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MINIMAL SURFACES NEAR SHORT GEODESICS IN HYPERBOLIC 3-MANIFOLDS

LAURENT MAZET AND HAROLD ROSENBERG

ABSTRACT. If M is a finite volume complete hyperbolic 3-manifold, the quantity $\mathcal{A}_1(M)$ is defined as the infimum of the areas of closed minimal surfaces in M . In this paper we study the continuity property of the functional \mathcal{A}_1 with respect to the geometric convergence of hyperbolic manifolds. We prove that it is lower semi-continuous and even continuous if $\mathcal{A}_1(M)$ is realized by a minimal surface satisfying some hypotheses. Understanding the interaction between minimal surfaces and short geodesics in M is the main theme of this paper

1. INTRODUCTION

The area of a closed minimal surface Σ in a complete hyperbolic 3-manifold is bounded above by $-2\pi\chi(\Sigma)$; this follows from the Gauss equation. Finding an optimal lower bound for the area is a more subtle question. Notice that in dimension 2, there is no lower bound for the length of a closed geodesic in a hyperbolic surface. However the Margulis lemma and the monotonicity formula does give a lower bound of $2\pi(\cosh(\bar{\varepsilon}) - 1)$, for the area of a properly immersed minimal surface in a complete hyperbolic 3-manifold; $\bar{\varepsilon}$ is the Margulis constant. According to explicit estimates of $\bar{\varepsilon}$, this number is at least 0.104 [12].

In a previous paper [11], the authors proved the area is at least 2π when Σ is a closed embedded minimal surface in a complete finite volume hyperbolic 3-manifold of Heegaard genus at least 6. If Σ is non-orientable the lower area bound is π . Perhaps the main goal of the present paper it to introduce techniques to resolve the remaining cases: $2 \leq \text{Heegaard genus} \leq 5$.

In our paper [11], we introduce the quantity $\mathcal{A}_1(M)$, where M is a compact orientable 3-manifold. If \mathcal{O} denotes the collection of all smooth orientable embedded closed minimal surfaces in M and \mathcal{U} the collection of all smooth non-orientable ones, $\mathcal{A}_1(M)$ is defined by

$$\mathcal{A}_1(M) = \inf(\{|\Sigma|, \Sigma \in \mathcal{O}\} \cup \{2|\Sigma|, \Sigma \in \mathcal{U}\})$$

so $\mathcal{A}_1(M)$ gives a lower bound for the area of any minimal surface in M .

The main result in [11] says that $\mathcal{A}_1(M)$ is the area (or twice the area) of some minimal surface in M . Moreover it gives some characterization of this minimal surface in terms of its index and its genus.

Let $(g_i)_i$ be a sequence of smooth Riemannian metrics on M which smoothly converge to \bar{g} . Because of the characterization of the minimal surface that

realizes $\mathcal{A}_1(M, g_i)$ and thanks to a compactness result by Sharp [16], it can be proved that $\liminf \mathcal{A}_1(M, g_i) \geq \mathcal{A}_1(M, \bar{g})$. Moreover, if $\mathcal{A}_1(M, \bar{g})$ is realized by a non degenerate minimal surface, $\lim \mathcal{A}_1(M, g_i) = \mathcal{A}_1(M, \bar{g})$. However one can produce examples where \mathcal{A}_1 is not upper semi-continuous (F. Morgan suggested examples of a 2-sphere looking like a pear).

Concerning hyperbolic manifolds, our study proves that, if M is hyperbolic and its Heegaard genus is at least 6, then $\mathcal{A}_1(M) \geq 2\pi$ which gives a universal lower bound for the area of a minimal surface in M . This reasoning can be adapted to the case M is a finite volume hyperbolic manifold (not necessarily compact).

In order to remove the hypothesis about the Heegaard genus, we ask the question of the continuity of \mathcal{A}_1 when the space of hyperbolic manifolds is endowed with the geometric convergence topology. Here the situation is not as above where we have a sequence of Riemannian metrics on a fixed manifold, here we have a sequence of manifolds M_i with changing topologies. Moreover, if $(M_i)_i$ is a non trivial converging sequence of hyperbolic manifolds then M_i contains a geodesic γ_i whose length goes to 0. As a consequence, an important question for our study is to understand the behaviour of a minimal surface intersecting a neighborhood of a short geodesic.

This question has been already studied by several authors. For example, Hass [8] and Huang and Wang [9] study the geometry of minimal surfaces near a short geodesic in order to construct hyperbolic manifolds that fiber over the circle but such that the fibers can not be made minimal.

Our study of minimal surfaces near short geodesics starts with a result of Meyerhoff [12]. Basically it says that a short geodesic in M of length ℓ has an embedded tubular neighborhood N_{R_ℓ} of radius R_ℓ and $\lim_{\ell \rightarrow 0} R_\ell = +\infty$.

We obtain two results concerning minimal surfaces in N_{R_ℓ} . The first one deals with stable minimal surfaces in tubular neighborhood of short geodesics (Corollary 7). Basically it says that such a stable minimal surface either stays far from the short geodesic or it intersects transversely the short geodesic. Moreover in the second case, the surface must have a very large area in the R_ℓ tubular neighborhood of the geodesic.

Our second result deals with general minimal surfaces (not assumed to be stable) (Proposition 9). It says that a minimal surface in the neighborhood of a short geodesic either stays very far from the core geodesic or comes very close to it (the estimate depending on the index of the minimal surface). As above in the second case, we obtain a lower bound for the area of a minimal surface coming close to the short geodesic.

Actually these two results are very similar to results we obtained with Collin and Hauswirth in [4] concerning the geometry of minimal surfaces in hyperbolic cusps. In both cases, the argument is based on the fact that the tubular neighborhoods are foliated by equidistant tori whose diameter are small. As a consequence, an embedded minimal surface with bounded curvature can not be tangent to these equidistant surfaces.

Once the behaviour of minimal surfaces close to short geodesics is understood, we study the continuity of \mathcal{A}_1 . A version of our result can be stated as follows. It is similar to the result that can be obtained for a fixed manifold with a converging sequence of metrics.

Theorem. *Let $M_i \rightarrow \overline{M}$ be a converging sequence of hyperbolic cusp manifolds. Then*

$$\mathcal{A}_1(\overline{M}) \leq \liminf \mathcal{A}_1(M_i).$$

If $\mathcal{A}_1(\overline{M})$ is not realized by the area of a stable-unstable separating minimal surface, then

$$\mathcal{A}_1(\overline{M}) = \lim \mathcal{A}_1(M_i).$$

Let us recall that "stable-unstable" means that the first eigenvalue of the stability operator is 0. Of course one can expect that the surface that realizes $\mathcal{A}_1(\overline{M})$ is never stable-unstable but we do not know how to prove this. Actually it is possible to expect that no minimal surface in a hyperbolic manifold is stable-unstable. In fact the above result is a combination of two propositions: Propositions 22 and 25

The main difficulty in the proof of Proposition 25 is to be able to control where is located a minimal surface Σ_i that realizes $\mathcal{A}_1(M_i)$. Actually, our study of minimal surfaces near short geodesics implies that Σ_i can not enter into a tubular neighborhood of a short geodesic. So it stays in a part of M_i where the convergence $M_i \rightarrow \overline{M}$ is just the smooth convergence of the metric tensor. Thus a compactness result by Sharp [16] gives the lower semicontinuity of \mathcal{A}_1 . Concerning Proposition 22, we first prove that $\limsup \mathcal{A}_1(M_i)$ is bounded. Thus if $\mathcal{A}_1(\overline{M})$ is not realized by a stable-unstable separating minimal surface Σ then Σ can be deformed into a minimal surface in M_i . This implies the second inequality.

Of course one can also think about hyperbolic manifolds with infinite volume and ask the following question. For which class of complete hyperbolic 3-manifolds of infinite volume can one hope for an area lower bound 2π ? There may not exist a closed minimal surface in M , but if $\mathcal{A}_1(M)$ is realized, can one expect it to be at least 2π ?

The paper is organized as follows. In Section 2.1, we recall some basic facts about the description of cusp and tubular ends of complete finite volume hyperbolic 3-manifolds. Section 3 studies the geometry of minimal surfaces with bounded curvature in tubular ends. In Section 4, we study the general behaviour of minimal surfaces in tubular ends. In Section 5 we recall some facts about the min-max theory for minimal surfaces that we will use in the next sections. Section 6 is devoted to recall the work we made in [11] and how it should be adapted to work with non compact hyperbolic manifolds. Sections 7 and 8 are devoted to the study of the lower and upper semicontinuity of the \mathcal{A}_1 functional. Finally in Appendix A, we prove some technical results and formulas.

Preliminary remarks. Let S be a smooth Riemannian surface, we will denote by $|S|$ its area.

Let $(T, d\sigma^2)$ be a flat torus. Its universal cover is a flat \mathbb{R}^2 so we have coordinates (x_1, x_2) such that the flat metric can be written $dx_1^2 + dx_2^2$. Then T is the quotient of \mathbb{R}^2 by some lattice Γ . We say that (x_1, x_2) is an orthonormal coordinate system on T .

Moreover, we can choose (x_1, x_2) such that Γ is generated by v_1, v_2 where $v_1 = (a_1, 0)$ and $v_2 = (a_2, b_2)$. We then say that (x_1, x_2) is a well oriented orthonormal coordinate system.

We notice that if $(T, d\sigma^2)$ has diameter δ then the lattice can be generated by vectors of length less than 2δ .

2. HYPERBOLIC MANIFOLDS

In this first section we recall some facts concerning the geometry of hyperbolic 3-manifolds with finite volume also called cusp manifolds. We refer to [2] for part of this description.

2.1. The cusp and tubular ends. Let M be a complete hyperbolic 3-manifold of finite volume. For any ε less than the Margulis constant, the manifold M can be split into two parts: the ε -thick part $M_{[\varepsilon, \infty)}$ which is connected, not empty (recall that $p \in M_{[\varepsilon, \infty)}$ if any non null homotopic closed loop at p has length at least ε) and the ε -thin part which may have a finite number of connected components. The connected components of the thin part are of two types: cusp ends and tubular neighborhoods of closed geodesics also called tubular ends.

Cusp ends are isometric to $E_0 = T \times \mathbb{R}_+$ endowed with a metric

$$g = e^{-2t} d\sigma^2 + dt^2$$

where $d\sigma^2$ is a flat metric on the 2-torus T . We define $E_t = T \times [t, +\infty)$. We notice that if E_0 is a component of the ε -thin part then E_t is a component of the $\delta\varepsilon$ -thin part with $e^{-2t} \leq \delta \leq e^{-t/2}$.

For tubular ends, let γ be a short geodesic in M and consider c a lift of γ to \mathbb{H}^3 . If R is small the R -tubular neighborhood N_R of γ in M is the quotient of the R tubular neighborhood V_R of c in \mathbb{H}^3 by some loxodromic transformation τ of axis c (see Figure 1).

In order to introduce some coordinate system, let z denote arclength along c and let $\vec{v}(z), \vec{\tau}(z)$ be parallel orthogonal unit normal vectorfields along γ , we introduce cylindrical coordinates in V_R by

$$F(z, \theta, r) = \exp_{c(z)}(r(\cos \theta \vec{v}(z) + \sin \theta \vec{\tau}(z)))$$

In these coordinates, the hyperbolic metric is

$$(1) \quad g = (\cosh^2 r) dz^2 + (\sinh^2 r) d\theta^2 + dr^2.$$

N_R can be viewed as the quotient of $M_R = \{(z, \theta, r) \in \mathbb{R}^2 \times [0, R]\}$ by the relations $(z, \theta, 0) \sim (z, \theta', 0)$, $(z, \theta, r) \sim (z, \theta + 2\pi, r)$ and $(z, \theta, r) \sim (z + \ell, \theta + \alpha, r)$ for some parameters $\ell > 0$ and α . ℓ is the length of the geodesic loop

γ and α is called the twist parameter of γ (it is the angle of the loxodromic transformation). As above, if N_R is a component of the ε -thin part, then N_r is a component of the $\delta\varepsilon$ -thin part some some $\delta \in [e^{2(r-R)}, e^{(r-R)/2}]$ if R and r are larger than some universal constant. In the following, we denote by $S_{\bar{r}} = \partial N_{\bar{r}}$ the torus $\{r = \bar{r}\}$.

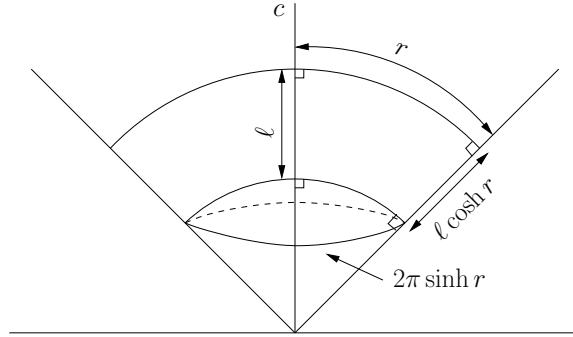


FIGURE 1. The r -tubular neighborhood V_r of a lift c of the geodesic γ of length ℓ

The above coordinates are called tubular coordinates. In order to be coherent with the coordinates we use on cusp ends, we will also use the coordinate system $(x_1, x_2, t) = (\theta, z, R - r)$ such that the metric can be written

$$g = \sinh^2(R - t)dx_1^2 + \cosh^2(R - t)dx_2^2 + dt^2 = d\sigma_t^2 + dt^2$$

on $T \times [0, R]$ where T is the quotient of \mathbb{R}^2 by the translations by $(2\pi, 0)$ and (α, ℓ) (notice that g is singular on $T \times \{R\}$).

The interest of these coordinates is that any part of a cusp or tubular end can be described as $T \times [a, b]$ with some metric $d\sigma_t^2 + dt^2$ where $d\sigma_t^2$ is a flat metric on the torus T . We denote by $T_t = T \times \{t\}$. So the family $(T_t)_t$ gives a foliation of the ends by tori.

If C is the torus in such an end that corresponds to $T \times \{\bar{t}\}$, the graph of a function $u : \Omega \subset C \rightarrow \mathbb{R}$ is just the surface parametrized by $\{(p, t) \in T \times \mathbb{R} \mid t = \bar{t} + u(p)\}$ (notice that we will often identify $C \subset M$ with $T_{\bar{t}} \in T \times \mathbb{R}$).

One question is to know what is the maximal radius R that can be considered in the above discussion (N_R being embedded). This has been estimated by Meyerhoff in [12] where the following result is proved.

