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REAL SPECTRUM VERSUS ℓ-SPECTRUM VIA BRUMFIEL SPECTRUM

FRIEDRICH WEHRUNG

Abstract. It is well known that the real spectrum of any commutative unital ring, and the ℓ-spectrum of any Abelian lattice-ordered group with order-unit, are all completely normal spectral spaces. We complete the existing list of containments and non-containments between the associated spectral spaces and their spectral subspaces, by proving the following results:

(1) Every real spectrum can be embedded, as a spectral subspace, into some ℓ-spectrum.
(2) Not every real spectrum is an ℓ-spectrum.
(3) A spectral subspace of a real spectrum may not be a real spectrum.
(4) Not every ℓ-spectrum can be embedded, as a spectral subspace, into a real spectrum.

The commutative unital rings and Abelian lattice-ordered groups constructed in (2), (3), (4) all have cardinality $\aleph_1$. Moreover, (3) solves a problem stated in 2012 by Mellor and Tressl.

1. Introduction

Denote by **SX** the class of all spectral subspaces of members of a class **X** of spectral spaces. Most of this paper is devoted to proving the not yet established containments and non-containments, between classes of spectral spaces, represented in Figure 1.1.

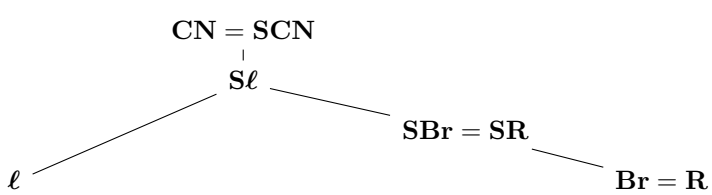

Figure 1.1. Classes of completely normal spectral spaces

The classes **X** in question are the following:

- **CN**, the class of all completely normal spectral spaces;
- $\boldsymbol{\ell}$, the class of ℓ-spectra of all Abelian ℓ-groups with order-unit;
- **Br**, the class of Brumfiel spectra of all commutative unital f-rings;
- **R**, the class of real spectra of all commutative unital rings.

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Key words and phrases. ℓ-group; f-ring; real-closed; condensate; ideal; prime; radical; spectral space; spectrum; ℓ-spectrum; Brumfiel spectrum; real spectrum; representable; sober; Stone duality; completely normal; distributive; lattice; closed map; convex map.

The context of our work is the following. The classical construction of the Zariski spectrum of a commutative unital ring (cf. Subsection 4.1) extends to many contexts, including distributive lattices, *lattice-ordered groups* (*ℓ-groups* for short), *partially ordered rings*, yielding *Stone duality*, the *ℓ-spectrum*, and the *real spectrum*, respectively. All the topological spaces thus obtained are *spectral spaces*, that is, sober spaces in which the compact open subsets are a basis of the topology, closed under finite intersection. Conversely, every spectral space is the spectrum of a bounded distributive lattice (Stone [29]) and also of a commutative unital ring (Hochster [15]).

The paper will focus on the *ℓ-spectrum* of an Abelian ℓ-group (cf. Subsection 4.2) and the *real spectrum* of a commutative unital ring (cf. Subsection 4.4). Those two frameworks are connected by the *Brumfiel spectrum* of a commutative f-ring (cf. Subsection 4.3). All the spectral spaces thus obtained are *completely normal*, that is, for all elements x and y in the closure of a singleton $\{z\}$, either x belongs to the closure of $\{y\}$ or y belongs to the closure of $\{x\}$.

Prior to the present paper, part of the picture (Figure 1.1) was already known:

- Delzell and Madden [8] proved that $\boldsymbol{\ell} \subsetneqq \mathbf{CN}$ and $\mathbf{R} \subsetneqq \mathbf{CN}$.
- Delzell and Madden's result got amplified in Mellor and Tressl [21], who established that *any class of spectral spaces containing* $\mathbf{R}$*, whose Stone dual lattices are definable by a class of $\mathscr{L}_{\infty,\lambda}$-formulas for some infinite cardinal λ, has a member outside* $\mathbf{SR}$. In particular, the class of all Stone duals of the spaces from $\mathbf{R}$ (resp., $\mathbf{SR}$) are not $\mathscr{L}_{\infty,\lambda}$-definable. Further, $\mathbf{SR} \subsetneqq \mathbf{CN}$.
- Delzell and Madden [9, Proposition 3.3] observed that $\mathbf{R} \subseteq \mathbf{Br}$.
- It follows easily from Madden and Schwartz [28] and Schwartz [27] that $\mathbf{Br} \subseteq \mathbf{R}$. Consequently, $\mathbf{Br} = \mathbf{R}$ (cf. Corollary 4.17).
- The author proved in [32] that *every second countable completely normal spectral space is in* $\boldsymbol{\ell}$, and that moreover, *the class of all Stone duals of spaces from* $\boldsymbol{\ell}$ *is not $\mathscr{L}_{\infty,\omega}$-definable.*
- The author provided an example in [32] showing that $\boldsymbol{\ell} \subsetneqq \mathbf{S}\boldsymbol{\ell}$.
- The present paper's first version, available on the HAL preprint server since 2017, established, with a counterexample with $\aleph_2$ compact open subsets, that $\mathbf{S}\boldsymbol{\ell} \subsetneqq \mathbf{CN}$. In the meantime, the preprint got lost in an editorial process and the result got superseded by the author's more recent paper [34], which provided a stronger counterexample. For more details see Section 7.

The missing pieces, all provided in the present paper, are the following:

- It is not hard to verify that every Brumfiel spectrum, thus also every real spectrum, can be embedded, as a spectral subspace, into some ℓ-spectrum. This is stated in Corollary 5.7. Hence, $\mathbf{SBr} \subseteq \mathbf{S}\boldsymbol{\ell}$.
- Not every real spectrum is an ℓ-spectrum. This is established in Theorem 5.4, *via* the construction of a *condensate*. Hence, $\mathbf{R} \nsubseteq \boldsymbol{\ell}$.
- A spectral subspace of a real spectrum may not be a Brumfiel spectrum (thus also not a real spectrum). This is stated in Corollary 5.10, *via* the construction of a condensate. It follows that $\mathbf{SR} \nsubseteq \mathbf{Br}$. This solves a problem of Mellor and Tressl [21].

- Not every ℓ-spectrum can be embedded, as a spectral subspace, into a Brumfiel spectrum (thus also not into a real spectrum). Hence, $\boldsymbol{\ell} \not\subseteq \mathbf{SBr}$. This is stated in Corollary 6.8.

We point out that although not every ℓ-spectrum is a real spectrum (cf. Corollary 6.8), there is a formally related problem, on the interaction between Abelian ℓ-groups and commutative rings, with a well known positive solution. Let D be an integral domain with group of units U and field of fractions K. Denote by $K^\times$ the multiplicative group of all nonzero elements of K. The *group of divisibility of* D (cf. Močkoř [23]) is the quotient group $K^\times/U$, endowed with the unique translation-invariant partial ordering with positive cone D/U. Every Abelian ℓ-group is the group of divisibility of some integral domain, which, in addition, can be taken a *Bezout domain* (cf. Anderson [1, page 4], where the result is credited to Krull, Jaffard, Kaplansky, and Ohm). Thus, Corollary 6.8 illustrates the gap between the group of divisibility and the real spectrum.

Since Stone duality is more conveniently stated with *bounded* distributive lattices, our results on spectral spaces are mostly formulated for commutative, *unital* rings and Abelian ℓ-groups *with order-unit*. On the other hand, most of our lattice-theoretical results are valid for lattices without top element, and thus formulated in that more general context.

Methodology. The underlying counterexamples of Theorems 5.4 and 5.9 are constructed with the one-arrow version, originating in the author's paper [31], of the general categorical model-theoretical construction of *condensates* developed in Gillibert and Wehrung [13]. A condensate of a diagram $\vec{A}$ is a special kind of directed colimit of finite products of vertices of $\vec{A}$. The condensate construction makes it possible to turn *diagram* counterexamples, to representation problems with respect to a given functor, to *object* counterexamples. In the one-arrow case required here, non-representable arrows are turned to non-representable objects, *via* the simplified condensate construction described in Definition 5.1 together with Lemma 5.3 (getting non-ℓ-representable distributive lattices) and its variant Lemma 5.8 (getting non-Brumfiel-representable distributive lattices).

In particular, the proof of Theorem 5.4, obtained by applying the condensate construction to an embedding $A \hookrightarrow K$ of countable real-closed rings, produces a real-closed ring of cardinality $\aleph_1$. Changing context, the proof of Theorem 5.9, obtained by applying the condensate construction to a surjective homomorphism between finite chains, produces a real spectrum, with $\aleph_1$ compact open subsets, with a spectral subspace which is not a real spectrum.

On the other hand, Theorem 6.4, producing the ℓ-spectrum of a unital Abelian ℓ-group of cardinality $\aleph_1$ which cannot be embedded as a spectral subspace into any real spectrum, is established *via* a direct argument, centered on the lexicographical product of a restricted Hahn power of the integers by the free Abelian ℓ-group on two positive generators.

2. Basic concepts

For ℓ-groups and f-rings, we refer the reader to Bigard, Keimel, and Wolfenstein [6] or Anderson and Feil [2]. For any partially ordered Abelian group G, we set $G^+ \stackrel{\text{def}}{=} \{x \in G \mid x \geq 0\}$ (the *positive cone* of G) and $G^{++} \stackrel{\text{def}}{=} \{x \in G \mid x > 0\}$. For $a, b \in G^+$, let $a \ll b$ hold if $ka \leq b$ for every positive integer k.

For partially ordered Abelian groups G and H, the *lexicographical product* of G by H, denoted $G \times_{\mathrm{lex}} H$, is the product group $G \times H$, endowed with the positive cone consisting of all pairs (x, y) with either $x > 0$ or ($x = 0$ and $y \geq 0$).

For any chain Λ, we denote by $\mathbb{Z}\langle\Lambda\rangle$ the restricted lexicographical power, of the chain $\mathbb{Z}$ of all integers, by Λ. Hence the elements of $\mathbb{Z}\langle\Lambda\rangle$ have the form $x = \sum_{i=1}^{n} k_i c_{\xi_i}$, where each $k_i \in \mathbb{Z} \setminus \{0\}$ and $\xi_1 < \cdots < \xi_n$ in Λ, and x belongs to the positive cone of $\mathbb{Z}\langle\Lambda\rangle$ iff either $n = 0$ (i.e., $x = 0$) or $k_n > 0$. This endows $\mathbb{Z}\langle\Lambda\rangle$ with a structure of a totally ordered Abelian group.

A *lattice-ordered group*, or *ℓ-group* for short, is a group endowed with a translation-invariant lattice ordering. All our ℓ-groups will be Abelian and will thus be denoted additively. Elements x and y, in an ℓ-group, are *orthogonal* if $x \wedge y = 0$.

