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# Multiscale method with patches for the propagation of localized uncertainties in (semi-)linear elliptic and parabolic equations

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joint work with Mathilde Chevreuil<sup>2</sup> and Anthony Nouy<sup>2</sup>

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Ecole Centrale Nantes, France

LMSSC Seminar, June 30, 2017, Conservatoire National des Arts et Métiers  
(CNAM), Paris, France

## Context

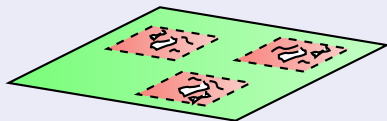
## Uncertainty quantification

- Linear transient model problem

$$c\dot{u} - \nabla \cdot B(u, \nabla u) + C(u, \nabla u) = f \quad \text{on } \Omega$$

with given initial conditions in  $\Omega$  and boundary conditions on  $\partial\Omega$ , and where  $\dot{u} = \frac{\partial u}{\partial t}$

Localized uncertainties  
and non-linearities



- Input variabilities
  - Source term  $f$
  - Evolution coefficient  $c$  and operators  $B, C$
  - Geometrical domain  $\Omega$  and boundary  $\partial\Omega$
- Parametric modeling of uncertainties
  - Finite set of random variables  $\xi = (\xi_1, \dots, \xi_m)$
  - Finite dimensional probability space  $(\Xi, \mathcal{B}(\Xi), \mu)$  with  $\Xi \subset \mathbb{R}^m$

## Goal

- Propagate the uncertainties through the transient model

$$c(\xi)\dot{u}(\xi) - \nabla \cdot B(u(\xi), \nabla u(\xi); \xi) + C(u(\xi), \nabla u(\xi); \xi) = f(\xi) \quad \text{on } \Omega(\xi)$$

# Context

## Stochastic spectral methods: Functional representation of random variables

- Functional expansion of the solution

$$u(\xi) \approx \sum_{\alpha \in \mathcal{A}} u_{\alpha} \psi_{\alpha}(\xi) \in L^p_{\mu}(\Xi; \mathcal{V})$$

- Stochastic approximation space
  - Continuous (generalized) polynomial chaos basis [Ghanem 1999; Ghanem and Spanos 1991; Soize and Ghanem 2004; Xiu and Karniadakis 2002]
  - Lagrange interpolation (or collocation) polynomial basis [Babuška, Nobile, and Tempone 2007; Xiu and Hesthaven 2005]
  - Piecewise polynomial basis (stochastic finite elements [Babuška, Tempone, and Zouraris 2004, 2005; Deb, Babuška, and Oden 2001; Wan and Karniadakis 2005], multi-wavelets [Le Maître et al. 2004a,b])

## Advantages

- **Explicit** and **accurate** prediction of the solution in terms of the random variables
- **Quick post-processing** / **Fast evaluation** of **quantities of interest**
  - Statistical moments
  - Probability of events
  - Quantiles
  - Confidence regions
  - Sensitivity indices
  - ...

# Motivation

## Multiscale problems with localized uncertainties on a subdomain called a patch

- Uncertain operators on local subdomains

$$c(\cdot; \xi) = \begin{cases} c(\cdot) & \text{on } \Omega \setminus \Lambda \text{ (deterministic)} \\ c(\cdot; \xi) & \text{on } \Lambda(\xi) \text{ (uncertain)} \end{cases}$$

$$B(\cdot; \xi) = \begin{cases} B(\cdot) & \text{on } \Omega \setminus \Lambda \\ B(\cdot; \xi) & \text{on } \Lambda(\xi) \end{cases}$$

$$C(\cdot; \xi) = \begin{cases} C(\cdot) & \text{on } \Omega \setminus \Lambda \\ C(\cdot; \xi) & \text{on } \Lambda(\xi) \end{cases}$$

- Dedicated strategy to handle high dimensionality and complexity

## Challenges

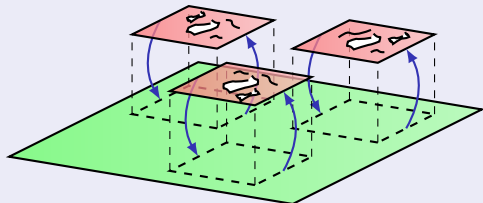
- Develop a multiscale method that exploits the localized side of uncertainties
  - Simple global problem
    - ↔ Traditional solvers / Commercial software packages available in the industry
  - More technical local problems
    - ↔ Specific solvers / In-house research codes

# Motivation

## Multiscale domain decomposition method with patches

- Multiscale method inspired from the **Semi-Schwarz-Lagrange method** [Chevreuil, Nouy, and Safatly 2013; Gendre, Allix, and Gosselet 2011; Gendre et al. 2009; Hager et al. 2012; Lozinski 2010] that belongs to the class of **global-local iterative methods** [Whitcomb 1991]

“Numerical zooms”



## Main ingredient - Separation of global and local problems

- Possible **refined local solutions** with **better approximation properties**
- **Simple global problems**
  - Linear
  - No geometrical variabilities
  - Deterministic operator
- **Appropriate approximation spaces** and **solvers** to solve efficiently both problems

# Outline

- 1 Context and motivation
- 2 Multiscale domain decomposition method
- 3 Computational aspects
- 4 Numerical illustrations
- 5 Conclusions and outlooks

# Outline

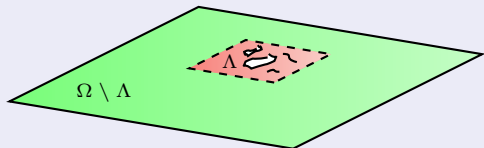
- 1 Context and motivation
- 2 Multiscale domain decomposition method**
  - Global-local formulation with patches
  - Global-local iterative algorithm
- 3 Computational aspects
- 4 Numerical illustrations
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# Initial formulation

## Domain featuring localized uncertainties and possible non-linearities

- Partition of **random** domain  $\Omega(\xi)$  into **random** patch  $\Lambda(\xi)$  and **deterministic** complementary subdomain  $\Omega \setminus \Lambda$



## Transient linear advection-diffusion-reaction stochastic problem - Formulation

Model problem with **random** material parameters and source terms

$$\begin{cases} c\dot{u} - \nabla \cdot (K\nabla u) + \alpha \cdot \nabla u + Ru = f & \text{on } (\Omega \setminus \Lambda) \times I \\ c(\xi)\dot{u} - \nabla \cdot (K(\xi)\nabla u) + \alpha(\xi) \cdot \nabla u + R(\xi)u = f(\xi) & \text{on } \Lambda(\xi) \times I \end{cases}$$

with possibly **random** boundary and initial conditions

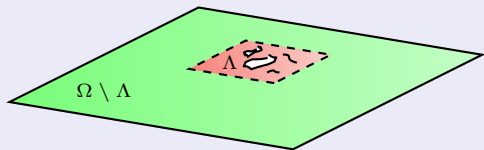
$$\begin{cases} u = 0 & \text{on } \Gamma_D \times I \\ K\nabla u \cdot n = g & \text{on } (\Gamma_N \setminus \Gamma_N^\Lambda) \times I \\ K(\xi)\nabla u \cdot n = g(\xi) & \text{on } \Gamma_N^\Lambda(\xi) \times I \end{cases} \text{ and } \begin{cases} u = u_0 & \text{on } (\Omega \setminus \Lambda) \times \{0\} \\ u = u_0(\xi) & \text{on } \Lambda(\xi) \times \{0\} \end{cases}$$

where  $\dot{u} = \frac{\partial u}{\partial t}$ .

# Initial formulation

## Domain featuring localized uncertainties and possible non-linearities

- Partition of **random** domain  $\Omega(\xi)$  into **random** patch  $\Lambda(\xi)$  and **deterministic** complementary subdomain  $\Omega \setminus \Lambda$



## Transient linear advection-diffusion-reaction stochastic problem - Space weak formulation

Find  $u \in \mathcal{V}^{\Xi}$  such that it holds almost surely

$$m_{\Omega}(\dot{u}(\xi), \delta u; \xi) + a_{\Omega}(u(\xi), \delta u; \xi) = \ell_{\Omega}(\delta u; \xi) \quad \forall \delta u \in \mathcal{V}$$

$$u|_{t=0+}(\xi) = u_0(\xi)$$

with **bilinear** forms  $a_{\Omega}(u, v; \xi) = \int_{\Omega} K(\xi) \nabla u \cdot \nabla v + \int_{\Omega} \alpha(\xi) \cdot \nabla u v + \int_{\Omega} R(\xi) uv$ ,

$m_{\Omega}(u, v; \xi) = \int_{\Omega} c(\xi) uv$  and **linear** form  $\ell_{\Omega}(v; \xi) = \int_{\Omega} f(\xi) v + \int_{\Gamma_N} g(\xi) v$ .

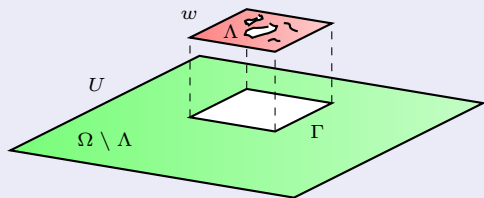
# Global-local formulation with patches

## Domain decomposition - Introduction of a patch

- Separation of the solution

$$u = \begin{cases} U & \text{on } \Omega \setminus \Lambda \\ w & \text{on } \Lambda \end{cases}$$

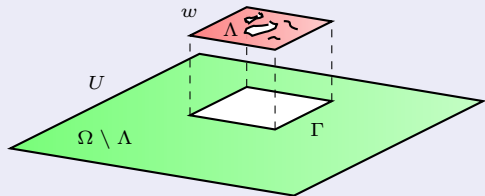
with  $U = w$  on  $\Gamma$  (weak sense)



# Global-local formulation with patches

## Domain decomposition - Introduction of a patch

- **Uncertain** domain  $\Omega$
- **Deterministic** subdomain  $\Omega \setminus \Lambda$
- **Uncertain** patch  $\Lambda$
- **Deterministic** interface  $\Gamma$



## Reformulation using Lagrange multiplier

Find  $(U, w, \lambda) \in \mathcal{U}^\Xi \times \mathcal{W}^\Xi \times \mathcal{M}^\Xi$  such that it holds almost surely

- **Global** equation on subdomain  $\Omega \setminus \Lambda$

$$m_{\Omega \setminus \Lambda}(\dot{U}(\xi), \delta U) + a_{\Omega \setminus \Lambda}(U(\xi), \delta U) + b_\Gamma(\lambda(\xi), \delta U) = \ell_{\Omega \setminus \Lambda}(\delta U; \xi) \quad \forall \delta U \in \mathcal{U}$$

- **Local** equation on patch  $\Lambda$

$$m_\Lambda(\dot{w}(\xi), \delta w; \xi) + a_\Lambda(w(\xi), \delta w; \xi) - b_\Gamma(\lambda(\xi), \delta w) = \ell_\Lambda(\delta w; \xi) \quad \forall \delta w \in \mathcal{W}$$

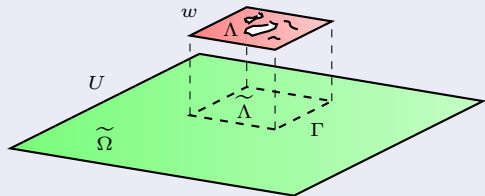
- **Coupling** equation on interface  $\Gamma$

$$b_\Gamma(\delta \lambda, U(\xi)) - b_\Gamma(\delta \lambda, w(\xi)) = 0 \quad \forall \delta \lambda \in \mathcal{M}$$

## Global-local formulation with patches

## Overlapping domain decomposition - Introduction of a fictitious patch

- **Deterministic fictitious** domain  $\tilde{\Omega}$
- **Deterministic fictitious** patch  $\tilde{\Lambda}$
- **Uncertain** patch  $\Lambda$
- **Deterministic** interface  $\Gamma$