Theorem 1. *Let γ be a geodesic loop in a complete hyperbolic 3-manifold. If the length ℓ of γ is less than $\frac{\sqrt{3}}{4\pi} \ln^2(\sqrt{2} + 1)$, then there exists an embedded tubular neighborhood around γ whose radius R satisfies*

$$\sinh^2 R = \frac{1}{2} \left(\frac{\sqrt{1 - 2k}}{k} - 1 \right)$$

where $k = \cosh \sqrt{\frac{4\pi\ell}{\sqrt{3}}} - 1$.

In the sequel we denote by R_ℓ the solution of $\sinh^2 R = \frac{1}{2} \left(\frac{\sqrt{1-2k}}{k} - 1 \right)$. When ℓ is small this implies that $\sinh^2 R_\ell \sim \cosh^2 R_\ell \sim \frac{\sqrt{3}}{4\pi\ell}$. For example, the area of S_{R_ℓ} goes to $\frac{\sqrt{3}}{2}$ as $\ell \rightarrow 0$.

Let us notice that the mean curvature of the torus S_{r_0} with respect to $-\partial_r$ is $(\tanh r_0 + \coth r_0)/2$.

2.2. The geometric convergence. The space of cusp manifolds with volume less than V_0 is compact for geometric convergence. This convergence is defined as follows (see Sections E.1 and E.2 in [2]). Let $\Pi_i : \mathbb{H}^3 \rightarrow M_i$ and $\bar{\Pi} : \mathbb{H}^3 \rightarrow \bar{M}$ be the universal covers and o a point in \mathbb{H}^3 . We say that the pointed manifolds $(M_i, \Pi_i(o))$ converge for the geometric convergence topology to $(\bar{M}, \bar{\Pi}(o))$ if, for any r , there are $f_i : B(o, r) \subset \mathbb{H}^3 \rightarrow \mathbb{H}^3$ which are equivariant ($\bar{\Pi}(z) = \bar{\Pi}(z') \Leftrightarrow \Pi_i(f_i(z)) = \Pi_i(f_i(z'))$) such that $(f_i)_i$ converges to the identity in the C^∞ topology (here $B(o, r)$ denotes the geodesic ball in \mathbb{H}^3). Actually defining φ_i by $\varphi_i(\bar{\Pi}(z)) = \Pi_i(f_i(z))$, we will often use the following consequence (see Lemma E.2.2 in [2]).

Lemma 2. *Let $(M_i)_i$ be a sequence of finite volume hyperbolic manifolds converging to \bar{M} in the geometric topology. Let $\varepsilon > 0$ be fixed, after eliminating some initial terms, there exists:*

- $(\sigma_i)_i$ with $\sigma_i > 0$ and $\sigma_i \rightarrow 0$,
- $(k_i)_i$ with $k_i > 1$ and $k_i \rightarrow 1$,
- for all i a k_i -quasi-isometric embedding φ_i from a neighborhood of $\bar{M}_{[\varepsilon, \infty)}$ into M_i .

with the following properties

- $\varphi_i(\bar{M}_{[\varepsilon, \infty)})$ is contained in the interior of $M_{i[\varepsilon - \sigma_i, \infty)}$ and
- $\varphi_i(\partial \bar{M}_{[\varepsilon, \infty)})$ does not meet an open neighborhood of $M_{i[\varepsilon + \sigma_i, \infty)}$.

Here k_i -quasi isometry must be understood as smooth maps φ_i such that

$$\frac{1}{k_i} d(p, q) \leq d(\varphi_i(p), \varphi_i(q)) \leq k_i d(p, q)$$

When we will use these properties, we will not forget that φ_i come from maps f_i that are C^∞ close to id.

Actually, ε is always chosen small enough such that the ε -thin part of \bar{M} contains only cusp ends. Moreover if ε is small enough each connected component of $M_{i[\varepsilon - \sigma_i, \varepsilon + \sigma_i]}$ contains exactly one component of $\varphi_i(\partial \bar{M}_{[\varepsilon, \infty)})$ (see Theorem E.2.4 in [2]). The description of this component of $\varphi_i(\partial \bar{M}_{[\varepsilon, \infty)})$ is given by the following result.

Lemma 3. *Let $(M_i)_i$, \bar{M} , $\varepsilon > 0$ and φ_i as above. Let C be a connected component of $\partial \bar{M}_{[\varepsilon, \infty)}$. Then for i large, $\varphi_i(C)$ is a graph of a function u_i over the corresponding component C_i of $\partial M_{i[\varepsilon, \infty)}$. Moreover $u_i \rightarrow 0$ and C and C_i are κ_i -quasi-isometric with $\kappa_i \rightarrow 1$.*

Proof. C is a surface with principal curvatures 1. Thus $\varphi_i(C)$ has principal curvatures close to 1 and between $1/2$ and $3/2$.

Since $\varphi_i(C) \subset A_i$ where A_i is a component of $M_{i[\varepsilon-\sigma_i, \varepsilon+\sigma_i]}$, $\varphi_i(C)$ is contained either in a cusp end of M_i or a neighborhood of a short geodesic of γ_i . In the second case, there is a smallest $\delta_i \leq \sigma_i$ such that $\varphi_i(C) \subset M_{i[\varepsilon-\sigma_i, \varepsilon+\delta_i]}$ so $\varphi_i(C)$ is tangent to a boundary torus of $\partial M_{i[\varepsilon+\delta_i, \infty)}$. The comparison of the mean curvature at this tangency point gives the mean curvature of $\partial M_{i[\varepsilon+\delta_i, \infty)}$ is close to 1. Thus the distance from γ_i to $\varphi_i(C)$ is very large and goes to $+\infty$.

In both cases, A_i is described as $T_i \times [-\alpha_i, \beta_i]$ with a metric $d\sigma_{i,t}^2 + dt^2$ with $\alpha_i, \beta_i \rightarrow 0$ and $C_i = T_i \times \{0\}$ is the boundary of $M_{i[\varepsilon, \infty)}$ in A_i .

Let γ be a geodesic in $\varphi_i(C)$. Since $\varphi_i(C)$ has curvature uniformly bounded, there is k_0 such that $|\partial_s(\gamma'(s), \partial_t)| \leq k_0$ so $|(\gamma'(s), \partial_t)| \geq |(\gamma'(0), \partial_t)|/2$ for $0 < s < s_0 = |(\gamma'(0), \partial_t)|/(2k_0)$. Looking at the t coordinate along γ , we then have $\beta_i + \alpha_i \geq |t(\gamma(s_0)) - t(\gamma(0))| \geq \frac{|(\gamma'(0), \partial_t)|^2}{4k_0}$. Since $\alpha_i + \beta_i \rightarrow 0$, this implies that the angle between $\varphi_i(C)$ and ∂_t goes to $\pi/2$ uniformly. Since $\varphi_i(C)$ is embedded this implies that $\varphi_i(C)$ is a graph over C_i : there is a function $u_i : C_i \rightarrow \mathbb{R}$ such that $\varphi_i(C) = \{(p, t) \in T \times \mathbb{R} | t = u_i(p)\}$. Since the angle between $\varphi_i(C)$ and ∂_t goes to $\pi/2$, the gradient of u_i goes to 0. Besides $\varphi_i(C) \subset A_i$ so $|u_i|$ is close to 0.

This implies that $(p, u_i(p)) \in \varphi_i(C) \mapsto (p, 0) \in C_i$ is a κ_i -quasi-isometry ($\kappa_i \rightarrow 1$) which can be composed with $\varphi_i : C \rightarrow \varphi_i(C)$ to obtain a $\kappa_i k_i$ -quasi-isometry. \square

As consequence, we have the following result.

Corollary 4. *Let V_0 be positive then there are $\ell_0, \delta_0, \mathfrak{s}_0$ such the following is true. Let M be a cusp manifold with volume less than V_0 and γ be a geodesic loop of length $\ell \leq \ell_0$. Then $S_{R_\ell} = \partial N_{R_\ell}$ has diameter less than δ_0 and systole larger than \mathfrak{s}_0 .*

The set of flat tori with diameter less than δ_0 and systole larger than \mathfrak{s}_0 is a compact subset of the set of flat tori.

Proof. If it not true there is a sequence of cusp manifolds M_i that converge to \overline{M} and in M_i there is a geodesic loop γ_i of length $\ell_i \rightarrow 0$ such that either the diameter of $S_{R_{\ell_i}}$ goes to ∞ or its systole goes to 0.

After taking a subsequence, we can assume that the tubular ends around γ_i converges to one cusp end in \overline{M} . Let $\varepsilon > 0$ be small and consider C the component of $\partial \overline{M}_{[\varepsilon, \infty)}$ inside this cusp end. Let C_i be the component of $\partial M_{i[\varepsilon, \infty)}$ inside the tubular end around γ_i . By the above lemma, C and C_i are 2 quasi-isometric. So the area C_i is close to that of C . Since the area of $S_{R_{\ell_i}}$ in M_i is close to $\sqrt{3}/2$ this implies that the distance between C_i and $S_{R_{\ell_i}}$ is uniformly bounded. Since the diameter and the systole of $S_{R_{\ell_i}}$ differ from those of C_i by at most a uniform factor. This contradicts that either the diameter goes to ∞ or the systole goes to 0. \square

Remark 1. Let us consider a particular one sided neighborhood of $\varphi_i(C)$ in M_i . Actually, let \bar{A} be the part of the 2-tubular neighborhood of C inside $\bar{M}_{[\varepsilon, \infty)}$. Thus $\varphi_i(\bar{A})$ is a one sided neighborhood of $\varphi_i(C)$.

\bar{A} can be parametrized by $T \times [-2, 0]$ with the metric $g = e^{-2t} d\bar{\sigma}^2 + dt^2$. Let $X : T \times [-2, 0] \rightarrow \bar{M}$ be this parametrization and (x_1, x_2) be orthonormal coordinates such that $g = e^{-2x_3} (dx_1^2 + dx_2^2) + dx_3^2$. Let us now estimate the metric $\varphi_i^* g_i$. We notice that X lifts to an equivariant map $\tilde{X} : \mathbb{R}^2 \times [-2, 0] \rightarrow \mathbb{H}^3$ i.e. $X = \bar{\Pi} \tilde{X}$. If $\tilde{g}_i = \tilde{g}_{i,kl} dx_k dx_l$ we have

$$\begin{aligned} \tilde{g}_{i,kl} &= \langle d\varphi_i X_{x_k}, d\varphi_i X_{x_l} \rangle_{M_i} \\ &= \langle d\varphi_i d\bar{\Pi} \tilde{X}_{x_k}, d\varphi_i d\bar{\Pi} \tilde{X}_{x_l} \rangle_{M_i} \\ &= \langle d\Pi_i df_i \tilde{X}_{x_k}, d\Pi_i df_i \tilde{X}_{x_l} \rangle_{M_i} \\ &= \langle df_i \tilde{X}_{x_k}, df_i \tilde{X}_{x_l} \rangle_{\mathbb{H}^3} \end{aligned}$$

since Π_i is a local isometry. Since f_i converges to the identity map in the C^∞ topology this implies that $\tilde{g}_i \rightarrow g$ in the C^∞ topology.

Remark 2. By Mostow rigidity theorem, the topology of a complete finite volume hyperbolic 3-manifold determines its hyperbolic structure. Thus if a converging sequence $M_i \rightarrow \bar{M}$ is not constant, there is a subsequence whose topologies are distinct from that of \bar{M} : there are short geodesics γ_i in M_i whose lengths converge to zero and whose maximal embedded tubular neighborhoods are converging to cusp ends of \bar{M} (see Figure 2).

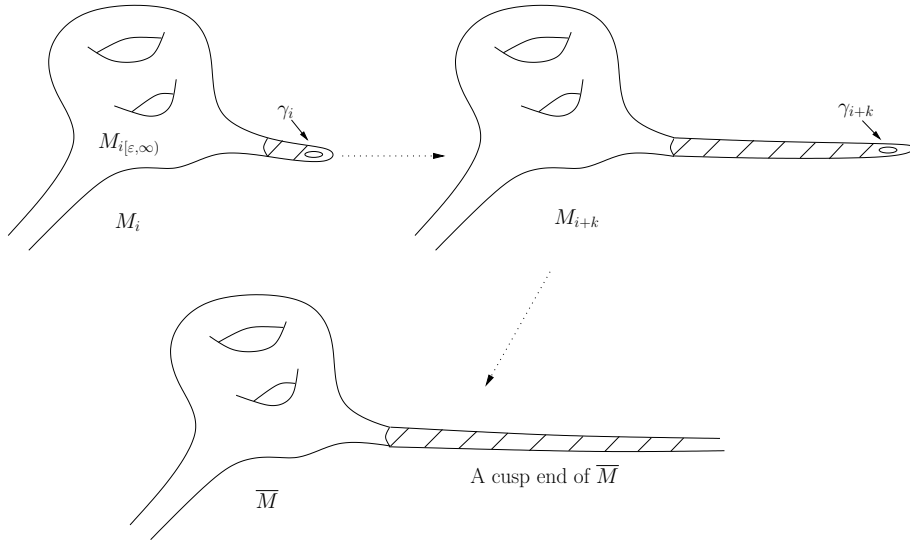


FIGURE 2. A schematic converging sequence $M_i \rightarrow \bar{M}$

3. TRANSVERSALITY IN TUBULAR ENDS

The aim of this section is to understand the behaviour of a minimal surface in a tubular end when we know a priori an upper bound on its curvature. A similar study was made for cusp ends in [4].

In this section, we use the tubular coordinates (z, θ, r) .

3.1. An intersection property. We recall that, if c is a geodesic in \mathbb{H}^3 , V_r denotes its tubular neighborhood of radius r . Moreover, for $r > 0$, we denote $B_r = \partial V_r$.

Lemma 5. *Let k_0 and ε_0 be positive, then there are r_0 and η_0 such that the following is true. Let c be a geodesic in \mathbb{H}^3 . Let $r \in [0, r_0]$ and $p_i = (z_i, \theta_i, r)$ ($i = 1, 2$) be two points in V_{r_0} such that $\theta_2 \in [\theta_1 + \frac{\pi}{3}, \theta_1 + \frac{2\pi}{3}]$ and $z_2 \in [z_1 - \eta_0, z_1 + \eta_0]$. Let Σ_i ($i = 1, 2$) be surfaces in V_{r_0} whose curvatures are bounded by k_0 , $p_i \in \Sigma_i$ and $d_{\Sigma_i}(p_i, \partial \Sigma_i) > \varepsilon_0$. If both Σ_i are tangent to B_r at p_i (if $r = 0$ we assume moreover that a unit normal vector to Σ_i at p_i is $\partial_r(z_i, \theta_i, 0)$) then Σ_1 and Σ_2 has non empty transversal intersection.*

Proof. We look for $r_0 \leq 2$. In V_2 the hyperbolic metric is $\cosh^2 r dz^2 + \sinh^2 r d\theta^2 + dr^2$. Let us change the metric in V_2 to the Euclidean metric $g_e = dz^2 + r^2 d\theta^2 + dr^2$. So there are constants \tilde{k}_0 and $\tilde{\varepsilon}_0$ depending only on k_0 and ε_0 such that, with g_e , Σ_1 and Σ_2 have curvature bounded by \tilde{k}_0 and $d_{\Sigma_i}(p_i, \partial \Sigma_i) > \tilde{\varepsilon}_0$.