A subset I in an Abelian ℓ-group G is an *ℓ-ideal* if it simultaneously a subgroup of G and an order-convex sublattice of G.

For any elements a and b in an Abelian ℓ-group G, we will set $a^+ \stackrel{\mathrm{def}}{=} a \vee 0$, $a^- \stackrel{\mathrm{def}}{=} (-a) \vee 0$, $|a| \stackrel{\mathrm{def}}{=} a \vee (-a)$, and $a \smallsetminus b \stackrel{\mathrm{def}}{=} (a - b)^+ = a - (a \wedge b)$.

A *lattice-ordered ring* is a ring endowed with a lattice ordering invariant under additive translations and preserved by multiplicative translations by positive elements. A lattice-ordered ring A is an *f-ring* if $x \wedge y = 0$ implies that $x \wedge yz = x \wedge zy = 0$ whenever $x, y, z \in A^+$ and $x \wedge y = 0$. Equivalently, A is a subdirect product of totally ordered (not necessarily unital) rings (cf. Bigard, Keimel, and Wolfenstein [6, Théorème 9.1.2]).

Lemma 2.1 (folklore)**.** *Let G be an Abelian ℓ-group and let $a, b, c \in G$. Then*

- $a \smallsetminus c \leq (a \smallsetminus b) + (b \smallsetminus c)$.
- $(a \smallsetminus b) \wedge (b \smallsetminus a) = 0$.
- *If, in addition, G is the underlying additive ℓ-group of an f-ring A and $c \in A^+$, then $ca \smallsetminus cb = c(a \smallsetminus b)$ and $ac \smallsetminus bc = (a \smallsetminus b)c$.*

A subset I in an f-ring A is an *ℓ-ideal* if it is, simultaneously, an ideal of the underlying ring of A and an order-convex sublattice of A.

Totally ordered rings are particular cases of f-rings. About those, we will need the following lemma.

Lemma 2.2. *Let A be a totally ordered (not necessarily unital) commutative domain and let I be a proper order-convex ideal of A. Then for every $x \in I$ and every $a \in A$, the relation $|xa| \ll |a|$ holds.*

Proof. We will use repeatedly the fact that for every $c \in A^{++}$, the assignment $t \mapsto tc$ defines an order-embedding of A into itself. Since A is totally ordered, we may assume that $a \geq 0$ and $x \geq 0$. Let $n < \omega$ and suppose that $nxa > a$ (so $a > 0$). Then for every $b \in A^+$, $nxab \geq ab \geq 0$, thus (as $a > 0$) $nxb \geq b \geq 0$. Since $x \in I$ and I is an ideal of A, we get $nxb \in I$. Since I is order-convex, it follows that $b \in I$. This holds for every $b \in A^+$, whence $I = A$, a contradiction. Since A is totally ordered, it follows that $nxa \leq a$. □

3. Stone duality between distributive lattices with zero and generalized spectral spaces

For lattice theory we refer the reader to Grätzer [14], Johnstone [17]. For any elements a and b in a distributive lattice D with zero, a *splitting* of (a, b) is a pair (x, y) of elements of D such that $a \vee b = a \vee y = x \vee b$ and $x \wedge y = 0$. Observe

that in that case, $x \leq a$ and $y \leq b$. We say that D is *completely normal* if every pair of elements in D has a splitting.

We denote by P^{op} the *opposite poset* of a poset P. For any functions f and g with common domain X, we set

$$[\![f \neq g]\!] \stackrel{\mathrm{def}}{=} \{x \in X \mid f(x) \neq g(x)\} \ .$$

For posets P and Q, a map $f\colon P \to Q$ is *cofinal* if every element of Q lies below some element of the range of f.

Definition 3.1. For a topological space X, we denote by $\mathring{\mathcal{K}}(X)$ the set of all compact[1] open subsets of X, ordered under set inclusion. We say that X is

- *sober*, if every join-irreducible member, of the lattice of all closed subsets of X, is the closure of a unique singleton[2] ;
- *generalized spectral*, if it is sober, $\mathring{\mathcal{K}}(X)$ is a basis of the topology of X, and $U \cap V$ is compact whenever U and V are compact open subsets of X;
- *spectral*, if it is simultaneously compact and generalized spectral.

The *specialization preorder* on X is defined by

$$x \leqslant y \text{ if } y \in \mathrm{cl}_X(\{x\}) \,, \quad \text{for all } x, y \in X \,.$$

A map $\varphi\colon X \to Y$, between generalized spectral spaces, is *spectral* if the inverse image under φ, of any compact open subset of Y, is a compact open subset of X.

The *spectrum* $\operatorname{Spec} D$, of a distributive lattice D with zero, is defined as the set of all (proper) prime ideals of D, endowed with the closed sets $\{P \in \operatorname{Spec} D \mid I \subseteq P\}$, for subsets (equivalently, ideals) I of D. The specialization order on $\operatorname{Spec} D$ is just set-theoretical inclusion. The assignments $D \mapsto \operatorname{Spec} D$, $X \mapsto \mathring{\mathcal{K}}(X)$ can be naturally extended to morphisms, yielding *Stone duality* (cf. Stone [29], Johnstone [17, § II.3], Grätzer [14, § II.5], and Rump and Yang [25, page 63] for the general, non necessarily unital case) between distributive lattices with zero with cofinal 0-lattice homomorphisms and generalized spectral spaces with spectral maps.

Remark 3.2. Let X and Y be generalized spectral spaces, with X contained (in the set-theoretical sense) in Y, and let $\varphi\colon X \hookrightarrow Y$ be the inclusion map. We say that X is a *spectral subspace* of Y if the topology of X is the topology induced by the topology of Y and the map φ is spectral. In that case, the dual map $\mathring{\mathcal{K}}(\varphi)\colon \mathring{\mathcal{K}}(Y) \to \mathring{\mathcal{K}}(X)$, $V \mapsto X \cap V$ is a surjective lattice homomorphism. Conversely, for every surjective lattice homomorphism $f\colon D \twoheadrightarrow E$, the spectral map $\operatorname{Spec} f\colon \operatorname{Spec} E \to \operatorname{Spec} D$ is a *spectral embedding*, that is, it embeds $\operatorname{Spec} E$ into $\operatorname{Spec} D$ as a spectral subspace. Hence, *spectral subspaces correspond,* via *Stone duality, to surjective lattice homomorphisms.*

The generalized spectral spaces X that we will consider in this paper will mostly be *completely normal.* By Monteiro [22, Théorème V.3.1], this is equivalent to saying that the dual lattice $\mathring{\mathcal{K}}(X)$ is completely normal (cf. Section 2 for the definition of completely normal lattices).

[1]Throughout the paper, "compact" means what some other references call "quasicompact"; in particular, it does not imply Hausdorff.

[2]Due to the uniqueness, every sober space is T_0 (not all references assume this).

We denote by $\mathfrak{P}(X)$ the powerset of any set X, ordered under set inclusion. We denote by ω_α, or $\aleph_\alpha$ according to the context ("ordinal versus cardinal"), the αth infinite cardinal, and we set $\omega \overset{\text{def}}{=} \omega_0 = \{0, 1, 2, \dots\}$.

Throughout the paper, "countable" means "at most countable".

4. Zariski, ℓ, Brumfiel, real: spectra and lattices

In this section we recall some well known facts on the various sorts of spectra and distributive lattices that will intervene in the paper. We also introduce some notation and include a few new results, such as Lemma 4.11. For more details and references, we refer the reader to Delzell and Madden [9], Johnstone [17, Chapter 5], Keimel [19], Coste and Roy [7], Dickmann [10, Chapter 6].

4.1. **Zariski spectrum.** The (*Zariski*) *spectrum* of a commutative unital ring A is defined as the set $\operatorname{Spec} A$ of all prime ideals of A, endowed with the topology whose closed sets are exactly the sets $\operatorname{Spec}(A, I) \overset{\text{def}}{=} \{P \in \operatorname{Spec} A \mid I \subseteq P\}$, for subsets (equivalently, radical ideals) I of A.

Denote by $\langle a_1, \dots, a_m \rangle^{\mathrm{r}}$ the radical ideal of A generated by elements $a_1, \dots, a_m$ of A, and denote[3] by $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A$ the set of all ideals of A of the form $\langle a_1, \dots, a_m \rangle^{\mathrm{r}}$ (*finitely generated radical ideals*), ordered by set inclusion. Due to the formulas

$$\langle a_1, \dots, a_m \rangle^{\mathrm{r}} \vee \langle b_1, \dots, b_n \rangle^{\mathrm{r}} = \langle a_1, \dots, a_m, b_1, \dots, b_n \rangle^{\mathrm{r}}, \tag{4.1}$$

$$\langle a_1, \dots, a_m \rangle^{\mathrm{r}} \cap \langle b_1, \dots, b_n \rangle^{\mathrm{r}} = \langle a_i b_j \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n \rangle^{\mathrm{r}}, \tag{4.2}$$

(where $\vee$ stands for the join in the lattice of all radical ideals of A), $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A$ is a 0-sublattice of the distributive lattice of all radical ideals of A.

Since every radical ideal of A is the intersection of all prime ideals containing it, $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A$ is the Stone dual of $\operatorname{Spec} A$ (cf. Delzell and Madden [9, page 115]):

Proposition 4.1. *The Zariski spectrum* $\operatorname{Spec} A$*, of a commutative unital ring* A*, is a spectral space, and the assignment* $I \mapsto \{P \in \operatorname{Spec} A \mid I \not\subseteq P\}$ *defines an isomorphism from* $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A$ *onto the Stone dual* $\overset{\circ}{\mathcal{K}}(\operatorname{Spec} A)$ *of* $\operatorname{Spec} A$*.*

Due to the following deep result by Hochster [15], there is no need to imagine a new name for the class of all lattices of the form $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A$.

Theorem 4.2 (Hochster)**.** *Every spectral space is homeomorphic to the Zariski spectrum of some commutative unital ring. Hence, every bounded distributive lattice is isomorphic to* $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A$ *for some commutative unital ring* A*.*

4.2. **ℓ-spectrum and ℓ-representable lattices.** The *ℓ-spectrum* of an Abelian ℓ-group G is defined as the set $\operatorname{Spec}_\ell G$ of all prime ℓ-ideals of G, endowed with the topology whose closed sets are exactly the $\{P \in \operatorname{Spec}_\ell G \mid I \subseteq P\}$, for subsets (equivalently, ℓ-ideals) I of G.