## Reformulation with overlapping domains

Find  $(U, w, \lambda) \in \tilde{\mathcal{U}}^{\Xi} \times \mathcal{W}^{\Xi} \times \mathcal{M}^{\Xi}$  such that it holds almost surely

- **Reformulated global** equation on **fictitious** domain  $\tilde{\Omega} = (\Omega \setminus \Lambda) \cup \tilde{\Lambda}$

$$n_{\tilde{\Omega}}(\dot{U}(\xi), \delta U; \xi) - n_{\tilde{\Lambda}}(\dot{U}(\xi), \delta U; \xi) + c_{\tilde{\Omega}}(U(\xi), \delta U; \xi) - c_{\tilde{\Lambda}}(U(\xi), \delta U; \xi) + b_{\Gamma}(\lambda(\xi), \delta U) = 0$$

- **Local** equation on patch  $\Lambda$

$$m_{\Lambda}(\dot{w}(\xi), \delta w; \xi) + a_{\Lambda}(w(\xi), \delta w; \xi) - b_{\Gamma}(\lambda(\xi), \delta w) = \ell_{\Lambda}(\delta w; \xi) \quad \forall \delta w \in \mathcal{W}$$

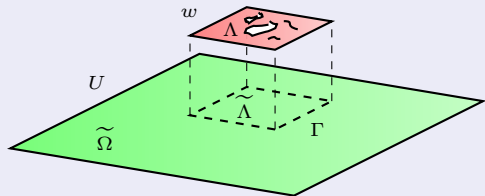
- **Coupling** equation on interface  $\Gamma$

$$b_{\Gamma}(\delta \lambda, U(\xi)) - b_{\Gamma}(\delta \lambda, w(\xi)) = 0 \quad \forall \delta \lambda \in \mathcal{M}$$

## Global-local formulation with patches

## Overlapping domain decomposition - Introduction of a fictitious patch

- Deterministic fictitious domain  $\tilde{\Omega}$
- Deterministic fictitious patch  $\tilde{\Lambda}$
- Uncertain patch  $\Lambda$
- Deterministic interface  $\Gamma$



## Reformulation with overlapping domains

Find  $(U, w, \lambda) \in \tilde{\mathcal{U}}^{\Xi} \times \mathcal{W}^{\Xi} \times \mathcal{M}^{\Xi}$  such that it holds almost surely

- Reformulated global equation on fictitious domain  $\tilde{\Omega} = (\Omega \setminus \Lambda) \cup \tilde{\Lambda}$

$$n_{\tilde{\Omega}}(\dot{U}(\xi), \delta U; \xi) - n_{\tilde{\Lambda}}(\dot{U}(\xi), \delta U; \xi) + c_{\tilde{\Omega}}(U(\xi), \delta U; \xi) - c_{\tilde{\Lambda}}(U(\xi), \delta U; \xi) + b_{\Gamma}(\lambda(\xi), \delta U) =$$

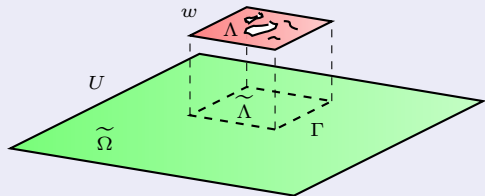
where bilinear forms  $n_{\mathcal{O}}(U, V; \xi) = \int_{\mathcal{O}} \tilde{c}(\xi) UV$  and

$$c_{\mathcal{O}}(U, V; \xi) = \int_{\mathcal{O}} \tilde{K}(\xi) \nabla U \cdot \nabla V + \int_{\mathcal{O}} \tilde{\alpha}(\xi) \cdot \nabla U V + \int_{\mathcal{O}} \tilde{R}(\xi) UV.$$

# Global-local formulation with patches

## Overlapping domain decomposition - Introduction of a fictitious patch

- **Deterministic fictitious** domain  $\tilde{\Omega}$
- **Deterministic fictitious** patch  $\tilde{\Lambda}$
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- **Deterministic** interface  $\Gamma$



## Reformulation with overlapping domains

Find  $(U, w, \lambda) \in \tilde{\mathcal{U}}^{\Xi} \times \mathcal{W}^{\Xi} \times \mathcal{M}^{\Xi}$  such that it holds almost surely

- **Reformulated global** equation on **fictitious** domain  $\tilde{\Omega} = (\Omega \setminus \Lambda) \cup \tilde{\Lambda}$

$$n_{\tilde{\Omega}}(\dot{U}(\xi), \delta U; \xi) - n_{\Lambda}(\dot{U}(\xi), \delta U; \xi) + c_{\tilde{\Omega}}(U(\xi), \delta U; \xi) - c_{\Lambda}(U(\xi), \delta U; \xi) + b_{\Gamma}(\lambda(\xi), \delta U) =$$

where **bilinear** forms  $n_{\mathcal{O}}(U, V; \xi) = \int_{\mathcal{O}} \tilde{c}(\xi) UV$  and

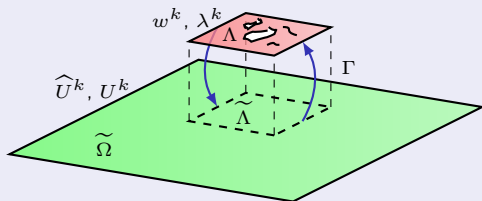
$$c_{\mathcal{O}}(U, V; \xi) = \int_{\mathcal{O}} \tilde{K}(\xi) \nabla U \cdot \nabla V + \int_{\mathcal{O}} \tilde{\alpha}(\xi) \cdot \nabla U V + \int_{\mathcal{O}} \tilde{R}(\xi) UV.$$

## Global-local iterative algorithm

## Iterative coupling procedure

1

Global step

 $\widehat{U}^k$ 

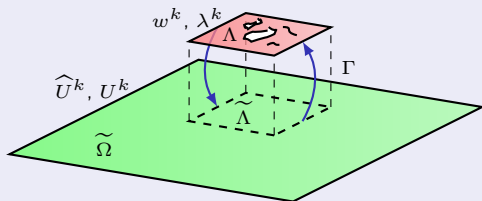
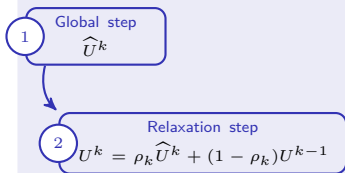
1 Global step: Compute  $\widehat{U}^k \in \tilde{U}^\Xi$  such that it satisfies almost surely

$$n_{\tilde{\Omega}}(\dot{\widehat{U}}^k(\xi), \delta U) + c_{\tilde{\Omega}}(\widehat{U}^k(\xi), \delta U) = n_{\tilde{\Lambda}}(\dot{U}^{k-1}(\xi), \delta U) + c_{\tilde{\Lambda}}(U^{k-1}(\xi), \delta U) - b_{\Gamma}(\lambda^{k-1}(\xi), \delta U) + \ell_{\Omega \setminus \Lambda}$$



## Global-local iterative algorithm

## Iterative coupling procedure



- ① **Global step:** Compute  $\widehat{U}^k \in \widetilde{U}^\Xi$  such that it satisfies almost surely

$$n_{\widetilde{\Omega}}(\dot{\widehat{U}}^k(\xi), \delta U) + c_{\widetilde{\Omega}}(\widehat{U}^k(\xi), \delta U) = n_{\widetilde{\Lambda}}(\dot{U}^{k-1}(\xi), \delta U) + c_{\widetilde{\Lambda}}(U^{k-1}(\xi), \delta U) - b_{\Gamma}(\lambda^{k-1}(\xi), \delta U) + \ell_{\Omega \setminus \Lambda}$$

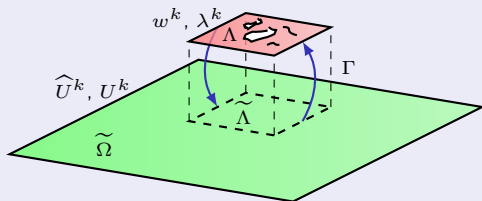
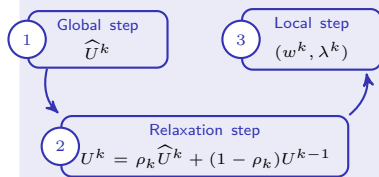
- ② **Relaxation step:** Define  $U^k \in \widetilde{U}^\Xi$  by

$$U^k(\xi) = \rho_k \widehat{U}^k(\xi) + (1 - \rho_k) U^{k-1}(\xi) \quad \text{with relaxation parameter } \rho_k > 0$$

↪ Significant influence on the convergence properties of the algorithm

## Global-local iterative algorithm

## Iterative coupling procedure



- 1 **Global step:** Compute  $\hat{U}^k \in \tilde{\mathcal{U}}^\Xi$  such that it satisfies almost surely

$$n_{\tilde{\Omega}}(\hat{U}^k(\xi), \delta U) + c_{\tilde{\Omega}}(\hat{U}^k(\xi), \delta U) = n_{\tilde{\Lambda}}(\hat{U}^{k-1}(\xi), \delta U) + c_{\tilde{\Lambda}}(U^{k-1}(\xi), \delta U) - b_{\Gamma}(\lambda^{k-1}(\xi), \delta U) + \ell_{\Omega \setminus \Lambda}$$

- 2 **Relaxation step:** Define  $U^k \in \tilde{\mathcal{U}}^\Xi$  by

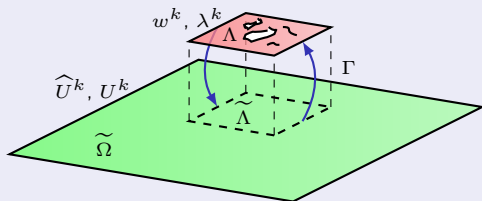
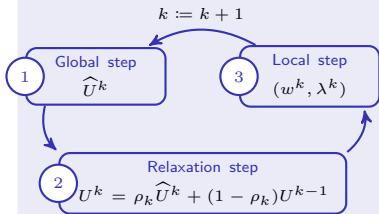
$$U^k(\xi) = \rho_k \hat{U}^k(\xi) + (1 - \rho_k) U^{k-1}(\xi) \quad \text{with relaxation parameter } \rho_k > 0$$

- 3 **Local step:** Compute  $(w^k, \lambda^k) \in \mathcal{W}^\Xi \times \mathcal{M}^\Xi$  such that it satisfies almost surely

$$\begin{cases} m_{\Lambda}(\dot{w}^k(\xi), \delta w; \xi) + a_{\Lambda}(w^k(\xi), \delta w; \xi) - b_{\Gamma}(\lambda^k(\xi), \delta w) = \ell_{\Lambda}(\delta w; \xi) & \forall \delta w \in \mathcal{W} \\ b_{\Gamma}(\delta \lambda, w^k(\xi)) = b_{\Gamma}(\delta \lambda, U^k(\xi)) & \forall \delta \lambda \in \mathcal{M} \end{cases}$$

## Global-local iterative algorithm

## Iterative coupling procedure



- ① **Global step:** Compute  $\widehat{U}^k \in \widetilde{U}^\Xi$  such that it satisfies almost surely

$$n_{\widetilde{\Omega}}(\dot{\widehat{U}}^k(\xi), \delta U) + c_{\widetilde{\Omega}}(\widehat{U}^k(\xi), \delta U) = n_{\widetilde{\Lambda}}(\dot{U}^{k-1}(\xi), \delta U) + c_{\widetilde{\Lambda}}(U^{k-1}(\xi), \delta U) - b_{\Gamma}(\lambda^{k-1}(\xi), \delta U) + \ell_{\Omega \setminus \Lambda}$$

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$$\begin{cases} m_{\Lambda}(\dot{w}^k(\xi), \delta w; \xi) + a_{\Lambda}(w^k(\xi), \delta w; \xi) - b_{\Gamma}(\lambda^k(\xi), \delta w) = \ell_{\Lambda}(\delta w; \xi) & \forall \delta w \in \mathcal{W} \\ b_{\Gamma}(\delta \lambda, w^k(\xi)) = b_{\Gamma}(\delta \lambda, U^k(\xi)) & \forall \delta \lambda \in \mathcal{M} \end{cases}$$

# Analysis of the algorithm (in the stationary case)

## Consistency

- There exists a **unique extension**  $\tilde{U}(\xi) \in \tilde{\mathcal{U}}_*$  of the global solution  $U(\xi) \in \tilde{\mathcal{U}}$  such that  $\tilde{U}(\xi) = U(\xi)$  on complementary subdomain  $\Omega \setminus \Lambda$  and the restriction of  $\tilde{U}(\xi)$  to fictitious patch  $\tilde{\Lambda}$  is uniquely defined by the trace of  $U(\xi)$  on interface  $\Gamma$ .
- The extended global solution  $U \in L^p_\mu(\Xi; \tilde{\mathcal{U}})$ , with exponent  $2 \leq p \leq +\infty$ .