Thus there is $\eta_1 > 0$ such that Σ_i can be described as a graph over the Euclidean disk of radius η_1 tangent to Σ_i at p_i (see Proposition 2.3 in [14]). Moreover if η_1 is chosen small enough, the gradient of the function parametrizing Σ_i is less than $1/10$.

Let $r_0 = \eta_0 = \eta_1/10$. With these choices, the tangent disks of radius η_1 tangent to Σ_i at p_i must intersect at an angle between $\pi/3$ and $2\pi/3$ (see the schematic figure 3). Moreover since each Σ_i is at a distance less than $\eta_1/10$ from its tangent disk, Σ_1 and Σ_2 must intersect and, as the gradient is less than $1/10$ and the angle between the disks is in $[\pi/3, 2\pi/3]$, the intersection is transverse. \square

3.2. The transversality result. The main result of the section is then the following. We recall that $S_r = \partial N_r$.

Proposition 6. *Let δ_0 , k_0 and ε_0 be positive, then there is $\ell_0 > 0$ and \bar{R} such that the following is true. Let $\ell \leq \ell_0$ and N_{R_ℓ} be the hyperbolic tubular neighborhood of a geodesic loop γ of length ℓ and such that the diameter of S_{R_ℓ} is less than δ_0 . Let Σ be an embedded minimal surface in N_{R_ℓ} whose curvature is bounded by k_0 . Let $\bar{r} < R_\ell - \bar{R}$ and p be a point in $\Sigma \cap S_{\bar{r}}$ such that $d_\Sigma(p, \partial \Sigma) > \varepsilon_0$. Then Σ is not tangent to $S_{\bar{r}}$ at p .*

We notice that for $\bar{r} = 0$, $S_{\bar{r}}$ is just the central geodesic γ so the proposition states that Σ can not be tangent to γ .

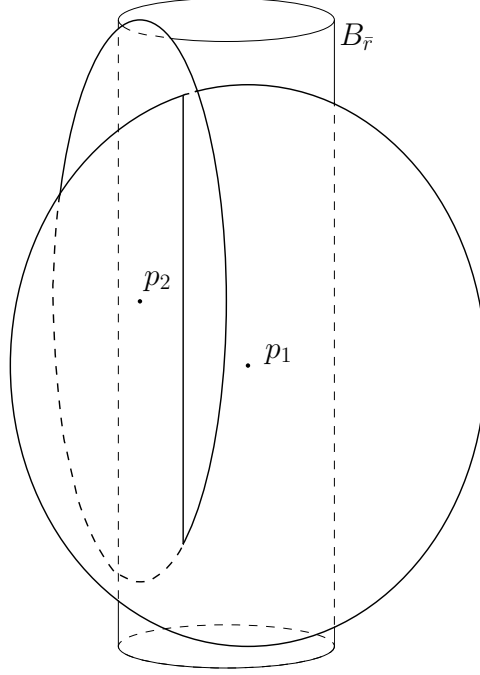


FIGURE 3.

Proof. We start with some ℓ_0 such that $R_\ell > 10$. Let $r_0 \leq 1$ and η_0 be given by Lemma 5 for k_0 and ε_0 (we assume $\varepsilon_0 \leq 1$). We first prove that the result is true if $\bar{r} \leq r_0$.

Let Σ be a minimal surface as in the statement of the proposition and assume that Σ is tangent at p to $S_{\bar{r}}$ for some \bar{r} . We consider the lift $\tilde{\Sigma}$ of Σ to \mathbb{H}^3 . $\tilde{\Sigma}$ is then contained in a solid cylinder V_{R_ℓ} .

The surface $\tilde{\Sigma}$ is then an embedded minimal surface (may be non connected) which is invariant by the action of the loxodromic transformation $\tau : (z, \theta, r) \mapsto (z + \ell, \theta + \alpha, r)$. Let p_1 be a lift of p . We can assume that $p_1 = (0, 0, \bar{r})$; if $\bar{r} = 0$, we assume that $\partial_r(0, 0, 0)$ is the unit normal vector to $\tilde{\Sigma}$.

S_{R_ℓ} has a diameter less than δ_0 . So, for any q in B_{R_ℓ} , the intrinsic disk of radius δ_0 in B_{R_ℓ} and center q must contain an image of $(0, 0, R_\ell)$ by some τ^n .

Let us consider the domain $A_r = \{(z, \theta, r) \in B_r \mid z \in [-\frac{\delta_0}{\cosh R_\ell}, \frac{\delta_0}{\cosh R_\ell}], \theta \in [\frac{\pi}{2} - \frac{\delta_0}{\sinh R_\ell}, \frac{\pi}{2} + \frac{\delta_0}{\sinh R_\ell}]\}$, A_{R_ℓ} is a square in B_{R_ℓ} whose edges have length $2\delta_0$. So A_{R_ℓ} contains an image of $(0, R_\ell, 0)$ by some τ^n . This implies that τ^n is the composition of a vertical translation by some $z' \in [-\frac{\delta_0}{\cosh R_\ell}, \frac{\delta_0}{\cosh R_\ell}]$ and a rotation by some $\theta' \in [\frac{\pi}{2} - \frac{\delta_0}{\sinh R_\ell}, \frac{\pi}{2} + \frac{\delta_0}{\sinh R_\ell}]$.

The point $p_2 = \tau^n(p_1) = (z_2, \theta_2, \bar{r})$ is another lift of p in $A_{\bar{r}}$. $\tilde{\Sigma}$ is then also tangent to $B_{\bar{r}}$ at p_2 . We have $|\theta_2 - \pi/2| \leq \delta_0/\sinh R_\ell$ and $|z_2| \leq \delta_0/\cosh R_\ell$. So we can choose ℓ_0 such that, for $\ell \leq \ell_0$, $\delta_0/\sinh R_\ell \leq \pi/6$ and $\delta_0/\cosh R_\ell \leq \eta_0$. Then we can apply Lemma 5 to the geodesic disks Σ_i of radius ε_0 in $\tilde{\Sigma}$ around p_i . Lemma 5 applies since, when $\bar{r} = 0$, the unit normal vector to Σ_2 at p_2 is $\partial_r(z_2, \theta_2, 0)$ with $|\theta_2 - \pi/2| \leq \delta_0/\sinh R_\ell$ (Σ_2 is the image of Σ_1 by τ^n). This gives that $\tilde{\Sigma}$ has transverse self-intersection which is impossible. So the result is proved for $\bar{r} \leq r_0$.

Let us now prove that we can extend this result to the region $r_0 \leq \bar{r} \leq R_L - \bar{R}$ for some $\bar{R} > 0$.

If the result is not true, for any $n > 0$, we can find a neighborhood $N_{R_{\ell_n}}$ of a closed geodesic γ_n of length $\ell_n \leq \frac{1}{n}$ and a minimal surface Σ_n in $N_{R_{\ell_n}}$ which is tangent to S_{r_n} at p_n for some $r_n \leq R_{\ell_n} - \frac{1}{4} \ln n$ (notice that $R_{\ell_n} - \frac{1}{4} \ln n > 0$). Actually because of the first part we can assume $r_n > r_0$. In the following we denote R_{ℓ_n} by R_n .

Let $\eta_1 = \min(r_0/10, \eta_0)$ and replace the sequence Σ_n by the sequence of η_1 -geodesic disks in Σ_n centered at p_n . So we can be sure that Σ_n never touches the central geodesic γ_n and stays outside of $N_{r_0 - \eta_1}$.

We lift Σ_n to M_{R_n} endowed with the metric (1). This gives us a minimal surface $\tilde{\Sigma}_n$ which is doubly periodic and may be non connected. $\tilde{\Sigma}_n$ is doubly periodic by translation in the (z, θ) parameters by two vectors v_1^n, v_2^n . Since T_{R_n} has diameter less than δ_0 we can choose v_1^n, v_2^n of Euclidean length less than $\frac{\delta_0}{\sinh R_n}$.

The point p_n lifts to some point \tilde{p}_n whose coordinates can be assumed to be $(0, 0, r_n)$ where $r_n \in (r_0, R_n - \frac{1}{2} \ln n)$. We can assume that either r_n converges to some \bar{r} or to $+\infty$. In the first case the ambient space around $(0, 0, \bar{r})$ is $M_\infty = \mathbb{R}^2 \times (0, +\infty)$ with the metric (1). If $r_n \rightarrow \infty$, we make the following change of coordinates $a = e^{r_n} z$, $b = e^{r_n} \theta$ and $\rho = r - r_n$. So the ambient space is now $\mathbb{R}^2 \times (r_0 - \eta_1 - r_n, R_n - r_n)$ with the metric

$$\cosh^2(\rho + r_n)e^{-2r_n} da^2 + \sinh^2(\rho + r_n)e^{-2r_n} db^2 + d\rho^2$$

As n goes to $+\infty$, these metrics converge smoothly to $\frac{e^{2\rho}}{4}(da^2 + db^2) + d\rho^2$ on \mathbb{R}^3 . In this model, the vectors v_1^n, v_2^n become $e^{r_n} v_1^n$ and $e^{r_n} v_2^n$ whose lengths are less than $\frac{\delta_0 e^{r_n}}{\sinh R_n} = O(e^{r_n - R_n}) = O(e^{-\frac{1}{2} \ln n}) \rightarrow 0$.

Actually, the cases $r_n \rightarrow \bar{r}$ and $r_n \rightarrow +\infty$ are very similar. Let us look first at the case $r_n \rightarrow \bar{r}$. We notice that the metric satisfies the hypotheses of Lemma 26 (Appendix A.1) for some parameter A and for $r \in [\bar{r} - \eta_1, R_n]$: we have $x_1 = z$, $x_2 = \theta$, $x_3 = r$ and $h = \sinh$. So there is a C and a function u_n defined on the Euclidean disk $\{(z, \theta) \in \mathbb{R}^2 | z^2 + \theta^2 \leq 2C^2/\sinh^2 r_n\}$ such that $(z, \theta) \mapsto (z, u_n(z, \theta), \theta)$ is a parametrization of a neighborhood of \tilde{p}_n in $\tilde{\Sigma}_n$. Moreover we have $u_n(0, 0) = r_n$, $\nabla u_n(0, 0) = 0$ and the estimates

$$\|u_n - r_n\| \leq A\varepsilon_0 \quad \|\nabla u_n\| \leq \sinh r_n \quad \|\text{Hess } u_n\| \leq \frac{1}{C} \sinh^2 r_n.$$

Here ∇ denote the Euclidean gradient operator.

So the sequence u_n is uniformly controlled in the C^2 topology and moreover u_n solves the minimal surface equation (2). Thus, after considering a subsequence, u_n converges to some u defined on $D_{\bar{r}} = \{(t, \theta) \in \mathbb{R}^2 | t^2 + \theta^2 \leq \frac{2C^2}{\sinh^2 \bar{r}}\}$ which solves the minimal surface equation.

If $r_n \rightarrow +\infty$, we apply the change of variables $a = e^{r_n}t$, $b = e^{r_n}\theta$ and $\rho = r - r_n$. So we get a new function $w_n(a, b) = u_n(e^{-r_n}a, e^{-r_n}b) - r_n$ defined on $\{(a, b) \in \mathbb{R}^2 | a^2 + b^2 \leq \frac{2C^2 e^{2r_n}}{\sinh^2 r_n}\}$. As above w_n satisfies the estimates

$$\|w_n\| \leq A\varepsilon_0 \quad \|\nabla w_n\| \leq e^{-r_n} \sinh r_n \quad \|\text{Hess } w_n\| \leq \frac{1}{C} e^{-2r_n} \sinh^2 r_n$$

and solves a minimal surface equation (2). So we can assume it converges to some function w defined on $\Delta = \{(a, b) \in \mathbb{R}^2 | a^2 + b^2 \leq 4C^2\}$.

Let us denote the surface $\{r = R\}$ by P_R . The surface $\tilde{\Sigma}_n$ is doubly periodic so it is tangent to P_{r_n} at any point of the form $(0, 0, r_n) + kv_1^n + lv_2^n$ for $(k, l) \in \mathbb{Z}^2$. Moreover, around these points, it is parametrized locally on $D_{r_n} + kv_1^n + lv_2^n$ by $(z, \theta) \mapsto (z, u_{n,k,l}(t, \theta), \theta)$ where $u_{n,k,l}(z, \theta) = u_n((z, \theta) - kv_1^n - lv_2^n)$. The surface $\tilde{\Sigma}_n$ is embedded, this implies that $u_n \leq u_{n,k,l}$ or $u_n \geq u_{n,k,l}$ on $D_n \cap (D_n + kv_1^n + lv_2^n)$ if it is non empty (notice that we can have $u_n \equiv u_{n,k,l}$ on the intersection).

If $r_n \rightarrow \bar{r}$, let v_0 be a vector in $D_{\bar{r}}$. Since $v_i^n \rightarrow 0$, there are sequences $(k_n)_n$ and $(l_n)_n$ such that $k_n v_1^n + l_n v_2^n \rightarrow v_0$. As $n \rightarrow \infty$, the sequence of functions u_{n,k_n,l_n} then converges to u_{v_0} on $D_{\bar{r}} + v_0$ where $u_{v_0}(\cdot) = u(\cdot - v_0)$. Because of $u_n \leq u_{n,k,l}$ or $u_n \geq u_{n,k,l}$, we get $u \leq u_{v_0}$ or $u \geq u_{v_0}$ on $D_{\bar{r}} \cap (D_{\bar{r}} + v_0)$.

If $r_n \rightarrow \infty$, we can do the same with the change of coordinates since $e^{r_n}v_i^n \rightarrow 0$. So for any $v_0 \in \Delta$, we have $w \leq w_{v_0}$ or $w \geq w_{v_0}$ on $\Delta \cap (\Delta + v_0)$ where $w_{v_0}(\cdot) = w(\cdot + v_0)$.

