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ON UNIQUE CONTINUATION FOR SOLUTIONS OF THE SCHRÖDINGER EQUATION ON TREES

AINGERU FERNÁNDEZ-BERTOLIN AND PHILIPPE JAMING

ABSTRACT. We prove that if a solution of the time-dependent Schrödinger equation on an homogeneous tree with bounded potential decays fast at two distinct times then the solution is trivial. For the free Schrödinger operator, we use the spectral theory of the Laplacian and complex analysis and obtain a characterization of the initial conditions that lead to a sharp decay at any time. We then use the recent spectral decomposition of the Schrödinger operator with compactly supported potential due to Colin de Verdière and Turc to extend our results in the presence of such potentials. Finally, we use real variable methods first introduced by Escauriza, Kenig, Ponce and Vegao to establish a general sharp result in the case of bounded potentials.

1. INTRODUCTION

The aim of the present paper is to study uniqueness results for Schrödinger equations with bounded potentials on homogeneous trees. These results can be seen as a version for homogeneous trees of a dynamical interpretation of the Hardy Uncertainty Principle.

The Schrödinger equation $\partial_t u = i(\Delta u + Vu)$ has been extensively studied by mathematicians and physicists. Those studies take place in various underlying spaces, both continuous (\mathbb{R}^d , manifolds,...) and discrete. In the discrete setting, on \mathbb{Z}^d , when the potential is chosen randomly on each $k \in \mathbb{Z}^d$, this corresponds to the celebrated Anderson model introduced by Anderson in [An] in order to describe the behavior of a quantum particle in disordered medium. In this paper, we will be dealing with the Schrödinger equation when the underlying space is an homogeneous tree (also known as a *Bethe lattice* in the physics community). The corresponding Anderson model has been introduced very early on by Abou-Chacra, Thouless and Anderson [ACTA]. This model allows to obtain closed form formulas for some models which

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is one of the reasons why it has also been extensively studied. (*see e.g.* the books [CL, PF, St] and the surveys [Ab, Wa] for more on the subject, for the tree case, one may further refer to [HES] and its extensive bibliography).

Our aim here is to show that solutions of the Schrödinger equation $\partial_t u = i(\Delta u + Vu)$ on an homogeneous tree can not be too sharply localized at 2 different times when the potential V is bounded. In our study, the potential is not random but the results directly apply to potentials that are chosen randomly on each vertex of the tree with bounded random variables (*e.g.* uniformly in some interval or Bernoulli random variable so that the results apply in the so-called Anderson-Bernoulli model). Our results may thus be seen as a dynamical version of the Uncertainty Principle. Before outlining our results more precisely, let us first explain what we mean by “localizing” and further explain our motivations in this paper.

Let us start by recalling Hardy’s uncertainty principle [Ha] on the real line: assume $f \in L^2(\mathbb{R})$ satisfies a decrease property like

$$(1.1) \quad |f(x)| \leq Ce^{-x^2/\beta^2}, \quad |\hat{f}(\xi)| \leq Ce^{-4\xi^2/\alpha^2}.$$

Then, if $\alpha\beta < 4$, $f \equiv 0$ while, in the end-point case, $\frac{1}{\alpha\beta} = \frac{1}{4}$, $f = Ce^{-x^2/\beta^2}$. In other words, a function and its Fourier transform can not both be localized below 2 sharply localized Gaussians.

Numerous authors have extended this result to higher dimensions, replacing the point-wise estimate (1.1) by integral or even distributional conditions [Ho, BDJ, BD, De] and also replacing the underlying space \mathbb{R}^d by various Lie groups (*see e.g.* the work of Baklouti, Kaniuth, Sitaram, Sundari, Thangavelu,... including [BK1, BK2, SST, Th1, Th2]. The survey [FS] and the book [Th3] may be taken as a starting point to further investigate the subject). Most of this work requires either complex analysis or a reduction to a real variable setting in which complex variable tools are available. A first difficulty appears here as the decrease in the space variable and in the Fourier variable can no longer be measured in the same way. This problem becomes even more striking in the discrete setting. For instance, for functions on \mathbb{Z} , the Fourier transform is a periodic function, so that there is no decrease at infinity.

To overcome this, one way is to consider a dynamical interpretation of the uncertainty principle. To explain what we mean by this, let us go back to the real line. Recall that the solution to the free Schrödinger

equation $\partial_t u = i\Delta u$, $u(0, x) = u_0(x)$ is given by the following representation formula:

$$u(x, t) = (4\pi it)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} u_0(y) dy = (2\pi it)^{-n/2} e^{\frac{i|x|^2}{4t}} \widehat{e^{i\frac{| \cdot |^2}{4t}} u_0} \left(\frac{x}{2t} \right).$$

Hence, the solution at a fixed time has, roughly speaking, the same size as the Fourier transform of the initial data, and we can translate decay properties of u_0 and \widehat{u}_0 into decay properties of u_0 and $u(x, T)$ for a fixed time T , to have

$$|u_0(x)| \leq C e^{-x^2/\beta^2}, \quad u(x, T) \leq C e^{-x^2/\alpha^2}, \quad \frac{T}{\alpha\beta} > \frac{1}{4} \implies u \equiv 0$$

and, if $\frac{T}{\alpha\beta} = \frac{1}{4}$, $u_0(x) = C e^{-x^2(1/\beta^2 + i/4T)}$.

This point of view has been used by Chanillo [Ch] to prove a dynamical uncertainty principle on complex semi-simple Lie groups by reducing the problem to Hardy's Uncertainty Principle on the real line. At the same time, Escauriaza, Kenig, Ponce and Vega started a series of papers [EKPV1, EKPV2, EKPV3] where they provide the first proof of the Hardy Uncertainty Principle, in its dynamical version and up to the end-point case, using real calculus. Their motivation is to consider solutions of general linear Schrödinger equations $\partial_t u = i(\Delta u + Vu)$, only assuming size conditions for the potential V . The robustness of both methods allows to extend their results to different settings, such as for covariant Schrödinger evolutions, [BFGRV, CF], or heat evolutions [EKPV4] but also to other underlying spaces [PS, BSTD, LuMu].

More recently, independently in [FB, FBV, JLMP], we began to extend the previous results to the discrete setting, understanding the Laplace operator as a finite-difference operator, acting on complex-valued functions $f : \mathbb{Z} \rightarrow \mathbb{C}$,

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n).$$

For the free evolution, or for the linear evolution with a bounded time-independent potential, as shown in [LyMa], one can use complex analysis tools to give a discrete version of the Hardy Uncertainty Principle. As in the continuous case, the critical decay is given by the discrete heat kernel, given in terms of modified Bessel functions. However, this similarity leads also to the main difference between both settings, because the critical decay is not Gaussian. More precisely, it is shown that for $\alpha < 1$ and u a $C^1([0, 1], \ell^2(\mathbb{Z}))$ -solution, also called

strong solution, of $\partial_t u = i\Delta_d u$, satisfying

$$|u(n, 0)| + |u(n, 1)| \leq CI_n(\alpha) \sim \frac{C}{\sqrt{|n|}} \left(\frac{e\alpha}{2|n|} \right)^{|n|}, \quad n \in \mathbb{Z} \setminus \{0\},$$

$u \equiv 0$. In the end-point case, $\alpha = 2$, $u(n, t) = Ci^{-n}e^{-2it}J_n(1 - 2t)$, where J_n is the Bessel function. This argument is also extended to other type of problems, as shown in [ART] for Jacobi operators.

In the case of linear Schrödinger equations, one can give a dynamical version of the Hardy Uncertainty Principle, only assuming that the potential is bounded, which makes another difference with the continuous case, since in the continuous case one requires to have some decay in the potential, and the result is still open for bounded potentials.

In this case, the decay condition to assure that there is no fast decaying solution, is, for $\mu > 1$,

$$\sum_{n \in \mathbb{Z}} e^{\mu(|n|+1)\log(|n|+1)} (|u(n, 0)|^2 + |u(n, 1)|^2) < \infty.$$

This result is sharp in μ , in the sense that the complex analytic part shows that there exist non-zero solutions that satisfy the previous condition with $\mu = 1$. A higher dimensional version of this result can be found in [FBV], although the rate of decay μ depends on the dimension and the sharp result is still open.

In this paper we extend both approaches to homogeneous trees of degree $q + 1$ (*Bethe lattices*), which we denote by \mathbf{T}_q . This is a connected graph with no loops, rooted in a point denoted by o , where every vertex is adjacent to $q + 1$ other vertices, relation denoted by $y \sim x$. Thus, one can see \mathbf{T}_q as a natural extension of the line \mathbb{Z} , which can be seen as a homogeneous tree of degree 2, and expect to have similar behavior for solutions to Schrödinger evolutions on \mathbf{T}_q . As in the line, we understand the Laplacian as the combinatorial Laplacian, that is a finite-difference operator \mathcal{L} only taking into account interactions between nearest-neighbors (see Section 2 for a precise definition).

It is our aim here is to contribute to the understanding of the behavior of solutions of Schrödinger equations on trees (*see e.g.* the recent papers [AS, Ed, CdVT] for other directions) as well as to establish Uncertainty Principles on trees (so far, we are only aware of one article [As] dealing with that issue). Finally, homogeneous trees can also be seen as a discrete analogs of hyperbolic spaces and more precisely 0-hyperbolic spaces in the sense of Gromov. The Schrödinger operator on real hyperbolic spaces has attracted a lot of attention recently (*see e.g.* [AP, Ba, BCS, IS]) and we hope that this work will also lead to new insight in that setting.

We are now in position to describe our results. First, since the spectral theory of the Laplacians on homogeneous trees is known (see [CS]), we have all the ingredients to give a dynamic interpretation of the Hardy Uncertainty Principle on \mathbf{T}_q when there is no potential:

Theorem A. *There exists a function U_q on \mathbf{T}_q such that, if u is a strong solution to the equation*

$$i\partial_t u(x, t) = \mathcal{L}u(x, t) = u(x, t) - \frac{1}{q+1} \sum_{y \sim x} u(y, t), \quad x \in \mathbf{T}_q$$

with $u(x, 0) = u_0(x)$ and if at times $t_0 = 0$ and $t_1 = 1$, there is a κ such that, for $x \neq o$

$$|u(x, t_i)| \leq \frac{\kappa}{\sqrt{|x|}} \left(\frac{e}{2(q+1)|x|} \right)^{|x|}$$

then $u_0 = \gamma U_q$ for some $\gamma \in \mathbb{C}$ with $|\gamma| \leq \kappa$.

The function U_q is explicitly given by an integral formula, see below.

Furthermore, in [CdVT], the authors develop the spectral theory for compactly supported hermitian perturbations

$$\mathcal{V}f(x) = \sum_{y \in \mathbf{T}_q} V(x, y)f(y).$$

Hence, we extend the previous theorem to compactly supported perturbations of the free Schrödinger equation. The spectral theory in [CdVT] is also extended to graphs isomorphic to a homogeneous tree at infinity. These graphs, outside a finite sub-graph, look like \mathbf{T}_q . Although we do not include the details, it can be checked that the same results are valid for this type of operators, which can be seen as an extension of [AR], where the case of a finite number of threads (lines) attached to a finite graph is studied.