## Convergence

- Assume that the sequence of relaxation parameters  $\{\rho_k\}_{k \in \mathbb{N}}$  is such that

$$0 < \rho_{\text{inf}} \leq \rho_k \leq \rho_{\text{sup}} < +\infty,$$

for some strictly positive constants  $\rho_{\text{inf}}$  and  $\rho_{\text{sup}}$  independent of  $\xi$  and  $k$ .

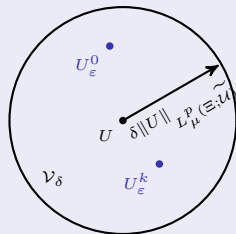
- Then, for  $\rho_{\text{sup}}$  **sufficiently small**, the sequence  $\{(U^k(\xi), w^k(\xi), \lambda^k(\xi))\}_{k \in \mathbb{N}}$  converges almost surely to the unique solution  $(U(\xi), w(\xi), \lambda(\xi))$  of the initial global-local problem in  $\tilde{\mathcal{U}}_*(\xi) \times \mathcal{W} \times \mathcal{M}$ .  $\hookrightarrow$  **Almost sure convergence**
- Also, the sequence  $\{(U^k, w^k, \lambda^k)\}_{k \in \mathbb{N}}$  converges to  $(U, w, \lambda)$  in  $L^p_\mu(\Xi; \tilde{\mathcal{U}}) \times L^p_\mu(\Xi; \mathcal{W}) \times L^p_\mu(\Xi; \mathcal{M})$ .  $\hookrightarrow$  **Convergence in  $L^p_\mu$**

# Analysis of the algorithm

## Robustness - Impact of approximations

- Analysis of the **sensitivity** of the iterative algorithm with respect to **approximations** introduced at the different steps of the algorithm
- **Control** of the **approximation error**  $U_\varepsilon^k - U^k$  between the approximate/perturbed global iterate  $U_\varepsilon^k$  and the unperturbed one  $U^k$  at iteration  $k$  in a  $L_\mu^p$ -norm
- Under certain assumptions on the unperturbed and perturbed iteration maps, if the initial global iterate  $U_\varepsilon^0$  is in a  $\delta$ -neighborhood  $\mathcal{V}_\delta$  of the exact global solution  $U$ , then all global iterates  $U_\varepsilon^k$  remain in  $\mathcal{V}_\delta$ , and the **approximate sequence**  $\{U_\varepsilon^k\}_{k \in \mathbb{N}}$  tends to a  $\gamma(\varepsilon)$ -neighborhood of  $U$  in  $L_\mu^p(\Xi; \tilde{\mathcal{U}})$  with  $\gamma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

- Open ball  $\mathcal{V}_\delta$  of radius  $\delta \|U\|_{L_\mu^p(\Xi; \tilde{\mathcal{U}})}$  centered at the exact global solution  $U$  and containing all global iterates  $U_\varepsilon^k$



# Outline

- 1 Context and motivation
- 2 Multiscale domain decomposition method
- 3 Computational aspects**
  - Adaptive sampling-based approach
  - Relaxation step
- 4 Numerical illustrations
- 5 Conclusions and outlooks

# Computational aspects

## Global problem

- Direct computation of global problem by solving a system of **uncoupled linear equations** with the **same deterministic** global finite element matrix
- Use of **traditional (parallel) solvers** available in **commercial deterministic finite element codes**

## Local problems

- Resolution of stochastic local problems using an **adaptive least-squares method**
- Sampling-based approaches require evaluations of the solution by **standard deterministic codes** and are well adapted to massively **parallel computation**
- Control of the **stability** of local approximations by **increasing the number of samples**
- Estimation of the **approximation error** using **cross-validation techniques**
- **Working sets strategies** proposed in [Chkifa, Cohen, and Schwab 2014, 2015; Chkifa et al. 2013] for the construction of an **increasing sequence of sparse approximation spaces**
- Algorithm with **adaptive random sampling** and an **adaptive selection of approximation spaces** for the construction of **sparse polynomial approximations** of local solutions [Nouy and Pled 2017]

# Least-squares method

## Least-squares approximation in a given subspace

- Least-squares approximation  $\mathbf{v} \in \mathbb{R}^n \otimes \mathcal{S}_{\mathcal{A}}$  of a random vector  $\mathbf{u} \in \mathbb{R}^n \otimes L^2_{\mu}(\Xi)$ :

$$\mathbf{v}(\xi) = \sum_{\alpha \in \mathcal{A}} \mathbf{v}_{\alpha} \psi_{\alpha}(\xi)$$

where  $\mathcal{S}_{\mathcal{A}} = \text{span}\{\psi_{\alpha}\}_{\alpha \in \mathcal{A}}$  is a polynomial space for a given subset  $\mathcal{A}$  of multi-indices  $\alpha$ .

- Set of coefficients  $\mathbf{V} = (\mathbf{v}_{\alpha})_{\alpha \in \mathcal{A}} \in \mathbb{R}^{n \times \#\mathcal{A}}$  solution of an unconstrained optimization problem:

$$\begin{aligned} \min_{(\mathbf{v}_{\alpha})_{\alpha \in \mathcal{A}}} \sum_{l=1}^N \|\mathbf{u}(\xi^l) - \sum_{\alpha \in \mathcal{A}} \mathbf{v}_{\alpha} \psi_{\alpha}(\xi^l)\|_2^2 &\iff \min_{\mathbf{V}} \|\mathbf{Y} - \Psi \mathbf{V}^T\|_F^2 \\ &\implies \mathbf{V}^T = (\Psi^T \Psi)^{-1} \Psi^T \mathbf{Y} \end{aligned}$$

where  $\Psi = (\psi_{\alpha}(\xi^l))_{1 \leq l \leq N, \alpha \in \mathcal{A}} \in \mathbb{R}^{N \times \#\mathcal{A}}$  and  $\mathbf{Y} = (\mathbf{u}(\xi^l))_{1 \leq l \leq N} \in \mathbb{R}^{N \times n}$ .

- In practice, **improvement of the stability** of the least-squares approximation  $\mathbf{v}$  by **increasing the number of samples  $N$**



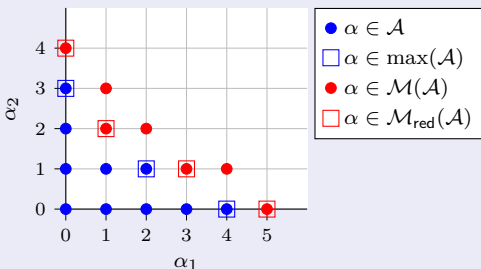
# Working set strategy for adaptive approximation

## Structure of monotone sets

- A finite subset  $\mathcal{A}$  of multi-indices  $\alpha$  is called **monotone** (or **lower** or **downward closed**) if it is such that  $(\beta \in \mathcal{A} \text{ and } \alpha \leq \beta) \implies \alpha \in \mathcal{A}$
- Margin  $\mathcal{M}(\mathcal{A})$  and reduced margin  $\mathcal{M}_{\text{red}}(\mathcal{A})$  of a monotone set  $\mathcal{A}$

$$\mathcal{M}(\mathcal{A}) = \{\alpha \notin \mathcal{A} ; \exists i \in \{1, \dots, m\} \text{ such that } \alpha_i \neq 0 \implies \alpha - e_i \in \mathcal{A}\}$$

$$\mathcal{M}_{\text{red}}(\mathcal{A}) = \{\alpha \notin \mathcal{A} ; \forall i \in \{1, \dots, m\} \text{ such that } \alpha_i \neq 0 \implies \alpha - e_i \in \mathcal{A}\}$$



**Figure:** Monotone multi-index set  $\mathcal{A}$ , its maximal elements  $\max(\mathcal{A})$ , its margin  $\mathcal{M}(\mathcal{A})$  and reduced margin  $\mathcal{M}_{\text{red}}(\mathcal{A})$

# Working set strategy for adaptive approximation

## Working set strategy - construction of an increasing sequence of monotone sets

- Polynomial function ( $d = 3$ ) [Sudret 2008]

$$u(\xi) = \frac{1}{2^d} \prod_{j=1}^d (3\xi_j^2 + 1)$$

where  $\xi_j \sim \mathcal{U}(0, 1)$  and  $\xi = (\xi_j)_{j=1}^d \in \Xi = (0, 1)^d$

# Working set strategy for adaptive approximation

## Working set strategy - construction of an increasing sequence of monotone sets

- Ishigami function [Blatman and Sudret 2010; Blatman and Sudret 2011; Saltelli Andrea and Scott 2000; Sudret 2008]

$$u(\xi) = \sin(\xi_1) + a \sin(\xi_2)^2 + b \xi_3^4 \sin(\xi_1) \quad \text{with } a = 7, b = 0.1$$

where  $\xi_j \sim \mathcal{U}(-\pi, \pi)$  and  $\xi = (\xi_j)_{j=1}^3 \in \Xi = (-\pi, \pi)^3$

# Working set strategy for adaptive approximation

## Working set strategy - construction of an increasing sequence of monotone sets

- Anisotropic function [Chkifa, Cohen, and Schwab 2014]

$$u(\xi) = \xi_3 \sin(\xi_4) + \xi_{16}$$

where  $\xi_j \sim \mathcal{U}(0, 1)$  and  $\xi = (\xi_j)_{j=1}^{16} \in \Xi = (0, 1)^{16}$

# Working set strategy for adaptive approximation

## Working set strategy - construction of an increasing sequence of monotone sets

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$$u(\xi) = \xi_3 \sin(\xi_4) + \xi_{16}$$

where  $\xi_j \sim \mathcal{U}(0, 1)$  and  $\xi = (\xi_j)_{j=1}^{16} \in \Xi = (0, 1)^{16}$

# Cross-validation technique

Leave-one-out cross-validation technique: statistical method based on random partitioning of the available sample set [Cawley and Talbot 2004]

**Input:** Coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  of the approximation  $\mathbf{v}(\xi) = \sum_{\alpha \in \mathcal{A}} \mathbf{v}_\alpha \psi_\alpha(\xi)$  of  $\mathbf{u}(\xi)$ , and matrices  $\Psi = (\psi_\alpha(\xi^l))_{1 \leq l \leq N, \alpha \in \mathcal{A}}$  and  $\mathbf{Y}^T = (\mathbf{u}(\xi^l))_{1 \leq l \leq N}$

**Output:** Vector  $\varepsilon = (\varepsilon_i)_{i \in \mathcal{I}}$ , where  $\varepsilon_i$  is an estimation of the error  $\mathbb{E}((u_i(\xi) - v_i(\xi))^2)$