Denote by $\langle a_1, \dots, a_m \rangle^{\ell}$, or $\langle a_1, \dots, a_m \rangle^{\ell}_G$ if G needs to be specified, the ℓ-ideal of G generated by elements $a_1, \dots, a_m$ of G, and denote by $\operatorname{Id}^{\ell}_{\mathrm{c}} G$ the set of all ℓ-ideals of G of the form $\langle a_1, \dots, a_m \rangle^{\mathrm{r}}$ (*finitely generated ℓ-ideals*), ordered by

[3] The subscript "c" stands for "compact", which is the lattice-theoretical formalization of "finitely generated". The superscript "r" stands for "radical".

set inclusion. Since $\langle a_1,\dots,a_m\rangle^\ell = \langle a\rangle^\ell$ where $a \overset{\text{def}}{=} \sum_{i=1}^m |a_i|$, we get $\operatorname{Id}_{\mathrm{c}}^\ell G = \left\{\langle a\rangle^\ell \mid a\in G^+\right\}$. Due to the formulas

$$\langle a\rangle^\ell \vee \langle b\rangle^\ell = \langle a+b\rangle^\ell \text{ and } \langle a\rangle^\ell \cap \langle b\rangle^\ell = \langle a\wedge b\rangle^\ell\,, \quad \text{for all } a,b\in G^+ \tag{4.3}$$

(where $\vee$ stands for the join in the lattice of all ℓ-ideals of A), $\operatorname{Id}_{\mathrm{c}}^\ell G$ is a 0-sublattice of the distributive lattice of all ℓ-ideals of G. It has a top element iff G has an order-unit.

Since every ℓ-ideal of G is the intersection of all prime ℓ-ideals containing it, $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} G$ is the Stone dual of $\operatorname{Spec}_\ell G$ (cf. Proposition 1.19, together with Theorem 1.10 and Lemma 1.20, in Keimel [18]):

Proposition 4.3. *The ℓ-spectrum $\operatorname{Spec}_\ell G$, of any Abelian ℓ-group G, is a generalized spectral space, and the assignment $I\mapsto \{P\in\operatorname{Spec} G \mid I\not\subseteq P\}$ defines an isomorphism from $\operatorname{Id}_{\mathrm{c}}^\ell G$ onto the Stone dual $\overset{\circ}{\mathcal{K}}(\operatorname{Spec}_\ell G)$ of $\operatorname{Spec}_\ell G$.*

Following terminology from Iberkleid, Martínez, and McGovern [16] and Wehrung [32], we recall the following definition.

Definition 4.4. For distributive lattices A and B, a map $f\colon A\to B$ is *closed* if for all $a_0,a_1\in A$ and all $b\in B$, if $f(a_0)\le f(a_1)\vee b$, then there exists $x\in A$ such that $a_0\le a_1\vee x$ and $f(x)\le b$.

The following lemma is established in Wehrung [32].

Lemma 4.5. *Let A and B be Abelian ℓ-groups and let $f\colon A\to B$ be an ℓ-homomorphism. Then the map $\operatorname{Id}_{\mathrm{c}}^\ell f\colon \operatorname{Id}_{\mathrm{c}}^\ell A\to \operatorname{Id}_{\mathrm{c}}^\ell B$, $\langle x\rangle^\ell\mapsto\langle f(x)\rangle^\ell$ is a closed 0-lattice homomorphism.*

In particular, the assignments $G\mapsto \operatorname{Id}_{\mathrm{c}}^\ell G$, $f\mapsto \operatorname{Id}_{\mathrm{c}}^\ell f$ define a functor, from the category of all Abelian ℓ-groups with ℓ-homomorphisms, to the category of all distributive lattices with zero with closed 0-lattice homomorphisms. It is well known that this functor preserves nonempty finite direct products and directed colimits.

Say that a lattice D is *ℓ-representable* if it is isomorphic to $\operatorname{Id}_{\mathrm{c}}^\ell G$ for some Abelian ℓ-group G. Equivalently, the spectrum of D is homeomorphic to the ℓ-spectrum of some Abelian ℓ-group. This terminology is extended to diagrams $\vec{D}$ of distributive lattices with zero and 0-lattice homomorphisms, by saying that $\vec{D}\cong \operatorname{Id}_{\mathrm{c}}^\ell \vec{G}$ for some diagram $\vec{G}$ of Abelian ℓ-groups and ℓ-homomorphisms.

It is well known that every ℓ-representable lattice is completely normal. The author established the following result in [32].

Theorem 4.6. *Every countable completely normal distributive lattice with zero is ℓ-representable. On the other hand, the class of all ℓ-representable lattices cannot be defined by a class of $\mathscr{L}_{\infty,\omega}$-formulas of lattice theory.*

The following easy lemma is established in Wehrung [32].

Lemma 4.7. *Let G be an Abelian ℓ-group, let S be a distributive lattice with zero, and let $\varphi\colon \operatorname{Id}_{\mathrm{c}}^\ell G\twoheadrightarrow S$ be a closed surjective 0-lattice homomorphism. Then $I\overset{\text{def}}{=}\left\{x\in G\mid \varphi(\langle x\rangle^\ell)=0\right\}$ is an ℓ-ideal of G, and there is a unique isomorphism $\psi\colon \operatorname{Id}_{\mathrm{c}}^\ell(G/I)\to S$ such that $\psi(\langle x+I\rangle^\ell)=\varphi(\langle x\rangle^\ell)$ for every $x\in G^+$.*

4.3. **Brumfiel spectrum and Brumfiel-representable lattices.** For any commutative f-ring A, we say that a (proper) ℓ-ideal P is *prime* if it is both an ℓ-ideal and prime as a ring ideal. Then P is also prime as an ℓ-ideal of the underlying additive ℓ-group of A, that is, $x \wedge y \in P$ implies that either $x \in P$ or $y \in P$, whenever $x, y \in A$.

The *Brumfiel spectrum* of a commutative f-ring A is defined as the set $\operatorname{Spec_B} A$ of all prime ℓ-ideals of A, endowed with the topology whose closed sets are exactly the $\{P \in \operatorname{Spec_B} A \mid I \subseteq P\}$, for subsets (equivalently, radical ℓ-ideals) I of A.

Denote by $\langle a_1, \dots, a_m \rangle^{\mathrm{r}}$, or $\langle a_1, \dots, a_m \rangle^{\mathrm{r}}_A$ if A needs to be specified, the radical ℓ-ideal of A generated by elements $a_1, \dots, a_m$ of A, and denote by $\operatorname{Id^r_c} A$ the set of all ideals of A of the form $\langle a_1, \dots, a_m \rangle^{\mathrm{r}}$ (*finitely generated radical ideals*),[4] ordered by set inclusion. Since $\langle a_1, \dots, a_m \rangle^{\mathrm{r}} = \langle a \rangle^{\ell}$ where $a \overset{\mathrm{def}}{=} \sum_{i=1}^{m} |a_i|$, we get $\operatorname{Id^r_c} A = \{\langle a \rangle^{\mathrm{r}} \mid a \in A^+\}$. Due to the formulas

$$\langle a \rangle^{\mathrm{r}} \vee \langle b \rangle^{\mathrm{r}} = \langle |a| + |b| \rangle^{\mathrm{r}} \text{ and } \langle a \rangle^{\mathrm{r}} \cap \langle b \rangle^{\mathrm{r}} = \langle |a| \wedge |b| \rangle^{\mathrm{r}} = \langle ab \rangle^{\mathrm{r}}, \quad \text{for all } a, b \in A \tag{4.4}$$

(where $\vee$ stands for the join in the lattice of all radical ℓ-ideals of A), $\operatorname{Id^r_c} A$ is a 0-sublattice of the distributive lattice of all radical ℓ-ideals of A. If A is unital, then $\operatorname{Id^r_c} A$ has a top element.

Since every radical ℓ-ideal of A is the intersection of all prime ℓ-ideals containing it, $\operatorname{Id^r_c} A$ is the Stone dual of $\operatorname{Spec_B} A$ (cf. Delzell and Madden [9, Proposition 4.2]):

Proposition 4.8. *The Brumfiel spectrum* $\operatorname{Spec_B} A$*, of a commutative f-ring* A*, is a generalized spectral space, and the assignment* $I \mapsto \{P \in \operatorname{Spec_B} A \mid I \not\subseteq P\}$ *defines an isomorphism from* $\operatorname{Id^r_c} A$ *onto the Stone dual* $\overset{\circ}{\mathcal{K}}(\operatorname{Spec_B} A)$ *of* $\operatorname{Spec_B} A$*.*

Say that a lattice D is *Brumfiel-representable* if it is isomorphic to $\operatorname{Id^r_c} A$ for some commutative f-ring A. Equivalently, the spectrum of D is homeomorphic to the Brumfiel spectrum of some commutative f-ring. As in Subsection 4.2, this terminology is extended to diagrams of lattices, in a standard fashion.

It is well known that every Brumfiel-representable lattice is completely normal. We will see, with Corollary 6.8 in the present paper, that not every ℓ-representable lattice (thus, *a fortiori*, not every completely normal distributive lattice) is Brumfiel-representable.

The following result is an analogue of Lemma 4.7 for f-rings. Its proof is similar and we omit it.

Lemma 4.9. *Let* A *be a commutative f-ring, let* S *be a distributive lattice with zero, and let* $\varphi\colon \operatorname{Id^r_c} A \twoheadrightarrow S$ *be a closed surjective* 0*-lattice homomorphism. Then* $I \overset{\mathrm{def}}{=} \{x \in A \mid \varphi(\langle x \rangle^{\mathrm{r}}) = 0\}$ *is a radical* ℓ*-ideal of* A*, and there is a unique isomorphism* $\psi\colon \operatorname{Id^r_c}(A/I) \to S$ *such that* $\psi(\langle x + I \rangle^{\mathrm{r}}) = \varphi(\langle x \rangle^{\mathrm{r}})$ *for every* $x \in A^+$*.*

Definition 4.10. Let A and B be distributive lattices with zero. A 0-lattice homomorphism $f\colon A \to B$ is *convex* if for all $P \in \operatorname{Spec} A$ and all $Q_0, Q_1 \in \operatorname{Spec} B$, if $Q_0 \subseteq Q_1$ and $f^{-1}[Q_0] \subseteq P \subseteq f^{-1}[Q_1]$, then there exists $Q \in \operatorname{Spec} B$ such that $Q_0 \subseteq Q \subseteq Q_1$ and $P = f^{-1}[Q]$.

Equivalently, the Stone dual map $\operatorname{Spec} f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is convex in the sense of Dickmann, Schwartz, and Tressl [12, Definition 13.3.10].

[4] Although we are using, for radical ℓ-ideals, the same notation as the one in Subsection 4.1 for radical ideals, the context will always make it clear which concept is used.

The following result extends to the Brumfiel spectrum functor a result originally established for the real spectrum functor in Korollar 4, pages 133–134 of Knebusch and Scheiderer [20]. Our proof is a straightforward modification of its analogue for real spectra, stated in the monograph Dickmann, Schwartz, and Tressl [12, Theorem 13.3.12]. We include it for convenience.