The second part of the paper is devoted to the study of the simplified problem $\partial_t u(x) = i(\mathcal{L}u(x) + V(x)u(x))$, with a bounded potential V , using real variable calculus. This approach combines the main techniques of [FBV, JLMP], to prove first that a fast decaying solution at two different times preserves this decay at any interior time, and, later, via a Carleman estimate with Gaussian weight, we give a lower bound for the ℓ^2 -norm of the solution in a region far from the origin. A combination of these two facts leads then to:

Theorem B (Uniqueness result). *Let $u \in C^1([0, 1] : \ell^2(\mathbf{T}_q))$ be a solution to $\partial_t u(x) = i(\mathcal{L}u(x) + V(x)u(x))$ with V a bounded potential.*

If for $\mu > 1$

$$\sum_{x \in \mathbf{T}_q} e^{2\mu|x| \log(|x|+1)} (|u(x, 0)|^2 + |u(x, 1)|^2) < +\infty,$$

then $u \equiv 0$.

It is worth to mention that the Carleman inequality (see Lemma 5.3) requires an extra term if we compare it to the case of the line. However, we can take advantage of the good properties of the Gaussian weight to neglect the effect of this extra term and conclude our result.

The paper is organized as follows: in Section 2 we introduce some notations and preliminaries from the theory of entire functions as well as a summary of the spectral theory of the adjacency matrix on \mathbf{T}_q . These notions can be found in [CdVT, Le], but we include them here to clarify our presentation. Section 3 studies the free Schrödinger equation and includes the proof of Theorem A. In Section 4 we use again complex analytic tools to extend Theorem A and cover the case of compactly supported potentials. Finally, Section 5 covers the real variable approach, proving Theorem B via a Carleman inequality and logarithmic convexity of ℓ^2 weighted norms.

2. NOTATIONS AND PRELIMINARIES

2.1. Entire functions of exponential type. As in [JLMP], we will use methods from complex analysis. For the reader's convenience, we begin by briefly outlining some definitions and facts on entire functions of exponential type that we need. Details can be found in [Le] (see in particular Lectures 8 and 9). Recall that an entire function f is said to be of exponential type if for some $k > 0$

$$(2.2) \quad |f(z)| \leq C \exp(k|z|).$$

In this case the type of an entire function f is defined by

$$(2.3) \quad \sigma = \limsup_{r \rightarrow \infty} \frac{\log \max\{|f(re^{i\phi})|; \phi \in [0, 2\pi]\}}{r} < \infty.$$

In particular, an entire function f is of zero exponential type if for any $k > 0$ there exists $C = C(k)$ such that (2.2) holds.

Let $f(z)$ be an entire function of exponential type, $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then the type of f can be expressed in terms of its Taylor coefficients as

$$(2.4) \quad \limsup_{n \rightarrow \infty} n |c_n|^{1/n} = e\sigma.$$

The growth of a function f of exponential type along different directions is described by the indicator function

$$h_f(\varphi) = \limsup_{r \rightarrow \infty} \frac{\log |f(re^{i\varphi})|}{r}.$$

This function is the support function of some convex compact set $I_f \subset \mathbb{C}$ which is called the indicator diagram of f . In particular

$$(2.5) \quad h_f(\varphi) + h_f(\pi + \varphi) \geq 0.$$

For example the indicator function of e^{az} for $a \in \mathbb{C}$ is $h(\varphi) = \Re(ae^{i\varphi})$ and its indicator diagram consists of a single point, \bar{a} .

Clearly, $h_{fg}(\varphi) \leq h_f(\varphi) + h_g(\varphi)$, implying that

$$I_{fg} \subset I_f + I_g := \{z = z_1 + z_2 : z_1 \in I_f, z_2 \in I_g\}.$$

2.2. Trees. Throughout this paper, q will be an integer, $q \geq 2$. We will denote by $\mathbf{T} = \mathbf{T}_q$ the homogeneous tree of degree $q + 1$. This means that the tree is formed by a connected graph with no loops where every vertex is adjacent to $q + 1$ other vertices, relation denoted by $y \sim x$.

A *geodesic path* (resp. *geodesic ray*, *infinite geodesic*) in \mathbf{T} is a finite (resp. one-sided infinite, resp. doubly infinite) sequence (x_n) such that two consecutive terms are adjacent, $x_n \sim x_{n-1}$ and that does not turn back $x_{n+1} \neq x_{n-1}$. We can then define the distance $d(x, y)$ as the number of points in the geodesic path which goes from x to y . In particular, in a geodesic, $d(x_n, x_m) = |n - m|$.

Moreover, we fix a vertex of the tree \mathbf{T} to be the root o and write $|x| = d(x, o)$. For an integer $\ell \geq 0$, we denote by $S_\ell = \{x \in \mathbf{T} : |x| = \ell\}$. The boundary $\partial\mathbf{T}$ of \mathbf{T} is defined as the set of infinite paths starting at the root o . Then, we define, for a point $x \in \mathbf{T}$ and $w \in \partial\mathbf{T}$, the confluence point of x and w , denoted by $x \wedge w$ as the last point lying on w in the geodesic path joining o and x . Attached to this confluence point we define the *Buseman function* h_w and the *Horocycles* \mathcal{H}_k^w , $k \in \mathbb{Z}$ by

$$h_w(x) = |x| - 2|x \wedge w| \quad , \text{ and } \quad \mathcal{H}_k^w = \{x \in \mathbf{T} : h_w(x) = k\}.$$

Every horocycle is infinite and every $x \in \mathcal{H}_k^w$ has one neighbor $x^- \in \mathcal{H}_{k-1}^w$ (its predecessor) and q neighbors in \mathcal{H}_{k+1}^w (its successors).

Now let $\psi_{\ell,k} = |S_\ell \cap \mathcal{H}_k^w|$ be the number of elements in an horocycle \mathcal{H}_k of length ℓ . When $k \geq 0$,

$$\psi_{\ell,k} = \begin{cases} q^k & \text{if } \ell = k \\ (q-1)q^{k+p-1} & \text{if } \ell = k + 2p, p \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and for $k \geq 1$,

$$\psi_{\ell, -k} = \begin{cases} 1 & \text{if } \ell = k \\ (q-1)q^{p-1} & \text{if } \ell = k + 2p, p \geq 1. \\ 0 & \text{otherwise} \end{cases}$$

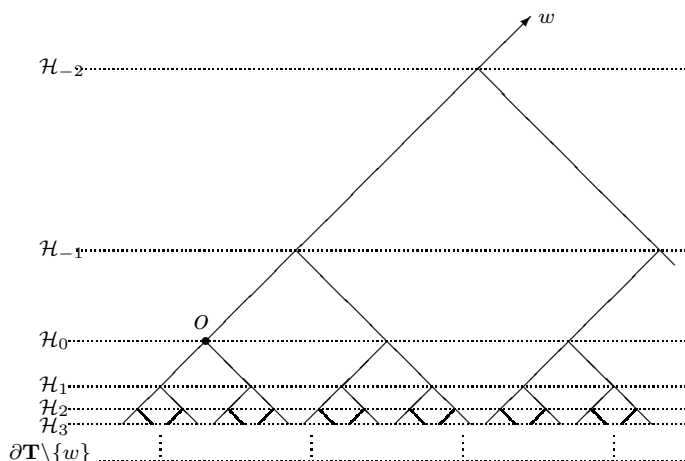


FIGURE 1. The tree \mathbf{T}_2 and horocycles.

The so called Helgason-Fourier transform (*see e.g.* [CS]) of a function f on the tree is defined by the formula

$$\mathcal{F}_{\mathbf{T}}[f](s, w) := \sum_{x \in \mathbf{T}} f(x) q^{-(1/2+is)h_w(x)}, \quad s \in \mathbb{T}, w \in \partial \mathbf{T},$$

where $\mathbb{T} = \mathbb{R}/\tau\mathbb{Z}$, usually identified with the interval $[-\tau/2, \tau/2)$, with $\tau = 2\pi/\log q$.

Moreover, the following inversion formula holds,

$$(2.6) \quad f(x) = \int_{\mathbb{T}} \int_{\partial \mathbf{T}} q^{-(1/2-is)h_w(x)} \mathcal{F}_{\mathbf{T}}[f](s, w) d\nu(w) d\mu(s), \quad x \in \mathbf{T}.$$

We refer to [CS] for the exact definitions of the measures ν and μ .

We will also need to distinguish between the neighbors and double neighbors of a vertex of the tree. More precisely, for $x \in \mathbf{T}$ with $|x| = n$ we set

— $x_f = \{y \in \mathbf{T} : |y| = n+1\}$ and, if $x \neq o$, x_p to be the unique $y \in \mathbf{T}$ such that $y \sim x$ and $|y| = n-1$. Note that $|o_f| = q+1$ and, if $x \neq o$, $|x_f| = q$.

— $x_{ff} = \{y \in \mathbf{T} : |y| = n+2, y_p \in x_f\}$ so that $|o_{ff}| = q(q+1)$ and, if $x \neq o$, $|x_{ff}| = q^2$.

— If $|x| \geq 2$, $x_{pp} = (x_p)_p$.

— If $x \neq o$, $x_{pf} = (x_p)_f \setminus \{x\}$ so that $|y| = |x|$ if $y \in x_{pf}$. Note that if $|x| = 1$, $|x_{pf}| = q$ while otherwise $|x_{pf}| = q - 1$.

In other words, x_f is the set of daughters of x , x_p the mother of x , x_{pp} is the grand-mother of x , x_{ff} the set of grand-daughters of x , x_{pf} the set of sisters of x .

Note that, for any function φ on \mathbf{T} , and any $n \geq 1$,

$$(2.7) \quad \sum_{|x|=n} \sum_{z \in x_{pf}} \varphi(z) = \begin{cases} q \sum_{|x|=1} \varphi(x) & \text{if } n = 1 \\ (q - 1) \sum_{|x|=n} \varphi(x) & \text{if } n \geq 2 \end{cases}.$$

Finally, we consider the adjacency operator A_0 and the Laplace operator \mathcal{L} on \mathbf{T} : for u a function on \mathbf{T} ,

$$A_0 u(x) = \sum_{y \sim x} u(y)$$

and

$$\begin{aligned} \mathcal{L}u(x) &= \left(I - \frac{1}{q+1} A_0 \right) u(x) = u(x) - \frac{1}{q+1} \sum_{y \sim x} u(y) \\ &= \frac{1}{q+1} \sum_{y \sim x} (u(x) - u(y)). \end{aligned}$$

We will denote by $\|\cdot\|_2$ the $\ell^2(\mathbf{T})$ -norm: if $u : \mathbf{T} \rightarrow \mathbb{C}$,

$$\|u\|_2^2 = \sum_{x \in \mathbf{T}} |u(x)|^2$$

and by $\|\cdot\|_{L_{x,t}^2}$ the $L_t^2 \ell^2$ -norm: if $u : [0, 1] \times \mathbf{T} \rightarrow \mathbb{C}$,

$$\|u\|_{L_{x,t}^2} = \int_0^1 \sum_{x \in \mathbf{T}} |u(t, x)|^2 dt.$$

3. SCHRÖDINGER EQUATION ON THE TREE

We want to study uniqueness properties of solutions to the Schrödinger equation $i\partial_t u = \mathcal{L}u$ assuming that they have fast decay at two different times. Adapting the method developed in [JLMP] in the case of the line to the tree, our main result in this Section is Theorem A from the introduction, that is

Theorem 3.1.