1: **for**  $i = 1, \dots, n$  **do**

2: Compute the **set of predicted residuals**  $\{\Delta_l^i\}_{l=1}^N$  associated with sample set  $\{\xi^l\}_{l=1}^N$  using Sherman-Morrison-Woodbury formula:

$$\Delta_l^i = \frac{r_{li}}{1 - h_l} \quad \text{with} \quad \begin{cases} r_{li} & \text{the } (l, i)\text{-th entry of matrix } \mathbf{R} = \Psi \mathbf{V}^T - \mathbf{Y} \\ h_l & \text{the } l\text{-th diagonal term in matrix } \mathbf{H} = \Psi (\Psi^T \Psi)^{-1} \Psi^T \end{cases}$$

3: Compute the **leave-one-out error**  $e_i = \frac{E_i}{\widehat{m}_2(\mathbf{Y}_i)}$ , where  $E_i = \frac{1}{N} \sum_{l=1}^N (\Delta_l^i)^2$  and

$\widehat{m}_2(\mathbf{Y}_i)$  is the empirical second moment of the  $i$ -th column  $\mathbf{Y}_i = (u_i(\xi^l))_{l=1}^N$  of matrix  $\mathbf{Y}$

4: Compute the **corrected leave-one-out error**  $\varepsilon_i = e_i \times T(\mathcal{A}, N)$ , where

$T(\mathcal{A}, N) = \left(1 - \frac{\#\mathcal{A}}{N}\right)^{-1} \left(1 + \frac{\text{tr}(\widehat{\mathbf{C}}^{-1})}{N}\right)$  is a correction factor allowing to reduce

the sensitivity to overfitting [Blatman and Sudret 2011; Chapelle, Vapnik, and Bengio 2002] and where  $\widehat{\mathbf{C}} = \frac{1}{N} \Psi^T \Psi$  is the empirical covariance matrix of  $(\psi_\alpha(\xi))_{\alpha \in \mathcal{A}}$

5: **end for**

## Adaptive sparse algorithm

Adaptive sparse least-squares solver with random sampling and working set strategy  
 [Nouy and Pled 2017]

**Input:** Initial number of samples  $N \geq 1$ , sampling factor  $p_{\text{add}} > 0$ , parameter  $\theta \in [0, 1]$

**Output:** Monotone set  $\mathcal{A}$  and coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  of the least-squares approximation

$$\mathbf{v}(\xi) = \sum_{\alpha \in \mathcal{A}} \mathbf{v}_\alpha \psi_\alpha(\xi) \text{ of } \mathbf{u}(\xi)$$

- 1: Start with null initial set  $\mathcal{A} = \{0_{\mathcal{F}}\}$
- 2: Generate the initial sample set  $\{\xi^l\}_{l=1}^N$  randomly
- 3: Compute the matrices  $\Psi_{\mathcal{A}} = (\psi_\alpha(\xi^l))_{1 \leq l \leq N, \alpha \in \mathcal{A}}$  and  $\mathbf{Y}^T = (\mathbf{u}(\xi^l))_{1 \leq l \leq N}$
- 4: **while** no convergence **do**
- 5:   // Adaptive random sampling
- 6:   **while** no convergence and no stagnation **do**
- 7:     Generate the additional sample set  $\{\xi^{N+l}\}_{l=1}^{N_{\text{add}}}$  randomly, with  $N_{\text{add}} = \text{ceil}(p_{\text{add}}N)$
- 8:     Compute the matrices  $\Psi_{\mathcal{A},\text{add}}$  and  $\mathbf{Y}_{\text{add}}^T$
- 9:     Update the number of samples  $N \leftarrow N + N_{\text{add}}$  and the matrices  
 $\Psi_{\mathcal{A}}^T \leftarrow (\Psi_{\mathcal{A}}^T, \Psi_{\mathcal{A},\text{add}}^T)$  and  $\mathbf{Y}^T \leftarrow (\mathbf{Y}^T, \mathbf{Y}_{\text{add}}^T)$
- 10:     Compute the coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  such that  $\mathbf{V}^T = (\Psi_{\mathcal{A}}^T \Psi_{\mathcal{A}})^{-1} \Psi_{\mathcal{A}}^T \mathbf{Y}$
- 11:   **end while**
- :
- :
- 12: **end while**

## Adaptive sparse algorithm

Adaptive sparse least-squares solver with random sampling and working set strategy  
 [Nouy and Pled 2017]

**Input:** Initial number of samples  $N \geq 1$ , sampling factor  $p_{\text{add}} > 0$ , parameter  $\theta \in [0, 1]$

**Output:** Monotone set  $\mathcal{A}$  and coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  of the least-squares approximation

$$\mathbf{v}(\xi) = \sum_{\alpha \in \mathcal{A}} \mathbf{v}_\alpha \psi_\alpha(\xi) \text{ of } \mathbf{u}(\xi)$$

- 1: Start with null initial set  $\mathcal{A} = \{0_{\mathcal{F}}\}$
- 2: Generate the initial sample set  $\{\xi^l\}_{l=1}^N$  randomly
- 3: Compute the matrices  $\Psi_{\mathcal{A}} = (\psi_\alpha(\xi^l))_{1 \leq l \leq N, \alpha \in \mathcal{A}}$  and  $\mathbf{Y}^T = (\mathbf{u}(\xi^l))_{1 \leq l \leq N}$
- 4: **while** no convergence **do**
- 5:   // Adaptive random sampling
- 6:   **while** no convergence and no stagnation **do**
- 7:     Compute the coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  such that  $\mathbf{V}^T = (\Psi_{\mathcal{A}}^T \Psi_{\mathcal{A}})^{-1} \Psi_{\mathcal{A}}^T \mathbf{Y}$   
       by taking into account additional samples
- 8:   **end while**
- :
- :
- 9: **end while**



## Adaptive sparse algorithm

Adaptive sparse least-squares solver with random sampling and working set strategy  
[Nouy and Pled 2017]

**Input:** Initial number of samples  $N \geq 1$ , sampling factor  $p_{\text{add}} > 0$ , parameter  $\theta \in [0, 1]$

**Output:** Monotone set  $\mathcal{A}$  and coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  of the least-squares approximation

$$\mathbf{v}(\xi) = \sum_{\alpha \in \mathcal{A}} \mathbf{v}_\alpha \psi_\alpha(\xi) \text{ of } \mathbf{u}(\xi)$$

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- 2: Generate the initial sample set  $\{\xi^l\}_{l=1}^N$  randomly
- 3: Compute the matrices  $\Psi_{\mathcal{A}} = (\psi_\alpha(\xi^l))_{1 \leq l \leq N, \alpha \in \mathcal{A}}$  and  $\mathbf{Y}^T = (\mathbf{u}(\xi^l))_{1 \leq l \leq N}$
- 4: **while** no convergence **do**
- 5:   // Adaptive random sampling
- 6:   **while** no convergence and no stagnation **do**
- 7:     Compute the coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  such that  $\mathbf{V}^T = (\Psi_{\mathcal{A}}^T \Psi_{\mathcal{A}})^{-1} \Psi_{\mathcal{A}}^T \mathbf{Y}$   
by taking into account additional samples
- 8:   **end while**
- 9:   // Working set strategy
- 10:   **while** no convergence and no overfitting **do**
- 11:     Compute the reduced margin  $\mathcal{M} = \mathcal{M}_{\text{red}}(\mathcal{A})$  of  $\mathcal{A}$  and the set  $\mathcal{T} = \mathcal{A} \cup \mathcal{M}$
- 12:     Compute the matrices  $\Psi_{\mathcal{M}}$  and  $\Psi_{\mathcal{T}} = (\Psi_{\mathcal{A}}, \Psi_{\mathcal{M}})$
- 13:     Compute the coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{T}}$  such that  $\mathbf{V}^T = (\Psi_{\mathcal{T}}^T \Psi_{\mathcal{T}})^{-1} \Psi_{\mathcal{T}}^T \mathbf{Y}$
- 14:     Define the smallest (monotone) subset  $\mathcal{N}$  of  $\mathcal{M}$  such that  $e(\mathcal{N}) \geq \theta e(\mathcal{M})$ , with  
 $e(\mathcal{N}) = \sum_{\alpha \in \mathcal{N}} \|\mathbf{v}_\alpha\|_2^2$  and  $e(\mathcal{M}) = \sum_{\alpha \in \mathcal{M}} \|\mathbf{v}_\alpha\|_2^2$
- 15:     Update the multi-index set  $\mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{N}$  and the matrix  $\Psi_{\mathcal{A}} \leftarrow (\Psi_{\mathcal{A}}, \Psi_{\mathcal{N}})$
- 16:     Compute the coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  such that  $\mathbf{V}^T = (\Psi_{\mathcal{A}}^T \Psi_{\mathcal{A}})^{-1} \Psi_{\mathcal{A}}^T \mathbf{Y}$
- 17:   **end while**
- 18: **end while**

## Adaptive sparse algorithm

Adaptive sparse least-squares solver with random sampling and working set strategy  
[Nouy and Pled 2017]

**Input:** Initial number of samples  $N \geq 1$ , sampling factor  $p_{\text{add}} > 0$ , parameter  $\theta \in [0, 1]$

**Output:** Monotone set  $\mathcal{A}$  and coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  of the least-squares approximation

$$\mathbf{v}(\xi) = \sum_{\alpha \in \mathcal{A}} \mathbf{v}_\alpha \psi_\alpha(\xi) \text{ of } \mathbf{u}(\xi)$$

- 1: Start with null initial set  $\mathcal{A} = \{0_{\mathcal{F}}\}$
- 2: Generate the initial sample set  $\{\xi^l\}_{l=1}^N$  randomly
- 3: Compute the matrices  $\Psi_{\mathcal{A}} = (\psi_\alpha(\xi^l))_{1 \leq l \leq N, \alpha \in \mathcal{A}}$  and  $\mathbf{Y}^T = (\mathbf{u}(\xi^l))_{1 \leq l \leq N}$
- 4: **while** no convergence **do**
- 5:   // Adaptive random sampling
- 6:   **while** no convergence and no stagnation **do**
- 7:     Compute the coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  such that  $\mathbf{V}^T = (\Psi_{\mathcal{A}}^T \Psi_{\mathcal{A}})^{-1} \Psi_{\mathcal{A}}^T \mathbf{Y}$   
by taking into account additional samples
- 8:   **end while**
- 9:   // Working set strategy
- 10:   **while** no convergence and no overfitting **do**
- 11:     Compute the coefficients  $\mathbf{V} = (\mathbf{v}_\alpha)_{\alpha \in \mathcal{A}}$  such that  $\mathbf{V}^T = (\Psi_{\mathcal{A}}^T \Psi_{\mathcal{A}})^{-1} \Psi_{\mathcal{A}}^T \mathbf{Y}$   
by updating the approximation basis
- 12:   **end while**
- 13: **end while**

# Relaxation step

## Impact of relaxation step on convergence

- The relaxation step may **affect** the **convergence rate** of the iterative algorithm.

## Simplest technique - Fixed relaxation parameter

- A **large** relaxation parameter may speed up the convergence but can lead to a divergence of the algorithm, while a **small** relaxation parameter enables to control the convergence but trigger more iterations in return.
- The computation of an **optimal** relaxation parameter is not obvious in the non-linear framework.
- Even in the linear case, the **optimal** fixed value of relaxation parameter  $\rho$  is problem-dependent and not known *a priori*.
- In the linear case and under some conditions on the iteration map, an optimal convergent rate of the global-local iterative algorithm can be achieved by computing the **optimal** relaxation parameter  $\rho_{\text{opt}}$  defined by [Xu 1992]

$$\rho_{\text{opt}} = \frac{2}{\lambda_{\min} + \lambda_{\max}},$$

where  $\lambda_{\min}$  and  $\lambda_{\max}$  are the lowest and largest eigenvalues of the iteration map (see [Chevreuil, Nouy, and Safatly 2013]).