Lemma 4.11. *Let A and B be commutative f-rings and let $f\colon A \to B$ be a homomorphism of f-rings. Then the map $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} f\colon \operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A \to \operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} B$ is convex.*

Proof. By Stone duality, it is sufficient to prove that for all $P \in \operatorname{Spec}_{\mathrm{B}} A$ and all $Q_0, Q_1 \in \operatorname{Spec}_{\mathrm{B}} B$, if $Q_0 \subseteq Q_1$ and $f^{-1}[Q_0] \subseteq P \subseteq f^{-1}[Q_1]$, then there exists $Q \in \operatorname{Spec}_{\mathrm{B}} B$ such that $Q_0 \subseteq Q \subseteq Q_1$ and $P = f^{-1}[Q]$. We may replace A by $A/f^{-1}[Q_0]$, B by B/Q_0, Q_1 by Q_1/Q_0, and f by the canonical embedding $A/f^{-1}[Q_0] \to B/Q_0$. Hence, we may assume that A is an ordered subring of a totally ordered (not necessarily unital) commutative domain B, $Q_0 = \{0\}$, f is the inclusion map from A into B, $P \in \operatorname{Spec}_{\mathrm{B}} A$, $Q_1 \in \operatorname{Spec}_{\mathrm{B}} B$, and $P \subseteq Q_1$. We must find $Q \in \operatorname{Spec}_{\mathrm{B}} B$ such that $P = Q \cap A$ and $Q \subseteq Q_1$. We set

$$Q \stackrel{\mathrm{def}}{=} \bigl\{y \in Q_1 \mid (\exists n \in \omega \setminus \{0\})(\exists x \in P)\bigl(|y|^n \leq x\bigr)\bigr\}\ .$$

We claim that Q is a prime ℓ-ideal of B. It is obvious that Q is an order-convex ℓ-subgroup of B. Now let $y \in Q$ and $b \in B$. We must prove that $yb \in Q$. Since B is totally ordered, we may assume that $\{y, b\} \subseteq B^+$. Since Q_1 is an ideal of B, $yb \in Q_1$. By assumption, there are $n \in \omega \setminus \{0\}$ and $x \in P$ such that $y^n \leq x$. It follows that $y^n \in Q_1$, thus, since Q_1 is a radical ideal of B, $y \in Q_1$, and thus $yb^{n+1} \in Q_1$. Since Q_1 is a proper ℓ-ideal of B, it follows, using Lemma 2.2, that $(yb)^{n+1} = y^n(yb^{n+1}) \leq y^n \leq x$, whence $yb \in Q$. This completes the proof that Q is an ℓ-ideal of B.

Let $x, y \in B$ such that $xy \in Q$, we must prove that $x \in Q$ or $y \in Q$. Since B is totally ordered, we may assume that $0 \leq x \leq y$. There are $n \in \omega \setminus \{0\}$ and $p \in P$ such that $(xy)^n \leq p$. It follows that $x^{2n} \leq (xy)^n \leq p$, whence $x \in Q$, thus completing the proof that Q is prime.

Now it is obvious that $Q \subseteq Q_1$ and $P = Q \cap A$. □

4.4. **Real spectrum and real-representable lattices.** Let A be a commutative unital ring. A subset C of A is a *cone* if it is both an additive and a multiplicative submonoid of A, containing all squares in A. A cone P of A is *prime* if $A = P \cup (-P)$ and the *support* $P \cap (-P)$ is a prime ideal of A. For a prime cone P, $-1 \notin P$ (otherwise $1 \in P \cap (-P)$, thus $P \cap (-P) = A$, a contradiction). We denote by $\operatorname{Spec}_{\mathrm{r}} A$ the set of all prime cones of A, endowed with the topology generated by all subsets of the form $\{P \in \operatorname{Spec}_{\mathrm{r}} A \mid a \notin P\}$, for $a \in A$, and we call $\operatorname{Spec}_{\mathrm{r}} A$ the *real spectrum of* A.

It is not so straightforward to describe directly the Stone dual of $\operatorname{Spec}_{\mathrm{r}} A$. However, it is possible to reduce it to the Brumfiel spectrum, as follows. The *universal f-ring* $\mathrm{F}(A)$ of A is a commutative unital f-ring. The first statement in the following result is established in Delzell and Madden [9, Proposition 3.3]. The second statement follows by using Proposition 4.8.

Theorem 4.12 (Delzell and Madden)**.** *Let A be a commutative unital ring. The canonical homomorphism $A \to \mathrm{F}(A)$ induces a homeomorphism between the real spectrum of A and the Brumfiel spectrum of $\mathrm{F}(A)$. Hence, the Stone dual of $\operatorname{Spec}_{\mathrm{r}} A$ is $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} \mathrm{F}(A)$.*

Say that a lattice is *real-representable* if it is isomorphic to the Stone dual of the real spectrum of some commutative unital ring. As in Subsection 4.2, this terminology is extended to diagrams of lattices, in a standard fashion. It follows from Theorem 4.12 that every real-representable lattice is isomorphic to $\mathrm{Id}_{\mathrm{c}}^{\mathrm{r}} A$ for some commutative unital f-ring A (thus it is Brumfiel-representable). We will see shortly that the converse holds (cf. Corollary 4.17).

Every real-representable lattice is completely normal. By Delzell and Madden [8], not every completely normal bounded distributive lattice can be represented in this way. In fact, Mellor and Tressl established in [21] the following result.

Theorem 4.13 (Mellor and Tressl)**.** *For every infinite cardinal λ, the class of all real-representable lattices cannot be defined by any class of $\mathscr{L}_{\infty,\lambda}$-formulas of lattice theory.*

The real spectrum can also be reduced to the Zariski spectrum, as follows. A commutative unital ring A is *real-closed* (cf. Schwartz [26, 27], Prestel and Schwartz [24]) if it has no nonzero nilpotent elements, the squares in A form the positive cone of a structure of f-ring on A, $0 \leq a \leq b$ implies that $a^2 \in Ab$, and for every prime ideal P of A, the quotient field $A(P)$ of A/P is real closed, and A/P is integrally closed in $A(P)$. Every commutative unital ring has a "real closure" $\mathrm{C_r}(A)$, which is a real-closed ring together with a unital ring homomorphism $A \to \mathrm{C_r}(A)$. The following result is contained in Theorem I.3.10, Propositions I.3.19 and I.3.23, and the top of page 27, in Schwartz [27].

Theorem 4.14 (Schwartz)**.** *For any commutative, unital ring A, the canonical homomorphism $A \to \mathrm{C_r}(A)$ induces a homeomorphism $\mathrm{Spec_r}\,\mathrm{C_r}(A) \to \mathrm{Spec_r}\,A$. Moreover, if A is real-closed, then the support map $P \mapsto P \cap (-P)$ induces a homeomorphism $\mathrm{Spec_r}\,A \to \mathrm{Spec}\,A$.*

Corollary 4.15. *For any commutative, unital ring A, the Stone dual of the real spectrum of A is isomorphic to the lattice $\mathrm{Id}_{\mathrm{c}}^{\mathrm{r}}\,\mathrm{C_r}(A)$ of all finitely generated radical ideals of $\mathrm{C_r}(A)$.*

Although the two following corollaries are probably well known, we could not find them explicitly stated anywhere, so we include proofs for convenience.

Corollary 4.16. *Every closed subspace of a real spectrum is a real spectrum.*

Proof. By Theorem 4.14, every real spectrum has the form $\mathrm{Spec}\,A$ for some real-closed ring A. By definition, any closed subspace of $\mathrm{Spec}\,A$ has the form $\mathrm{Spec}(A, I) \stackrel{\mathrm{def}}{=} \{P \in \mathrm{Spec}\,A \mid I \subseteq P\}$, for a subset I of A, which we may assume to be a radical ideal of A. It follows that the assignment $P \mapsto P/I$ defines a homeomorphism from $\mathrm{Spec}(A, I)$ onto $\mathrm{Spec}(A/I)$. Now it follows from Schwartz [27, Theorem I.4.5] that the ring A/I is real-closed. By the second part of Theorem 4.14, it follows that $\mathrm{Spec}(A/I)$ is homeomorphic to $\mathrm{Spec_r}(A/I)$. □

Corollary 4.17. *The class of real spectra of all commutative unital rings and the class of Brumfiel spectra of all commutative unital f-rings are identical.*

Proof. It follows from Theorem 4.12 that every real spectrum is the Brumfiel spectrum of some commutative unital f-ring. Conversely, for every commutative unital f-ring A, the assignment $P \mapsto A^+ + P$ defines a homeomorphism from $\mathrm{Spec_B}\,A$ onto $\mathrm{Spec_r}(A, A^+) \stackrel{\mathrm{def}}{=} \{Q \in \mathrm{Spec_r}\,A \mid A^+ \subseteq Q\}$, with inverse the support map $Q \mapsto$

$Q \cap (-Q)$ (cf. Madden and Schwartz [28, page 49]). Since $\operatorname{Spec}_{\mathrm{r}}(A, A^+)$ is, by definition, a closed subspace of $\operatorname{Spec}_{\mathrm{r}} A$, it follows form Corollary 4.16 that $\operatorname{Spec}_{\mathrm{B}} A$ is the real spectrum of some commutative unital ring. $\square$

While real-representability makes sense only for bounded lattices, Brumfiel-representability can also be defined for unbounded lattices. According to the following corollary, the two concepts agree on bounded lattices, so there is no ambiguity.