Assume that u is a strong solution to the equation

$$(3.8) \quad i\partial_t u(x, t) = \mathcal{L}u(x, t) = u(x, t) - \frac{1}{q+1} \sum_{y \sim x} u(y, t), \quad x \in \mathbf{T}$$

with $u(x, 0) = u_0(x)$. Assume that there is a κ such that, at times $t_0 = 0$ and $t_1 = 1$, for $x \neq 0$

$$(3.9) \quad |u(x, t_i)| \leq \frac{\kappa}{\sqrt{|x|}} \left(\frac{e}{2(q+1)|x|} \right)^{|x|}.$$

Then there exists a constant C such that u_0 is the function that only depends on $|x|$ given by the integral representation formula

$$u_0(x) = \frac{C}{q^{-|x|/2}} \int_0^\pi \exp\left(-i \frac{q^{1/2}}{q+1} \cos(z)\right) \varphi_{|x|}(z) \sin(z) dz$$

where

$$\varphi_j(z) = \frac{q^{1/2} \sin(z(j+1)) - q^{-1/2} \sin(z(j-1))}{q + q^{-1} - 2 \cos(2z)}.$$

Remark 3.2. A change of variable allows us to write u_0 as

$$u(|x|, 0) = \frac{C}{q^{-|x|/2}} \mathcal{F}[\psi_{|x|}] \left(\frac{q^{1/2}}{q+1} \right)$$

where \mathcal{F} is the Fourier transform on \mathbb{R} and

$$\psi_j(s) = \frac{q^{1/2} \sin((j+1) \arccos s) - q^{-1/2} \sin((j-1) \arccos s)}{q + q^{-1} + 1 - 2s^2}$$

on $(-1, 1)$ and $\psi_j = 0$ on $\mathbb{R} \setminus (-1, 1)$.

Proof. Let us fix a ray $w = 0y_1y_2 \dots$. Let $k \in \mathbb{Z}$. As we already noticed, if $x \in \mathcal{H}_k^w$, it has exactly one predecessor in \mathcal{H}_{k-1}^w and q successors in \mathcal{H}_{k+1}^w . Therefore,

$$\mathcal{L}\left(q^{-(1/2+is)h_w(x)}\right) = \left(1 - \frac{q^{1/2+is}}{q+1} - q \frac{q^{-1/2-is}}{q+1}\right) q^{-(1/2+is)h_w(x)}.$$

For a solution u to (3.8), we consider the Fourier-Helgason transform $\tilde{u}(s, w, t) = \mathcal{F}_{\mathbf{T}}[u(\cdot, t)]$, whose evolution is given by

$$i\partial_t \tilde{u} = \left(1 - \frac{q^{1/2}}{q+1}(q^{is} + q^{-is})\right) \tilde{u}.$$

Hence, if we set $\sigma = \frac{q^{1/2}}{2(q+1)}$,

$$(3.10) \quad \tilde{u}(s, w, t) = e^{-i(1-2\sigma(q^{is}+q^{-is}))t} \tilde{u}(s, w, 0).$$

Now we decompose,

$$\begin{aligned}
\tilde{u}(s, w, t) &= \sum_{x \in \mathbf{T}, h_w(x) > 0} u(x, t) q^{-h_w(x)/2} q^{-ish_w(x)} \\
&\quad + \sum_{x \in \mathbf{T}, h_w(x) \leq 0} u(x, t) q^{-h_w(x)/2} q^{-ish_w(x)} \\
&= \sum_{k=0}^{+\infty} \frac{1}{q^{k/2}} \left(\sum_{x \in \mathcal{H}_k^w} u(x, t) \right) \xi^k \\
&\quad + \sum_{k=1}^{+\infty} q^{k/2} \left(\sum_{x \in \mathcal{H}_{-k}^w} u(x, t) \right) \left(\frac{1}{\xi} \right)^k
\end{aligned}$$

where $\xi = q^{-is}$.

Now write $b_0 = 1$ and, for $\ell \geq 1$, $b_\ell = \frac{1}{\sqrt{\ell}} \left(\frac{e}{2(q+1)\ell} \right)^\ell$ so that if $t_j \in \{0, 1\}$,

$$\begin{aligned}
\left| \sum_{x \in \mathcal{H}_k^w} u(x, t_j) \right| &= \left| \sum_{\ell=0}^{\infty} \sum_{x \in \mathcal{H}_k^w \cap S_\ell} u(x, t_j) \right| \leq \kappa \sum_{\ell=0}^{\infty} \psi_{\ell, k} b_\ell \\
&\leq \begin{cases} \kappa q^k \left(b_k + (q-1) \sum_{p=1}^{\infty} q^{p-1} b_{k+2p} \right) & \text{for } k \geq 0 \\ \kappa \left(b_{-k} + (q-1) \sum_{p=1}^{\infty} q^{p-1} b_{-k+2p} \right) & \text{for } k \leq -1 \end{cases}.
\end{aligned}$$

Using that $(k+2p)^{k+2p+1/2} \geq k^{k+1/2}$ when $k, p \geq 1$, and that $(2p)^{2p+1/2} \geq 4$ we get that, for $k \geq 1$,

$$\begin{aligned}
\sum_{p=1}^{\infty} q^{p-1} b_{k+2p} &= \left(\frac{e}{2(q+1)} \right)^k \frac{1}{q} \sum_{p=1}^{\infty} \left(\frac{e\sqrt{q}}{2(q+1)} \right)^{2p} \frac{1}{(k+2p)^{k+2p+1/2}} \\
&\leq \frac{1}{\sqrt{k}} \left(\frac{e}{2(q+1)k} \right)^k \frac{e^2}{4(q+1)^2 \left(1 - \frac{e^2 q}{4(q+1)^2} \right)} \leq b_k
\end{aligned}$$

and the same bound holds for $k = 0$, thus

$$\left| \sum_{x \in \mathcal{H}_k^w} u(x, t_j) \right| \leq \begin{cases} \kappa q & \text{for } k = 0 \\ \kappa q^{k+1} b_k = \kappa q \frac{1}{\sqrt{k}} \left(\frac{eq}{2(q+1)k} \right)^k & \text{when } k \geq 1 \\ \kappa q b_{|k|} = \kappa q \frac{1}{\sqrt{|k|}} \left(\frac{e}{2(q+1)|k|} \right)^{|k|} & \text{when } k \leq -1 \end{cases}.$$

It follows that

$$\phi_j^+(\xi, w) := \sum_{k=0}^{+\infty} q^{-k/2} \left(\sum_{x \in \mathcal{H}_k^w} u(x, t_j) \right) \xi^k$$

extends into an entire function in ξ of exponential type σ . Its indicator diagram I_j^+ is therefore included in the closed disc $\bar{D}(0, \sigma)$. On the other hand

$$(3.11) \quad \phi_j^-(\zeta, w) := \sum_{k=1}^{+\infty} q^{k/2} \left(\sum_{x \in \mathcal{H}_{-k}^w} u(x, t_j) \right) \zeta^k$$

extends into an entire function in ζ of exponential type σ as well and its indicator diagram I_j^- is therefore also included in the disc $\bar{D}(0, \sigma)$. Actually, a little more is shown, namely that

$$(3.12) \quad |\phi_j^\pm(\xi, w)| \leq C_q \kappa e^{\sigma|\xi|}.$$

Let us now turn back to (3.10) which we write as

$$\tilde{u}(s, w, t) = e^{-i(1-2\sigma(\xi+\xi^{-1}))t} (\phi_0^-(\xi^{-1}, w) + \phi_0^+(\xi, w)).$$

This holds a priori for $\xi = q^{-is}$ and thus extends to $\xi \in \mathbb{C} \setminus \{0\}$ and every t . We write $\tilde{u}(\xi, w, t)$ for the corresponding extension.

For $t = 1$ we obtain

$$\begin{aligned} \phi_1^\pm(\xi, w) &= -\phi_1^\mp(\xi^{-1}, w) \\ &\quad + e^{-i} \exp(2i\sigma(\xi + \xi^{-1})) (\phi_0^\pm(\xi, w) + \phi_0^\mp(\xi^{-1}, w)). \end{aligned}$$

It follows that $I_1^\pm \subset I_0^\pm + 2i\sigma$ which in turn implies that I_1^\pm is reduced to $i\sigma$ and I_0^\pm is reduced to $-i\sigma$.

Let us now take $t = 1/2$. Then

$$\tilde{u}(\xi, w, 1/2) = e^{-i/2} e^{i\sigma(\xi+\xi^{-1})} (\phi_0^-(\xi^{-1}, w) + \phi_0^+(\xi, w)).$$

Write $\tilde{u}(\xi, w, 1/2) = u_+(\xi) + u_-(\xi^{-1})$ where u_+ (resp. u_-) contains all terms of positive (resp. negative) exponent in the Laurent series of \tilde{u} . The indicator diagram of those functions coincide with $\{0\}$ thus u_\pm are entire functions of 0 exponential type. On the other hand, (3.12) shows

that u_{\pm} are bounded. According to the Phragmen-Lindelöf principle (see e.g. [Le, Lecture 6]) u_{+} and u_{-} are constant and thus u is a constant as well.

It follows that

$$\tilde{u}(s, w, 0) = C_w \exp(i(1 - \sigma(q^{is} + q^{-is})))$$

for some constant C_w that depends on the ray w . But, by definition, for $\xi = q^{-is}$

$$\tilde{u}(\xi, w, 0) = \sum_{x \in \mathbf{T}} u(x, 0) \left(\frac{\xi}{\sqrt{q}} \right)^{h_w(x)}$$

and this extends to all $\xi \in \mathbb{C} \setminus \{0\}$, in particular to $\xi = \sqrt{q}$. This shows that

$$C_w = \exp(-i(1 - \sigma(q^{1/2} + q^{-1/2}))) \sum_{x \in \mathbf{T}} u(x, 0)$$

does not depend on w . We thus write $C_w = C$.

The integral formula for $u(|x|, 0)$ then comes from the inversion formula (2.6) and (see [CS])

$$\int_{\partial \mathbf{T}} q^{-(1/2-is)h_w(x)} d\nu(w) = c(-s)q^{(-is-1/2)|x|} + c(-s)q^{(is-1/2)|x|},$$

where $c(s) = \frac{q^{1/2}}{q+1} \frac{q^{1/2+is} - q^{-1/2-is}}{q^{is} - q^{-is}}$. \square

As an immediate corollary, we have the following uniqueness property for strong solutions to (3.8):

Corollary 3.3.

Assume that u is a strong solution to the equation (3.8). Assume that there exists $\epsilon > 0$ and κ such that, for $x \neq 0$

$$|u(x, t_i)| \leq \frac{\kappa}{\sqrt{|x|}} \left(\frac{e}{(2 + \epsilon)(q + 1)|x|} \right)^{|x|}, \quad t_0 = 0, \quad t_1 = 1.$$

Then $u \equiv 0$.

4. UNIQUENESS FOR PERTURBED PROBLEMS: THE COMPACT SUPPORT CASE

In this section we want to apply similar techniques to solutions to equations of the form

$$(4.13) \quad i\partial_t u(x, t) = \mathcal{L}u(x, t) + \mathcal{V}u(x, t), \quad x \in \mathbf{T},$$

where $\mathcal{V}u(x, t) = \sum_{y \in \mathbf{T}} V(x, y)u(y, t)$ and V is a compactly supported hermitian potential, $V(y, x) = \overline{V(x, y)}$. We denote its support as the set $K \times K$, where K is included in $B_R = \{x \in \mathbf{T}, |x| \leq R\}$ for some R .

