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► **To cite this version:**

| Hassan Jolany. Fiberwise Clabi-Yau Foliation is not foliation. 2017. hal-01551080

**HAL Id: hal-01551080**

**<https://hal.science/hal-01551080>**

Preprint submitted on 29 Jun 2017

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# Fiberwise Calabi-Yau Foliation is not foliation

Hassan Jolany

June 27, 2017

Fiberwise Calabi-Yau metric introduced by Greene, Shapire, Vafa, and Yau [3] in the context of string theory. The study of fiberwise Calabi-Yau metric on a fibration play an important role in the mirror symmetry, and also study of finding canonical metric on varieties with an intermediate Kodaira dimension [1][2][4]. One of applications of fiberwise Calabi-Yau metric is in positivity theory and study of Gromov-Hausdorff limit for degeneration of Calabi-Yau varieties along relative Kähler Ricci flow. In this letter by inspiring the works of S.T.Yau [3], Song-Tian [1], Bedford-Kalka [15], Chen-Tian[11], Tsuji [14] we introduce fiberwise Calabi-Yau foliation which relies in the context of generalized notion of foliation in complex geometry(not exactly foliation). In fact fiberwise Calabi-Yau foliation is foliation in fiber direction and it may not be foliation in horizontal direction and that's why this type of bundle generalize the notion of foliation in context of complex geometry and moduli space of Calabi-Yau manifolds.

We recall basic facts and definitions concerning regular foliations and also singular foliations.[15]

**Definition (Foliation).** A foliation on a normal variety  $X$  is a coherent subsheaf  $\mathcal{F} \subset TX$  such that

- 1)  $\mathcal{F}$  is closed under the Lie bracket, and
- 2)  $\mathcal{F}$  is saturated in  $TX$ . In other words, the quotient  $TX/\mathcal{F}$  is torsion free.

The rank  $r$  of  $\mathcal{F}$  is the generic rank of  $\mathcal{F}$ . The codimension of  $\mathcal{F}$  is defined as  $q := \dim X - r$ . Let  $U \subset X_{reg}$  be the open set where  $\mathcal{F}|_{X_{reg}}$  is a subbundle of  $T_{X_{reg}}$ . We say that  $\mathcal{F}$  is regular if  $U = X_{reg} = X$

A **leaf** of  $\mathcal{F}$  is a connected, locally closed holomorphic submanifold  $L \subset U$  such that  $T_L = \mathcal{F}|_L$

**Definition .** Let  $\pi : X \rightarrow Y$  be a dominant morphism of normal varieties. Suppose that  $\pi$  is equidimensional. relative canonical bundle can be defined as follows

$$K_{X/Y} := K_X - \pi^* K_Y$$

Let  $\mathcal{F}$  be the foliation on  $X$  induced by  $\pi$ , then

$$K_{\mathcal{F}} = K_{X/Y} - R(\pi)$$

where  $R(\pi) = \cup_D ((\pi)^* D - ((\pi)^* D)_{red})$  is the ramification divisor of  $\pi$ . Here  $D$  runs through all prime divisors on  $Y$ . The canonical class  $K_{\mathcal{F}}$  of  $\mathcal{F}$  is any Weil divisor on  $X$  such that  $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(\mathcal{F}) := (\wedge^r \mathcal{F})^{**}$

Now, because we are in deal with singularities, so we use of  $(1, 1)$ -current instead of  $(1, 1)$ -forms which is singular version of forms. See [9][10][12]

**Definition .** A current is a differential form with distribution coefficients. We recall a singular metric  $h_{\text{sing}}$  on a Line bundle  $L$  which locally can be written as  $h_{\text{sing}} = e^{\phi}h$  where  $h$  is a smooth metric, and  $\phi$  is an integrable function. Then one can define locally the closed current  $T_{L, h_{\text{sing}}}$  by the following formula

$$T_{L, h_{\text{sing}}} = \omega_{L, h} + \frac{1}{2i\pi} \partial \bar{\partial} \log \phi$$

If  $T$  is a positive  $(1, 1)$ -current then locally one can find a plurisubharmonic function  $u$  such that

$$\sqrt{-1} \partial \bar{\partial} u = T$$

Now we define semi-Ricci flat metric or fiberwise Calabi-Yau metric  $\omega_{SRF}$ . [3][6]

**Definition .** Let  $\pi : X \rightarrow Y$  be a smooth holomorphic fibration of Kähler manifolds  $X$  and  $Y$  such that the fibers are Calabi-Yau manifolds. A fiberwise Calabi-Yau metric is a unique  $(1, 1)$ -current  $\omega_{SRF}$  such that its restriction on each fiber  $X_y$  is smooth Ricci flat metric.