Corollary 4.18. *A bounded distributive lattice is real-representable iff it is Brumfiel-representable.*

Proof. By Stone duality and Corollary 4.17, it suffices to prove that if a *bounded* distributive lattice is Brumfiel-representable, then it can be represented by a commutative *unital* f-ring. Let A be a commutative f-ring such that $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$ has a top element; that is, $A = \langle u \rangle_A^{\mathrm{r}}$ for some $u \in A^+$. The localization $A[u^{-1}]$ of A with respect to the multiplicative subset $S \stackrel{\mathrm{def}}{=} \{u^n \mid 0 < n < \omega\}$, endowed with the positive cone $A[u^{-1}]^+ \stackrel{\mathrm{def}}{=} \{x/u^n \mid x \in A^+\,,\ 0 < n < \omega\}$, is a commutative unital[5] f-ring, for which the canonical homomorphism $\kappa\colon A \to A[u^{-1}]$, $x \mapsto xu/u$ is an f-ring homomorphism. Obviously, the induced map $\boldsymbol{\kappa}\colon \operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A \to \operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A[u^{-1}]$ is surjective. We claim that $\boldsymbol{\kappa}$ is one-to-one. Let $x, y \in A^+$ such that $\boldsymbol{\kappa}\bigl(\langle x \rangle_A^{\mathrm{r}}\bigr) \subseteq \boldsymbol{\kappa}\bigl(\langle y \rangle_A^{\mathrm{r}}\bigr)$. This means that $\langle \kappa(x) \rangle_{A[u^{-1}]}^{\mathrm{r}} \subseteq \langle \kappa(y) \rangle_{A[u^{-1}]}^{\mathrm{r}}$, that is, there are $z \in A^+$ and positive integers m, n such that $\kappa(x)^m \leq (z/u^n)\kappa(y)$. It follows that there are positive integers k, l such that $x^m u^k \leq zyu^l$ within A. Since $\langle u^k \rangle_A^{\mathrm{r}} = \langle u \rangle_A^{\mathrm{r}} = A$, we obtain, using (4.4),

$$\langle x \rangle_A^{\mathrm{r}} = \langle x^m \rangle_A^{\mathrm{r}} = \langle x^m \rangle_A^{\mathrm{r}} \cap \langle u^k \rangle_A^{\mathrm{r}} = \langle x^m u^k \rangle_A^{\mathrm{r}} \subseteq \langle zyu^l \rangle_A^{\mathrm{r}} \subseteq \langle y \rangle_A^{\mathrm{r}}\,,$$

as required. Therefore, $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A \cong \operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A[u^{-1}]$, with $A[u^{-1}]$ a commutative unital f-ring. $\square$

5. Counterexamples constructed from condensates

In the present section we shall apply the construction of a *condensate*, put to use in the author's paper [31] for one arrow and studied in depth in Gillibert and Wehrung [13] for more complicated diagrams. This construction enables us to construct non-representable *objects* from non-representable *arrows* (cf. Theorems 5.4 and 5.9), and it runs as follows.

Definition 5.1. Let A and B be universal algebras (in a given similarity type) and let I be a set. The *I-condensate* of a homomorphism $\varphi\colon A \to B$ is the following subalgebra of $A \times B^I$:

$$\operatorname{Cond}(\varphi, I) \stackrel{\mathrm{def}}{=} \bigl\{(x, y) \in A \times B^I \mid y_i = \varphi(x) \text{ for all but finitely many } i\bigr\}\,.$$

Lemma 5.2. *The condensate* $\operatorname{Cond}(\varphi, I)$ *is a directed union of copies of algebras of the form* $A \times B^J$ *for finite* $J \subseteq I$.

Proof. For each finite $J \subseteq I$, denote by C_J the subalgebra of $\operatorname{Cond}(\varphi, I)$ consisting of all pairs $(x, y) \in A \times B^I$ such that y is constant on $I \setminus J$, with value $\varphi(x)$. Then $\operatorname{Cond}(\varphi, I)$ is the directed union of all C_J. Clearly, $C_J \cong A \times B^J$. $\square$

[5] This also applies to the degenerate case where u is nilpotent, in which case $A[u^{-1}]$ collapses to $\{0\}$.

In particular, it follows from Lemma 5.2 that if A and B are members of a class $\mathcal{C}$ of algebras, closed under nonempty finite direct products and directed colimits, then so is the condensate $\operatorname{Cond}(\varphi, I)$. For example:

- Whenever A and B are distributive lattices with zero and φ is a 0-lattice homomorphism, the structure $\operatorname{Cond}(\varphi, I)$ is a distributive lattice with zero.
- Whenever A and B are Abelian ℓ-groups and φ is an ℓ-homomorphism, the structure $\operatorname{Cond}(\varphi, I)$ is an Abelian ℓ-group.
- Whenever A and B are real-closed rings and φ is a unital ring homomorphism, the structure $\operatorname{Cond}(\varphi, I)$ is a real-closed ring.

The proof of the following lemma is an extension of the one of Wehrung [31, Theorem 9.3]. It is also an instance of a much more general result, called the *Condensate Lifting Lemma* (*CLL*), established in Gillibert and Wehrung [13], that enables to infer representability of lattice-indexed diagrams from representability of certain larger objects, called there (for the first time) condensates. In particular, as we will see shortly, Lemma 5.3 can be extended to (many) other functors than $\operatorname{Id}_{\mathrm{c}}^{\ell}$. The use of CLL requires quite an amount of technical steps and although it may be unavoidable for complicated diagrams, the case of one arrow can be handled directly; thus we include here a self-contained proof for convenience.

Lemma 5.3. *Let A and B be distributive lattices with zero, with A countable, and let $\varphi\colon A \to B$ be a 0-lattice homomorphism. Then for any set I, if the lattice homomorphism φ is ℓ-representable, then the distributive lattice $\operatorname{Cond}(\varphi, I)$ is ℓ-representable. If I is uncountable, then the converse holds.*

Proof. Suppose first that φ is ℓ-representable. This means that there are Abelian ℓ-groups G and H, together with an ℓ-homomorphism $f\colon G \to H$, such that $\operatorname{Id}_{\mathrm{c}}^{\ell} f \cong \varphi$ (as arrows). Since the functor $\operatorname{Id}_{\mathrm{c}}^{\ell}$ commutes with nonempty finite direct products and with directed colimits, it sends the representation

$$\operatorname{Cond}(f, I) = \varinjlim_{J \subset I \text{ finite}} (G \times H^J)\,,$$

given by Lemma 5.2, to a representation

$$\operatorname{Id}_{\mathrm{c}}^{\ell} \operatorname{Cond}(f, I) \cong \varinjlim_{J \subset I \text{ finite}} (A \times B^J) \cong \operatorname{Cond}(\varphi, I)\,.$$

Hence, $\operatorname{Cond}(\varphi, I)$ is ℓ-representable.

Suppose, conversely, that I is uncountable and the distributive lattice $E \stackrel{\text{def}}{=} \operatorname{Cond}(\varphi, I)$ is ℓ-representable. This means that there are an Abelian ℓ-group G and an isomorphism $\varepsilon\colon \operatorname{Id}_{\mathrm{c}}^{\ell} G \to E$. Denote by $p\colon E \twoheadrightarrow A$ and $q_i\colon E \twoheadrightarrow B$, for $i \in I$, the canonical projections from E. The set $U \stackrel{\text{def}}{=} \left\{x \in G \mid (p \circ \varepsilon)(\langle x \rangle_G^{\ell}) = 0\right\}$ is an ℓ-ideal of G; denote by $u\colon G \twoheadrightarrow G/U$ the canonical projection. Since p is easily seen to be a closed map (cf. Definition 4.4), it follows from Lemma 4.7 that there is a unique isomorphism $\alpha\colon \operatorname{Id}_{\mathrm{c}}^{\ell}(G/U) \to A$ such that

$$p \circ \varepsilon = \alpha \circ (\operatorname{Id}_{\mathrm{c}}^{\ell} u)\,. \tag{5.1}$$

Likewise, for every $i \in I$, the set $V_i \stackrel{\text{def}}{=} \left\{x \in G \mid (q_i \circ \varepsilon)(\langle x \rangle_G^{\ell}) = 0\right\}$ is an ℓ-ideal of G; denote by $v_i\colon G \twoheadrightarrow G/V_i$ the canonical projection. As above for U and α, there is a unique isomorphism $\beta_i\colon \operatorname{Id}_{\mathrm{c}}^{\ell}(G/V_i) \to B$ such that

$$q_i \circ \varepsilon = \beta_i \circ (\operatorname{Id}_{\mathrm{c}}^{\ell} v_i)\,. \tag{5.2}$$

Since A is countable, there exists a countable ℓ-subgroup H of G such that α induces an isomorphism $\alpha'\colon \operatorname{Id}_{\mathrm{c}}^{\ell}(H/U) \to A$ (where we set $H/U \overset{\text{def}}{=} \{x+U \mid x \in H\}$). We are going to argue that for every index i outside a certain countable subset of I, the ℓ-homomorphism $f\colon H/U \to G/V_i$, $x+U \mapsto x+V_i$ is well-defined and represents the lattice homomorphism $\varphi\colon A \to B$. Our argument is partly illustrated in Figure 5.1. The diagram of Figure 5.1 is not commutative, because $\varphi \circ p \neq q_i$ as a rule.

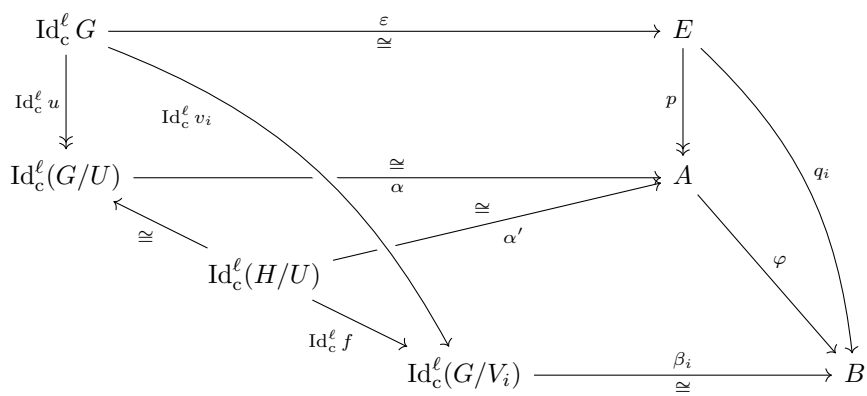

FIGURE 5.1. Illustrating the proof of Lemma 5.3

For each $x \in H$, the element $\varepsilon(\langle x\rangle_G^{\ell})$ belongs to E, that is, $(\varphi \circ p)\bigl(\varepsilon(\langle x\rangle_G^{\ell})\bigr) = q_i\bigl(\varepsilon(\langle x\rangle_G^{\ell})\bigr)$ for all but finitely many i. Since H is countable, there exists a countable subset J of I such that

$$(\varphi \circ p)\bigl(\varepsilon(\langle x\rangle_G^{\ell})\bigr) = q_i\bigl(\varepsilon(\langle x\rangle_G^{\ell})\bigr) \text{ for all } x \in H \text{ and all } i \in I \setminus J\,. \tag{5.3}$$

Pick $i \in I \setminus J$. We claim that $H \cap U \subseteq V_i$. Every $x \in H \cap U$ satisfies the equations

$$(\operatorname{Id}_{\mathrm{c}}^{\ell} u)\bigl(\langle x\rangle_G^{\ell}\bigr) = \langle x+U\rangle_{G/U}^{\ell} = 0\,,$$

thus, using (5.1), $p\bigl(\varepsilon(\langle x\rangle_G^{\ell})\bigr) = (\alpha \circ \operatorname{Id}_{\mathrm{c}}^{\ell} u)\bigl(\langle x\rangle_G^{\ell}\bigr) = 0$. By (5.3), it follows that $q_i\bigl(\varepsilon(\langle x\rangle_G^{\ell})\bigr) = 0$. By (5.2), this means that $\beta_i\bigl(\langle x+V_i\rangle_{G/V_i}^{\ell}\bigr) = 0$, that is, since β_i is an isomorphism, $x \in V_i$, thus completing the proof of our claim.