In [CdVT] there is an extension of the Helgason-Fourier transformation in this context, and, for the sake of completeness, we recall here the main features.

We define the operator

$$Af(x) = \sum_{y \sim x} f(x) + \mathcal{W}f(x) = A_0f(x) + \mathcal{W}f(x),$$

where, $\mathcal{W} = -(q+1)\mathcal{V}$, and we recall the Green's functions of the operator A_0 . For $s \in \mathbb{T} \times i\mathbb{R}^+$,

$$G_0(\lambda_s)f(x) = \sum_{y \in \mathbf{T}} G_0(\lambda_s, x, y)f(y), \quad G_0(\lambda_s, x, y) = \frac{q^{(-1/2+is)d(x,y)}}{q^{1/2-is} - q^{-1/2+is}}.$$

We define, for $s \in \mathbb{T}$ and $w \in \partial\mathbf{T}$ the functions $a(s, w, x)$ and $e(s, w, x)$ as the solutions to the following problems

$$(4.14) \quad a(s, w, x) = \chi_K e_0(s, w, x) + \chi_K G_0(\lambda_s)[\mathcal{W}a(s, w)](x),$$

$$(4.15) \quad e(s, w, x) = e_0(s, w, x) + G_0(\lambda_s)[\mathcal{W}a(s, w)](x),$$

where $e_0(s, w, x) = q^{-(1/2-is)h_w(x)}$. With this notation, the deformed Fourier-Helgason transformation is defined as

$$\tilde{\mathcal{F}}_{\mathbf{T}}[f](s, w) = \sum_{x \in \mathbf{T}} f(x) \overline{e(s, w, x)}.$$

In [CdVT] it is shown that this formula is well defined for $\ell^2(\mathbf{T})$ functions and that it can be holomorphically extended to $s \in S^+ = \mathbb{T} \times i\mathbb{R}^+$. Here again, $\mathbb{T} = \mathbb{R}/\tau\mathbb{Z}$, identified with the interval $[-\tau/2, \tau/2)$, and $\tau = 2\pi/\log q$.

Further, there is a decomposition of $\ell^2(\mathbf{T}) = \mathcal{H}_{ac} \oplus \mathcal{H}_{pp}$ where

— the space \mathcal{H}_{pp} is finite dimensional, admits an orthonormal basis of $\ell^2(\mathbf{T})$ eigenfunctions associated to a finite set of eigenvalues.

— $\tilde{\mathcal{F}}_{\mathbf{T}}[f] = 0$ then $f \in \mathcal{H}_{pp}$.

The actual statement [CdVT, Theorem 4.3] is stronger, but this is enough for our needs.

Thanks to the deformed Fourier-Helgason transformation, we are able to prove the main result of this section.

Theorem 4.1.

Let V be a bounded and compactly supported hermitian potential and \mathcal{V} be defined by $\mathcal{V}u(x) = \sum_{y \in \mathbf{T}} V(x, y)u(y)$.

Let $u_0 \in \mathcal{H}_{ac}$ and let $u \in C^1([0, 1], \ell^2(\mathbf{T}))$ be a solution to

$$i\partial_t u(x, t) = \mathcal{L}u(x, t) + \mathcal{V}u(x, t), \quad x \in \mathbf{T}, \quad t \in [0, 1]$$

with initial condition $u(\cdot, 0) = u_0$. Assume that, for some $\epsilon > 0$, at times $t_0 = 0$ and $t_1 = 1$ the solution satisfies the bound

$$(4.16) \quad |u(x, t_j)| \leq C \frac{1}{\sqrt{|x|}} \left(\frac{e}{(2 + \epsilon)(q + 1)|x|} \right)^{|x|}, \quad j = 0, 1.$$

Then $u \equiv 0$.

Remark 4.2. Alternatively, we may impose the bounds (4.16) on $\pi_{ac}u$, the projection of u on \mathcal{H}_{ac} and conclude that $\pi_{ac}u = 0$.

On the other hand, if $u_0 \in \mathcal{H}_{pp}$, $\mathcal{L}u + \mathcal{V}u = \lambda u$, then $u(x, t) = e^{-i\lambda t}u_0(x)$. Such a solution has therefore the same decrease rate at any time. As shown in [CdVT], for certain \mathcal{V} 's, \mathcal{H}_{pp} may contain finitely supported functions on \mathbf{T} so that our theorem can not hold without the restriction $u_0 \in \mathcal{H}_{ac}$. However, in the case every element of \mathcal{H}_{pp} is finitely supported, then if u satisfies (4.16) so thus $\pi_{ac}u$. As a consequence, the theorem remains valid provided we replace the conclusion $u \equiv 0$ by $u_0 \in \mathcal{H}_{pp}$. Recall that this space is finite dimensional.

Note also that when \mathcal{V} is diagonal, *i.e.* $\mathcal{V}u(x) = V(x)u(x)$ then there are no compactly supported eigenfunctions.

Proof. First note that if $u_0 \in \mathcal{H}_{ac}$ then $u(\cdot, t) \in \mathcal{H}_{ac}$ for all t .

As we did in the free case, for a fixed $w \in \partial\mathbf{T}$ we consider \tilde{u}_{sc} as the deformed Fourier-Helgason transform

$$\tilde{u}_{sc}(s, w, t) = \sum_{x \in \mathbf{T}} u(x, t) \overline{e(s, w, x)}.$$

As u takes values in \mathcal{H}_{ac} , it is enough to show that $\tilde{u}_{sc} = 0$.

From [CdVT], we have again that

$$(4.17) \quad \tilde{u}_{sc}(s, w, t) = e^{-i[1-2\sigma(q^{is}+q^{-is})]t} \tilde{u}_{sc}(s, w, 0).$$

According to (4.14) we have

$$\begin{aligned} \tilde{u}_{sc}(s, w, t) &= \sum_{x \in \mathbf{T}} u(x, t) q^{-h_w(x)/2} q^{-ish_w(x)} \\ &\quad + \sum_{x \in \mathbf{T}} u(x, t) \sum_{y \in K} \frac{q^{-(1/2+is)d(x,y)}}{q^{1/2+is} - q^{-1/2-is}} [\mathcal{W}a(s, w)](y) \\ &= \sum_{x \in \mathbf{T}} u(x, t) q^{-h_w(x)/2} \xi^{h_w(x)} \\ &\quad + \sum_{x \in \mathbf{T}} u(x, t) \sum_{y \in K} \frac{q^{-d(x,y)/2} \xi^{d(x,y)}}{q^{1/2} \xi^{-1} - q^{-1/2} \xi} [\mathcal{W}a(\xi, w)](y), \end{aligned}$$

with $\xi = q^{-is}$. We are going to split the second sum in the regions $x \in B_R$ and $x \notin B_R$. For the second region, notice that, if we denote by x_K the closest point in K to x , then for $y \in K$, $d(x, y) = d(x, x_K) + d(x_K, y)$. Hence, if $x \notin B_R$, by (4.14),

$$\begin{aligned} & \sum_{y \in K} \frac{(q^{-1/2}\xi)^{d(x,y)}}{(q^{-1/2}\xi)^{-1} - q^{-1/2}\xi} \overline{[\mathcal{W}a(\xi, w)](y)} \\ &= (q^{-1/2}\xi)^{d(x, x_K)} \left(\overline{a(\xi, w, x_K)} - (q^{-1/2}\xi)^{-h_w(x_K)} \right). \end{aligned}$$

Notice that, from (4.14), it is easy to check that $a(\xi, w, x)$ is a rational polynomial in the variable ξ , so, since $x_K \in K$, there exists $M = M(R)$ such that, for $|\xi|$ large enough, to avoid the (finite number of) poles of $\overline{a(\xi, w, x_K)} - (q^{-1/2}\xi)^{-h_w(x_K)}$, we have

$$\left| \overline{a(\xi, w, x_K)} - (q^{-1/2}\xi)^{-h_w(x_K)} \right| \leq C|\xi|^M.$$

Moreover, thanks to the decay hypothesis, at $t = 0$ and $t = 1$ the three different sums are holomorphic in the region $\{|\xi| > 1\}$ except perhaps at a finite number of poles. Let us study separately each sum in that region. First,

$$\begin{aligned} |\Phi_1(\xi, w, t)| &:= \left| \sum_{x \in \mathbf{T}} u(x, t) (q^{-1/2}\xi)^{-h_w(x)} \right| \\ &\leq C + \sum_{x \in \mathbf{T}, h_w(x) < 0} |u(x, t)| |q^{-1/2}\xi|^{-h_w(x)}. \end{aligned}$$

Following the same reasoning as in the free case for (3.11), we get that this sum extends to an exponential function of type $\frac{\sqrt{q}}{(2+\epsilon)(q+1)}$. Hence, at $t = 0$ and $t = 1$, for any α ,

$$\limsup_{r \rightarrow \infty} \frac{\log |\Phi_1(re^{i\alpha}, w, t)|}{r} \leq \frac{\sqrt{q}}{(2+\epsilon)(q+1)}.$$

Now, since for $x \in B_R$, $y \in K$ we have $|d(x, y)| \leq 2R$, there exists $M = M(R)$ such that

$$\begin{aligned} |\Phi_2(\xi, \omega, t)| &:= \left| \sum_{x \in B_R} u(x, t) \sum_{y \in K} \frac{(q^{-1/2}\xi)^{d(x,y)}}{(q^{-1/2}\xi)^{-1} - q^{-1/2}\xi} \overline{[\mathcal{W}a(\xi, w)](y)} \right| \\ &\leq C \|\mathcal{V}\|_\infty |\xi|^M \sum_{x \in B_R} |u(x, t)|. \end{aligned}$$

Thus, since $u \in \ell^2(\mathbf{T})$, at $t = 0$ and $t = 1$, for any α ,

$$\limsup_{r \rightarrow \infty} \frac{\log |\Phi_2(re^{i\alpha}, w, t)|}{r} = 0.$$

Finally, we have to study $|\Phi_3(\xi, w, t)|$

$$\begin{aligned} &:= \left| \sum_{x \notin B_R} u(x, t) (q^{-1/2} \xi)^{d(x, x_K)} \left(\overline{a(\xi, w, x_K)} - (q^{-1/2} \xi)^{-h_w(x_K)} \right) \right| \\ &\leq |\xi|^M \sum_{x \notin B_R} |u(x, t)| |q^{-1/2} \xi|^{d(x, x_K)}. \end{aligned}$$

Let us consider, for $j \geq 1$, the points $x \in \mathbf{T}$ such that $|x| = R + j$. Notice that there are $q^{R+j-1}(q+1)$ such points and that, for each such point, $d(x, x_K) \leq j + 2R$. Hence, setting $b_\ell = \frac{1}{\sqrt{\ell}} \left(\frac{e}{(2+\epsilon)(q+1)\ell} \right)^\ell$,

$$\begin{aligned} \sum_{x \notin B_R} |u(x, t)| |q^{-1/2} \xi|^{d(x, x_K)} &\leq \sum_{j \geq 1} q^{R+j-1} (q+1) |b_{R+j}| |q^{-1/2} \xi|^{j+2R} \\ &= \frac{q+1}{q} |q^{-1/2} \xi|^R \sum_{j \geq 1} |b_{R+j}| |q^{1/2} \xi|^{R+j}. \end{aligned}$$