# Relaxation step

## Convergence acceleration technique - Aitken's dynamic relaxation

- Aitken's Delta-Squared method [Irons and Tuck 1969; Macleod 1986]
  - Simple and efficient convergence acceleration technique
  - Improvement of the current solution by using information gained at two previous iterations
- Update Aitken's formula for the relaxation parameter [Duval et al. 2014; Küttler and Wall 2008; Liu, Sun, and Fan 2014]

$$\begin{cases} \rho_1 = \rho_2 = 1 \\ \rho_k = -\rho_{k-1} \frac{\langle \delta^k(\xi) - \delta^{k-1}(\xi), \delta^{k-1}(\xi) \rangle_{\tilde{\mathcal{U}}}}{\|\delta^k(\xi) - \delta^{k-1}(\xi)\|_{\tilde{\mathcal{U}}}^2}, \end{cases} \quad \text{with } \delta^k(\xi) = \widehat{U}^k(\xi) - U^{k-1}(\xi)$$

- Adapted recursive formula to ensure convergence of the algorithm [Nouy and Pled 2017]

$$\rho_k = T_{[\rho_{\text{inf}}, \rho_{\text{sup}}]} \left( -\rho_{k-1} \frac{\langle \delta^k(\xi) - \delta^{k-1}(\xi), \delta^{k-1}(\xi) \rangle_{\tilde{\mathcal{U}}}}{\|\delta^k(\xi) - \delta^{k-1}(\xi)\|_{\tilde{\mathcal{U}}}^2} \right),$$

where  $T_{[\rho_{\text{inf}}, \rho_{\text{sup}}]}(\rho)$  is the projection of  $\rho$  on the interval  $[\rho_{\text{inf}}, \rho_{\text{sup}}]$

# Outline

- 1 Context and motivation
- 2 Multiscale domain decomposition method
- 3 Computational aspects
- 4 Numerical illustrations**
  - Stationary non-linear diffusion-reaction stochastic problem
  - Transient linear advection-diffusion-reaction deterministic problem
- 5 Conclusions and outlooks

## Stationary non-linear diffusion-reaction stochastic problem

## Non-linear diffusion-reaction problem with localized random material heterogeneities

[Nouy and Pled 2017]

$$-\nabla \cdot (K \nabla u) + Ru^3 = 1 \quad \text{on } \Omega = (0, 2) \times (0, 16)$$

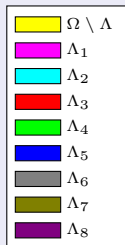
$$u = 0 \quad \text{on } \Gamma_D = \partial\Omega$$

- Random diffusion and reaction fields

$$K(x, \xi) = \begin{cases} 1 & \text{on } \Omega \setminus \left( \bigcup_{q=1}^8 \Lambda_q \right) \\ 1 + \gamma_q \xi_{2q-1} \chi_q(x) & \text{on } \Lambda_q \end{cases}$$

$$R(x, \xi) = \begin{cases} 0 & \text{on } \Omega \setminus \left( \bigcup_{q=1}^8 \Lambda_q \right) \\ \gamma_q \xi_{2q} \chi_q(x) & \text{on } \Lambda_q \end{cases}$$

- Uniform random variables  $\xi = (\xi_i)_{i=1}^{16}$  on  $\Xi_i = (0, 1)$
- Uniform weights  $\gamma_q = 1$



## Stationary non-linear diffusion-reaction stochastic problem

Non-linear diffusion-reaction problem with localized random material heterogeneities  
 [Nouy and Pled 2017]

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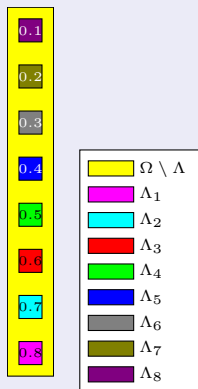
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- Uniform random variables  $\xi = (\xi_i)_{i=1}^{16}$  on  $\Xi_i = (0, 1)$
- Heterogeneous weights  $\gamma_q = 1 - 0.1(q + 1)$



# Stationary non-linear diffusion-reaction stochastic problem

## Non-linear diffusion-reaction problem with localized random material heterogeneities [Nouy and Pled 2017]

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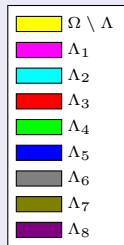
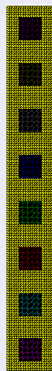
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$$K(x, \xi) = \begin{cases} 1 & \text{on } \Omega \setminus \left( \bigcup_{q=1}^8 \Lambda_q \right) \\ 1 + \gamma_q \xi_{2q-1} \chi_q(x) & \text{on } \Lambda_q \end{cases}$$

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- **Uniform** random variables  $\xi = (\xi_i)_{i=1}^{16}$  on  $\Xi_i = (0, 1)$
- **Heterogeneous** weights  $\gamma_q = 1 - 0.1(q + 1)$



## Approximation spaces

- **Spatial level:** nested global and local finite element approximation spaces
  - $\dim(\tilde{\mathcal{U}}_H) = 3381$  on  $\tilde{\Omega}$
  - $\dim(\mathcal{W}_h^q) = 441$  on  $\Lambda_q$
- **Stochastic level:** adaptive construction of sparse Legendre polynomial approximation spaces using working sets strategies [Chkifa et al. 2013]



# Stationary non-linear diffusion-reaction stochastic problem

## Convergence analysis

- Control of the accuracy of global approximations  $U^k$  with respect to a global reference solution  $U^{\text{ref}}$  using the relative error indicator

$$\varepsilon_{\Xi, \Omega \setminus \Lambda}(U^k; U^{\text{ref}}) = \frac{\|U^k - U^{\text{ref}}\|_{L^2_{\mu}(\Xi; L^2(\Omega \setminus \Lambda))}}{\|U^{\text{ref}}\|_{L^2_{\mu}(\Xi; L^2(\Omega \setminus \Lambda))}}$$

with  $\|U\|_{L^2_{\mu}(\Xi; L^2(\Omega \setminus \Lambda))}^2 = \mathbb{E}(\|U(\xi)\|_{L^2(\Omega \setminus \Lambda)}^2)$

- Reference solution  $(U^{\text{ref}}, w_q^{\text{ref}}, \lambda_q^{\text{ref}})$  of the full-scale coupled problem obtained using an adaptive sparse least-squares method
  - Spatial level:** deterministic non-linear problems solved using a tangent-Newton iterative algorithm with a tolerance  $\epsilon = 10^{-12}$
  - Stochastic level:** control of the number of samples  $N^{\text{ref}}$  and approximation spaces  $\mathcal{S}_{\mathcal{A}^{\text{ref}}}$  using a cross-validation procedure with a tolerance  $\varepsilon_{\text{cv}}^{\text{ref}} = 10^{-6}$
  - Isotropic case:**
    - $N^{\text{ref}} = 795$  samples
    - $\#\mathcal{A}^{\text{ref}} = 384$  for  $U^{\text{ref}}$
  - Anisotropic case:**
    - $N^{\text{ref}} = 491$  samples
    - $\#\mathcal{A}^{\text{ref}} = 173$  for  $U^{\text{ref}}$

# Stationary non-linear diffusion-reaction stochastic problem

## Convergence analysis - Influence of relaxation parameter $\rho_k$

- Fixed cross-validation tolerance  $\varepsilon_{cv} = 10^{-3}$
- Evolution of relative error indicator  $\varepsilon_{\Xi, \Omega \setminus \Lambda}(U^k; U^{ref})$  with respect to iteration  $\#k$

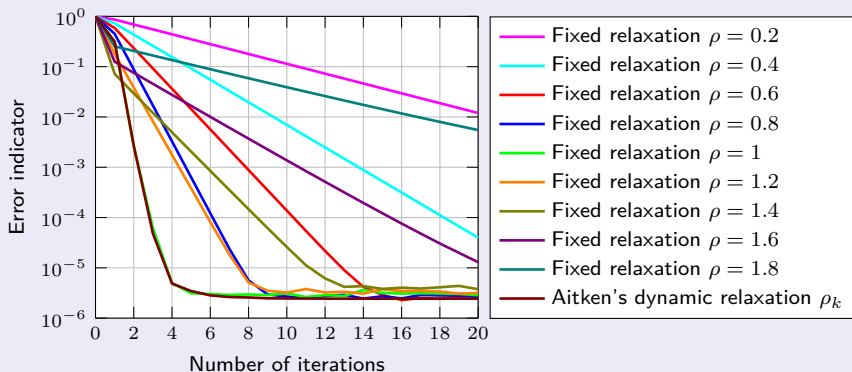


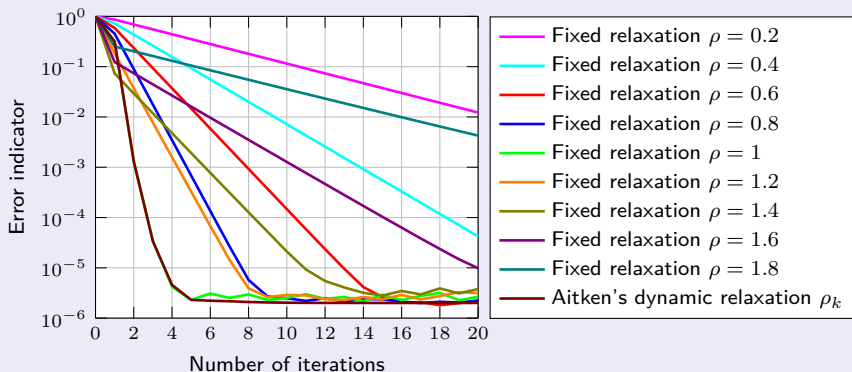
Figure: Isotropic case

↪ Strong influence of relaxation parameter  $\rho_k$  on the convergence properties

# Stationary non-linear diffusion-reaction stochastic problem

## Convergence analysis - Influence of relaxation parameter $\rho_k$

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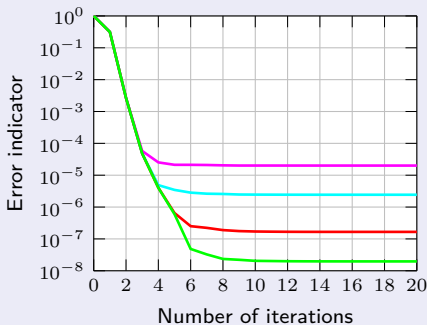


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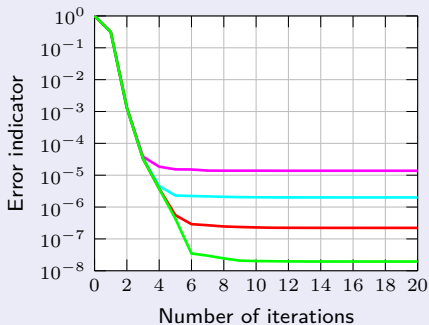
# Stationary non-linear diffusion-reaction stochastic problem

## Convergence analysis - Influence of cross-validation tolerance $\varepsilon_{cv}$

- Relaxation parameter  $\rho_k$  dynamically updated using **Aitken's Delta-Squared method**
- Evolution of relative **error indicator**  $\varepsilon_{\Xi, \Omega \setminus \Lambda}(U^k; U^{ref})$  with respect to iteration  $\#k$



(a) Isotropic case



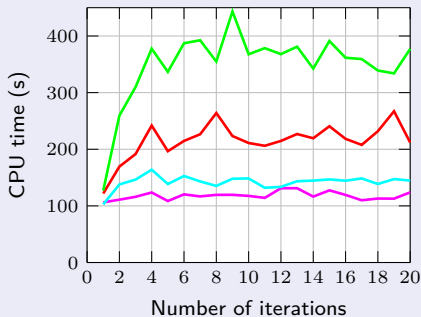
(b) Anisotropic case

—  $\varepsilon_{cv} = 10^{-2}$ 
—  $\varepsilon_{cv} = 10^{-3}$ 
—  $\varepsilon_{cv} = 10^{-4}$ 
—  $\varepsilon_{cv} = 10^{-5}$