It follows that there exists a unique ℓ-homomorphism $f\colon H/U \to G/V_i$ such that $f(x+U) = x+V_i$ for every $x \in H$. For every $x \in H$,

$$\begin{aligned}
(\varphi \circ \alpha')\bigl(\langle x+U\rangle_{H/U}^{\ell}\bigr) &= (\varphi \circ \alpha)\bigl(\langle x+U\rangle_{G/U}^{\ell}\bigr) \\
&= (\varphi \circ \alpha \circ \operatorname{Id}_{\mathrm{c}}^{\ell} u)\bigl(\langle x\rangle_G^{\ell}\bigr) \\
&= (\varphi \circ p \circ \varepsilon)\bigl(\langle x\rangle_G^{\ell}\bigr) && \text{(use (5.1))} \\
&= (q_i \circ \varepsilon)\bigl(\langle x\rangle_G^{\ell}\bigr) && \text{(use (5.3))} \\
&= (\beta_i \circ \operatorname{Id}_{\mathrm{c}}^{\ell} v_i)\bigl(\langle x\rangle_G^{\ell}\bigr) && \text{(use (5.2))} \\
&= \beta_i(\langle x+V_i\rangle_{G/V_i}^{\ell}) \\
&= (\beta_i \circ \operatorname{Id}_{\mathrm{c}}^{\ell} f)\bigl(\langle x+U\rangle_{H/U}^{\ell}\bigr)\,,
\end{aligned}$$

so $\varphi \circ \alpha' = \beta_i \circ \operatorname{Id}_{\mathrm{c}}^{\ell} f$. Therefore, f represents φ. $\square$

Theorem 5.4. *There exists a real-closed ring, of cardinality $\aleph_1$, whose real spectrum is not homeomorphic to the ℓ-spectrum of any Abelian ℓ-group.*

Proof. Let K be any countable, non-Archimedean real-closed field. The subset

$$A \stackrel{\text{def}}{=} \{x \in K \mid (\exists n < \omega)(-n \cdot 1_K \leq x \leq n \cdot 1_K)\}$$

is an order-convex unital subring of K, thus it is a real-closed ring. Hence, denoting by $\varepsilon\colon A \hookrightarrow K$ the inclusion map, it follows from Lemma 5.2 that the condensate $R \stackrel{\text{def}}{=} \operatorname{Cond}(\varepsilon, \omega_1)$ is a real-closed ring. Observe that the cardinality of R is $\aleph_1$.

Suppose that the real spectrum of R is homeomorphic to the ℓ-spectrum of an Abelian ℓ-group G. By Stone duality, it follows that $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} R \cong \operatorname{Id}_{\mathrm{c}}^{\ell} G$. Since the functor $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}}$ commutes with nonempty finite direct products and directed colimits, it follows from Lemma 5.2 that $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} R \cong \operatorname{Cond}(\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} \varepsilon, \omega_1)$. In particular, the distributive lattice $\operatorname{Cond}(\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} \varepsilon, \omega_1)$ is ℓ-representable. By Lemma 5.3, it follows that the lattice homomorphism $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} \varepsilon$ is ℓ-representable. By Lemma 4.5, it follows that the map $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} \operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} \varepsilon$ is closed.

However, $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$ is a chain with more than two elements, $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} K$ is the two-element chain, and $\boldsymbol{\varepsilon}$ is the unique zero-separating map $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A \to \operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} K$. In particular, if $0 < u < 1$ in $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$, then $\boldsymbol{\varepsilon}(1) = \boldsymbol{\varepsilon}(u) \vee 0$ but there is no $x \in \operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$ such that $1 \leq u \vee x$ and $\boldsymbol{\varepsilon}(x) \leq 0$. Hence, $\boldsymbol{\varepsilon}$ is not closed. $\square$

Remark 5.5. By (the proof of) Dickmann, Gluschankof, and Lucas [11, Proposition 1.1], the field K and the ring A, of the proof of Theorem 5.4, can be constructed in such a way that $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$ is the three-element chain. By a simple Löwenheim-Skolem type argument, K may be taken countable. Then the lattice $\operatorname{Cond}(\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} \varepsilon, \omega_1)$ is isomorphic to the lattice $\boldsymbol{D}_{\omega_1}$ introduced in Wehrung [32]. As observed in the final example of Wehrung [32], $\boldsymbol{D}_{\omega_1}$ is a homomorphic image of an ℓ-representable distributive lattice, without being itself ℓ-representable. By Stone duality, this means that the spectrum of $\boldsymbol{D}_{\omega_1}$ can be embedded, as a spectral subspace, into the ℓ-spectrum of an Abelian ℓ-group, without being itself an ℓ-spectrum.

The negative property satisfied by the counterexample R of Theorem 5.4 cannot be strengthened further by replacing "ℓ-spectrum" by "spectral subspace of an ℓ-spectrum". The reason for this is the following easy observation, which ought to be well known but for which we could not locate a reference.

Proposition 5.6. *For every commutative f-ring A, there are an Abelian ℓ-group G, which can be taken with order-unit if A is unital, together with a surjective lattice homomorphism $\mu\colon \operatorname{Id}_{\mathrm{c}}^{\ell} G \twoheadrightarrow \operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$. Hence, every Brumfiel-representable lattice is a homomorphic image of some ℓ-representable lattice.*

Proof. Denote by G the underlying additive ℓ-group of A. It is easy to verify that the map $\operatorname{Id}_{\mathrm{c}}^{\ell} G \twoheadrightarrow \operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$, $\langle x \rangle^{\ell} \mapsto \langle x \rangle^{\mathrm{r}}$ is a well defined lattice homomorphism (use (4.3) and (4.4)). It is, of course, surjective.

Now suppose that A is unital. Since the multiplicative unit 1 of A may not be an order-unit of A, the construction of G above must be modified. To this end, define G as the underlying ℓ-group of $\{x \in A \mid (\exists n < \omega)(-n \cdot 1 \leq x \leq n \cdot 1)\}$, and define again $\mu\colon \langle x \rangle^{\ell} \mapsto \langle x \rangle^{\mathrm{r}}$. We need to prove that μ is surjective. For every $x \in A^+$, the relations $x = (x \vee 1) \cdot (x \wedge 1)$ and $0 \leq x \wedge 1 \leq x$ imply that x and $x \wedge 1$ generate the same ℓ-ideal, and thus, *a fortiori*, $\langle x \rangle^{\mathrm{r}} = \langle x \wedge 1 \rangle^{\mathrm{r}} = \mu(\langle x \wedge 1 \rangle^{\ell})$. $\square$

By applying Stone duality to Proposition 5.6, we thus get

Corollary 5.7. *The Brumfiel spectrum of any commutative unital f-ring* (*thus also every real spectrum*) *is a spectral subspace of the ℓ-spectrum of some Abelian ℓ-group with order-unit.*

Moving to f-rings, a *mutatis mutandis* modification of the proof of Lemma 5.3, using Lemma 4.9 instead of Lemma 4.7, leads to the following result.

Lemma 5.8. *Let A and B be distributive lattices with zero, with A countable, let $\varphi\colon A \to B$ be a 0-lattice homomorphism, and let I be a set. If φ is Brumfiel-representable, then the distributive lattice* $\mathrm{Cond}(\varphi, I)$ *is Brumfiel-representable. If I is uncountable, then the converse holds.*

Theorem 5.9. *There exists a real-representable distributive lattice E, of cardinality $\aleph_1$, with a non-Brumfiel-representable* (*thus non-real-representable*) *homomorphic image.*

Proof. It follows from Dickmann, Gluschankof, and Lucas [11, Proposition 1.1] that there exists a real-closed domain A with exactly three prime ideals $\{0\} \subset P_1 \subset P_2$. Hence, $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\, A$ is isomorphic to the four-element chain $\mathbf{4} \stackrel{\mathrm{def}}{=} \{0, 1, 2, 3\}$. By an easy Löwenheim-Skolem type argument, A may be taken countable. Denote by R the ring of all almost constant families $(x_\xi \mid \xi < \omega_1)$ of elements of A. Then the lattice E of all almost constant ω_1-sequences of elements in $\mathbf{4}$ is isomorphic to $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\, R$, thus it is real-representable.

Now consider the chain $\mathbf{3} \stackrel{\mathrm{def}}{=} \{0, 1, 2\}$. The map $\{1, 2\} \to \{1, 2, 3\}$, sending 1 to 1 and 2 to 3, is not convex in the sense of [12, Definition 13.3.10], which means that its Stone dual $\varphi\colon \mathbf{4} \twoheadrightarrow \mathbf{3}$ is not convex in the sense of Definition 4.10. By Lemma 4.11, φ is not Brumfiel-representable. By Lemma 5.8, it follows that the lattice $\mathrm{Cond}(\varphi, \omega_1)$ is not Brumfiel-representable.

On the other hand, the assignment $(x_\xi \mid \xi < \omega_1) \mapsto \bigl(x_\infty, (\varphi(x_\xi) \mid \xi < \omega_1)\bigr)$ defines a surjective lattice homomorphism from E onto $\mathrm{Cond}(\varphi, \omega_1)$. $\square$

By Stone duality, it follows that the spectrum of the bounded distributive lattice $\mathrm{Cond}(\varphi, \omega_1)$ witnesses the following corollary.

Corollary 5.10. *There exists a real spectrum with a spectral subspace which is not a Brumfiel spectrum* (*thus also not a real spectrum*).

6. An ℓ-representable, non-Brumfiel representable lattice

Although the proof of the present section's main negative result, Theorem 6.4, arises from a lattice-theoretical investigation of the argument of Delzell and Madden [8, Lemma 2], the construction of its counterexample, which is the ℓ-group that we will denote by $G_{\omega_1^{\mathrm{op}}}$, is somehow simpler. Moreover, the constructions of Delzell and Madden [8], Mellor and Tressl [21] yield lattices with $2^{\aleph_1}$ elements *a priori*, while our construction yields the smaller size $\aleph_1$.

Notation 6.1. We denote by F the free Abelian ℓ-group defined by generators a and b, subjected to the relations $a \geq 0$ and $b \geq 0$. Moreover, we set $G_\Lambda \stackrel{\mathrm{def}}{=} \mathbb{Z}\langle\Lambda\rangle \times_{\mathrm{lex}} F$ (lexicographical product), for any chain Λ.

Since $\mathbb{Z}\langle\Lambda\rangle$ is a totally ordered group and F is an Abelian ℓ-group, G_Λ is also an Abelian ℓ-group. It has an order-unit iff Λ has a largest element. We will occasionally identify F with the ℓ-ideal $\{0\}\times F$ of G_Λ.