Finally, the sum on the right-hand side is bounded by $e^{\frac{\sqrt{q}|\xi|}{(2+\epsilon)(q+1)}}$, so we get

$$(4.18) \quad \limsup_{r \rightarrow \infty} \frac{\log |\Phi_3(re^{i\alpha}, w, t)|}{r} \leq \frac{\sqrt{q}}{(2+\epsilon)(q+1)}.$$

Gathering the results for Φ_i , $i = 1, 2, 3$ we conclude that, at $t = 0$ and $t = 1$, for any α ,

$$\limsup_{r \rightarrow \infty} \frac{\log |\tilde{u}_{sc}(re^{i\alpha}, w, t)|}{r} \leq \frac{\sqrt{q}}{(2+\epsilon)(q+1)}.$$

Using (2.5), if $\tilde{u}_{sc} \neq 0$, we get that, at $t = 0$ and $t = 1$, for any α ,

$$\limsup_{r \rightarrow \infty} \frac{\log |\tilde{u}_{sc}(re^{i\alpha}, w, t)|}{r} \geq -\frac{\sqrt{q}}{(2+\epsilon)(q+1)}.$$

Now, let us recall that, from (4.17), if $g(\xi) = \exp\{-i(1-2\sigma(\xi+\xi^{-1}))\}$

$$\tilde{u}_{sc}(\xi, w, 1) = g(\xi) \tilde{u}_{sc}(\xi, w, 0)$$

and, furthermore,

$$\limsup_{y \rightarrow +\infty} \frac{\log |g(-iy)|}{y} = 2\sigma = \frac{\sqrt{q}}{(q+1)}.$$

We then have

$$\begin{aligned} \limsup_{y \rightarrow +\infty} \frac{\log |\tilde{u}_{sc}(-iy, w, 1)|}{y} &= \frac{\sqrt{q}}{(q+1)} + \limsup_{y \rightarrow +\infty} \frac{\log |\tilde{u}_{sc}(-iy, w, 0)|}{y} \\ &\geq \frac{\sqrt{q}}{(q+1)} - \frac{\sqrt{q}}{(2+\epsilon)(q+1)} \\ &> \frac{\sqrt{q}}{(2+\epsilon)(q+1)}, \end{aligned}$$

which contradicts (4.18). Thus $\tilde{u}_{sc} \equiv 0$ thus $u \equiv 0$. \square

5. UNIQUENESS FOR PERTURBED PROBLEMS USING CARLEMAN ESTIMATES

In this section we consider the simplified problem

$$(5.19) \quad \partial_t u = i(\mathcal{L}u + Vu)$$

where $V = V(x, t)$ is a bounded potential.

We are going to begin this section by pointing out that a fast decaying solution at times $t = 0$ and $t = 1$ extends the fast decay to the whole interval $[0, 1]$. This is given by an immediate extension of part of the results in [JLMP]. We first need an auxiliary lemma:

Lemma 5.1.

Let $u \in C^1([0, T], \mathbf{T})$ satisfy

$$\partial_t u(x, t) = i(\mathcal{L}u(x, t) + V(x, t)u + F(x, t)),$$

where V and F are complex valued functions in $\mathbf{T} \times [0, T]$ and bounded. Let

$$\psi_\alpha(x, t) = (1 + |x|)^{\alpha|x|/(1+t)}, \quad \alpha \in (0, 1].$$

Then, for $T > 0$,

$$\|\psi_\alpha(T)u(T)\|_2^2 \leq e^{CT} \left(\|\psi_\alpha(0)u(0)\|_2^2 + \int_0^T \|\psi_\alpha(s)F(s)\|_2^2 ds \right),$$

provided the right-hand side is finite.

Proof. Define $f(x, t) = \psi_\alpha(x, t)u(x, t)$ and $H(t) = \|f(t)\|_2^2$ for a fixed α . We will just write $\psi = \psi_\alpha$. Notice that ψ only depends on $|x|$, so for $|x| = n$ we write $\psi(x) = \psi(n)$.

Formally,

$$\partial_t f = i\psi\mathcal{L}(\psi^{-1}f) + \phi_t f + iVf + i\psi F = \mathcal{S}f + \mathcal{A}f + iVf + i\psi F,$$

where $\phi = \log \psi$ and

$$\begin{aligned}\mathcal{S}f &= \phi_t f + \frac{i}{q+1} \sum_{y \sim x} \sinh(\phi(x, t) - \phi(y, t)) f(y) \\ \mathcal{A}f &= \frac{i}{q+1} \sum_{y \sim x} \cosh(\phi(x, t) - \phi(y, t)) f(y) - if(x).\end{aligned}$$

are symmetric and skew-symmetric operators respectively. Since

$$\partial_t H(t) = 2\Re\langle \partial_t f, f \rangle,$$

it is easy to check that $\partial_t H(t)$ is

$$\begin{aligned}&\leq 2\|\psi F\|_2 \|f\|_2 + \|V\|_\infty \|f\|_2 \\ &+ \left(2\phi_t(0) + \frac{2}{\sqrt{q}} |\sinh(\phi(1) - \phi(0))| \right) |f(o)|^2 \\ &+ \sum_{n \geq 1, |x|=n} \left(2\phi_t(n) + \frac{2\sqrt{q}}{q+1} |\sinh(\phi(n) - \phi(n-1))| \right) |f(x)|^2 \\ &+ \frac{2\sqrt{q}}{q+1} \sum_{n \geq 1, |x|=n} |\sinh(\phi(n+1) - \phi(n))| |f(x)|^2.\end{aligned}$$

The result follows after proving that the last three terms are bounded by $C\|f\|_2$, in the same spirit as in [JLMP]. To justify this formal argument, we can prove again the same result (now rigorously) for a truncated weight ψ_N and then let $N \rightarrow \infty$ (See [JLMP] for this argument in the line). \square

This result shows that if we have a solution of (5.19) with fast decay at time $t = 0$, the solution has fast decay at any future time, although the decay gets worse with time. Our aim now is to use also the fast decay at time $t = 1$ to improve the decay at future times.

Theorem 5.2.

Let $\gamma > 0$ and V a bounded potential. Let u be a strong solution of (5.19) and assume that at times $t = 0$ and $t = 1$,

$$\|(1 + |x|)^{\gamma(1+|x|)} u(x, t)\|_2 < +\infty, \quad t \in \{0, 1\}.$$

Then, for all $t \in [0, 1]$, $\|(1 + |x|)^{\gamma(1+|x|)} u(t)\|_2 < +\infty$.

Proof. For $1/2 < b < 1$, let $\phi_b(n) = \gamma(1+n) \log^b(1+n)$, $n \in \mathbb{N} \cup \{0\}$. Set $f = e^{\phi_b(|x|)} u$ and, as before $H(t) = \|f(t)\|_2^2$. The previous lemma shows that $H(t)$ is finite for all t , so the subsequent formal computations are

justified. We will show that, for some $C > 0$,

$$\begin{aligned} H_b(t) &\leq e^{Ct(1-t)} H_b(0)^{1-t} H_b(1)^t \\ &\leq e^{Ct(1-t)} \|(1+|x|)^{\gamma(1+|x|)} u(0)\|_2^{2(1-t)} \|(1+|x|)^{\gamma(1+|x|)} u(1)\|_2^{2t}. \end{aligned}$$

The result will follow by letting $b \rightarrow 1$ and applying the monotone convergence theorem.

In order to prove our claim, we write again $\partial_t f = \mathcal{S}f + \mathcal{A}f + iVf$ and, as shown in [JLMP], the claim follows from a lower bound

$$(5.20) \quad \langle [\mathcal{S}, \mathcal{A}]f, f \rangle \geq -C\|f\|^2,$$

with \mathcal{S}, \mathcal{A} the operators defined in the previous lemma, in this case for the weight e^{ϕ_b} . Since ϕ_b does not depend on t , it is easy to check that $(q+1)^2 \langle [\mathcal{S}, \mathcal{A}]f, f \rangle$ is

$$\begin{aligned} &= \sum_{x \in \mathbf{T}} \sum_{y \sim x} \sum_{z \sim y} \sinh(2\phi_b(|y|) - \phi_b(|x|) - \phi_b(|z|)) f(z) \overline{f(x)} \\ &= \sinh(2\phi_b(1) - 2\phi_b(0)) |f(o)|^2 \\ &\quad + 2 \sinh(2\phi_b(1) - \phi_b(0) - \phi_b(2)) \Re \sum_{z \in o_{ff}} f(z) \overline{f(o)} \\ &\quad + \sum_{x \in \mathbf{T} \setminus \{0\}} \sinh(2\phi_b(|x| - 1) - 2\phi_b(|x|)) \sum_{z \in x_{pf}} f(z) \overline{f(x)} \\ &\quad + \sum_{x \in \mathbf{T} \setminus \{0\}} \sinh(2\phi_b(|x| - 1) - 2\phi_b(|x|)) |f(x)|^2 \\ &\quad + 2 \Re \sum_{x \in \mathbf{T} \setminus \{0\}} \sinh(2\phi_b(|x| + 1) - \phi_b(|x|) - \phi_b(|x| + 2)) \sum_{z \in x_{ff}} f(z) \overline{f(x)} \\ &\quad + \sum_{x \in \mathbf{T} \setminus \{0\}} q \sinh(2\phi_b(|x| + 1) - 2\phi_b(|x|)) |f(x)|^2 \\ &= S_1 + \cdots + S_6. \end{aligned}$$

As for each n , there exists γ_n such that, for every $1/2 < b < 1$, $|\Phi_b(n)| \leq \gamma_n$, there exists a constant C such that $S_1, S_2 \geq -C\|f\|^2$.

As in [JLMP], there exists a constant κ such that, for every n , $|\sinh(2\phi_b(n+1) - \phi_b(n) - \phi_b(n+2))| \leq \kappa$. Further, $|x_{ff}| = q^2$ so that Cauchy-Schwarz shows that there is a constant C such that $S_5 \geq -C\|f\|^2$.

Next, if $|x| \geq 2$,

$$\begin{aligned} \left| \sum_{z \in x_{pf}} f(z) \overline{f(x)} \right| &\leq \frac{1}{2} \sum_{z \in x_{pf}} (|f(z)|^2 + |f(x)|^2) \\ &= \frac{q-1}{2} |f(x)|^2 + \frac{1}{2} \sum_{z \in x_{pf}} |f(z)|^2, \end{aligned}$$

while if $|x| = 1$,

$$\left| \sum_{z \in x_{pf}} f(z) \overline{f(x)} \right| \leq \frac{q}{2} |f(x)|^2 + \frac{1}{2} \sum_{z \in x_{pf}} |f(z)|^2,$$

But then

$$\begin{aligned} S_3 &\geq -\frac{1}{2} \sum_{x \in \mathbf{T} \setminus \{0\}} \sinh(2\phi_b(|x|) - 2\phi_b(|x| - 1)) \sum_{z \in x_{pf}} |f(z)|^2 \\ &\quad - \frac{q-1}{2} \sum_{|x| \geq 2} \sinh(2\phi_b(|x|) - 2\phi_b(|x| - 1)) |f(x)|^2 \\ &\quad - \frac{q}{2} \sum_{|x|=1} \sinh(2\phi_b(1) - 2\phi_b(0)) |f(x)|^2 \\ &= -(q-1) \sum_{x \in \mathbf{T} \setminus \{0\}} \sinh(2\phi_b(|x|) - 2\phi_b(|x| - 1)) |f(x)|^2 \\ &\quad - \sum_{|x|=1} \sinh(2\phi_b(1) - 2\phi_b(0)) |f(x)|^2 \\ &= S_3^a + S_3^b \end{aligned}$$

since each $x \in \mathbf{T} \setminus \{0\}$ appears $q-1$ or q times in the first sum if $|x| \geq 2$ or $|x| = 1$. It follows that $S_3^b \geq -C\|f\|^2$ and

$$S_3^a + S_4 + S_6 \geq q \sum_{x \in \mathbf{T} \setminus \{0\}} \psi_b(|x|) |f(x)|^2 \geq 0$$

where

$$\psi_b(n) = \sinh(2\phi_b(n+1) - 2\phi_b(n)) - \sinh(2\phi_b(n) - 2\phi_b(n-1)) \geq 0$$

due to the properties of the function $(1+x) \log^b(1+x)$ for $x > 0$ and $1/2 < b < 1$, see [JLMP]. \square

As it happens in the continuous case, or in the line, uniqueness holds from an argument related to Carleman inequalities. Here we prove the following Carleman inequality

Lemma 5.3 (Carleman inequality on the tree).