We now consider the case $r_n \rightarrow \bar{r}$ (the second one is similar). Let G be the totally geodesic surface in M_∞ tangent to $P_{\bar{r}}$ at $(0, \bar{r}, 0)$. As $\tilde{\Sigma}$, G can be described as the graph of a function h over $D_{\bar{r}}$. We have $h(0) = \bar{r}$ and there is some $\alpha > 0$ such that, over $D_{\bar{r}}$, $h(z, \theta) \geq \bar{r} + \alpha(z^2 + \theta^2)$. This second property comes from the fact that the principal curvatures of $P_{\bar{r}}$ with respect to ∂_r are $-\tanh \bar{r} < 0$ and $-\coth \bar{r} < 0$. The functions u and h are two solutions of the minimal surface equation (2) with the same value and the same gradient at the origin. So by Bers theorem, the function $u - h$ looks like a harmonic polynomial of degree at least 2.

If the degree of the polynomial is 2, one can find $v_0 \in D_{\bar{r}} \setminus \{(0, 0)\}$ such that $(u - h)(v_0) > 0$ and $(u - h)(-v_0) > 0$. Then we have

$$\begin{aligned} u(v_0) &> h(v_0) > h(0, 0) = u(0, 0) = u_{v_0}(v_0) \\ u_{v_0}(0, 0) &= u(-v_0) > h(-v_0) > h(0, 0) = u(0, 0) \end{aligned}$$

So this contradicts $u \leq u_{v_0}$ or $u \geq u_{v_0}$ on the whole $D_{\bar{r}} \cap (D_{\bar{r}} + v_0)$

If the degree is at least 3, the growth at the origin of h implies that $u \geq \bar{r}$ on a smaller disk $D' \subset D_{\bar{r}}$ and $u > \bar{r}$ on $D' \setminus \{(0, 0)\}$. So if $v_0 \in D' \setminus \{(0, 0)\}$

we have

$$u(v_0) > \bar{r} = u_{v_0}(v_0) \text{ and } u_{v_0}(0, 0) = u(-v_0) > \bar{r} = f(0, 0)$$

Once again, this contradicts $u \leq u_{v_0}$ or $u \geq u_{v_0}$ on the whole $D_{\bar{r}} \cap (D_{\bar{r}} + v_0)$

If $r_n \rightarrow \infty$, the same argument can be done with a totally geodesic surface tangent to the horosphere. \square

3.3. A first area estimate. The preceding result allows us to estimate the area of a minimal surface with bounded curvature in a tubular end.

Corollary 7. *Let δ_0 and k_0 be positive, then there is ℓ_0 and \bar{R} such that the following is true. Let $\ell \leq \ell_0$ and N_{R_ℓ} be the hyperbolic tubular neighborhood of a geodesic loop γ of length ℓ and such that the diameter of S_{R_ℓ} is less than δ_0 . Let $0 < R \leq R_\ell - \bar{R}$ and Σ be a compact embedded minimal surface in N_{R+1} whose curvature is bounded by k_0 and $\partial\Sigma \subset S_{R+1}$. Then one of the following possibilities occurs*

- (1) $\Sigma \cap N_R = \emptyset$
- (2) $\Sigma \cap N_R$ is a finite union of minimal disks. Each of these disks has boundary curve homotopic to a parallel of $S_R = \partial N_R$ and $|\Sigma \cap N_R| \geq 2\pi(\cosh R - 1)$.

A parallel of S_R is a curve $\{z = \text{const.}\}$ in the tubular coordinates.

Proof. Let ℓ_0 and \bar{R} be given by Proposition 6 for δ_0 , k_0 and $\varepsilon_0 = 1$. Let Σ be as in the statement of the corollary and assume $\Sigma \cap N_R \neq \emptyset$. By Proposition 6, Σ is transverse to the foliation $(S_r)_r$ of N_R . So any connected component of $\Sigma \cap N_R$ intersects the geodesic loop γ transversely. This implies that in N_ε for ε small each connected component of $\Sigma \cap N_\varepsilon$ is a disk whose boundary is homotopic to a parallel. Thus this description extends by transversality to $\Sigma \cap N_R$. Let Π be the geodesic projection from N_R to a geodesic parallel disk Δ (*i.e.* the map $(z, \theta, r) \mapsto (z_0, \theta, r)$ for some z_0). This map is a contraction mapping and it is surjective on any disk component of $\Sigma \cap N_R$ since the boundary of such a disk is homotopic to a parallel. As a consequence the area of such a disk component is at least that of Δ , *i.e.* $2\pi(\cosh(R) - 1)$. \square

4. A MAXIMUM PRINCIPLE

One aim of this section is to study some aspect of the behavior of minimal surfaces in a tubular end. Actually we need to study this in a more general setting. So we consider the ambient space $C = T \times [a, b]$ endowed with some reference metric $\bar{g} = h^2(x_3)d\bar{\sigma}^2 + dx_3^2$ where $d\bar{\sigma}^2$ is a flat metric on the torus T . We consider orthonormal coordinates (x_1, x_2) on T associated to $d\bar{\sigma}^2$; so $\bar{g} = h^2(x_3)(dx_1^2 + dx_2^2) + dx_3^2$.

On C , we also consider a second metric $g = a_{kl}(x_1, x_2, x_3)dx_k dx_l$. For $s \in [a, b]$, we denote $C_s = T \times [s, b]$ and $T_s = T \times \{s\}$. We are going to make several hypotheses on the metrics \bar{g} and g . In order to formulate them, we

need the following notation: for $k_1, k_2, k_3, k_4, k_5 \in \{1, 2, 3\}$ and $p \leq 5$, we define

$$n_p(k_1, \dots, k_p) = \#\{i \in \{1, \dots, p\} | k_i \in \{1, 2\}\}.$$

The hypotheses on \bar{g} and g are: there is $A \geq 1$ such that

$$\text{H1 } \frac{1}{A^2} \bar{g} \leq g \leq A^2 \bar{g}$$

$$\text{H2 } \frac{|h'|}{h} \leq A, \frac{|h''|}{h} \leq A \text{ and } \frac{|h'''|}{h} \leq A.$$

$$\text{H3 } |a_{kl}| \leq Ah^{n_2(k,l)}(x_3), |\partial_i a_{kl}| \leq Ah^{n_3(k,l,i)}(x_3), |\partial_i \partial_j a_{kl}| \leq Ah^{n_4(k,l,i,j)}(x_3) \\ \text{and } |\partial_i \partial_j \partial_m a_{kl}| \leq Ah^{n_5(k,l,i,j,m)}(x_3).$$

$$\text{H4 } h' \leq 0 \text{ and the mean curvature vector of } T_s \text{ with respect to } g \text{ points} \\ \text{in the } \partial_{x_3} \text{ direction (this is also true for the metric } \bar{g} \text{ since } h' \leq 0)$$

One consequence of H1 and H2 is that the sectional curvatures of \bar{g} are uniformly bounded. Actually by H1 and H3 the sectional curvatures of g are also uniformly bounded. We also notice that these hypotheses does not depend on the choice of the orthonormal coordinates on $(T, d\bar{\sigma}^2)$.

4.1. The maximum principle. We have the following maximum principle for embedded minimal surfaces in C endowed with the metric g .

Proposition 8. *Let $i_0 \in \mathbb{N}$, then there is h_0 such the following is true. Assume that $h(a) \leq h_0$ and let Σ be an embedded minimal surface in (C, g) whose non empty boundary is inside T_a and its index is less than i_0 . Then $\Sigma \cap C_{a+1/2} = \emptyset$.*

We notice that h_0 will depend on i_0 , A and the metric $d\bar{\sigma}^2$. We also notice that this control on h is actually a control on the size of the torus T_a .

Proof. If the proposition is not true there is a sequence of functions h_n with $h_n(a) \rightarrow 0$ and minimal surfaces $S_n \subset (C, g_n)$ ($g_n = a_{n,kl}(x_1, x_2, x_3)dx_k dx_l$) such that $\partial S_n \subset T_a$, its index is less than i_0 and $\Sigma \cap C_{a+1/2} \neq \emptyset$.

Let s_n be the maximum of the x_3 coordinate on S_n , $x_3 \geq a + 1/2$. Let us define $\lambda_n = (h_n(s_n))^{-1}$. Then we change the coordinates by $y_1 = x_1$, $y_2 = x_2$ and $y_3 = \lambda_n(x_3 - s_n)$ and blow up the metric by a factor λ_n . This gives us a minimal surface Σ_n in $T \times [\lambda_n(a - s_n), 0]$ that touches T_0 and with boundary in $T_{\lambda_n(a-s_n)}$ (we notice that $\lambda_n(a - s_n) \rightarrow -\infty$). The ambient metric is then $\tilde{g}_n = b_{n,kl}(y_1, y_2, y_3)dy_k dy_l$ where

$$b_{n,kl}(y_1, y_2, y_3) = a_{n,kl}(y_1, y_2, y_3/\lambda_n + s_n)\lambda_n^{n_2(k,l)}$$

The reference metric becomes

$$\frac{h_n^2(y_3/\lambda_n + s_n)}{h_n^2(s_n)}(dy_1^2 + dy_2^2) + dy_3^2$$

Because of the hypothesis H2, considering a subsequence, this metric converges to the flat metric $d\bar{\sigma}^2 + dy_3^2$ in $C^{2,\alpha}$ topology. Because of H3, considering a new subsequence, the metrics \tilde{g}_n converges to a flat metric $\bar{h} = \bar{b}_{kl}dy_k dy_l$ in $C^{2,\alpha}$ topology. For example we have

$$\partial_i b_{n,kl} = \partial_i a_{n,kl}(y_1, y_2, y_3/\lambda_n + s_n)\lambda_n^{n_{k,l,i}-1}$$

So

$$|\partial_i b_{n,kl}| \leq A \frac{h_n^{n_3(k,l,i)}(y_3/\lambda_n + s_n)}{h_n^{n_3(k,l,i)}(s_n)} h_n(s_n) \rightarrow 0$$

Once this is known, the arguments in order to conclude use the fact that Σ_n converges to a minimal lamination in $T^2 \times \mathbb{R}_-$ endowed with the flat metric \bar{h} : the precise argument can be found in the proof of Proposition 1 in [5]. \square

Remark 3. We notice that h_0 can be chosen uniformly if $d\bar{\sigma}^2$ lies in a compact subset of flat metrics on T .

4.2. Some applications. In this section, we will see some consequences of the above result.

The case of cusp ends $E = T \times \mathbb{R}_+$ endowed with $\bar{g} = e^{-2x_3} d\bar{\sigma}^2 + dx_3^2$ is the simplest one. Indeed in this case the metric g is the reference metric \bar{g} and $h(x_3) = e^{-x_3}$. Then hypotheses H1 to H4 are satisfied. So Proposition 8 yields: if ∂E has small diameter, then no compact embedded minimal surface with boundary inside ∂E and index less than 1 can enter in $E_{1/2}$. As a consequence, in a cusp manifold M , there is $\varepsilon > 0$ such that any compact embedded minimal surface in M with index less than 1 is contained in $M_{[\varepsilon, \infty)}$.

The second case of interest concerns the tubular ends.

Proposition 9. *Let K be a compact set of flat tori T . Then there are ℓ_0 and \bar{R} such the following is true. Let $\ell \leq \ell_0$ and N_{R_ℓ} be a hyperbolic tubular neighborhood of a geodesic loop of length ℓ such that S_{R_ℓ} belongs in K . Let $0 < R \leq R_\ell - \bar{R}$ and Σ be a compact embedded minimal surface in N_{R+1} with $\partial\Sigma \subset S_{R+1}$ with index less than 1. Then one of the following possibilities occurs*

- (1) $\Sigma \cap N_R = \emptyset$
- (2) $\Sigma \cap N_1 \neq \emptyset$

Moreover there is a universal constant κ such that, in the second case and for any $3 \leq R \leq R_\ell - \bar{R}$, $|\Sigma \cap N_R| \geq \kappa \mathfrak{s}_0 e^{R-R_\ell}$ where $\mathfrak{s}_0 \leq 1$ is a lower bound on the systole of T_{R_ℓ} .

Proof. We first prove that $\Sigma \cap N_R = \emptyset$ or $\Sigma \cap N_1 \neq \emptyset$. We have seen in Section 2.1 that we can consider, on N_{R_ℓ} , a coordinate system $C = T \times [0, R_\ell]$ endowed with the metric $g = \sinh^2(R_\ell - x_3) dx_1^2 + \cosh^2(R_\ell - x_3) dx_2^2 + dx_3^2$ ((x_1, x_2) are orthonormal coordinates on T).

In order to fit with the notations of the preceding section we should introduce the coordinates $y_i = \sinh(R_\ell) x_i$ $i = 1, 2$ and $y_3 = x_3$. The first two are orthonormal coordinates on $(T, d\bar{\sigma}^2) = \sinh(R_\ell)^2 (dx_1^2 + dx_2^2)$. So we define $\bar{g} = h^2(x_3)(dy_1^2 + dy_2^2) + dy_3^2$ with $h(x) = \sinh(R_\ell - x)/\sinh(R_\ell)$ and in these coordinates the metric g can be written

$$\frac{1}{\sinh^2(R_\ell)} (\sinh^2(R_\ell - y_3) dy_1^2 + \cosh^2(R_\ell - y_3) dy_2^2) + dy_3^2$$

There is $A > 0$ (that does not depend on ℓ) such that g and \bar{g} satisfy hypotheses H1, H2, H3 and H4 on $T \times [0, R_\ell - \frac{1}{2})$. Moreover we notice that, since S_{R_ℓ} belongs to K , $(T, d\bar{\sigma}^2)$ belongs to a compact set of flat tori.

Let h_0 be given by Proposition 8 and let \bar{R} be such that $\sinh(R_\ell - \bar{R}) \leq h_0 \sinh(R_\ell)$. Consider $0 < R \leq R_\ell - \bar{R}$ and let Σ be an embedded minimal surface in $N_{R+1} \setminus N_{\frac{1}{2}}$ with $\partial\Sigma \subset S_{R+1}$ and index less than 1. If $\Sigma \cap N_1 = \emptyset$, Σ can be seen as a minimal surface in (C, g) with boundary in T_s where $s = R_\ell - (R + 1)$. Since $h(s) \leq h_0$, Proposition 8 gives $\Sigma \cap C_{s+1/2} = \emptyset$. So in the tubular coordinates, we have $\Sigma \cap N_R = \emptyset$.

In the second case we now prove the area estimate. For this we use the tubular coordinates. We notice that Σ must meet all the tori S_r for $1 \leq r \leq R + 1$.