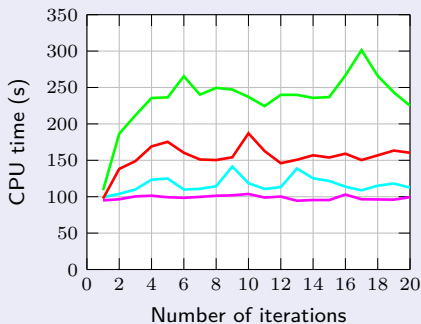
# Stationary non-linear diffusion-reaction stochastic problem

## Computational efficiency - Influence of cross-validation tolerance $\epsilon_{cv}$

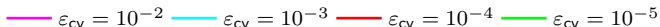
- Relaxation parameter  $\rho_k$  dynamically updated using **Aitken's Delta-Squared method**
- Evolution of **CPU time per iteration** with respect to iteration  $\#k$



(a) Isotropic case



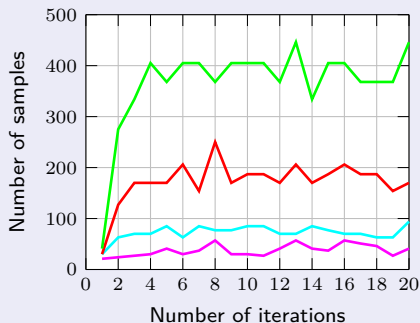
(b) Anisotropic case



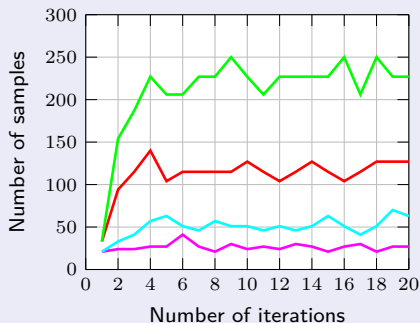
# Stationary non-linear diffusion-reaction stochastic problem

## Adaptive sampling

- Initial sample size  $N = 1$ , Sampling factor  $p_{\text{add}} = 0.1$
- Evolution of **sample size**  $N_q$  for local solution  $(w_q^k, \lambda_q^k)$  within patch  $\Lambda_q$  with respect to iteration  $\#k$ , for patch  $\#q = 4$



(a) Isotropic case



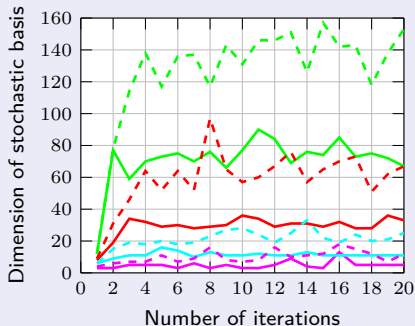
(b) Anisotropic case



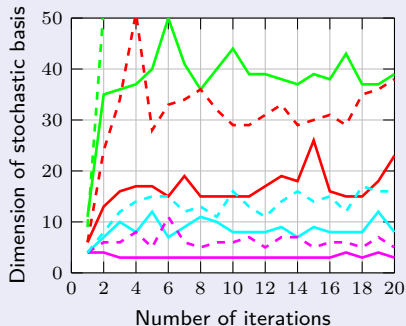
# Stationary non-linear diffusion-reaction stochastic problem

## Adaptive construction of sparse polynomial approximation spaces

- Initial dimension  $\#\mathcal{A}_q = 0$
- Evolution of the **dimension**  $\#\mathcal{A}_q$  of the **stochastic approximation basis** for local solution  $w_q^k$  (solid lines) and Lagrange multiplier  $\lambda_q^k$  (dashed lines) with respect to iteration  $\#k$ , for patch  $\#q = 4$



(a) Isotropic case



(b) Anisotropic case



## Stationary non-linear diffusion-reaction stochastic problem

## Adaptive construction of sparse polynomial approximation spaces

- Partial polynomial degrees  $p_i$  (in each random variable  $\xi_i$ ) and dimension  $\#\mathcal{A}$  of approximation spaces  $\mathcal{S}_{\mathcal{A}}$  for global solution  $U$  and local solutions  $(w^q, \lambda^q)$

|             | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ | $p_{11}$ | $p_{12}$ | $p_{13}$ | $p_{14}$ | $p_{15}$ | $p_{16}$ | $\#\mathcal{A}$ |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|----------|----------|----------|----------|----------|-----------------|
| $U$         | 5     | 2     | 4     | 3     | 4     | 2     | 4     | 3     | 4     | 3        | 4        | 3        | 4        | 2        | 5        | 2        | 192             |
| $w_1$       | 3     | 1     | 1     | 1     | 1     | 0     | 0     | 0     | 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 9               |
| $\lambda_1$ | 4     | 2     | 2     | 2     | 1     | 1     | 0     | 0     | 0     | 0        | 1        | 0        | 0        | 0        | 0        | 0        | 21              |
| $w_2$       | 2     | 1     | 3     | 2     | 1     | 1     | 0     | 0     | 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 13              |
| $\lambda_2$ | 3     | 1     | 3     | 2     | 2     | 2     | 0     | 1     | 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 19              |
| $w_3$       | 0     | 0     | 1     | 1     | 3     | 2     | 1     | 1     | 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 11              |
| $\lambda_3$ | 1     | 1     | 2     | 1     | 3     | 2     | 2     | 1     | 0     | 1        | 0        | 0        | 0        | 0        | 1        | 1        | 21              |
| $w_4$       | 0     | 0     | 0     | 0     | 1     | 1     | 3     | 2     | 1     | 1        | 0        | 0        | 0        | 0        | 0        | 0        | 11              |
| $\lambda_4$ | 0     | 0     | 0     | 1     | 2     | 2     | 4     | 2     | 2     | 1        | 1        | 1        | 0        | 1        | 1        | 0        | 25              |
| $w_5$       | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 1     | 3     | 2        | 1        | 1        | 0        | 0        | 0        | 0        | 11              |
| $\lambda_5$ | 0     | 0     | 0     | 0     | 0     | 1     | 1     | 2     | 3     | 2        | 2        | 1        | 1        | 1        | 0        | 0        | 20              |
| $w_6$       | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 1        | 3        | 2        | 1        | 1        | 0        | 0        | 11              |
| $\lambda_6$ | 0     | 0     | 1     | 0     | 0     | 0     | 1     | 1     | 2     | 2        | 4        | 2        | 2        | 2        | 1        | 1        | 28              |
| $w_7$       | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 1        | 1        | 3        | 2        | 2        | 1        | 13              |
| $\lambda_7$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 1        | 2        | 2        | 3        | 2        | 3        | 1        | 20              |
| $w_8$       | 0     | 1     | 1     | 0     | 0     | 0     | 0     | 0     | 1     | 0        | 1        | 0        | 1        | 1        | 3        | 1        | 12              |
| $\lambda_8$ | 0     | 1     | 1     | 0     | 0     | 0     | 0     | 0     | 1     | 1        | 1        | 1        | 3        | 2        | 5        | 2        | 26              |

(a) Isotropic case



## Stationary non-linear diffusion-reaction stochastic problem

## Adaptive construction of sparse polynomial approximation spaces

- Partial polynomial degrees  $p_i$  (in each random variable  $\xi_i$ ) and dimension  $\#\mathcal{A}$  of approximation spaces  $\mathcal{S}_{\mathcal{A}}$  for global solution  $U$  and local solutions  $(w^q, \lambda^q)$

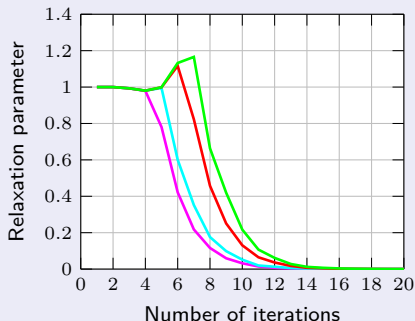
|             | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ | $p_{11}$ | $p_{12}$ | $p_{13}$ | $p_{14}$ | $p_{15}$ | $p_{16}$ | $\#\mathcal{A}$ |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|----------|----------|----------|----------|----------|-----------------|
| $U$         | 5     | 3     | 3     | 2     | 3     | 2     | 3     | 2     | 3     | 2        | 3        | 1        | 2        | 3        | 2        | 1        | 112             |
| $w_1$       | 3     | 1     | 1     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 8               |
| $\lambda_1$ | 4     | 2     | 2     | 1     | 1     | 1     | 1     | 0     | 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 18              |
| $w_2$       | 1     | 1     | 2     | 1     | 1     | 1     | 0     | 0     | 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 9               |
| $\lambda_2$ | 2     | 1     | 3     | 2     | 1     | 1     | 0     | 1     | 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 15              |
| $w_3$       | 1     | 0     | 1     | 1     | 2     | 1     | 1     | 2     | 0     | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 13              |
| $\lambda_3$ | 1     | 1     | 2     | 1     | 3     | 2     | 1     | 2     | 0     | 1        | 0        | 0        | 0        | 0        | 0        | 0        | 18              |
| $w_4$       | 0     | 0     | 0     | 0     | 1     | 1     | 2     | 1     | 1     | 1        | 0        | 0        | 0        | 0        | 0        | 0        | 8               |
| $\lambda_4$ | 0     | 0     | 0     | 1     | 1     | 1     | 2     | 2     | 1     | 2        | 0        | 1        | 0        | 0        | 1        | 0        | 16              |
| $w_5$       | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 1     | 2     | 1        | 0        | 1        | 0        | 0        | 0        | 0        | 7               |
| $\lambda_5$ | 0     | 0     | 0     | 0     | 0     | 1     | 1     | 1     | 2     | 2        | 1        | 1        | 0        | 0        | 0        | 0        | 11              |
| $w_6$       | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 1        | 2        | 1        | 0        | 1        | 0        | 0        | 7               |
| $\lambda_6$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 1        | 2        | 1        | 1        | 1        | 0        | 0        | 8               |
| $w_7$       | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        | 1        | 1        | 1        | 1        | 0        | 5               |
| $\lambda_7$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 1        | 1        | 2        | 1        | 1        | 1        | 8               |
| $w_8$       | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        | 0        | 0        | 1        | 1        | 1        | 4               |
| $\lambda_8$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        | 1        | 1        | 1        | 2        | 1        | 7               |

(a) Anisotropic case

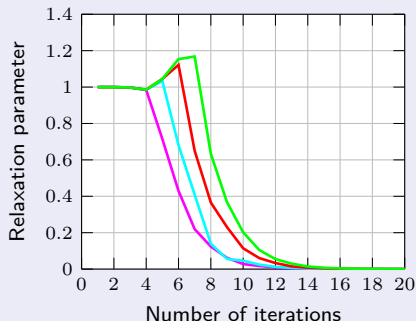
# Stationary non-linear diffusion-reaction stochastic problem

## Aitken's dynamic relaxation

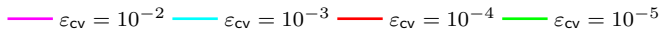
- Evolution of relaxation parameter  $\rho_k$  with respect to iteration  $\#k$



(a) Isotropic case



(b) Anisotropic case



# Stationary non-linear diffusion-reaction stochastic problem

## Statistical moments of global, local and multiscale solutions at convergence

- Mean  $\mathbb{E}$  and variance  $\mathbb{V}$  of global solution  $U$ , local solutions  $w_q$  and multiscale solution  $u$