Let $\varphi : [0, 1] \rightarrow \mathbb{R}$ be a smooth function, $\beta > 0$ and $\gamma > \frac{1}{2\beta}$. There exists $R_0 = R_0(\|\varphi\|_\infty + \|\varphi'\|_\infty + \|\varphi''\|_\infty, \beta, \gamma)$ such that, if $R > R_0$, $\alpha \geq \gamma R \log R$ and if g is a function on $\mathbf{T} \times [0, 1]$, $g \in C_0^1([0, 1], \ell^2(\mathbf{T}))$ has its support contained in the set

$$\{(x, t) : |x|/R + \varphi(t) \geq \beta\},$$

then

$$\begin{aligned} & \sinh \frac{2\alpha}{R^2} \cosh \frac{4\alpha\beta}{R} \|e^{\alpha(\frac{|x|}{R} + \varphi)} g \mathbf{1}_{|x| \geq 1}\|_{L_{x,t}^2}^2 \\ & \leq (q+1)^2 \|e^{\alpha(\frac{|x|}{R} + \varphi(t))} (i\partial_t + \mathcal{L})g\|_{L_{x,t}^2}^2 \\ & \quad + \int_0^1 \sinh \frac{4\alpha}{R} \left(\frac{1}{2R} + \varphi \right) \sum_{|x|=1} \left| e^{\alpha(\frac{1}{R} + \varphi(t))} g(x) \right|^2 dt. \end{aligned}$$

Proof. Let ϕ be defined by $\phi(n) = \alpha \left(\frac{n}{R} + \varphi(t) \right)^2$. For $f = e^\phi g$ we have,

$$e^\phi (i\partial_t + \mathcal{L})g = \mathcal{S}f + \mathcal{A}f,$$

where

$$\begin{aligned} \mathcal{S}f &= i\partial_t f + \frac{1}{q+1} \sum_{y \sim x} \cosh(\phi(x, t) - \phi(y, t)) f(y, t) - f, \\ \mathcal{A}f &= -i\phi_t f + \frac{1}{q+1} \sum_{y \sim x} \sinh(\phi(x, t) - \phi(y, t)) f(y, t). \end{aligned}$$

We need to give a lower bound for the commutator, which immediately implies the result using the fact that

$$\|e^{\alpha(\frac{|x|}{R} + \varphi(t))} (i\partial_t + \mathcal{L})g\|_{L_{x,t}^2}^2 \geq \langle [\mathcal{S}, \mathcal{A}]f, f \rangle.$$

To simplify notation, we will not explicitly write the dependence of f on the time variable t so that $f(x)$ means $f(x, t)$, $x \in \mathbf{T}$, $t \in [0, 1]$. A simple computation shows that

$$(5.21) \quad \langle [\mathcal{S}, \mathcal{A}]f, f \rangle = \int_0^1 S(t) dt$$

where

$$(5.22) \quad S(t) := \sum_{x \in \mathbf{T}} \phi_{tt}(x) |f(x)|^2 \\ + \frac{2}{q+1} \sum_{x \in \mathbf{T}} \sum_{y \sim x} (\phi_t(x) - \phi_t(y)) \cosh(\phi(x) - \phi(y)) f(y) \overline{f(x)} \\ + \frac{1}{(q+1)^2} \sum_{x \in \mathbf{T}} \sum_{y \sim x} \sum_{z \sim y} \sinh(2\phi(y) - \phi(x) - \phi(z)) f(z) \overline{f(x)}.$$

As in the previous proof, we split them into sums over mothers and daughters. Recall that the root has only daughters while the rest of the points in the tree have a single mother and q daughters. Further, the function $\phi(x, t)$ only depends on $|x|$, the distance of a point in the tree to the root o . We therefore decompose the sums in (5.22) as follows: $S(t) = S_1 + \dots + S_7$ where

— The first sum in (5.22) is $S_1 = \sum_{n \geq 0} \sum_{|x|=n} \phi_{tt}(n) |f(x)|^2$.

— For the second sum in (5.22), each pair $x \sim y$ appears twice, once $|x| = |y| + 1$, once with $|x| = |y| - 1$. Therefore, this sum can be rewritten as

$$S_2 = \frac{4}{q+1} \operatorname{Im} \sum_{n \geq 1} \sum_{|x|=n} (\phi_t(n) - \phi_t(n-1)) \cosh(\phi(n) - \phi(n-1)) f(x_p) \overline{f(x)}.$$

— For the last sum in (5.22), we need to distinguish more cases:

a) $x = 0$, y any daughter and $z = 0$. This happens $q + 1$ times and leads to

$$S_3 = \frac{1}{q+1} \sinh 2(\phi(1) - \phi(0)) |f(o)|^2;$$

b) $x \in \mathbf{T} \setminus \{0\}$, *i.e.* $n := |x| \geq 1$ y is one of the q daughters of x and $z = x$ which leads to

$$S_4 = \frac{q}{(q+1)^2} \sum_{n \geq 1} \sum_{|x|=n} \sinh 2(\phi(n+1) - \phi(n)) |f(x)|^2$$

while if y is the mother of x and $z = x$, we get

$$S_5 = \frac{1}{(q+1)^2} \sum_{n \geq 1} \sum_{|x|=n} \sinh 2(\phi(n-1) - \phi(n)) |f(x)|^2;$$

c) $x \in \mathbf{T} \setminus \{0\}$, *i.e.* $n := |x| \geq 1$, y is the mother of x and z is any of the sisters of x , we get

$$S_6 = \frac{1}{(q+1)^2} \sum_{n \geq 1} \sum_{|x|=n} \sum_{z \in x_{pf}} \sinh 2(\phi(n-1) - \phi(n)) f(z) \overline{f(x)};$$

— Finally, all other terms x is the grand-mother of z and each such couple (x, z) appears twice. As $|z| \geq 2$, this may be written as

$$S_7 = \frac{2}{(q+1)^2} \Re \sum_{n \geq 2} \sum_{|x|=n} \sinh(2\phi(n-1) - \phi(n) - \phi(n-2)) f(x) \overline{f(x_{pp})}.$$

Before estimating those quantities, as $\phi(n) = \alpha \left(\frac{n}{R} + \varphi(t)\right)^2$, we obtain

$$\begin{aligned} \phi_t(n) &= 2\alpha \left(\frac{n}{R} + \varphi\right) \varphi' \\ \phi_{tt}(n) &= 2\alpha \left[\left(\frac{n}{R} + \varphi\right) \varphi'' + (\varphi')^2 \right] \\ \phi_t(n) - \phi_t(n-1) &= \frac{2\alpha}{R} \varphi' \\ \phi(n-1) - \phi(n) &= -\frac{2\alpha}{R} \left(\frac{n-1/2}{R} + \varphi\right) \\ \phi(n) + \phi(n+2) - 2\phi(n+1) &= \frac{2\alpha}{R^2}. \end{aligned}$$

Let us now estimate S_1 to S_7 . We will treat them from the simplest to the most involved one rather than the order in which they appeared in the above decomposition. We start with S_1 , which can be bounded by

$$(5.23) \quad S_1 \geq -2\|\varphi''\|_\infty \alpha \sum_{n \geq 0} \left| \frac{n}{R} + \varphi \right| \sum_{|x|=n} |f(x)|^2.$$

To estimate S_7 , we write $2\Re(f(x)\overline{f(x_{pp})}) = -|f(x) - f(x_{pp})|^2 + |f(x)|^2 + |f(x_{pp})|^2$, then

$$\begin{aligned} S_7 &= \frac{\sinh \frac{2\alpha}{R^2}}{(q+1)^2} \sum_{n \geq 2} \sum_{|x|=n} (|f(x) - f(x_{pp})|^2 - |f(x)|^2 - |f(x_{pp})|^2) \\ &\geq -\frac{\sinh \frac{2\alpha}{R^2}}{(q+1)^2} \left(\sum_{n \geq 2} \sum_{|x|=n} |f(x)|^2 + \sum_{n \geq 2} \sum_{|x|=n} |f(x_{pp})|^2 \right) \\ &\geq -\frac{\sinh \frac{2\alpha}{R^2}}{(q+1)^2} \left(q(q+1)|f(o)|^2 + q^2 \sum_{|x|=1} |f(x)|^2 \right. \\ (5.24) \quad &\left. + (q^2 + 1) \sum_{n \geq 2} \sum_{|x|=n} |f(x)|^2 \right) \end{aligned}$$

since o has $q(q+1)$ grand-daughters and thus appears $q(q+1)$ times as an x_{pp} , if $|x| \geq 1$, it has q^2 grand-daughters and thus will appear q^2 times in the second sum.

Next, for S_6 , we use that $f(z)\overline{f(x)} \geq -\frac{1}{2}(|f(x)|^2 + |f(z)|^2)$ to obtain

$$\begin{aligned} S_6 &\geq -\frac{1}{2(q+1)^2} \sum_{n \geq 1} |\sinh 2(\phi(n-1) - \phi(n))| \times \\ &\quad \times \sum_{|x|=n} \sum_{z \in x_{pf}} (|f(x)|^2 + |f(z)|^2) \\ &= -\frac{1}{(q+1)^2} \left(q |\sinh 2(\phi(0) - \phi(1))| \sum_{|x|=1} |f(x)|^2 \right. \\ &\quad \left. + (q-1) \sum_{n \geq 1} |\sinh 2(\phi(n-1) - \phi(n))| \sum_{|x|=n} |f(x)|^2 \right). \end{aligned}$$

Here we use the fact that x_{pf} has q elements if $|x| = 1$ and $q-1$ elements otherwise for $\sum_{|x|=n} \sum_{z \in x_{pf}} |f(x)|^2$ and we use (2.7) for the second sum. Finally, using the expression of ϕ , we get

$$(5.25) \quad S_6 \geq -\frac{1}{(q+1)^2} \left(q \sinh \frac{4\alpha}{R} \left(\frac{1}{2R} + \varphi \right) \sum_{|x|=1} |f(x)|^2 \right. \\ \left. + (q-1) \sum_{n \geq 2} \sinh \frac{4\alpha}{R} \left(\frac{n-1/2}{R} + \varphi \right) \sum_{|x|=n} |f(x)|^2 \right).$$