Since $g \leq \cosh^2 r(dz^2 + d\theta^2) + dr^2$ and the systole of T_{R_ℓ} is at least \mathfrak{s}_0 , the disk $\{(z + z_0, \theta + \theta_0, R_\ell) \mid z^2 + \theta^2 \leq \frac{\mathfrak{s}_0^2}{4 \cosh^2 R_\ell}\}$ is embedded in S_{R_ℓ} for any t_0, θ_0 . For any $\rho \in [3/2, R_\ell]$, let us define $a = \frac{\sinh \rho}{\cosh R_\ell} \frac{\mathfrak{s}_0}{4} \leq \frac{1}{4}$. The cylinder $Y_\rho = \{(z + z_0, \theta + \theta_0, r) \mid r \in [\rho - 2a, \rho] \text{ and } z^2 + \theta^2 \leq \frac{\mathfrak{s}_0^2}{4 \cosh^2 R_\ell}\}$ is embedded in N_ρ . Y_ρ contains the geodesic ball of center $(z_0, \theta_0, \rho - a)$ and radius a which is then embedded in N_ρ . Indeed, in the cylinder, we have

$$g \geq \sinh^2(\rho - 2a)(dz^2 + d\theta^2) + dr^2 \geq \frac{1}{4} \sinh^2 \rho (dz^2 + d\theta^2) + dr^2$$

So the geodesic ball is contained in $\{(z + z_0, \theta + \theta_0, r + \rho - a) \mid \frac{1}{4} \sinh^2 \rho (z^2 + \theta^2) + r^2 \leq a^2\}$ which is a subset of Y_ρ .

Since Σ meets any S_r for $r \geq 1$, for any ρ we can select z_0, θ_0 such that $(z_0, \theta_0, \rho - a) \subset \Sigma$. So by the monotonicity formula in \mathbb{H}^3 , $|\Sigma \cap Y_\rho| \geq \pi a^2$. We are going to sum over all these contributions to estimate the area of Σ .

Let $c(s) = (z(s), \theta(s), R_\ell)$ be a parametrization of a systole of S_{R_ℓ} and consider the surface S in N_{R_ℓ} parametrized by $X : (s, r) \in \mathbb{S}^1 \times [1, R_\ell] \mapsto (z(s), \theta(s), r)$. So, for $\rho_1 < \rho_2$, we can estimate

$$\begin{aligned} |S \cap (N_{\rho_2} \setminus N_{\rho_1})| &\leq \int_{\rho_1}^{\rho_2} \int_{\mathbb{S}^1} (\cosh^2(r) z'^2(s) + \sinh^2(r) \theta'^2(s))^{1/2} dr ds \\ &\leq \int_{\rho_1}^{\rho_2} \int_{\mathbb{S}^1} \frac{\cosh r}{\sinh R_\ell} (\cosh^2(R_\ell) z'^2(s) + \sinh^2(R_\ell) \theta'^2(s))^{1/2} dr ds \\ &\leq \frac{\mathfrak{s}_0}{\sinh R_\ell} (\sinh \rho_2 - \sinh \rho_1) \\ &\leq \frac{2\mathfrak{s}_0}{\sinh R_\ell} \cosh \frac{\rho_1 + \rho_2}{2} \sinh \frac{\rho_2 - \rho_1}{2} \end{aligned}$$

So in $N_\rho \setminus N_{\rho-2a}$

$$\begin{aligned}
 |S \cap (N_\rho \setminus N_{\rho-2a})| &\leq \frac{2\mathfrak{s}_0}{\sinh R_\ell} \cosh(\rho - a) \sinh a \\
 &\leq \frac{2\kappa\mathfrak{s}_0}{\sinh R_\ell} \cosh(\rho)a \\
 &\leq \frac{2\kappa'\mathfrak{s}_0}{\cosh R_\ell} \sinh(\rho)a \\
 &\leq 8\kappa'a^2 \leq \frac{8\kappa'}{\pi} |\Sigma \cap Y_\rho| \leq \frac{8\kappa'}{\pi} |\Sigma \cap (N_\rho \setminus N_{\rho-2a})|
 \end{aligned}$$

for some universal constant κ and κ' .

So considering a disjoint union of $N_\rho \setminus N_{\rho-2a}$ in $N_R \setminus N_1$ that covers $N_R \setminus N_{3/2}$, we obtain

$$\begin{aligned}
 |\Sigma \cap N_R| &\geq \frac{\pi}{8\kappa'} |S \cap (N_R \setminus N_{3/2})| \\
 &\geq \frac{\pi}{8\kappa'} \int_{3/2}^R \int_{\mathbb{S}^1} (\cosh^2(r)y'^2(s) + \sinh^2(r)\theta'^2(s))^{1/2} dr ds \\
 &\geq \frac{\pi}{8\kappa''} \frac{\mathfrak{s}_0}{\cosh R_\ell} (\cosh R - \cosh 3/2) \geq \kappa''' \mathfrak{s}_0 e^{R-R_\ell}
 \end{aligned}$$

for any $R \geq 3$ and some universal constant κ''' . □

5. THE MIN-MAX THEORY

In this section we recall some definitions and results of the min-max theory for minimal surfaces. There are basically two settings for this theory: the discrete and the continuous one. We recall the main points that we will use in the sequel.

5.1. The discrete setting. The discrete setting for the min-max theory was developed by Almgren and Pitts (see [1, 13]).

Let M be a compact orientable 3-manifold with no boundary. The Almgren-Pitts min-max theory deals with discrete families of elements in $\mathcal{Z}_2(M)$ i.e. integral rectifiable 2-currents in M with no boundary.

If $I = [0, 1]$, we define some cell complex structure on I and I^2 .

Definition 10. Let j be an integer. $I(1, j)$ is the cell complex on I whose 0-cells are points $[\frac{i}{3j}]$ and 1-cells are intervals $[\frac{i}{3j}, \frac{i+1}{3j}]$.

The cell complex $I(2, j)$ on I^2 is $I(2, j) = I(1, j) \otimes I(1, j)$.

For these cell complexes we can associate some notations

- $I(m, j)_0$ denotes the set of 0-cells of $I(m, j)$.
- $I_0(1, j)$ denotes the set of 0-cells $[0], [1]$.

- The distance between two elements of $I(m, j)_0$ is

$$\mathbf{d} : I(m, j)_0 \times I(m, j)_0 \rightarrow \mathbb{N} ; (x, y) \mapsto 3^j \sum_{i=1}^m |x_i - y_i|$$

- The projection map $n(i, j) : I(m, i)_0 \rightarrow I(m, j)_0$ is defined by $n(i, j)(x)$ is the unique element in $I(m, j)_0$ such that

$$\mathbf{d}(x, n(i, j)(x)) = \inf\{\mathbf{d}(x, y), y \in I(1, j)_0\}.$$

Let $\varphi : I(m, j)_0 \rightarrow \mathcal{Z}_2(M)$ be a map. The fineness of φ is defined by

$$\mathbf{f}(\varphi) = \sup \left\{ \frac{\mathbf{M}(\varphi(x) - \varphi(y))}{\mathbf{d}(x, y)}, x, y \in I(m, j)_0 \text{ and } x \neq y \right\}$$

where \mathbf{M} is the mass of a current.

We write $\varphi : I(1, j)_0 \rightarrow (\mathcal{Z}_2(M), \{0\})$ to mean $\varphi(I(1, j)_0) \subset \mathcal{Z}_2(M)$ and $\varphi(I_0(1, j)) = \{0\}$.

Definition 11. Let δ be positive and $\varphi_i : I(1, k_i)_0 \rightarrow (\mathcal{Z}_2(M), \{0\})$ for $i = 1, 2$. φ_1 and φ_2 are 1-homotopic in $(\mathcal{Z}_2(M), \{0\})$ with fineness δ if there is $k_3 \in \mathbb{N}$, $\max(k_1, k_2) \leq k_3$ and a map

$$\psi : I(2, k_3)_0 \rightarrow \mathcal{Z}_2(M)$$

such that

- $\mathbf{f}(\psi) \leq \delta$;
- $\psi([i-1], x) = \varphi_i(n(k_3, k_i)(x))$ for $x \in I(1, k_3)_0$;
- $\psi(I(1, k_3)_0 \times \{[0], [1]\}) = 0$.

The main objects in the discrete min-max theory are the $(1, \mathbf{M})$ -homotopy sequences.

Definition 12. A $(1, \mathbf{M})$ -homotopy sequence of maps into $(\mathcal{Z}_2(M), \{0\})$ is a sequence of maps $\{\varphi_i\}_{i \in \mathbb{N}}$,

$$\varphi_i : I(1, k_i)_0 \rightarrow (\mathcal{Z}_2(M), \{0\}),$$

such that φ_i is 1-homotopic to φ_{i+1} in $(\mathcal{Z}_2(M), \{0\})$ with fineness δ_i and

- $\lim_{i \rightarrow \infty} \delta_i = 0$;
- $\sup_i \{\mathbf{M}(\varphi_i(x)), x \in I(1, k_i)_0\} < +\infty$.

Moreover we have a notion of discrete homotopy between $(1, \mathbf{M})$ -homotopy sequences

Definition 13. Let $S_j = \{\varphi_i^j\}_{i \in \mathbb{N}}$ ($j = 1, 2$) be two $(1, \mathbf{M})$ -homotopy sequences of maps into $(\mathcal{Z}_2(M), \{0\})$. S_1 is homotopic to S_2 if there is a sequence $\{\delta_i\}_{i \in \mathbb{N}}$ such that

- $\lim \delta_i = 0$;
- φ_i^1 is 1-homotopic to φ_i^2 in $(\mathcal{Z}_2(M), 0)$ with fineness δ_i .

This notion defines an equivalence relation between $(1, \mathbf{M})$ -homotopy sequences. The set of equivalence classes is denoted by $\pi_1^\#(\mathcal{Z}_2(M), \mathbf{M}, \{0\})$. The Almgren-Pitts theory says that $\pi_1^\#(\mathcal{Z}_2(M), \mathbf{M}, \{0\})$ is naturally isomorphic to the homology group $H_3(M, \mathbb{Z})$ (see Theorem 4.6 in [13] and [1]). We denote by Π_M the element of $\pi_1^\#(\mathcal{Z}_2(M), \mathbf{M}, \{0\})$ that corresponds to the fundamental class in $H_3(M)$. If $S \in \Pi_M$ we say that S is a discrete sweep-out of M .

For $S = \{\varphi_i\}_i$ a $(1, \mathbf{M})$ -homotopy sequence we define

$$\mathbf{L}(S) = \limsup_{i \rightarrow \infty} \max\{\mathbf{M}(\varphi_i(x)), x \in I(1, k_i)_0\}$$

If $\Pi \in \pi_1^\#(\mathcal{Z}_2(M), \mathbf{M}, \{0\})$ is an equivalence class, we define the width associated to Π by

$$W(\Pi) = \inf\{\mathbf{L}(S), S \in \Pi\}$$

For $\Pi = \Pi_M$, the number $W_M = W(\Pi_M)$ is call the width of the manifold M . The Almgren-Pitts theory says that this number is positive and is $\mathbf{L}(S)$ for some particular $S \in \Pi_M$. If $S = \{\varphi_i\}_i$ we say that $\varphi_{i_j}(x_j)_j$ ($x_j \in I(1, k_{i_j})$) is a min-max sequence if $\mathbf{M}(\varphi_{i_j}(x_j)) \rightarrow W_M$.

Theorem 14 (Pitts [13]). *Let M be a closed 3-manifold, then there is $S = \{\varphi_i\}_i \in \Pi_M$ with $\mathbf{L}(S) = W_M$ and a min-max sequence $\{\varphi_{i_j}(x_j)\}_j$ that converges (in the varifold sense) to an integral varifold whose support is a finite collection of embedded connected disjoint minimal surfaces $\{S_i\}_i$ of M . So there are positive numbers $\{n_i\}_i$ such that*

$$W_M = \sum_{i=1}^p n_i |S_i|$$

A consequence of this result is that there is always a minimal surface S in M such that $|S| \leq W_M$. Actually, Zhou [19] proved that, if S_i is a non orientable minimal surface produced by the above theorem, then n_i is even.

5.2. The continuous setting. The continuous setting was developed by Colding and De Lellis [3]. Here we present it with the modifications made by Song in [18].

Let M be Riemannian 3-manifold and consider $N \subset M$ a bounded open subset whose boundary ∂N , when non empty, is a rectifiable surface of finite \mathcal{H}^2 -measure. Moreover when $\partial N \neq \emptyset$, we assume that each connected component C of ∂N separates M .

If $a < b \in \mathbb{R}$, we then have the following definitions.

Definition 15. *A family of \mathcal{H}^2 -measurable closed subsets $\{\Gamma_t\}_{t \in [a, b]}$ in $N \cup \partial N$ with finite \mathcal{H}^2 -measure is called a generalized smooth family if*

- for each t there is a finite set $P_t \in N$ such that $\Gamma_t \cap N$ is a smooth surface in $N \setminus P_t$ or the empty set;
- $\mathcal{H}^2(\Gamma_t)$ depends continuously in t and $t \mapsto \Gamma_t$ is continuous in the Hausdorff sense;

- on any $U \subset\subset N \setminus P_{t_0}$, $\Gamma_t \xrightarrow{t \rightarrow t_0} \Gamma_{t_0}$ smoothly in U .

We notice the smoothness hypothesis is only made on $\Gamma_t \cap N$ so this allows ∂N to be non smooth. We now define the notion of continuous sweep-out in this setting.

Definition 16. *If $\partial N = \emptyset$, a generalized smooth family $\{\Gamma_t\}_{t \in [a,b]}$ is called a continuous sweep-out of N if there exists a family of open subsets $\{\Omega_t\}_{t \in [a,b]}$ of N such that*

(sw1) $(\Gamma_t \setminus \partial\Omega_t) \subset P_t$ for any t ;

(sw2) $\mathcal{H}^3(\Omega_t \triangle \Omega_s) \rightarrow 0$ as $t \rightarrow s$ (where \triangle denotes the symmetric difference of subsets).

(sw3) $\Omega_a = \emptyset$ and $\Omega_b = N$;

If $\partial N \neq \emptyset$, for a open subset $\Omega \subset N$ we denote $\partial_*\Omega = \partial\Omega \cap N$. A continuous sweep-out of N is then required to satisfy (sw1) and (sw2) above except that ∂ is replaced by ∂_* and $t > a$ in (sw1). Moreover (sw3) is replaced by

(sw3') $\Omega_a = \emptyset$, $\Omega_b = N$, $\Sigma_a = \partial N$ and $\Sigma_t \subset N$ for $t > a$.

