an order-monomorphic image in the set  $\prod_{\alpha} Z_2 = \{0,1\}^{\alpha}$  and in the lexicographic ordering; where the  $\alpha$  is the cardinality of  $X$ , thus an initial ordinal number.

The  $\{0,1\}^{\alpha}$  is a level of the full binary tree  $D_{\alpha}$  of height  $\alpha$ .

The ordering of the level can be taken to be the lexicographical ordering Q.E.D.

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## **Key Words**

Lattices,

general logic

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Fuzzy-subsets lattices, logic

transfinite real numbers

**Note on the relation of the Fuzzy subsets, Postean,**

**Boolean lattices. The  $\lambda$ -rainbow lattices.**

**The transfinite (real, complex, quaternion) Fuzzy subsets.**

**By Dr Constantine E. Kyritsis**

**Abstract**

In this short note the main results are:

1) They are introduced axiomatically the  $\lambda$ -rainbow lattices. It is proved that the  $\lambda$ -Fuzzy subsets lattices are distributive  $\lambda$ -rainbow lattices and other relations of these two classes of lattices.

2) They are introduced the transfinite real, complex, quaternion. Fuzzy subsets. They can be considered as a kind of Arithmetisation of the abstract  $\lambda$ -Fuzzy subsets lattices. The transfinite real Fuzzy subsets are special  $\lambda$ -Fuzzy subsets with good properties.

3) It is proved a mutual isomorphic representation of the Boolean (2-valued logic) Postean (3-valued logic) and Fuzzy-subsets (Fuzzy-logic) lattices.

extremes 0,1 of the lattice. We define the function:  $h:L \rightarrow \prod_{\text{Spec}L} Z_2$  by  $h(x)=a_x$

where  $a_x(t) = \begin{cases} 1 & \text{if } x \in t \\ 0 & \text{if } x \notin t \end{cases} \quad t \in \text{Spec}L.$

We shall prove that the  $h$  is a lattice isomorphism of  $L$  and its range

$$h(L) \leq \prod_{\text{Spec}L} Z_2.$$

a) We shall prove that  $h(x \wedge y) = h(x) \wedge h(y)$ . Let  $h(x) = a_x$  and  $h(y) = a_y$ .

Let us suppose that  $x \notin t$  and  $y \notin t$  for some  $t \in \text{Spec}L$ . Then  $a_x = 1 = a_y$ . If  $x \wedge y \in t$  and because the  $t$  is a prime ideal we deduce that either  $x \in t$  or  $y \in t$ , contradiction.

Thus  $x \wedge y \notin t$ , hence  $a_{x \wedge y} = 1$  and  $a_{x \wedge y} = 1 = 1 \wedge 1 = a_x \wedge a_y$ . The other case is that either  $x \in t$  or  $y \in t$ . Then either  $a_x = 0$  or  $a_y = 0$ . In both cases, because the  $t$  is an ideal  $x \wedge y \in t$  and thus  $a_{x \wedge y} = 0$ ; and  $a_{x \wedge y} = 0 = 0 \wedge (0 \text{ or } 1) = a_x \wedge a_y$ .

b) We shall prove that  $h(x \vee y) = h(x) \vee h(y)$ . Let us suppose that  $x \in t$  and  $y \in t$  for some  $t \in \text{Spec}L$ . Then  $a_x(t) = 1 = a_y(t)$ . If  $x \vee y \in t$  and because the  $t$  is an ideal we deduce that  $x, y \in t$ , which is a contradiction.

Thus  $x \vee y \notin t$  and  $a_x \vee a_y = 1$ . Thus  $a_{x \vee y}(t) = 1 = 1 \vee 1 = a_x(t) \vee a_y(t)$ . The other case is that either  $x \in t$  or  $y \in t$ . Then either  $a_x(t) = 0$  or  $a_y(t) = 0$ . And if  $x \vee y \in t$  then because the  $t$  is an ideal we deduce that  $x, y \in t$  thus  $a_{x \vee y}(t) = 0 = 0 \vee 0 = a_x(t) \vee a_y(t)$ . If  $x \vee y \notin t$ , then either  $x \in t$  or  $y \in t$  (in the contrary  $x \vee y \in t$  which is a contradiction). Thus either  $x \in t$  and  $y \in t$  or  $x \in t$  and  $y \notin t$ . In both cases  $a_{x \vee y}(t) = 1 = 1 \vee 0 = 0 \vee 1 = a_x(t) \vee a_y(t)$ .

c) We shall prove that if  $x \neq y$  then  $h(x) \neq h(y)$ .

If  $x \neq y$  because the  $\text{Spec}L$  is separating there is  $t \in \text{Spec}L$  such that  $x \in t$  and  $y \notin t$ .

Thus  $a_x(t) = 0$  and  $a_y(t) = 1$  hence  $a_x \neq a_y$  or  $h(x) \neq h(y)$ . By the a), b), c) we conclude

that the  $h$  is a monomorphism of the lattices  $L$  and  $h(L) \leq \prod_{\text{Spec} L} \mathbb{Z}_2$ . But in the

category of lattices a monomorphism and epimorphism is also an isomorphism

Q.E.D.

**Corollary 21.** (The extended Birkhoff-Stone theorem).

A lattice is with separating spectrum if and only if it is distributive.

Proof: By the Birkhoff-Stone theorem we get that any distributive lattice is with separating spectrum. By the extended Stone's theorem 20 we get that any lattice with separating spectrum is isomorphic with a sublattice of a Boolean lattice. But any sublattice of a Boolean lattice is a distributive lattice.

Q.E.D.

Remark: We conclude that any distributive lattice has a representation with sets: This seems that it is already known (see [Sirkoski,R 1969] Appendix §3 first page of the paragraph). Our proof does not nevertheless make use of any distributive assumption for the lattice. With our technique it is proved something more: The inverse of the Birkhoff-Stone theorem.