Now, for S_2 , let us first introduce

$$\Psi(n) = \cosh(\phi(n) - \phi(n-1))$$

and

$$\Sigma_n = \sum_{|x|=n} (q^{1/2}|f(x)|^2 + q^{-1/2}|f(x_p)|^2).$$

We use the expression of ϕ_t and the fact that

$$2|f(x)f(x_p)| \leq q^{1/2}|f(x)|^2 + q^{-1/2}|f(x_p)|^2$$

to bound S_2 by

$$\begin{aligned} &\geq -\frac{4\alpha|\varphi'|}{(q+1)R} \sum_{n \geq 1} \Psi(n)\Sigma_n \\ &= -\frac{4\alpha|\varphi'|}{q^{1/2}R} \Psi(1)|f(o)|^2 - \frac{4q^{1/2}\alpha|\varphi'|}{(q+1)R} \sum_{n \geq 1} \sum_{|x|=n} [\Psi(n) + \Psi(n+1)] |f(x)|^2 \end{aligned}$$

since o will appear $q + 1$ times as an x_p and each x with $|x| \geq 1$ will appear once as an x and q times as an x_p . Using the expression of ϕ we conclude that

$$(5.26) \quad S_2 \geq -\frac{4\alpha\|\varphi'\|_\infty}{Rq^{1/2}} \cosh \frac{2\alpha}{R} \left(\frac{1}{2R} + \varphi \right) |f(o)|^2 \\ - \frac{4q^{1/2}\alpha\|\varphi'\|_\infty}{(q+1)R} \cosh \frac{\alpha}{R^2} \sum_{n \geq 1} \cosh \frac{2\alpha}{R} \left(\frac{n}{R} + \varphi \right) \sum_{|x|=n} |f(x)|^2.$$

Next, write

$$S_4 = \frac{q-1}{(q+1)^2} \sum_{n \geq 1} \sum_{|x|=n} \sinh 2(\phi(n+1) - \phi(n)) |f(x)|^2 \\ + \frac{1}{(q+1)^2} \sum_{n \geq 1} \sum_{|x|=n} \sinh 2(\phi(n+1) - \phi(n)) |f(x)|^2 \\ = S_4^a + S_4^b.$$

We will group S_4^b and S_5 noticing that

$$\sinh \frac{4\alpha}{R} \left(\frac{n+1/2}{R} + \varphi \right) - \sinh \frac{4\alpha}{R} \left(\frac{n-1/2}{R} + \varphi \right) \\ = 2 \cosh \frac{4\alpha}{R} \left(\frac{n}{R} + \varphi \right) \sinh \frac{2\alpha}{R^2}.$$

This leads to

$$(5.27) \quad S_4^b + S_5 \geq \frac{2}{(q+1)^2} \sinh \frac{2\alpha}{R^2} \sum_{n \geq 1} \cosh \frac{4\alpha}{R} \left(\frac{n}{R} + \varphi \right) \sum_{|x|=n} |f(x)|^2.$$

We are now in position to estimate $S_1 + \dots + S_7$. Let us first isolate all terms containing $|f(o)|^2$. They appear in (5.23), S_3 , (5.24) and (5.26).

The factor of $|f(o)|^2$ is

$$A := -2\alpha\|\varphi\|_\infty\|\varphi''\|_\infty + \frac{1}{q+1} \sinh \frac{4\alpha}{R} \left(\frac{1}{2R} + \varphi \right) \\ - \frac{q \sinh \frac{2\alpha}{R^2}}{q+1} - \frac{4\alpha\|\varphi'\|_\infty}{Rq^{1/2}} \cosh \frac{2\alpha}{R} \left(\frac{1}{2R} + \varphi \right).$$

Now, the hypothesis of the lemma show that, if $f(o) \neq 0$, then $\varphi \geq \beta > 0$. Further, as $\alpha > \frac{1}{2\beta} R \log R$, it is easy to see that the dominating term in A is the second one and that the other three can be absorbed in it provided R is large enough. Thus $A \geq 0$ if R is large enough (depending on q , $\|\varphi\|_\infty$, $\|\varphi'\|_\infty$, $\|\varphi''\|_\infty$ and β).

Next, we compute the factor of $\sum_{|x|=1} |f(x)|^2$. The one stemming from S_4^a and the one appearing in (5.25) give

$$\begin{aligned} & \left[\frac{q-1}{(q+1)^2} \sinh \frac{4\alpha}{R} \left(\frac{3}{2R} + \varphi \right) - \frac{q}{(q+1)^2} \sinh \frac{4\alpha}{R} \left(\frac{1}{2R} + \varphi \right) \right] \\ & \geq -\frac{1}{(q+1)^2} \sinh \frac{4\alpha}{R} \left(\frac{1}{2R} + \varphi \right). \end{aligned}$$

The remaining terms for $|x|=1$ come from (5.27), (5.26), (5.24) and (5.23). The factor of $\sum_{|x|=1} |f(x)|^2$ stemming from those terms is

$$\begin{aligned} & \frac{2}{(q+1)^2} \sinh \frac{2\alpha}{R^2} \cosh \frac{4\alpha}{R} \left(\frac{1}{R} + \varphi \right) \\ & \quad - \frac{4q^{1/2}\alpha\|\varphi'\|_\infty}{(q+1)R} \cosh \frac{\alpha}{R^2} \cosh \frac{2\alpha}{R} \left(\frac{1}{R} + \varphi \right) \\ & \quad - \frac{q^2 \sinh \frac{2\alpha}{R^2}}{(q+1)^2} - 2\|\varphi''\|_\infty \alpha \left| \frac{1}{R} + \varphi \right|. \end{aligned}$$

The three last terms are again absorbed in the first one (see [FBV] for details). We are thus left with

$$\begin{aligned} & \frac{1}{(q+1)^2} \sinh \frac{2\alpha}{R^2} \cosh \frac{4\alpha}{R} \left(\frac{1}{R} + \varphi \right) \sum_{|x|=1} |f(x)|^2 \\ & \geq \frac{1}{(q+1)^2} \sinh \frac{2\alpha}{R^2} \cosh \frac{4\alpha\beta}{R} \sum_{|x|=1} |f(x)|^2 \end{aligned}$$

because of the support property of f .

For $n \geq 2$ the factor of $\sum_{|x|=n} |f(x)|^2$ come from

— first those from S_4^a and from (5.25) which now are

$$\frac{q-1}{(q+1)^2} \sinh \frac{4\alpha}{R} \left(\frac{n+1/2}{R} + \varphi \right) - \frac{q-1}{(q+1)^2} \sinh \frac{4\alpha}{R} \left(\frac{n-1/2}{R} + \varphi \right) \geq 0,$$

— the remaining ones coming from (5.23), (5.24), (5.26) and (5.27)

$$\begin{aligned} & - 2\|\varphi''\|_\infty \alpha \left| \frac{n}{R} + \varphi \right| - \frac{q^2 + 1}{(q+1)^2} \sinh \frac{2\alpha}{R^2} \\ & \quad - \frac{4q^{1/2}\alpha\|\varphi'\|_\infty}{(q+1)R} \cosh \frac{\alpha}{R^2} \cosh \frac{2\alpha}{R} \left(\frac{n}{R} + \varphi \right) \\ & \quad \quad + \frac{2}{(q+1)^2} \sinh \frac{2\alpha}{R^2} \cosh \frac{4\alpha}{R} \left(\frac{n}{R} + \varphi \right). \end{aligned}$$

The first three terms are again absorbed in the last one (see [FBV] for details). We are thus left with

$$\begin{aligned} & \frac{1}{(q+1)^2} \sinh \frac{2\alpha}{R^2} \sum_{n \geq 2} \cosh \frac{4\alpha}{R} \left(\frac{n}{R} + \varphi \right) \sum_{|x|=n} |f(x)|^2 \\ & \geq \frac{1}{(q+1)^2} \sinh \frac{2\alpha}{R^2} \cosh \frac{4\alpha\beta}{R} \sum_{n \geq 2} \sum_{|x|=n} |f(x)|^2 \end{aligned}$$

because of the support property of f .

In summary, if R is large enough,

$$\begin{aligned} \langle [\mathcal{S}, \mathcal{A}]f, f \rangle & \geq - \int_0^1 \frac{1}{(q+1)^2} \sinh \frac{4\alpha}{R} \left(\frac{1}{2R} + \varphi \right) \sum_{|x|=1} |f(x)|^2 dt \\ & \quad - \frac{1}{(q+1)^2} \sinh \frac{2\alpha}{R^2} \cosh \frac{4\alpha\beta}{R} \int_0^1 \sum_{n \geq 1} \sum_{|x|=n} |f(x)|^2 dt \end{aligned}$$

as announced. \square

Even though we need a correction term in order to give the Carleman estimate, we can adapt the argument in [FBV] to give again a lower bound for solutions to Schrödinger evolutions on trees.

Theorem 5.4 (Lower bound for solutions to Schrödinger equations). *Let $q \geq 2$, $A, L, \eta > 0$ then there exists $R_0 = R_0(q, A, L) > 0$ and $c = c(q, \eta)$ such that*

— if V is a bounded function on \mathbf{T} with

$$\|V\|_\infty = \sup_{t \in [0,1], x \in \mathbf{T}} \{|V(x, t)|\} \leq L,$$

— and $u \in C^1([0, 1] : \ell^2(\mathbf{T}))$ is a strong solution of

$$\partial_t u = i(\mathcal{L}u + Vu)$$

that satisfies the bounds

$$\int_0^1 \sum_{x \in \mathbf{T}} |u(x, t)|^2 dt \leq A^2 \quad , \quad \int_{1/2-1/8}^{1/2+1/8} |u(x_0, t)|^2 dt \geq 1$$

for some x_0 with $|x_0| = 2$.

Then for $R \geq R_0$,

$$\lambda(R) \equiv \left(\int_0^1 \sum_{\lfloor R \rfloor - 1 \leq |x| \leq \lfloor R \rfloor + 1} |u(x, t)|^2 dt \right)^{1/2} \geq ce^{-(1+\eta)R \log R}.$$

Proof. For $\epsilon > 0$ fixed let us define the following cut-off functions:

— we define θ^R, μ to be $C^\infty(\mathbb{R})$ functions such that $0 \leq \theta^R, \mu \leq 1$ and

$$(5.28) \quad \theta^R(x) = \begin{cases} 1, & |x| \leq R - 1 \\ 0, & |x| \geq R \end{cases} \quad \mu(x) = \begin{cases} 1, & |x| \geq \epsilon^{-1} + 1 \\ 0, & |x| \leq \epsilon^{-1} \end{cases}.$$

— and a $C^\infty([0, 1])$ function φ such that $0 \leq \varphi \leq 2 + \epsilon^{-1}$ and

$$(5.29) \quad \varphi(t) = \begin{cases} 2 + \epsilon^{-1}, & t \in [\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}] \\ 0, & t \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1] \end{cases}.$$

We are going to apply the previous lemma to

$$g(x, t) := \theta^R(|x|) \mu \left(\frac{|x|}{R} + \varphi(t) \right) u(x, t), \quad x \in \mathbf{T}, \quad t \in [0, 1].$$

Notice that the evolution of g is given by the expression

$$\begin{aligned} (i\partial_t + \mathcal{L})g &= \theta^R \mu \left(\frac{|x|}{R} + \varphi \right) (i\partial_t u + \mathcal{L}u) + i\varphi' \theta^R(x) \mu' \left(\frac{|x|}{R} + \varphi \right) u \\ &\quad + \theta^R(x) \frac{1}{q+1} \sum_{y \sim x} \left(\mu \left(\frac{|y|}{R} + \varphi \right) - \mu \left(\frac{|x|}{R} + \varphi \right) \right) u(y, t) \\ &\quad + \frac{1}{q+1} \sum_{y \sim x} (\theta^R(|y|) - \theta^R(|x|)) \mu \left(\frac{|y|}{R} + \varphi \right) u(y, t). \end{aligned}$$