For a continuous sweep-out as above $\{\Gamma_t\}_{t \in [a,b]}$, we define the quantity $\mathbf{L}(\{\Gamma_t\}) = \max_{t \in [a,b]} \mathcal{H}^2(\Gamma_t)$. When ∂N is a smooth surface, constructing a continuous sweep-out can be done in the following way. Let $f : N \rightarrow [0, 1]$ be a Morse function such that $\{f^{-1}(0)\} = \partial N$. Then if $\Gamma_t = f^{-1}(t)$ for $t \in [0, 1]$, $\{\Gamma_t\}_{t \in [0,1]}$ is a sweep-out of N .

Two continuous sweep-outs $\{\Gamma_t^1\}_{t \in [a,b]}$ and $\{\Gamma_t^2\}_{t \in [a,b]}$ are said to be homotopic if, informally, they can be continuously deformed one to the other (the precise definition is Definition 8 in [18]). Then a family Λ of sweep-outs is called homotopically closed if it contains the homotopy class of each of its elements. For such a family Λ , we can define the width associated to Λ as

$$W(N, \partial N, \Lambda) = \inf_{\{\Gamma_t\} \in \Lambda} \mathbf{L}(\{\Gamma_t\})$$

As in the discrete setting this number can be realized as the mass of some varifold supported by smooth disjoint minimal surfaces (see Theorem 12 in [18]).

5.3. From continuous to discrete. In order to construct discrete sweep-outs of a closed orientable 3-manifold M , we will use a result obtained by Zhou (see Theorem 5.1 in [20]). We denote by $\mathcal{C}(M)$ the space of subsets in M with finite perimeter. Let \mathcal{F} denote the flat metric on $\mathcal{Z}_2(M)$.

Theorem 17. *Let $\Phi : [0, 1] \rightarrow (\mathcal{Z}_2(M), \mathcal{F})$ be a continuous map such that*

- $\Phi(0) = 0 = \Phi(1)$;
- $\Phi(x) = \partial[[\Omega_x]]$, $\Omega_x \in \mathcal{C}(M)$ for all $x \in [0, 1]$;
- $\Omega_0 = \emptyset$ and $\Omega_1 = M$
- $\sup_x \mathbf{M}(\Phi(x)) < +\infty$.

Then there exists a discrete sweep-out S such that

$$\mathbf{L}(S) \leq \sup_{x \in [0,1]} \mathbf{M}(\Phi(x))$$

Here $[[\Omega]]$ denotes the element of $\mathcal{Z}_3(M)$ corresponding to Ω .

Let us notice that if $\{\Gamma_t\}_{t \in [0,1]}$ is a continuous sweep-out of a compact orientable Riemannian 3-manifold then $\Phi : t \mapsto [[\Gamma_t]] \in \mathcal{Z}_2(M)$ satisfies the hypotheses of the above theorem (as above $[[\Gamma_t]]$ denotes the element of $\mathcal{Z}_2(M)$ corresponding to Γ_t).

6. THE QUANTITY $\mathcal{A}_1(M)$

In this section we recall some results the authors obtained in [11] (see also [5]).

6.1. The quantity $\mathcal{A}_1(M)$ for compact M . If M is a closed orientable Riemannian 3-manifold, we denote by \mathcal{O} the collection of all smooth orientable embedded closed minimal surfaces in M and \mathcal{U} the collection of all smooth non-orientable ones. We then define

$$\mathcal{A}_1(M) = \inf(\{|\Sigma|, \Sigma \in \mathcal{O}\} \cup \{2|\Sigma|, \Sigma \in \mathcal{U}\})$$

One result of [11] is the following theorem (Theorem B in [11])

Theorem 18. *Let M be an oriented closed Riemannian 3-manifold. Then $\mathcal{A}_1(M)$ is equal to one of the following possibilities.*

- (1) $|\Sigma|$ where $\Sigma \in \mathcal{O}$ is a min-max surface of M associated to the fundamental class of $H_3(M)$, Σ has index 1, is separating and $\mathcal{A}_1(M) = W_M$.
- (2) $|\Sigma|$ where $\Sigma \in \mathcal{O}$ is stable.
- (3) $2|\Sigma|$ where $\Sigma \in \mathcal{U}$ is stable and its orientable 2-sheeted cover has index 0.

Moreover, if $\Sigma \in \mathcal{O}$ satisfies $|\Sigma| = \mathcal{A}_1(M)$, then Σ is of type 1 or 2 and if $\Sigma \in \mathcal{U}$ satisfies $2|\Sigma| = \mathcal{A}_1(M)$, then Σ is of type 3.

Actually in [11], the case (3) mentions the possibility for the orientable 2-sheeted cover to have index 0 or 1. In fact, the index 1 case can be ruled out thanks to the work of Ketover, Marques and Neves [10].

If \mathcal{S} denotes the collection of all smooth embedded stable minimal surfaces, we define $\mathcal{A}_{\mathcal{S}}(M) = \inf(\{|\Sigma|, \Sigma \in \mathcal{O} \cap \mathcal{S}\} \cup \{2|\Sigma|, \Sigma \in \mathcal{U} \cap \mathcal{S}\})$. Actually we proved in [11] that $\mathcal{A}_1(M) = \min(W_M, \mathcal{A}_{\mathcal{S}}(M))$. In order to simplify some notations, we will denote $a(\Sigma) = |\Sigma|$ if $\Sigma \in \mathcal{O}$ and $a(\Sigma) = 2|\Sigma|$ if $\Sigma \in \mathcal{U}$.

When M is not compact, one can still define \mathcal{O} and \mathcal{U} for M by considering only compact embedded minimal surfaces in M . Of course these collections could be empty but if it is not $\mathcal{A}_1(M)$ is well defined. If M is a cusp manifold this can be done as we show in section 6.3.

6.2. The filler. We want to study $\mathcal{A}_1(M)$ when M is a cusp manifold. In order to do that the idea is to change M into a compact manifold $D(M)$ that contains all the compact minimal surfaces of M . To do this the main tool are the fillers.

Definition 19. *Let $(T, d\sigma^2)$ be a flat torus and $L > 10$ be a real number. A filler F associated to T and L is a solid torus endowed with a Riemannian metric g with the following properties.*

- (i) *Let T_t be the set of points at distance t from ∂F . For $t \in [0, L + 1)$, T_t is a smooth flat torus and T_{L+1} is a closed geodesic.*
- (ii) *The diameter of T_t is a decreasing function and the mean curvature vector points in the ∂_t direction.*
- (iii) *For $t \in [0, 1]$, T_t has the metric $e^{-2t}d\sigma^2$.*
- (iv) *Any minimal surface Σ that meets all the T_t for $0 \leq t \leq L - 1$ has area at least κL where κ is a constant depending on the systole of $(T, d\sigma^2)$.*

Proposition 20. *Let $(T, d\sigma^2)$ be a flat torus and $L > 10$. There exists a filler associated to T and L . Moreover, let δ and \mathfrak{s} be the diameter and the systole of $(T, d\sigma^2)$ and $K \leq \mathfrak{s}/\delta$. Then there is $\delta_0 > 0$ that depends only on K such that*

- (v) *if δ is less than δ_0 , then any minimal surface Σ with $\partial\Sigma \subset \partial F$ and index at most 1 satisfies either $\Sigma \cap T_1 = \emptyset$ or $\Sigma \cap T_t \neq \emptyset$ for any $t \leq L - 1$.*

Proof. We construct F as $T \times [L + 1]$ with a Riemannian metric which is singular on $T_{L+1} = T \times \{L + 1\}$ in order for T_{L+1} to be a geodesic. We use the notation $F_t = T \times [t, L + 1]$.

Let $f : [0, L + 1] \rightarrow \mathbb{R}$ be a function satisfying the following properties

- $f(t) = t$ on $[0, 1]$,
- $f' > 0$ on $[0, L + 2/3)$ and $f' = 0$ on $[L + 2/3, L + 1]$.
- $f \leq 3$.
- f' and f'' are bounded independently of L .

On $F \setminus F_L$, we define the metric $g = e^{-2f(t)}d\sigma^2 + dt^2$. Since $f(t) = t$ on $[0, 1]$, (iii) is satisfied.

In order to define the metric on F_L , we consider a well oriented orthonormal coordinate system on $(T, d\sigma^2)$ such that T is the quotient of \mathbb{R}^2 by the translations by $(\alpha, 0)$ and (β, ℓ) .

Let $\eta : [0, 1] \rightarrow [0, 1]$ be a non increasing function such that η is decreasing on $[1/2, 1]$, $\eta = 1$ near 0, and $\eta(x) = (1 - x)^{\frac{2\pi}{\alpha}}e^{f(L+1)}$ near 1. On F_L we extend the definition of the metric g by

$$g = e^{-2f(t)}(\eta^2(t - L)dx_1^2 + dx_2^2) + dt^2$$

Since $\eta(1) = 0$, the metric is singular at $t = L + 1$. Let D be the unit disk and consider the solid torus \mathbb{T} constructed as the quotient of $D \times [0, \ell]$ by the relation $(p, 0) \sim (R_\beta(p), \ell)$ where R_β is the rotation of angle β . If

(ρ, θ) are the polar coordinates on D and $h : \mathbb{T} \rightarrow F_L$ is the map $(\rho, \theta, z) \mapsto (\frac{\alpha}{2\pi}\theta, v, L + 1 - \rho)$ the metric h^*g is given by

$$h^*g = e^{-2f(L+1-\rho)}(\eta^2(1-\rho)\frac{\alpha^2}{4\pi^2}\theta^2 + dz^2) + d\rho^2$$

so, near $\rho = 0$ (i.e. $t = L+1$), it is equal to $h^*g = \rho^2 d\theta^2 + e^{-2f(L+1)} dz^2 + d\rho^2$ which is a smooth metric on \mathbb{T} near the core circle $\{\rho = 0\}$. So F is a smooth solid torus with a smooth metric and (i) is satisfied.

Because of the monotonicity of f and η , (ii) is satisfied. Moreover the curvature of g is uniformly controlled on $F \setminus F_L$. If ρ_0 is the minimum of 1 and half the systole of $(T, d\sigma^2)$, then for any $p \in F_1 \setminus F_{L-1}$ the geodesic ball of center ρ and radius $e^{-3}\rho_0$ is embedded in $F \setminus F_L$.

Let Σ be a minimal surface that meets all the T_t for $t \in [0, L]$. Consider $t_n = 1 + 2e^{-3}\rho_0 n$ and, for any $n \in \{0, \dots, n_0\}$ where $t_{n_0} \leq L + 2 \leq t_{n_0+1}$, let p_n be in T_{t_n} with $p_n \in T_{t_n} \cap \Sigma$. Then by the monotonicity formula, the area of Σ in the ball of radius $e^{-3}\rho_0$ and center p_n is at least $ce^{-6}\rho_0^2$ for some universal constant c . Since these balls are disjoint, the area of Σ in $F \setminus F_L$ is at least

$$(n_0 + 1)ce^{-6}\rho_0^2 \geq \frac{L-3}{2}ce^{-3}\rho_0 \geq \kappa L$$

if $L \geq 10$ and where κ only depends on the systole of $(T, d\sigma^2)$.

For item (v), we notice that $(T, \frac{1}{\delta^2}d\Sigma^2)$ belongs to a compact subset of flat tori fixed by K . So Proposition 8 applies to $(F \setminus F_L, \delta^2 e^{-2f(t)} \frac{d\sigma^2}{\delta^2} + dt^2)$ to prove that if Σ has index at most 1 and $\Sigma \subset T \times [0, L-1]$ then $\Sigma \subset T \times [0, 1]$. \square

6.3. The quantity $\mathcal{A}_1(M)$ for cusp manifolds. In this section we recall the study of compact minimal surfaces inside orientable cusp manifolds we made in [4, 11].

Let M be a cusp manifold. First we prove that M contains a compact embedded minimal surface. Let ε be such that the ε -thin part is only made of cusp ends. Since $\partial M_{[\varepsilon, \infty)}$ is smooth there is a homotopically closed family Λ of sweep-outs associated to a Morse function on $M_{[\varepsilon, \infty)}$ (we recall that the tori components of $\partial M_{[\varepsilon, \infty)}$ are leaves of the sweep-outs). If $\varepsilon' < \varepsilon$, $M_{[\varepsilon', \varepsilon]}$ is foliated by tori that can be used to extend any continuous sweep-out in Λ into a sweep-out of $M_{[\varepsilon', \infty)}$ that belongs to a homotopically closed family Λ' . Since $W(M_{[\varepsilon, \infty)}, \partial M_{[\varepsilon, \infty)}, \Lambda) \geq |\partial M_{[\varepsilon, \infty)}|$ we obtain

$$W(M_{[\varepsilon, \infty)}, \partial M_{[\varepsilon, \infty)}, \Lambda) \geq W(M_{[\varepsilon', \infty)}, \partial M_{[\varepsilon', \infty)}, \Lambda')$$

So there is $W_0 > 0$ such that $W_0 \geq W(M_{[\varepsilon, \infty)}, \partial M_{[\varepsilon, \infty)}, \Lambda)$ for any ε . Besides a continuous sweep-out of $M_{[\varepsilon, \infty)}$ must sweep out also a fixed geodesic ball in M . So there is w_0 such that $W(M_{[\varepsilon, \infty)}, \partial M_{[\varepsilon, \infty)}, \Lambda) \geq w_0$ for any ε .

Thus we can choose ε small such that any flat tori C in $\partial M_{[\varepsilon, \infty)}$ has small diameter and $w_0 > |\partial M_{[\varepsilon, \infty)}|$. For each C , we consider a filler F^C associated to the flat torus C and L that will be chosen later. ε is chosen

small enough such that item (v) of Proposition 20 is satisfied. Since there are a finite number of C , item (iv) of Definition 19 gives some constant $\kappa > 0$ independent of C . Then L is chosen such that $\kappa L \geq W_0 + 1$.