Lemma 6.2. *The ℓ-ideal $\langle a+b\rangle^\ell$ is directly indecomposable in the lattice $\operatorname{Id}^\ell_{\mathrm{c}} G_\Lambda$. That is, there are no nonzero $\boldsymbol{x},\boldsymbol{y}\in\operatorname{Id}^\ell_{\mathrm{c}} G_\Lambda$ such that $\langle a+b\rangle^\ell=\boldsymbol{x}\vee\boldsymbol{y}$ and $\boldsymbol{x}\cap\boldsymbol{y}=\{0\}$.*

Proof. Since $a,b\in F$ and F is an ideal of G_Λ, it suffices to prove that $\langle a+b\rangle^\ell$ is directly indecomposable in the lattice $\operatorname{Id}^\ell_{\mathrm{c}} F$. The right closed upper quadrant

$$\Omega\overset{\mathrm{def}}{=}\bigl\{(x,y)\in\mathbb{R}^2\mid x\geq 0\text{ and }y\geq 0\bigr\}$$

is a convex subset of $\mathbb{R}^2$. Further, by the Baker-Beynon duality (cf. Baker [3], Beynon [4, 5]), F is isomorphic to the ℓ-subgroup of $\mathbb{R}^\Omega$ generated by the canonical projections $a\colon(x,y)\mapsto x$ and $b\colon(x,y)\mapsto y$, and there exists a unique lattice embedding ι from $\operatorname{Id}^\ell_{\mathrm{c}} F$ to the lattice of all relative open subsets of Ω such that $\iota(\langle x\rangle^\ell)=[\![x\neq 0]\!]$ whenever $x\in F$. Hence, in order to prove that $\langle a+b\rangle^\ell$ is indecomposable in $\operatorname{Id}^\ell_{\mathrm{c}} F$, it suffices to prove that $[\![a+b\neq 0]\!]$ is a connected subset of $\mathbb{R}^2$. This, in turn, follows from the relation $[\![a+b\neq 0]\!]=\{(x,y)\in\Omega\mid x+y>0\}$, which implies that $[\![a+b\neq 0]\!]$ is a convex, thus connected, subset of $\mathbb{R}^2$. □

Lemma 6.3. *Let Λ be a chain. Then every pairwise orthogonal subset of G^{++}_Λ is countable.*

Proof. Since $\mathbb{Z}\langle\Lambda\rangle$ is a chain, every pairwise orthogonal subset X of G^{++}_Λ, with more than one element, is a subset of F. The latter being countable, X is countable. □

Theorem 6.4. *There are no commutative f-ring A and no surjective lattice homomorphism $\mu\colon\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A\twoheadrightarrow\operatorname{Id}^\ell_{\mathrm{c}} G_{\omega_1^{\mathrm{op}}}$.*

Until the end of the proof of Theorem 6.4, we shall assume, by way of contradiction, that there are a commutative f-ring A and a surjective lattice homomorphism $\mu\colon\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A\twoheadrightarrow\operatorname{Id}^\ell_{\mathrm{c}} G_{\omega_1^{\mathrm{op}}}$.

Pick $x,y\in A^+$ such that $\langle a\rangle^\ell=\mu\langle x\rangle^{\mathrm{r}}$ and $\langle b\rangle^\ell=\mu\langle y\rangle^{\mathrm{r}}$. Moreover, for each $\xi<\omega_1$, pick $z_\xi\in A^+$ such that

$$\langle c_\xi\rangle^\ell=\mu\langle z_\xi\rangle^{\mathrm{r}}\,. \tag{6.1}$$

In particular,

$$\langle a+b\rangle^\ell\subseteq\mu\langle z_\xi\rangle^{\mathrm{r}}\,,\quad\text{whenever }\xi<\omega_1\,. \tag{6.2}$$

Let $\xi<\eta<\omega_1$. Since $z_\xi\leq z_\eta+(z_\xi\smallsetminus z_\eta)$, with z_ξ, z_η, and $z_\xi\smallsetminus z_\eta$ all in A^+, we get

$$\langle z_\xi\rangle^{\mathrm{r}}\subseteq\langle z_\eta\rangle^{\mathrm{r}}\vee\langle z_\xi\smallsetminus z_\eta\rangle^{\mathrm{r}}\,,$$

whence, applying the lattice homomorphism μ and by (6.1),

$$c_\xi\in\langle c_\eta\rangle^\ell\vee\mu\langle z_\xi\smallsetminus z_\eta\rangle^{\mathrm{r}}\,.$$

Since $c_\eta\ll c_\xi$ (within $G_{\omega_1^{\mathrm{op}}}$), it follows that

$$c_\xi\in\mu\langle z_\xi\smallsetminus z_\eta\rangle^{\mathrm{r}}\,.$$

Hence we obtain, *a fortiori*, that

$$\langle a+b\rangle^\ell\subseteq\mu\langle z_\xi\smallsetminus z_\eta\rangle^{\mathrm{r}}\,,\quad\text{whenever }\xi<\eta<\omega_1\,. \tag{6.3}$$

Lemma 6.5. *For all $\xi \leq \eta < \omega_1$, the relation $\langle a+b\rangle^{\ell} \cap \mu\langle z_\eta \smallsetminus z_\xi\rangle^{\mathrm{r}} = \{0\}$ holds.*

Proof. The conclusion is trivial if $\xi = \eta$. Now suppose that $\xi < \eta$. We compute:

$$\begin{aligned}
\langle a+b\rangle^{\ell} \cap \mu\langle z_\eta \smallsetminus z_\xi\rangle^{\mathrm{r}} &\subseteq \mu\langle z_\xi \smallsetminus z_\eta\rangle^{\mathrm{r}} \cap \mu\langle z_\eta \smallsetminus z_\xi\rangle^{\mathrm{r}} && \text{(use (6.3))}\\
&= \mu\big(\langle z_\xi \smallsetminus z_\eta\rangle^{\mathrm{r}} \cap \langle z_\eta \smallsetminus z_\xi\rangle^{\mathrm{r}}\big)\\
&= \mu(\{0\}) && \text{(use (4.4) and Lemma 2.1)}\\
&= \{0\}\ .
\end{aligned}$$

□

For each $\xi < \omega_1$, we set

$$\begin{aligned}
x_\xi &\stackrel{\text{def}}{=} (x+y) \wedge \big(xz_0 \smallsetminus (x+y)z_{\xi+1}\big)\,,\\
y_\xi &\stackrel{\text{def}}{=} (x+y) \wedge \big((x+y)z_\xi \smallsetminus xz_0\big)\,,\\
\boldsymbol{x}_\xi &\stackrel{\text{def}}{=} \mu\langle x_\xi\rangle^{\mathrm{r}}\,,\\
\boldsymbol{y}_\xi &\stackrel{\text{def}}{=} \mu\langle y_\xi\rangle^{\mathrm{r}}\,.
\end{aligned}$$

Lemma 6.6. *The following relations hold, whenever $\xi < \eta < \omega_1$:*

(1) $\boldsymbol{x}_\xi \subseteq \langle a\rangle^{\ell}$;
(2) $\boldsymbol{y}_\xi \subseteq \langle b\rangle^{\ell}$;
(3) $\langle a+b\rangle^{\ell} = \boldsymbol{x}_\xi \vee \boldsymbol{y}_\xi$;
(4) $\boldsymbol{x}_\xi \cap \boldsymbol{y}_\eta = \{0\}$.

Proof. Ad (1). From $x+y \geq 0$ and $z_\xi \geq 0$ it follows that $0 \leq x_\xi \leq xz_0$, whence $\langle x_\xi\rangle^{\mathrm{r}} \subseteq \langle xz_0\rangle^{\mathrm{r}} \subseteq \langle x\rangle^{\mathrm{r}}$. Apply the homomorphism μ.

Ad (2). From (6.3) it follows that $\langle a+b\rangle \subseteq \mu\langle z_0 \smallsetminus z_\xi\rangle^{\mathrm{r}}$. Hence,

$$\langle a+b\rangle \cap \mu\langle xz_\xi \smallsetminus xz_0\rangle^{\mathrm{r}} \subseteq \langle a+b\rangle \cap \mu\langle z_\xi \smallsetminus z_0\rangle^{\mathrm{r}} \tag{6.4}$$

$$= \{0\} \qquad \text{by Lemma 6.5}\,. \tag{6.5}$$

Using Lemma 2.1, we get $(x+y)z_\xi \smallsetminus xz_0 \leq yz_\xi + (xz_\xi \smallsetminus xz_0)$, thus

$$y_\xi \leq (x+y) \wedge \big(yz_\xi + (xz_\xi \smallsetminus xz_0)\big)\,,$$

and thus, applying the homomorphism μ together with (6.4)–(6.5),

$$\boldsymbol{y}_\xi = \mu\langle y_\xi\rangle^{\mathrm{r}} \subseteq \mu\langle (x+y) \wedge yz_\xi\rangle^{\mathrm{r}} \vee \Big(\langle a+b\rangle^{\ell} \cap \mu\langle xz_\xi \smallsetminus xz_0\rangle^{\mathrm{r}}\Big) \subseteq \mu\langle y\rangle^{\mathrm{r}} = \langle b\rangle^{\ell}\,.$$

Ad (3). We compute:

$$\begin{aligned}
\langle a+b\rangle^{\ell} &= \mu\big(\langle x+y\rangle^{\mathrm{r}} \cap \langle z_\xi \smallsetminus z_{\xi+1}\rangle^{\mathrm{r}}\big) && \text{(apply (6.3))}\\
&= \mu\langle (x+y)(z_\xi \smallsetminus z_{\xi+1})\rangle^{\mathrm{r}} && \text{(use (4.4))}\\
&= \mu\langle (x+y)z_\xi \smallsetminus (x+y)z_{\xi+1}\rangle^{\mathrm{r}} && \text{(use Lemma 2.1)}\\
&\subseteq \mu\langle (x+y)z_\xi \smallsetminus xz_0\rangle^{\mathrm{r}} \vee \mu\langle xz_0 \smallsetminus (x+y)z_{\xi+1}\rangle^{\mathrm{r}} && \text{(use Lemma 2.1)}\,.
\end{aligned}$$

Since $\langle a+b\rangle^{\ell} = \mu\langle x+y\rangle^{\mathrm{r}}$, the desired conclusion follows from the distributivity of the lattice $\mathrm{Id}^{\ell}_{\mathrm{c}}\, G_{\omega_1^{\mathrm{op}}}$.