It is still open conjecture that such fiberwise Kähler metric is semi-positive. In fact it is positive in fiber direction and we don't know yet such metric to be semi-positive in horizontal direction

Now we need the definition of relative differential.

**Definition .** Let  $\pi : X \rightarrow Y$  be a morphism of schemes. The *sheaf of differentials*  $\Omega_{X/Y}$  of  $X$  over  $Y$  is the sheaf of differentials of  $f$  viewed as a morphism of ringed spaces equipped with its *universal  $Y$ -derivation*

$$d_{X/S} : \mathcal{O}_X \longrightarrow \Omega_{X/S}.$$

Now take a  $C^\infty$   $(1, 1)$ -form  $\omega$  on a complex manifold  $X$  of complex dimension  $n$  and let

$$\text{ann}(\omega) = \{W \in TX \mid \omega(W, \bar{V}) = 0, \forall V \in TX\}$$

Now we have the following lemma due to Schwarz inequality [16]

**Lemma .** If  $\omega$  is non-negative then we can write,

$$\text{ann}(\omega) = \{W \in TX \mid \omega(W, \bar{W}) = 0, \forall W \in TX\}$$

Moreover, if we assume  $\omega^{n-1} \neq 0$  and  $\omega^n = 0$  then  $\text{ann}(\omega)$  is subbundle of  $TX$ .

Furthermore, we have the following straightforward lemma which make  $\text{ann}(\omega)$  to be as foliation

**Lemma .** If  $\omega$  is non-negative,  $\omega^{n-1} \neq 0$ ,  $\omega^n = 0$ , and  $d\omega = 0$ , then

$$\mathcal{F} = \text{ann}(\omega) = \{W \in TX | \omega(W, \bar{W}) = 0, \forall W \in TX\}$$

define a foliation  $\mathcal{F}$  on  $X$  and each leaf of  $\mathcal{F}$  being a Riemann surface

Now Tsuji [7][14] took relative form  $\omega_{X/Y}$  instead  $\omega$  in previous lemma and wrote it as a foliation. In my opinion Tsuji's foliation is fail to be right foliation and we need to revise it. First of all we don't know such metric  $\omega_{SRF}$  is non-negative and second we must take  $W \in T_{X/Y}$  in relative tangent bundle and we don't have in general  $d\omega_{SRF} = 0$ , In fact we know just that  $d_{X/Y}\omega_{SRF} = 0$ . Moreover  $\omega_{SRF}$  is not smooth in general and it is a  $(1, 1)$ -current with log pole singularities.

Hence on Calabi-Yau fibration, we can introduce the following bundle

$$\mathcal{F} = \text{ann}(\omega_{SRF}) = \{W \in T_{X/Y} | \omega_{SRF}(W, \bar{W}) = 0, \forall W \in T_{X/Y}\}$$

in general is the right bundle to be considered and not something Tsuji wrote in [14]. It is not a foliation in general. In fact it is a foliation is fiber direction and may not be a foliation in horizontal direction, but it generalize the notion of foliation. The correct solution of it as Monge-Ampere foliation still remained as open problem.

In the fibre direction,  $\mathcal{F}$  is a foliation and we have the following straightforward theorem due to Bedford-Kalka.[13][15]

**Theorem .** Let  $\mathcal{L}$  be a leaf of  $f_*\mathcal{F}$ , then  $\mathcal{L}$  is a closed complex submanifold and the leaf  $\mathcal{L}$  can be seen as fiber on the moduli map

$$\eta : \mathcal{Y} \rightarrow \mathcal{M}_{CY}^D$$

where  $\mathcal{M}_{CY}$  is the moduli space of calabi-Yau fibers with at worst canonical singularities and

$$\mathcal{Y} = \{y \in Y_{reg} | X_y \text{ has Kawamata log terminal singularities}\}$$

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