We can glue each filler F^C along C to obtain a compact manifold without boundary denoted $D(M)$ with some metric. The construction of $D(M)$ depends on two parameters ε and L , so sometimes we will write $D_{\varepsilon,L}(\overline{M})$ (actually it also depends on the choice of some coordinates on F). We will use this construction in the following sections. $D(M)$ contains isometrically a 1-tubular neighborhood of $M_{[\varepsilon,\infty)}$. Let $\{\Gamma_t\}_{t \in [0,1]}$ be a continuous sweep-out of $M_{[\varepsilon,\infty)}$ with $\mathbf{L}(\Gamma_t) \leq W(M_{[\varepsilon,\infty)}, \partial M_{[\varepsilon,\infty)}, \Lambda) + 1/2$. We can extend $\{\Gamma\}_{t \in [0,1]}$ to a continuous sweep-out $\{\tilde{\Gamma}_t\}_{t \in [-L-1,1]}$ of $D(M)$ by considering $\tilde{\Gamma}_t = \cup_C \partial F_{-t}^C$ for $t \in [-L-1, 0]$. Since, for $t < 0$, $|\tilde{\Gamma}_t| \leq |\partial M_{[\varepsilon,\infty)}|$ we have $\mathbf{L}(\tilde{\Gamma}_t) \leq \mathbf{L}(\Gamma_t)$. By Theorem 17, the width $W_{D(M)}$ is then less than $W_0 + 1/2$. Thus by Theorem 18, there is a minimal surface Σ in $D(M)$ with index at most 1 such that $a(\Sigma) \leq W_0 + 1/2$.

Because of our choice of ε and items (iv) and (v), if Σ enters in some F_1^C then $a(\Sigma) \geq |\Sigma \cap F^C| \geq \kappa L \geq W_0 + 1$; this is impossible. So Σ stays outside of F_1^C for any C so Σ is embedded in the part isometric to the 1-tubular neighborhood of $M_{[\varepsilon,\infty)} \subset M$: Σ is a compact minimal surface in M .

Now we know that M contains compact embedded minimal surfaces and we can define the number $\mathcal{A}_1(M)$. In order to prove that $\mathcal{A}_1(M)$ is realized as in Theorem 18, we have the following argument. Let S be a compact minimal surface in M . We construct $D(M)$ as above with an extra hypothesis on ε which is $M_{[\varepsilon,\infty)}$ contains S and all compact embedded minimal surfaces in M with index at most 1. The above construction gives $\mathcal{A}_1(M) \leq W_0 + 1/2$.

Let Σ is a minimal surface that realizes $\mathcal{A}_1(D(M))$ (it has index at most 1), we have $a(\Sigma) \leq a(S)$. If Σ enters in into F_1^C for some C , we have

$$a(\Sigma) \geq |\Sigma \cap F^C| \geq \kappa L \geq W_0 + 1 \geq \mathcal{A}_1(M) + 1/2$$

So Σ does not enter into such a filler: Σ belongs to M . Thus $\mathcal{A}_1(D(M)) = \mathcal{A}_1(M)$ and $\mathcal{A}_1(D(M))$ is realized by a minimal surface as in Theorem 18.

The remainder of this paper is devoted to the study of the continuity of the \mathcal{A}_1 functional over the collection of orientable cusp manifolds. We are going to study the lower and the upper semi-continuity of \mathcal{A}_1 .

7. THE UPPER SEMI-CONTINUITY STUDY

In this section, we consider $(M_i)_i$ a sequence of cusp manifolds that converges to \overline{M} for the geometric convergence. The first step and the main step of the upper semi-continuity study is to prove that the sequence $(\mathcal{A}_1(M_i))_i$ is bounded. The following proposition answers this question.

Proposition 21. *Let $M_i \rightarrow \overline{M}$ be a converging sequence of cusp manifolds. Then for small ε and large L , $\limsup \mathcal{A}_1(M_i) \leq W_{D(\overline{M})}$.*

Proof. The idea of the proof consist in constructing a Riemannian manifold (N_i, \tilde{g}_i) which is κ_i -quasi isometric to $D(\overline{M})$ with $\kappa_i \rightarrow 1$ and such that a large part N_i^1 of N_i is isometric to a large part of M_i . Moreover N_i is such that any minimal surface with index at most 1 that gets out of N_i^1 has area at least $W_{D(\overline{M})} + 1/2$. As a consequence, a minimal surface S_i in N_i realizing $\mathcal{A}_1(N_i)$ satisfies $a(S_i) \leq \kappa_i^2 W_{D(\overline{M})} < W_{D(\overline{M})} + 1/2$ for large i and is contained in N_i^1 . Thus $\limsup \mathcal{A}_1(M_i) \leq \limsup a(S_i) \leq \limsup \kappa_i^2 W_{D(\overline{M})} = W_{D(\overline{M})}$.

We choose ε small such that the ε -thin part of \overline{M} is made only of cusp ends. The convergence $M_i \rightarrow \overline{M}$ gives us $\varphi_i : \overline{M}_{[\varepsilon, \infty)} \rightarrow M_i$ as in Subsection 2.2. From Section 6.3, we know that there is $W_0 > 0$ independent of ε and L such that $W_{D_{\varepsilon, L}(\overline{M})} \leq W_0 + 1/2$.

Let C be one boundary component of $\overline{M}_{[\varepsilon, \infty)}$ and \overline{A} the part of the 2-tubular neighborhood of C inside $\overline{M}_{[\varepsilon, \infty)}$ (the rest of the proof is written as if there is only one C in $\partial\overline{M}_{[\varepsilon, \infty)}$, actually we need to repeat the argument for each C). \overline{A} can be parametrized by $T \times [-2, 0]$ with the metric $\bar{g} = e^{-2x_3}(dx_1^2 + dx_2^2) + dx_3^2$ where $(x_1, x_2) \in T$ are orthonormal coordinates on C .

By Subsection 2.2, $\varphi_i(\overline{A})$ is a one sided neighborhood of $\varphi_i(C)$ in $\varphi_i(\overline{M}) \subset M_i$. On $\varphi_i(\overline{A})$ we have the coordinates $T \times [-2, 0]$ with the metric $g_i = a_{i,kl} dx_k dx_l$ that C^∞ converges to \bar{g} .

Let N_i^1 be $\varphi_i(\overline{M}_{[\varepsilon, \infty)}^1)$ with the metric g_i where $\overline{M}_{[\varepsilon, \infty)}^1$ is the set of points in $\overline{M}_{[\varepsilon, \infty)}$ at distance at least 1 from $\partial\overline{M}_{[\varepsilon, \infty)}$. We notice that N_i^1 contains the part of $\varphi_i(\overline{A})$ parametrized by $T \times [-2, -1]$. We are going to modify the metric g_i on $T \times [-1, 0]$ in order to define a new metric \tilde{g}_i on $\varphi_i(\overline{M}_{[\varepsilon, \infty)})$ which will be the Riemannian manifold N_i^2 .

Let $\chi : [-1, 0] \rightarrow \mathbb{R}$ be C^∞ such that $0 \leq \chi \leq 1$, $\chi = 1$ near -1 and $\chi = 0$ near 0 . We then define $\tilde{g}_i = \chi(x_3)g_i + (1 - \chi(x_3))\bar{g}$. Since g_i and \bar{g} are C^∞ close. \tilde{g}_i is also C^∞ close to \bar{g} . As explained above, \tilde{g}_i turns $\varphi_i(\overline{M}_{[\varepsilon, \infty)})$ into a new Riemannian manifold (N_i^2, \tilde{g}_i) . The map $\varphi_i : \overline{M}_{[\varepsilon, \infty)} \rightarrow N_i^2$ is still well defined and since the metrics \tilde{g}_i converge in the C^∞ topology to \bar{g} . φ_i is a κ'_i quasi-isometry where $\kappa'_i \rightarrow 1$. Moreover φ_i is an isometry close to $\partial\overline{M}_{[\varepsilon, \infty)}$.

Let L be large and consider a filler F associated to T and L . Since N_i^2 and $\overline{M}_{[\varepsilon, \infty)}$ are isometric close to their boundary we can glue to all of them the filler F to produce $(D_{\varepsilon, L}(\overline{M}), \tilde{g})$ and (N_i, \tilde{g}_i) and extend the definition of φ_i to a map $D(\overline{M}) \rightarrow N_i$ which is the identity on the filler. As a consequence $\varphi_i : D(\overline{M}) \rightarrow N_i$ is a κ'_i quasi-isometry.

Let us estimate the area of a minimal surface $S \subset N_i$ with index at most 1 that is not contained in N_i^1 . Thus S must enter in some part of N_i which is isometric to $T \times [-2, L]$ endowed with the metric $\tilde{g}_i = \tilde{a}_{i,kl} dx_k dx_l$ which is C^∞ close to $\tilde{g} = \bar{g}$ on $T \times [-2, 0]$ and is equal to $\tilde{g} = e^{-2f(x_3)}(dx_1^2 + dx_2^2) + dx_3^2$ on $T \times [0, L]$ (f is introduced in Section 6.2). Because of our choice of f

function, the metrics \tilde{g}_i and \tilde{g} satisfy the hypotheses H1 to H4 of Section 4 for a uniform constant A .

If S does not meet all the tori T_s for $s \in [-2, L]$ then, by Proposition 8, S must stay outside of $T \times [-3/2, L + 1]$, so $S \subset N_i^1$. Since we assume $S_i \not\subset N_i^1$, it must meet all the tori T_s for $s \in [0, L]$. Then by Proposition 20, $|S| \geq \kappa L$ for some κ that only depends on the injectivity radius of T_0 . Now, we choose L large enough such that $\kappa L > W_0 + 1$. We obtain $|S| \geq \kappa L > W_0 + 1 \geq W_{D(\overline{M})} + 1/2$. This finishes the construction of N_i and then $\limsup \mathcal{A}_1(M_i) \leq W_{D(\overline{M})}$. \square

We know that for ε small and L large we have $\mathcal{A}_1(\overline{M}) = \mathcal{A}_1(D(\overline{M})) = \min(\mathcal{A}_S(\overline{M}), W_{D(\overline{M})})$. So the above result gives us a first upper semi-continuity property.

Proposition 22. *Let $M_i \rightarrow \overline{M}$ be a converging sequence of cusp manifolds. If one of the following hypotheses is satisfied then $\limsup \mathcal{A}_1(M_i) \leq \mathcal{A}_1(\overline{M})$.*

- $\mathcal{A}_1(\overline{M}) = W_{D(\overline{M})}$
- $\mathcal{A}_1(\overline{M})$ is realized by a stable non separating minimal surface Σ
- $\mathcal{A}_1(\overline{M})$ is realized by a stable non degenerate minimal surface Σ

Proof. The first case comes directly from the above proposition.

Let Σ be a non separating minimal surface that realizes $\mathcal{A}_1(\overline{M})$. Let ε be small such that Σ is contained in the ε -thick part of \overline{M} . Let $\varphi_i : \overline{M}_{[\varepsilon, \infty)} \rightarrow M_i$ be the κ_i quasi-isometry associated to the convergence $M_i \rightarrow \overline{M}$. Then $\varphi_i(\Sigma)$ is a surface in M_i with $a(\varphi_i(\Sigma)) \leq \kappa_i^2 a(\Sigma)$. Because of the topology of the ε -thin part (cusps or solid tori), $\varphi_i(\Sigma)$ is non separating in M_i . So taking a small ε_i and a large L_i we can see $\varphi_i(\Sigma)$ as a non separating surface in $D_{\varepsilon_i, L_i}(M_i)$. So minimizing the area in the non vanishing homology class of Σ (see [6, 17]) there is a minimal surface S_i in $D(M_i)$ with $a(S_i) \leq a(\varphi_i(\Sigma)) \leq \kappa_i^2 a(\Sigma) = \kappa_i^2 \mathcal{A}_1(\overline{M})$. Thus

$$\mathcal{A}_1(M_i) = \mathcal{A}_1(D(M_i)) \leq a(S_i) \leq \kappa_i^2 \mathcal{A}_1(\overline{M})$$

and this gives the result.

Concerning the last case, as above, let ε be small such that Σ is contained in the ε -thick part of \overline{M} and $\varphi_i : \overline{M}_{[\varepsilon, \infty)} \rightarrow M_i$. Let $h_i = \varphi_i^* g_i$. Since $M_i \rightarrow \overline{M}$, the metrics h_i converge in the C^∞ topology to \bar{g} . Since Σ is a non degenerate surface, for large i , Σ can be deformed to a minimal surface S_i in $(\overline{M}_{[\varepsilon, \infty)}, h_i)$. So $\varphi_i(S_i)$ is a minimal surface in M_i and

$$\limsup \mathcal{A}_1(M_i) \leq \limsup a(S_i) = a(\Sigma) = \mathcal{A}_1(\overline{M})$$

\square

Remark 4. We notice that the hypothesis $\mathcal{A}_1(\overline{M}) = W_{D(\overline{M})}$ is satisfied if $\mathcal{A}_1(\overline{M})$ is realized by an index 1 minimal surface.

The second case is realized if $\mathcal{A}_1(\overline{M})$ is realized by a non orientable minimal surface.

8. THE LOWER SEMI-CONTINUITY STUDY

In this section we are going to prove that the \mathcal{A}_1 functional is lower semi-continuous.

8.1. An exclusion property. Let S be a two-sided embedded surface. Let ν be a choice of a unit normal vectorfield along S and $f : S \rightarrow \mathbb{R}$ be a smooth function. Then we can define

$$\exp_{S,f} : S \rightarrow M; p \mapsto \exp_p(f(p)\nu(p))$$

If f is sufficiently small, $\exp_{S,f}(S)$ is an embedded surface which inherits from S a natural unit normal vector still denoted by ν . The lemma below is inspired by Lemma 16 in [18]

Lemma 23. *Let S be a two-sided embedded surface and U be a subset of S such that the mean curvature of S vanishes on U . If $S \setminus U$ has non empty interior, there is a positive function f and $\tau > 0$ such that $\exp_{S,sf}(S)$ has positive mean curvature on $\exp_{S,sf}(U)$ with respect to the naturally induced unit normal vector for $0 < s < \tau$.*

Proof. Let S and U be as in the statement. Let q be a function on S such that $q = Ric(\nu, \nu) + \|A\|^2$ on U and $\mathcal{L} = -\Delta - q$ has negative first eigenvalue on S . It is enough to assume q is large enough somewhere in $\Sigma \setminus U$ to ensure that the first eigenvalue λ_1 is negative. Let $f > 0$ be the first eigenfunction of \mathcal{L} . Consider $S_t = \exp_{S,tf}(S)$ and $H_t(p)$ be the mean curvature of S_t at $\exp_{S,tf}(p)$. It is known that $2\partial_t H_t|_{t=0} = \Delta f + (Ric(\nu, \nu) + \|A\|^2)f = -\lambda_1 f + (Ric(\nu, \nu) + \|A\|^2 - q)f > 0$ on U . Thus, there is $\tau > 0$ such that $H_t(p) > 0$ for any $t \in (0, \tau]$ and $p \in U$. \square

Using the above lemma we can prove the following result.