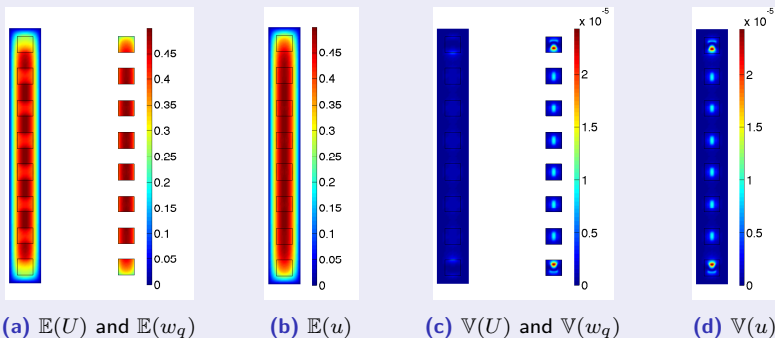


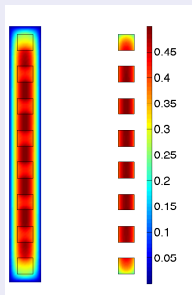
Figure: Isotropic problem

↔ The highest spatial contributions to the variance  $\mathbb{V}(u)$  are captured by the local solution  $w_q$  within every patch  $\Lambda_q$

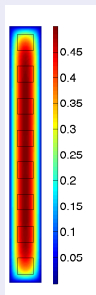
# Stationary non-linear diffusion-reaction stochastic problem

## Statistical moments of global, local and multiscale solutions at convergence

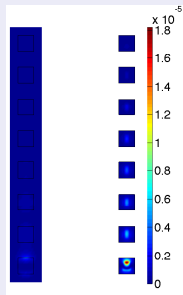
- Mean  $\mathbb{E}$  and variance  $\mathbb{V}$  of global solution  $U$ , local solutions  $w_q$  and multiscale solution  $u$



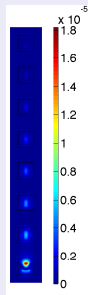
(a)  $\mathbb{E}(U)$  and  $\mathbb{E}(w_q)$



(b)  $\mathbb{E}(u)$



(c)  $\mathbb{V}(U)$  and  $\mathbb{V}(w_q)$



(d)  $\mathbb{V}(u)$

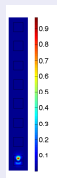
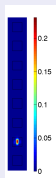
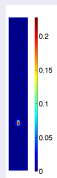
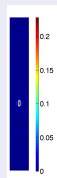
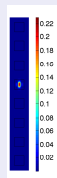
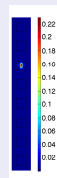
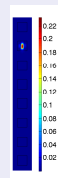
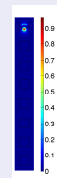
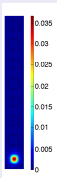
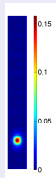
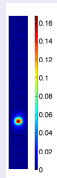
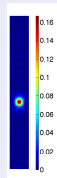
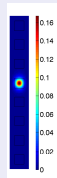
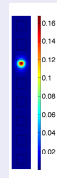
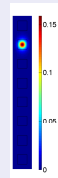
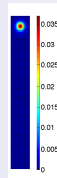
**Figure:** Anisotropic problem

↔ The **highest** spatial contributions to the **variance**  $\mathbb{V}(u)$  are **localized** in the patches exhibiting the **highest input variabilities**

# Stationary non-linear diffusion-reaction stochastic problem

## Global sensitivity indices of multiscale solution at convergence

- Sensitivity indices  $\tilde{S}_i(u) = \frac{\mathbb{V}(\mathbb{E}(u|\xi_i))}{\max_{x \in \Omega} \mathbb{V}(u)}$  quantifying the impact of each random variable  $\xi_i$  on the variance  $\mathbb{V}(u)$  of multiscale solution  $u$

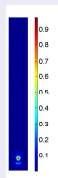
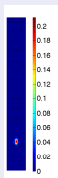
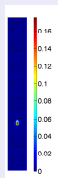
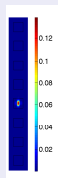
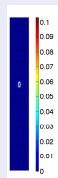
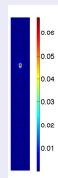
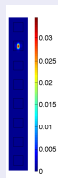
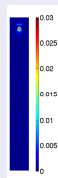
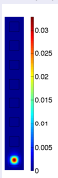
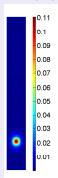
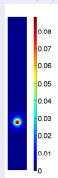
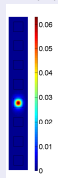
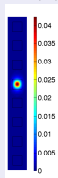
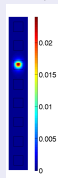
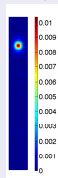
 $\tilde{S}_1(u)$  $\tilde{S}_3(u)$  $\tilde{S}_5(u)$  $\tilde{S}_7(u)$  $\tilde{S}_9(u)$  $\tilde{S}_{11}(u)$  $\tilde{S}_{13}(u)$  $\tilde{S}_{15}(u)$  $\tilde{S}_2(u)$  $\tilde{S}_4(u)$  $\tilde{S}_6(u)$  $\tilde{S}_8(u)$  $\tilde{S}_{10}(u)$  $\tilde{S}_{12}(u)$  $\tilde{S}_{14}(u)$  $\tilde{S}_{16}(u)$ 

(a) Isotropic case

## Stationary non-linear diffusion-reaction stochastic problem

## Global sensitivity indices of multiscale solution at convergence

- Sensitivity indices  $\tilde{S}_i(u) = \frac{\mathbb{V}(\mathbb{E}(u|\xi_i))}{\max_{x \in \Omega} \mathbb{V}(u)}$  quantifying the impact of each random variable  $\xi_i$  on the variance  $\mathbb{V}(u)$  of multiscale solution  $u$

 $\tilde{S}_1(u)$  $\tilde{S}_3(u)$  $\tilde{S}_5(u)$  $\tilde{S}_7(u)$  $\tilde{S}_9(u)$  $\tilde{S}_{11}(u)$  $\tilde{S}_{13}(u)$  $\tilde{S}_{15}(u)$  $\tilde{S}_2(u)$  $\tilde{S}_4(u)$  $\tilde{S}_6(u)$  $\tilde{S}_8(u)$  $\tilde{S}_{10}(u)$  $\tilde{S}_{12}(u)$  $\tilde{S}_{14}(u)$  $\tilde{S}_{16}(u)$ 

(a) Anisotropic case

# Transient linear advection-diffusion-reaction deterministic problem

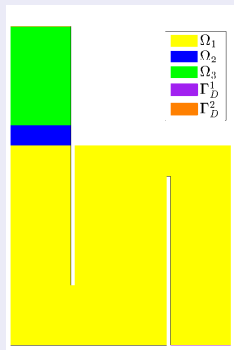
Linear advection-diffusion-reaction problem - Transport of pollutant inside an active carbon filter [Nouy 2010; Parés, Díez, and Huerta 2008]

$$c\dot{u} - \nabla \cdot (K\nabla u) + \alpha \cdot \nabla u + Ru = 0 \quad \text{on } \Omega \times [0, T]$$

$$u = 1 \text{ on } \Gamma_D^1, \quad u = 0 \text{ on } \Gamma_D^2, \quad u = 0 \text{ elsewhere on } \partial\Omega$$

$$u_0 = 1 \text{ on } \Gamma_D^1, \quad u_0 = 0 \text{ elsewhere on } \Omega$$

- Space domain  $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$
- Time interval  $I = [0, T]$  with  $T = 2 \text{ s}$
- Thermal capacity  $c = 1$
- Diffusion coefficient  $K = 0.01$
- Reaction parameter  $R = \begin{cases} 0.1 & \text{on } \Omega_1 \\ 10 & \text{on } \Omega_2 \cup \Omega_3 \end{cases}$
- Piecewise linear advection field  $\alpha = \nabla\psi$



# Transient linear advection-diffusion-reaction deterministic problem

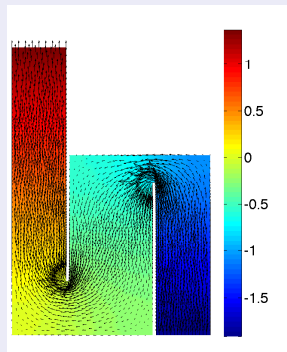
Linear advection-diffusion-reaction problem - Transport of pollutant inside an active carbon filter [Nouy 2010; Parés, Díez, and Huerta 2008]

$$c\dot{u} - \nabla \cdot (K\nabla u) + \alpha \cdot \nabla u + Ru = 0 \quad \text{on } \Omega \times [0, T]$$

$$u = 1 \text{ on } \Gamma_D^1, u = 0 \text{ on } \Gamma_D^2, u = 0 \text{ elsewhere on } \partial\Omega$$

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- Potential flow  $\psi$



# Transient linear advection-diffusion-reaction deterministic problem

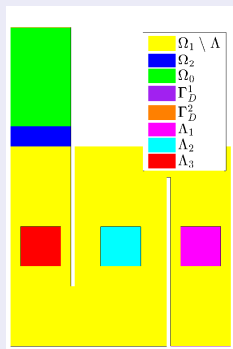
Linear advection-diffusion-reaction problem - Transport of pollutant inside an active carbon filter [Nouy 2010; Parés, Díez, and Huerta 2008]

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# Transient linear advection-diffusion-reaction deterministic problem

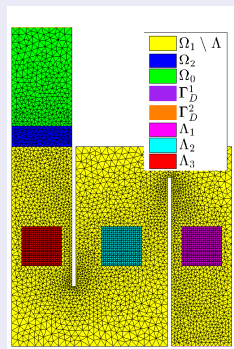
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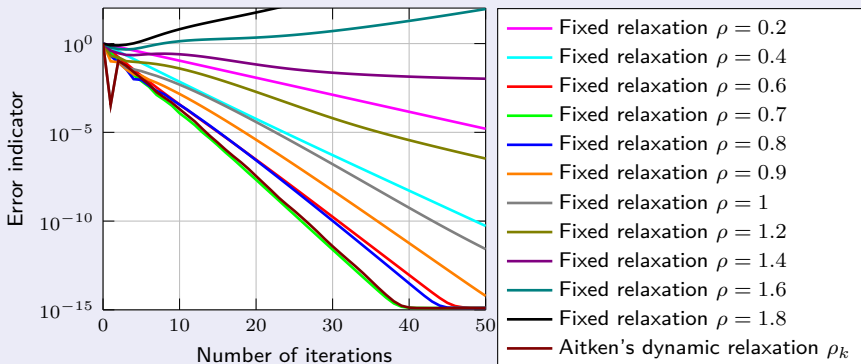
## Approximation spaces

- **Spatial level:** coarse global and fine local finite element approximation spaces
  - $\dim(\tilde{\mathcal{U}}_H) = 3416$  on  $\tilde{\Omega}$
  - $\dim(\mathcal{W}_h^q) = 441$  on  $\Lambda_q$
- **Time level:** implicit Euler time scheme with  $N_t = 100$  time steps

# Transient linear advection-diffusion-reaction deterministic problem

## Convergence analysis - Influence of relaxation parameter $\rho_k$

- Evolution of relative error indicator  $\varepsilon_{\Omega \setminus \Lambda, I}(U^k; U^{\text{ref}})$  with respect to iteration  $\#k$



- ↪ Strong influence of relaxation parameter  $\rho_k$  on the convergence properties
- ↪ Aitken's acceleration technique provides similar results as those obtained with an optimal fixed relaxation parameter

# Transient linear advection-diffusion-reaction deterministic problem

## Global, local and multiscale solutions at convergence

- Time evolutions of **global** solution  $U$ , **local** solutions  $w_q$  and **multiscale** solution  $u$

**Local** solutions  $w_q$

**Global** solution  $U$

# Transient linear advection-diffusion-reaction deterministic problem

## Global, local and multiscale solutions at convergence

- Time evolutions of **global** solution  $U$ , **local** solutions  $w_q$  and **multiscale** solution  $u$   
Multiscale solution  $u$