Using the bounds on the cut-off functions and the fact that $|i\partial_t u + \mathcal{L}u| = |Vu| \leq \|V\|_\infty |u|$ we get

$$\begin{aligned} |(i\partial_t + \mathcal{L})g| &\leq \|V\|_\infty |u| + C_\varphi \left| \mu' \left(\frac{|x|}{R} + \varphi \right) \right| |u| \\ &\quad + \frac{1}{q+1} \left| \sum_{y \sim x} \left(\mu \left(\frac{|y|}{R} + \varphi \right) - \mu \left(\frac{|x|}{R} + \varphi \right) \right) u(y, t) \right| \\ &\quad + \frac{1}{q+1} \sum_{y \sim x} |\theta^R(|y|) - \theta^R(|x|)| |u(y, t)|. \end{aligned}$$

Thus, by means of the Carleman estimate with $\beta = 1/\varepsilon$ and R large enough,

$$\begin{aligned} (5.30) \quad &\sinh(2\alpha/R^2) \cosh(4\alpha/\varepsilon R) \|e^{\alpha(\frac{|x|}{R} + \varphi)^2} g \mathbf{1}_{|x| \geq 1}\|_{L^2_{x,t}} \\ &\leq c \|V\|_\infty^2 \|e^{\alpha(\frac{|x|}{R} + \varphi)^2} g\|_{L^2_{x,t}}^2 \\ &\quad + c \int_0^1 \sum_{n \geq 0, |x|=n} e^{2\alpha(\frac{n}{R} + \varphi)^2} \left| \mu' \left(\frac{n}{R} + \varphi \right) \right|^2 |u(x, t)|^2 dt \\ &+ c \int_0^1 \sum_{n \geq 0, |x|=n} \sum_{y \sim x} e^{2\alpha(\frac{n}{R} + \varphi)^2} \left| \mu \left(\frac{|y|}{R} + \varphi \right) - \mu \left(\frac{|x|}{R} + \varphi \right) \right|^2 |u(y, t)|^2 dt \\ &\quad + c \int_0^1 \sum_{n \geq 0, |x|=n} \sum_{y \sim x} e^{2\alpha(\frac{n}{R} + \varphi)^2} |\theta^R(|y|) - \theta^R(|x|)| |u(y, t)|^2 dt \\ &\quad + \int_0^1 \sinh \frac{4\alpha}{R} \left(\frac{1}{2} + \varphi \right) \sum_{|x|=1} e^{2\alpha(\frac{1}{R} + \varphi)^2} |g(x, t)|^2 dt. \end{aligned}$$

Note that we used Cauchy-Schwarz in the third and fourth sums in the

$$\text{form } \sum_{x \in \mathbf{T}} \left| \sum_{y \sim x} \psi(y) \right|^2 \leq (q+1) \sum_{x \in \mathbf{T}} \sum_{y \sim x} |\psi(y)|^2.$$

We now study carefully the support of each term.

For the first term involving V : by taking $\alpha = cR \log R$ with $c \geq \varepsilon/2$

$$(5.31) \quad \sinh(2\alpha/R^2) \cosh(4\alpha/\varepsilon R) \geq 2cR^{\frac{4c}{\varepsilon}-1} \log R,$$

so that, when R large enough (depending on L also now), the term on the right, up to the term involving root o , is absorbed in the left-hand side. Further, the remaining term is bounded by $ce^{2\alpha(2+\varepsilon^{-1})} L^2 A^2$.

For the term involving the derivative of the function μ , we easily see that $\frac{n}{R} + \varphi \leq 1 + \epsilon^{-1}$, and, therefore

$$\int_0^1 \sum_{n \geq 0, |x|=n} e^{2\alpha(\frac{n}{R} + \varphi)^2} \left| \mu' \left(\frac{n}{R} + \varphi \right) \right|^2 |u(x, t)|^2 dt \leq c e^{2\alpha(1 + \epsilon^{-1})} A^2.$$

Next we study the term involving the difference of μ functions, which is similar to the last one. It is easy to check that if $\frac{n}{R} + \varphi \geq \epsilon^{-1} + 1 + \frac{1}{R}$ both functions μ , the one evaluated at x and the one evaluated at one neighbor of x are 0. Hence,

$$\begin{aligned} \int_0^1 \sum_{n \geq 0, |x|=n} \sum_{y \sim x} e^{2\alpha(\frac{n}{R} + \varphi)^2} \left| \mu \left(\frac{|y|}{R} + \varphi \right) - \mu \left(\frac{|x|}{R} + \varphi \right) \right|^2 |u(y, t)|^2 dt \\ \leq e^{2\alpha(\epsilon^{-1} + 1 + 1/R)^2} A^2. \end{aligned}$$

Now we focus on the term with difference of θ functions. In this case, the only possibilities where the difference is not zero are summarize as

- $|x| = \lfloor R \rfloor - 1$ and y a future neighbor, $|y| = \lfloor R \rfloor$.
- $|x| = \lfloor R \rfloor$ and y any neighbor of x .
- $|x| = \lfloor R \rfloor + 1$ and y the past neighbor, $|y| = \lfloor R \rfloor$.

Thus,

$$\begin{aligned} \int_0^1 \sum_{n \geq 0, |x|=n} \sum_{y \sim x} e^{2\alpha(\frac{n}{R} + \varphi)^2} |\theta^R(|y|) - \theta^R(|x|)| |u(y, t)|^2 dt \\ \leq c e^{2\alpha(3 + \epsilon^{-1} + 1/R)^2} \lambda^2(R). \end{aligned}$$

For the last term in the right-hand side, we just bound the function φ to put all the functions out of the sum. Now, by the definition of θ^R and μ , we see that if $x = x_0$ and $t \in [1/2 - 1/8, 1/2 + 1/8]$ then $\left| \frac{|x_0|}{R} + \varphi e_1 \right| = 2 + \epsilon^{-1} + 2/R$, so the cut-off functions are 1 and $g(x_0, t) = u(x_0, t)$. This allows us to bound the left-hand side of the Carleman inequality of the lemma by

$$\| e^{\alpha(\frac{|x|}{R} + \varphi)^2} g \mathbf{1}_{|x| \geq 1} \|_{L_{x,t}^2}^2 \geq e^{(2 + \epsilon^{-1} + 2/R)^2 2\alpha},$$

since $\int_{1/2 - 1/8}^{1/2 + 1/8} |u(x_0, t)|^2 \geq 1$.

Gathering all these results we have,

$$\begin{aligned} & \sinh\left(\frac{2\alpha}{R^2}\right) \cosh\left(\frac{4\alpha}{\epsilon R}\right) e^{2\alpha(2+\epsilon^{-1}+2/R)^2} \\ & \leq e^{2\alpha(2+\epsilon^{-1})} A^2 L^2 + e^{2\alpha(1+\epsilon^{-1}+1/R)^2} A^2 \\ & + \sinh\frac{4\alpha}{R} \left(\frac{1/2}{R} + 2 + \epsilon^{-1}\right) e^{2\alpha(2+\epsilon^{-1}+1/R)^2} A^2 \\ & \qquad \qquad \qquad + e^{2\alpha(3+\epsilon^{-1}+1/R)^2} \lambda^2(R). \end{aligned}$$

It is clear that the first two terms in the right-hand side are smaller than the third term. Let us see that the third term can be hidden in the left-hand side, for R large enough, depending on A (recall that before we showed that R depends on L as well) and ϵ , which is a fixed number. Indeed, taking into account that $\alpha = cR \log R$ with $c > \frac{\epsilon}{2}$, we have

$$\begin{aligned} & \sinh\left(\frac{2\alpha}{R^2}\right) \cosh\left(\frac{4\alpha}{\epsilon R}\right) e^{2\alpha(2+\epsilon^{-1}+2/R)^2} \\ & \sim 2c \log R R e^{2cR(2+\epsilon^{-1})^2+8c(2+\epsilon^{-1})+4c\epsilon^{-1}-1+8c/R} \end{aligned}$$

and

$$\begin{aligned} & \sinh\frac{4\alpha}{R} \left(\frac{1/2}{R} + 2 + \epsilon^{-1}\right) e^{2\alpha(2+\epsilon^{-1}+1/R)^2} A^2 \\ & \sim A^2 R e^{2cR(2+\epsilon^{-1})^2+8c(2+\epsilon^{-1})+4c/R}, \end{aligned}$$

which proves our claim.

Finally, we conclude that

$$1 \leq 2c \log R R e^{\frac{4c}{\epsilon}-1} \leq c_\epsilon e^{(5+2\epsilon^{-1})2cR \log R - (2+2\epsilon^{-1})2c \log R} \lambda^2(R),$$

so

$$\lambda(R) \geq c_\epsilon e^{-(5+2\epsilon^{-1})cR \log R + (2+2\epsilon^{-1})c \log R}.$$

We just finish this result by taking $c = \epsilon/2 + \epsilon^2$, to have

$$(5.32) \quad \lambda(R) \geq c_\epsilon e^{-(1+9\epsilon/2+5\epsilon^2)R \log R + (1+3\epsilon+2\epsilon^2) \log R}$$

which is of the desired form. \square

Once we have the lower bound, since the previous log-convexity properties derive upper bounds for the term $\lambda(R)$, we are in position to prove Theorem B from the introduction, that is

Theorem 5.5 (Uniqueness result).

Let $u \in C^1([0, 1] : \ell^2(\mathbf{T}))$ be a solution of (5.19) with V a bounded potential. If for $\mu > 1$

$$\sum_{x \in \mathbf{T}} e^{2\mu|x| \log(|x|+1)} (|u(x, 0)|^2 + |u(x, 1)|^2) < +\infty,$$

then $u \equiv 0$.

Proof. Let $\eta > 0$ be such that $\mu > 1 + \eta > 1$.

If u is not zero, by translation and dilation, we may assume that u satisfies, for some $x_0 \in \mathbf{T}$ with $|x_0| = 2$,

$$\int_{1/2-1/8}^{1/2+1/8} |u(x_0, t)|^2 dt \geq 1,$$

so that we can apply the previous theorem to find a lower bound for $\lambda(R)$. More precisely, we know that $\lambda(R)$ satisfies (5.32). On the other hand, by Theorem 5.2 we have

$$\sup_{t \in [0, 1]} \sum_{x \in \mathbf{T}} |u(x, t)|^2 e^{2\mu|x| \log|x|} < +\infty.$$

Hence $\lambda(R) \leq ce^{-\mu R \log R}$. Combining both bounds,

$$ce^{-\mu R \log R} \geq \lambda(R) \geq ce^{-(1+\eta)R \log R}.$$

We get a contradiction letting $R \rightarrow \infty$. \square

Remark 5.6. If one wish to study the full general problem in the tree

$$\partial_t u = i(\mathcal{L}u + \mathcal{V}u),$$

with $\mathcal{V}u(x) = \sum_{y \in \mathbf{T}} V(x, y)u(y)$, a bounded potential seems not enough

to conclude uniqueness and some decay or support conditions should be required for V , since, in order to use the same approach, one needs to study the operator $\psi \mathcal{V}(\psi^{-1}f)$ for the previous weights.

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