Ad (4). It follows from Lemma 6.5 that $\langle a+b\rangle \cap \mu\langle z_\eta \smallsetminus z_{\xi+1}\rangle^{\mathrm{r}} = \{0\}$ and hence, *a fortiori*, that

$$\langle a+b\rangle \cap \mu\langle (x+y)z_\eta \smallsetminus (x+y)z_{\xi+1}\rangle^{\mathrm{r}} = \{0\}\ . \tag{6.6}$$

Hence, using Lemma 2.1,

$$\begin{aligned}\boldsymbol{y}_\eta &= \langle a+b\rangle \cap \mu\langle (x+y)z_\eta \smallsetminus xz_0\rangle^{\mathrm{r}}\\ &\subseteq \langle a+b\rangle \cap \Bigl(\mu\langle (x+y)z_\eta \smallsetminus (x+y)z_{\xi+1}\rangle^{\mathrm{r}} \vee \mu\langle (x+y)z_{\xi+1} \smallsetminus xz_0\rangle^{\mathrm{r}}\Bigr)\\ &= \langle a+b\rangle \cap \mu\langle (x+y)z_{\xi+1} \smallsetminus xz_0\rangle^{\mathrm{r}} \qquad \text{(use (6.6))}\,.\end{aligned}$$

It follows that

$$\begin{aligned}\boldsymbol{x}_\xi \cap \boldsymbol{y}_\eta &\subseteq \mu\langle xz_0 \smallsetminus (x+y)z_{\xi+1}\rangle^{\mathrm{r}} \cap \mu\langle (x+y)z_{\xi+1} \smallsetminus xz_0\rangle^{\mathrm{r}}\\ &= \mu(\{0\}) \qquad \text{(use (4.4) and Lemma 2.1)}\\ &= \{0\}\,.\end{aligned}$$

□

Set $u_\xi \overset{\mathrm{def}}{=} x_\xi \wedge y_\xi$ and $\boldsymbol{u}_\xi \overset{\mathrm{def}}{=} \mu\langle u_\xi\rangle^{\mathrm{r}}$, for all $\xi < \omega_1$.

Lemma 6.7. *The principal ℓ-ideals $\boldsymbol{u}_\xi$ of $G_{\omega_1^{\mathrm{op}}}$, for $\xi < \omega_1$, are all nonzero. Moreover, they are pairwise orthogonal.*

Proof. The statement of pairwise orthogonality follows from Lemma 6.6(4). If $\boldsymbol{u}_\xi = \{0\}$, then $\boldsymbol{x}_\xi \cap \boldsymbol{y}_\xi = \{0\}$, thus, by Lemmas 6.2 and 6.6(3), either $\boldsymbol{x}_\xi = \{0\}$ or $\boldsymbol{y}_\xi = \{0\}$, thus either $\boldsymbol{x}_\xi = \langle a+b\rangle^\ell$ or $\boldsymbol{y}_\xi = \langle a+b\rangle^\ell$, and thus, by items (1) and (2) of Lemma 6.6, either $a+b \in \langle a\rangle^\ell$ or $a+b \in \langle b\rangle^\ell$, a contradiction. □

End of the proof of Theorem 6.4. By Lemma 6.7, $\mathrm{Id}_{\mathrm{c}}^\ell\, G_{\omega_1^{\mathrm{op}}}$ has an uncountable, pairwise orthogonal set of nonzero ideals. Picking positive generators of the $\boldsymbol{u}_\xi$, we get an uncountable, pairwise orthogonal set of nonzero elements in $G_{\omega_1^{\mathrm{op}}}$, in contradiction with Lemma 6.3. □

As mentioned before, the real spectrum of any commutative unital ring A is homeomorphic to the Brumfiel spectrum of the universal f-ring of A (cf. Theorem 4.12). By Stone duality, we obtain the following corollary.

Corollary 6.8. *The ℓ-spectrum of the unital Abelian ℓ-group $G_{\omega_1^{\mathrm{op}}}$ cannot be embedded, as a spectral subspace, into the Brumfiel spectrum of any commutative f-ring. Hence, it also cannot be embedded, as a spectral subspace, into the real spectrum of any commutative unital ring.*

7. Omitting homomorphic images of ℓ-representable lattices

Although it is well known, since Delzell and Madden [8], that there are non-ℓ-representable completely normal bounded distributive lattices, the corresponding result for *homomorphic images* of ℓ-representable lattices was not known until June 2017, where we posted online the first version of the present paper. In that preprint, we solved the problem negatively, by constructing a counterexample: namely, a completely normal bounded distributive lattice D, of cardinality $\aleph_2$, which is not a homomorphic image of any ℓ-representable lattice. That structure D was defined there as the lattice reduct of the free object on $\aleph_2$ generators, within the variety of all bounded distributive lattices (in the language $(\vee, \wedge, 0, 1)$) with an additional binary operation $\smallsetminus$ subjected to the identities

$$(\mathsf{x} \wedge \mathsf{y}) \vee (\mathsf{x} \smallsetminus \mathsf{y}) = \mathsf{x}\,; \quad (\mathsf{x} \smallsetminus \mathsf{y}) \wedge (\mathsf{y} \smallsetminus \mathsf{x}) = 0\,. \tag{7.1}$$

The method used there, in particular the part involving infinite combinatorics (more specifically, Kuratowski's Free Set Theorem), originates in the author's paper [30]

(living in the world of the nonstable K-theory of von Neumann regular rings). The algebraic core of that construction was contained in the commutative diagram $\vec{D}$, of bounded distributive lattices, represented in Figure 7.1, where $\mathbf{2} \overset{\text{def}}{=} \{0,1\}$, $\mathbf{3} \overset{\text{def}}{=} \{0,1,2\}$, $\overline{0} = 0$, $\overline{1} = \overline{2} = 2$, $\boldsymbol{e}(0) = 0$, $\boldsymbol{e}(1) = 2$, $\boldsymbol{r}(0) = 0$, $\boldsymbol{r}(1) = \boldsymbol{r}(2) = 1$, and for all $x, y \in \mathbf{3}$,

$$\boldsymbol{f}(x) = (\overline{x}, x) \quad \text{and} \quad \boldsymbol{g}(x) = (x, \overline{x})\,,$$

$$\begin{aligned}\boldsymbol{a}(x,y) &= (\overline{x}, x, y, \boldsymbol{r}(y))\,,\\ \boldsymbol{b}(x,y) &= (x, \overline{x}, y, \boldsymbol{r}(x))\,,\\ \boldsymbol{c}(x,y) &= (x, y, \overline{y}, \boldsymbol{r}(y))\,.\end{aligned}$$

$$\begin{tikzcd}
 & & \mathbf{3}^3 \times \mathbf{2} & & \\
\mathbf{3}^2 \arrow[urr, "\boldsymbol{a}"] & & \mathbf{3}^2 \arrow[u, "\boldsymbol{b}"] & & \mathbf{3}^2 \arrow[ull, "\boldsymbol{c}"'] \\
\mathbf{3} \arrow[u, "\boldsymbol{f}"] \arrow[urr, "\boldsymbol{g}"] & & \mathbf{3} \arrow[ul, "\boldsymbol{f}"] \arrow[ur, "\boldsymbol{g}"'] & & \mathbf{3} \arrow[u, "\boldsymbol{g}"'] \arrow[ull, "\boldsymbol{f}"] \\
 & & \mathbf{2} \arrow[ull, "\boldsymbol{e}"] \arrow[u, "\boldsymbol{e}"] \arrow[urr, "\boldsymbol{e}"'] & &
\end{tikzcd}$$

FIGURE 7.1. A cube of $0,1$-lattice embeddings

Roughly speaking, the key property of the diagram $\vec{D}$ was that it could not be represented as the image of a cube of ℓ-groups and ℓ-homomorphisms under a natural transformation all of whose components are surjective ℓ-homomorphisms.

In the meantime, the paper got lost for years in an editorial process and the above-summarized result got superseded by the author's following result, contained in [34, Corollary 7.3].

Theorem 7.1. *There exists a bounded completely normal distributive lattice B, of cardinality $\aleph_2$, with no binary operation $\smallsetminus$ satisfying the identities $\mathsf{x} \leq \mathsf{y} \vee (\mathsf{x} \smallsetminus \mathsf{y})$, $(\mathsf{x} \smallsetminus \mathsf{y}) \wedge (\mathsf{y} \smallsetminus \mathsf{x}) = 0$, and $\mathsf{x} \smallsetminus \mathsf{z} \leq (\mathsf{x} \smallsetminus \mathsf{y}) \vee (\mathsf{y} \smallsetminus \mathsf{z})$ (in* [34] *we call such lattices* Cevian*). In particular, B is not a homomorphic image of $\mathrm{Id}^{\ell}_{\mathrm{c}}\, G$ for any representable*[6] *ℓ-group G.*

The lattice B constructed in the proof of Theorem 7.1 satisfies one more property than the lattice D above, called *countably based differences* and enabling the author to establish in [33], *via* more general variants of the *condensates* introduced in [13], that *the class of all ℓ-representable lattices is not closed under $\mathscr{L}_{\infty\lambda}$-elementary equivalence, for any infinite cardinal λ* (thus extending, *via* a completely different method, Mellor and Tressl's Theorem 4.13 to ℓ-spectra of Abelian ℓ-groups). This improvement forces the argument leading to the negative property of B, in Theorem 7.1, to be far more elaborate than the one leading to the negative property of the original D, in particular involving a noticeably more sophisticated variant of the diagram $\vec{D}$ of Figure 7.1 where, unlike $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ above, all the transition maps are closed homomorphisms (cf. Definition 4.4).

[6]Recall that an ℓ-group is *representable* if it is a subdirect product of totally ordered ℓ-groups. In particular, every Abelian ℓ-group is representable.

By Stone duality, Theorem 7.1 contains the following result.

Corollary 7.2. *The spectrum Ω of B is a completely normal spectral space, which cannot be embedded, as a spectral subspace, into any ℓ-spectrum.*

Final note: the countable case and the $\aleph_1$ case. While all containments represented in Figure 1.1 are valid for arbitrary cardinalities, the counterexamples underlying the non-containments all have cardinality $\aleph_1$, except for the proper containment $\mathbf{S}\boldsymbol{\ell} \subsetneqq \mathbf{CN}$ discussed above in this section, which involves a bounded distributive lattice of cardinality $\aleph_2$. This raises the obvious question about what happens for *countable* structures (i.e., second countable spectral spaces). The answer, provided by the author's papers [32] (for Abelian ℓ-groups) and [35] (for real spectra), is that *in the countable case the diagram collapses*: that is, every second countable, completely normal spectral space is both an ℓ-spectrum [32] and a real spectrum [35].

This leaves the following problem open:

Problem. Is every completely normal spectral space, with $\aleph_1$ compact open members, a spectral subspace of the ℓ-spectrum of some Abelian ℓ-group?

Equivalently, is every completely normal bounded distributive lattice of cardinality $\aleph_1$ a homomorphic image of the principal ℓ-ideal lattice of some Abelian ℓ-group?

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