# Transient linear advection-diffusion-reaction deterministic problem

## Global, local and multiscale solutions at convergence

- Time evolutions of **global** solution  $\dot{U}$ , **local** solutions  $\dot{w}_q$  and **multiscale** solution  $\dot{u}$   
**Local** solutions  $\dot{w}_q$

**Global** solution  $\dot{U}$

# Transient linear advection-diffusion-reaction deterministic problem

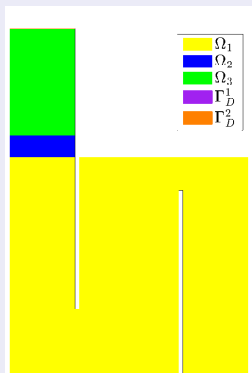
## Global, local and multiscale solutions at convergence

- Time evolutions of **global** solution  $\dot{U}$ , **local** solutions  $\dot{w}_q$  and **multiscale** solution  $\dot{u}$   
Multiscale solution  $\dot{u}$

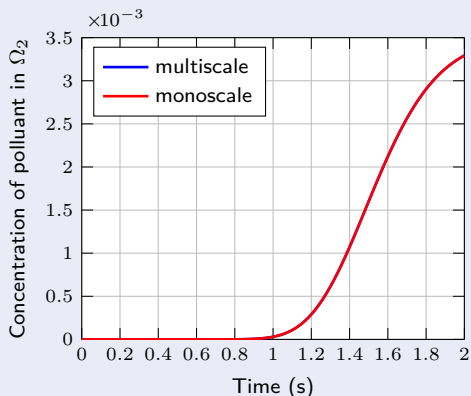
# Transient linear advection-diffusion-reaction deterministic problem

Quantity of interest: concentration of pollutant captured by the trap domain

- Evolution of the concentration of pollutant  $\ell_2(u; t) = \int_{\Omega_2} u(x, y, t) d\Omega$  captured by the trap domain  $\Omega_2$  with respect to time  $t$



(a) Pollutant trap  $\Omega_2$



(b) Concentration of pollutant  $\ell_2(u; t)$



## Transient linear advection-diffusion-reaction deterministic problem

Quantity of interest: concentration of pollutant captured by the trap domain

- Total concentration of pollutant captured by the trap domain  $\Omega_2$  along the complete time evolution  $I = [0, T]$ :

$$L_2(u) = \int_0^T \ell_2(u; t) dt = \int_0^T \int_{\Omega_2} u(x, y, t) d\Omega dt = 1.60876 \times 10^{-3}$$

$$L_2(u^{\text{ref}}) = \int_0^T \ell_2(u^{\text{ref}}; t) dt = \int_0^T \int_{\Omega_2} u^{\text{ref}}(x, y, t) d\Omega dt = 1.60876 \times 10^{-3}$$

with a relative error

$$\varepsilon_L = \frac{L_2(u) - L_2(u^{\text{ref}})}{L_2(u^{\text{ref}})} \approx 3 \times 10^{-15} \quad (\text{machine precision})$$

↔ The canister is considered to work properly if  $L_2(u)$  is **small enough**.

↔ If the outcome of pollutant exceeds a threshold value, the canister breaks and the **design is not admissible**.

# Outline

- 1 Context and motivation
- 2 Multiscale domain decomposition method
- 3 Computational aspects
- 4 Numerical illustrations
- 5 Conclusions and outlooks**

# Conclusions and outlooks


## Conclusions

- A multiscale method dedicated to **localized uncertainties** and possible **localized non-linearities**
  - Domain decomposition method with patches
  - Global-local iterative algorithm with proven convergence properties
- **Analysis** of the **consistency**, **convergence** and **robustness** properties of the algorithm under general assumptions
- Application to a broad class of problems
  - **linear** [Chevreuil, Nouy, and Safatly 2013] and **semi-linear** [Nouy and Pled 2017] **elliptic** equations (linear elasticity, stationary (non-)linear diffusion-reaction)
  - linear **parabolic** equations (transient linear advection-diffusion-reaction)

## On-going works

- Extension to linear **hyperbolic** equations (linear elasticity structural dynamics)
- Validation on **industrial applications** with 3D complex geometry
- Extension to **non-linear dynamic problems** by incorporating **non-linear effects**
  - geometrical or material non-linearities
  - contact with friction (see [Hager et al. 2012] for a 3D dynamic tire problem with non-linear material and frictional contact)
- Improved sampling-based methods for solving time-dependent local problems

Thanks for your attention !

 Nouy, A. & F. Pled [2017]. “A multiscale method for semi-linear elliptic equations with localized uncertainties and non-linearities”. In: *ESAIM: M2AN (submitted)*. 37 pages.

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## References I



Babuška, I., F. Nobile, and R. Tempone (2007). “A Stochastic Collocation Method for Elliptic Partial Differential Equations with Random Input Data”. In: *SIAM Journal on Numerical Analysis* 45.3, pp. 1005–1034.



Babuška, I., R. Tempone, and G. E. Zouraris (2004). “Galerkin Finite Element Approximations of Stochastic Elliptic Partial Differential Equations”. In: *SIAM Journal on Numerical Analysis* 42.2, pp. 800–825.



— (2005). “Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation”. In: *Computer Methods in Applied Mechanics and Engineering* 194.12-16, pp. 1251–1294.



Blatman, G. and B. Sudret (2010). “An adaptive algorithm to build up sparse polynomial chaos expansions for stochastic finite element analysis”. In: *Probabilistic Engineering Mechanics* 25.2, pp. 183–197.



Blatman, G. and B. Sudret (2011). “Adaptive sparse polynomial chaos expansion based on least angle regression”. In: *Journal of Computational Physics* 230.6, pp. 2345–2367.



Cawley, G. C. and N. L. Talbot (2004). “Fast exact leave-one-out cross-validation of sparse least-squares support vector machines”. In: *Neural Networks* 17.10, pp. 1467–1475.



Chapelle, O., V. Vapnik, and Y. Bengio (2002). “Model Selection for Small Sample Regression”. English. In: *Machine Learning* 48.1-3, pp. 9–23.



Chevreuril, M., A. Nouy, and E. Safatly (2013). “A multiscale method with patch for the solution of stochastic partial differential equations with localized uncertainties”. In: *Computer Methods in Applied Mechanics and Engineering* 255.0, pp. 255–274.

## References II



Chkifa, A., A. Cohen, and C. Schwab (2014). “High-Dimensional Adaptive Sparse Polynomial Interpolation and Applications to Parametric PDEs”. English. In: *Foundations of Computational Mathematics* 14.4, pp. 601–633.



— (2015). “Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs”. In: *Journal de Mathématiques Pures et Appliquées* 103.2, pp. 400–428.



Chkifa, A. et al. (2013). “Sparse adaptive Taylor approximation algorithms for parametric and stochastic elliptic PDEs”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 47 (01), pp. 253–280.



Deb, M. K., I. M. Babuška, and J. Oden (2001). “Solution of stochastic partial differential equations using Galerkin finite element techniques”. In: *Computer Methods in Applied Mechanics and Engineering* 190.48, pp. 6359–6372.



Duval, M. et al. (2014). “Non-intrusive Coupling: Recent Advances and Scalable Nonlinear Domain Decomposition”. English. In: *Archives of Computational Methods in Engineering*, pp. 1–22.



Gendre, L., O. Allix, and P. Gosselet (2011). “A two-scale approximation of the Schur complement and its use for non-intrusive coupling”. In: *International Journal for Numerical Methods in Engineering* 87.9, pp. 889–905.



Gendre, L. et al. (2009). “Non-intrusive and exact global/local techniques for structural problems with local plasticity”. English. In: *Computational Mechanics* 44 (2), pp. 233–245.



Ghanem, R. (1999). “Stochastic Finite Elements with Multiple Random Non-Gaussian Properties”. In: *Journal of Engineering Mechanics* 125.1, pp. 26–40.



Ghanem, R. G. and P. D. Spanos (1991). *Stochastic Finite Elements: A Spectral Approach*. New York, NY, USA: Springer-Verlag New York, Inc.



Hager, C. et al. (2012). "Solving dynamic contact problems with local refinement in space and time". In: *Computer Methods in Applied Mechanics and Engineering* 201-204.0, pp. 25–41.



Irons, B. M. and R. C. Tuck (1969). "A version of the Aitken accelerator for computer iteration". In: *International Journal for Numerical Methods in Engineering* 1.3, pp. 275–277.



Küttler, U. and W. A. Wall (2008). "Fixed-point fluid–structure interaction solvers with dynamic relaxation". English. In: *Computational Mechanics* 43.1, pp. 61–72.



Le Maître, O. et al. (2004a). "Multi-resolution analysis of Wiener-type uncertainty propagation schemes". In: *Journal of Computational Physics* 197.2, pp. 502–531.



Le Maître, O. et al. (2004b). "Uncertainty propagation using Wiener–Haar expansions". In: *Journal of Computational Physics* 197.1, pp. 28–57.



Liu, Y., Q. Sun, and X. Fan (2014). "A non-intrusive global/local algorithm with non-matching interface: Derivation and numerical validation". In: *Computer Methods in Applied Mechanics and Engineering* 277.0, pp. 81–103.



Lozinski, A. (2010). "Méthodes numériques et modélisation pour certains problèmes multi-échelles". Habilitation à diriger des recherches. Toulouse 3, France: Université Paul Sabatier.



Macleod, A. J. (1986). "Acceleration of vector sequences by multi-dimensional  $\Delta^2$  methods". In: *Communications in Applied Numerical Methods* 2.4, pp. 385–392.

## References IV



Nouy, A. (2010). "A priori model reduction through Proper Generalized Decomposition for solving time-dependent partial differential equations". In: *Computer Methods in Applied Mechanics and Engineering* 199.23-24, pp. 1603–1626.



Nouy, A. and F. Pled (2017). "A multiscale method for semi-linear elliptic equations with localized uncertainties and non-linearities". In: *ESAIM: M2AN (submitted)*. 37 pages.



Parés, N., P. Díez, and A. Huerta (2008). "Bounds of functional outputs for parabolic problems. Part I: Exact bounds of the discontinuous Galerkin time discretization". In: *Computer Methods in Applied Mechanics and Engineering* 197.19-20, pp. 1641–1660.



Saltelli Andrea, K. C. and E. M. Scott (2000). *Sensitivity Analysis*. New York: John Wiley & Sons.



Soize, C. and R. Ghanem (2004). "Physical Systems with Random Uncertainties: Chaos Representations with Arbitrary Probability Measure". In: *SIAM Journal on Scientific Computing* 26.2, pp. 395–410.



Sudret, B. (2008). "Global sensitivity analysis using polynomial chaos expansions". In: *Reliability Engineering & System Safety* 93.7, pp. 964–979.



Wan, X. and G. E. Karniadakis (2005). "An adaptive multi-element generalized polynomial chaos method for stochastic differential equations". In: *Journal of Computational Physics* 209.2, pp. 617–642.



Whitcomb, J. (1991). "Iterative global/local finite element analysis". In: *Computers & Structures* 40.4, pp. 1027–1031.



Xiu, D. and J. Hesthaven (2005). "High-Order Collocation Methods for Differential Equations with Random Inputs". In: *SIAM Journal on Scientific Computing* 27.3, pp. 1118–1139.



 Xiu, D. and G. Karniadakis (2002). “The Wiener–Askey Polynomial Chaos for Stochastic Differential Equations”. In: *SIAM Journal on Scientific Computing* 24.2, pp. 619–644.

 Xu, J. (1992). “Iterative Methods by Space Decomposition and Subspace Correction”. In: *SIAM Review* 34.4, pp. 581–613.