Proposition 24. *Let A_0, δ_0 and $\mathfrak{s}_0 \leq 1$ be positive. Then there is ℓ_0 and \bar{R} such the following is true. Let $\ell \leq \ell_0$ and M be a cusp manifold such that $\mathcal{A}_1(M) \leq A_0$ and M contains a tubular end N_{R_ℓ} of a geodesic loop of length ℓ and such that S_{R_ℓ} has diameter less than δ_0 and systole larger than \mathfrak{s}_0 . Let Σ be an embedded minimal surface that realizes $\mathcal{A}_1(M)$ then $\Sigma \cap N_{R_\ell - \bar{R}}$ is empty.*

Proof. We first assume that Σ is stable. By Schoen curvature estimate [15], this implies that there is k_0 such that Σ has curvature bounded by k_0 . So by Corollary 7, there is ℓ_0 and \bar{R} such that, if $\ell \leq \ell_0$, either $\Sigma \cap N_{R_\ell - \bar{R}}$ is empty or $\Sigma \cap N_{R_\ell - \bar{R}}$ has area at least $2\pi(\cosh(R_\ell - \bar{R}) - 1)$. Actually, if ℓ_0 is chosen such that $2\pi(\cosh(R_{\ell_0} - \bar{R}) - 1) \geq A_0$, the second case can not occur and $\Sigma \cap N_{R_\ell - \bar{R}}$ is empty.

So we can assume that Σ is separating and has index 1. By Proposition 9, there is \bar{R}, ℓ_0 and κ such that $\Sigma \cap N_{R_\ell - \bar{R}} = \emptyset$ or $|\Sigma \cap N_R| \geq \kappa \mathfrak{s}_0 e^{R - R_\ell}$ for any $R \in [3, R_\ell - \bar{R}]$. Moreover ℓ_0 and \bar{R} can be chosen such that the preceding paragraph is still true.

Let us assume that $\Sigma \cap N_{R_\ell - \bar{R}}$ is not empty. Let us notice that $|S_R| = \pi \ell \sinh(2R) \leq \sqrt{3} \frac{\sinh(2R)}{\sinh(2R_\ell)} \leq \sqrt{3} e^{2(R-R_\ell)}$. So choosing $R_* \geq \bar{R}$ such that $e^{-R_*} \leq \frac{\kappa s_0}{4\sqrt{3}}$ we obtain

$$|\Sigma \cap N_R| \geq 4|S_R|.$$

for any $3 \leq R \leq R_\ell - R_*$.

Let ε be small such that the ε -thin part of M contains only cusp ends. For L large we consider $D_{\varepsilon, L}(M) = D(M)$ such that $\mathcal{A}_1(M) = \mathcal{A}_1(D(M))$. So Σ is a separating index 1 minimal surface in $D(M)$ that realizes $\mathcal{A}_1(D(M))$. The idea is now to construct a discrete sweep-out S of $D(M)$ such that $\mathbf{L}(S) < |\Sigma|$ which contradicts Σ realizes $\mathcal{A}_1(M)$. We notice that N_{R_ℓ} is still isometrically included in $D(M)$.

Σ separates M into two connected components Ω_1 and Ω_2 . Let $R \in [R_\ell - R_* - 1, R_\ell - R_*]$ such that S_R is transverse to Σ . We define $\Gamma = \Sigma \cap S_R$. The subset $\Omega_i \cap N_R$ has mean convex boundary made of pieces of S_R and Σ . We can find a least area minimal surface $\Sigma_i \subset \Omega_i \cap N_R$ with $\partial \Sigma_i = \Gamma$ and homologous to $S_R \cap \Omega_i$ [6, 17]. Since $S_R \cap \Omega_i$ is a surface with boundary Γ , $|\Sigma_i| \leq |S_R \cap \Omega_i| \leq |S_R|$. Besides $\Sigma_i \cup (S_R \cap \Omega_i)$ bounds a subset D_i of $N_R \cap \Omega_i$. By boundary regularity of solutions to the Plateau problem [7], Σ_i is a smooth surface up to its boundary Γ and, by the maximum principle, Σ_i is transverse to Σ along Γ . We notice that since Σ_i is smooth up to its boundary we can slightly extend Σ_i across Γ to Σ'_i . Σ'_i is not assumed to be minimal outside of Σ_i and $\partial \Sigma'_i$ is assumed to be outside Ω_i .

Let us fix $i \in \{1, 2\}$ and consider ν the unit normal along Σ pointing into Ω_i . Since Σ has index 1, there is $\tau > 0$ and $f_i > 0$ on Σ such that $\exp_{\Sigma, s f_i}(\Sigma)$ is an embedded surface with positive mean curvature for any $s \in (0, \tau]$. Moreover, we assume $f_i > 1$. If ν_i denote the unit normal along Σ'_i pointing into D_i along Σ_i , by Lemma 23, there is $g_i > 0$ on Σ'_i such that $\exp_{\Sigma'_i, s g_i}(\Sigma'_i)$ is an embedded surface with positive mean curvature on $\exp_{\Sigma'_i, s g_i}(\Sigma_i)$ for any $s \in (0, \tau]$. Moreover, we assume $g_i < 1$ and $\exp_{\Sigma'_i, s g_i}(\partial \Sigma'_i) \not\subset \Omega_i$ for any $s \in [0, \tau]$.

Let us define $U_{i,0} = D_i \cup (\Omega_i \setminus N_R)$. For $s \leq \tau$ we define

$$V_{i,s} = \left(\bigcup_{0 \leq s' \leq s} \exp_{\Sigma, s' f_i}(\Sigma) \right) \cup \left(\bigcup_{0 \leq s' \leq s} \exp_{\Sigma'_i, s' g_i}(\Sigma'_i) \right) \cap U_{i,0},$$

$$U_{i,s} = U_{i,0} \setminus V_{i,s}$$

We postpone the precise description of $\partial U_{i,s}$ to the end of the proof. Actually we are going to prove that there is a smaller τ such that, for $s \in [0, \tau]$, $\partial U_{i,s}$ is $\exp_{\Sigma, s f_i}(A_{i,s}) \cup \exp_{\Sigma'_i, s g_i}(B_{i,s})$ where $A_{i,s}$ is a smooth subdomain in Σ and $B_{i,s}$ is a smooth subdomain of Σ'_i . Moreover both components of $\partial U_{i,s}$ are transverse. We also have $B_{i,s} \subset \Sigma_i$ and $A_{i,s} \subset \Sigma \setminus N_{R-1}$. $s \mapsto \partial U_{i,s}$ is then a continuous map with values in $\mathcal{Z}_2(M)$. Moreover $\mathbf{M}(\partial U_{i,s}) \leq \mathbf{M}(\partial U_{i,0})$. We then have

$$\mathbf{M}(\partial U_{i,s}) \leq \mathbf{M}(\partial U_{i,0}) = |\Sigma_i| + |\Sigma \setminus N_R| \leq |S_R| + |\Sigma| - |\Sigma \cap N_R| \leq |\Sigma| - 3|S_R|$$

Besides we notice that, since $A_{i,s} \subset \Sigma \setminus N_{R-1}$ and $B_{i,s} \subset \Sigma'_i$, $\partial U_{i,s}$ is piecewise smooth mean convex in the sense of Definition 10 in [18].

Using the work of Song in [18], we can adapt the work of the authors in [11] to the case $\partial U_{i,\tau}$ is not smooth and prove the following statement.

Claim 1. *There is a continuous sweep-out $\{\partial U_{i,s}\}_{s \in [\tau,1]}$ of $U_{i,\tau}$ such that*

$$\mathbf{L}(\{\partial U_{i,s}\}_{s \in [\tau,1]}) \leq |\Sigma| - 2|S_R|.$$

Proof. Because of the Appendix in [18], we know that there exists a homotopically closed family Λ of sweepouts in $U_{i,\tau}$. Let us assume that $W(U_{i,\tau}, \partial U_{i,\tau}, \Lambda) \geq |\Sigma| - 5/2|S_R| > |\partial U_{i,\tau}|$.

Thus by Theorem 12 in [18], there is a closed minimal surface S in $U_{i,\tau}$. As in the proof of Lemma 20 in [18], $N = U_{i,\tau} \setminus S$ is then a mean convex subset such that $\partial U_{i,\tau}$ has non vanishing homology class in N . Thus we can minimize the area in this homology class [6, 17] in order to get S' a stable minimal surface such that $|S'| \leq |\partial U_{i,\tau}|$. But this implies $\mathcal{A}_S(D(M)) \leq |\partial U_{i,\tau}| \leq |\Sigma| - 3|T_R| < \mathcal{A}_1(D(M)) \leq \mathcal{A}_S(D(M))$ which is contradictory. So $W(U_{i,\tau}, \partial U_{i,\tau}, \Lambda) \leq |\Sigma| - 5/2|S_R|$ and there is $\{\partial U_{i,s}\}_{s \in [\tau,1]}$ with

$$\mathbf{L}(\{\partial U_{i,s}\}_{s \in [\tau,1]}) \leq |\Sigma| - 2|S_R|.$$

□

Using these two sweep-outs, we can define a family G_s (see Figure 4) of open subsets of M by

$$G_s = \begin{cases} U_{1,1-s} & \text{if } 0 \leq s \leq 1 \\ U_{1,0} \cup N_{s-1} & \text{if } 1 \leq s \leq R+1 \\ (\Omega_1 \cup N_R) \setminus \overline{(D_2 \setminus N_{2R+1-s})} & \text{if } R+1 \leq s \leq 2R+1 \\ M \setminus \overline{U_{2,s-2R-1}} & \text{if } 2R+1 \leq s \leq 2R+2 \end{cases}$$

The family G_s satisfies $\mathcal{H}^3(G_s \Delta G_t) \rightarrow 0$ as $s \rightarrow t$. Moreover, ∂G_s is rectifiable so $\Phi : s \mapsto \partial G_s$ is a continuous path in $\mathcal{Z}_2(M)$ for the flat topology. Moreover $G_0 = \emptyset$ and $G_{2R+2} = M$. Let us now study $\mathbf{M}(\Phi(s))$. For $s \in [0, 1]$ we have $\mathbf{M}(\Phi(s)) = \mathbf{M}(\partial U_{1,1-s}) \leq |\Sigma| - 2|S_R|$. For $s \in [1, R+1]$, ∂G_s is contained in $\partial U_{1,0} \cup S_{s-1}$ so $\mathbf{M}(\Phi(s)) \leq \mathbf{M}(\partial U_{1,0}) + |S_{s-1}| \leq |\Sigma| - 2|S_R|$. If $s \in [R+1, 2R+1]$, ∂G_s is contained in $\partial U_{2,0} \cup S_{2R+1-s}$ so $\mathbf{M}(\Phi(s)) \leq \mathbf{M}(\partial U_{2,0}) + |S_{2R+1-s}| \leq |\Sigma| - 2|S_R|$. Finally for $s \in [2R+1, 2R+2]$, $\mathbf{M}(\Phi(s)) = \mathbf{M}(\partial U_{2,s-2R-1}) \leq |\Sigma| - 2|S_R|$.

After a reparametrization, we have then constructed a continuous map $\Phi : [0, 1] \rightarrow \mathcal{Z}_2((D(M)), \mathcal{F})$ satisfying all the hypotheses of Theorem 17 with $\sup \mathbf{M}(\Phi) \leq |\Sigma| - 2|T_R| < |\Sigma|$. So by Theorem 17, we have $\mathcal{A}_1(M) = \mathcal{A}_1(D(M)) \leq W_{D(M)} \leq \sup \mathbf{M}(\Phi) \leq |\Sigma| - 2|T_R| < |\Sigma| = \mathcal{A}_1(M)$. This gives a contradiction with $\Sigma \cap N_{R_L - \bar{R}} \neq \emptyset$ and finishes the proof.

Let us come back to the study of $\partial U_{i,s}$ for small s and check the properties we announced. Clearly this boundary is contained in $\Sigma'_{i,s} = \exp_{\Sigma'_{i,sg_i}}(\Sigma'_i)$ and $\Sigma_s = \exp_{\Sigma_s, sf_i}(\Sigma)$. We need to understand the intersection of these two surfaces when s is small.

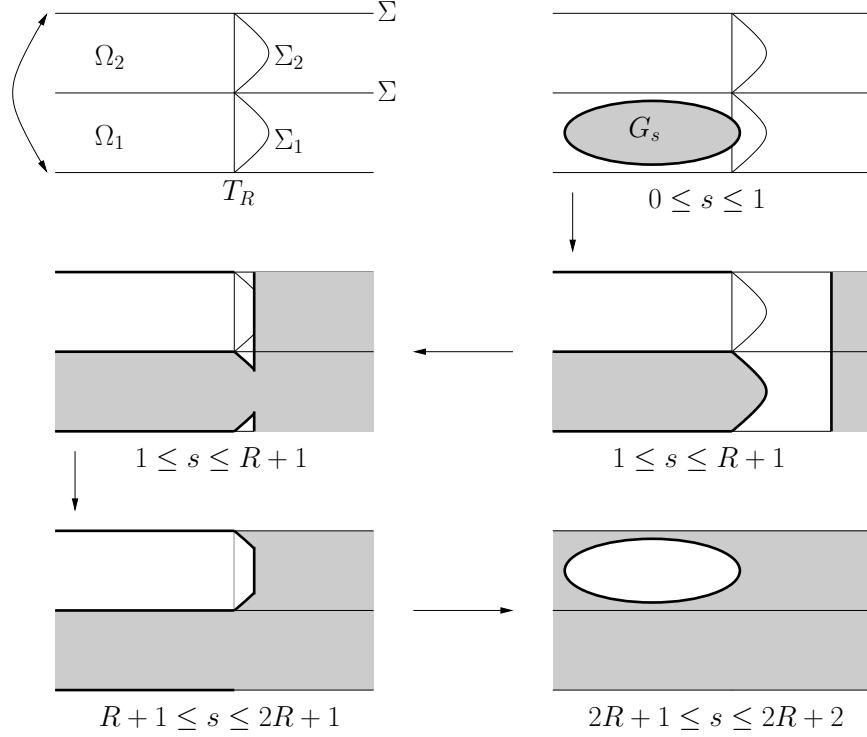


FIGURE 4. A schematic view of G_s (the bottom and top line are identified)

We define $F_i : (p, s) \in \Sigma \times (-\tau, \tau) \mapsto \exp_{\Sigma, s f_i}(p) \in M$ and $G_i : (p, s) \in \Sigma'_i \times (-\tau, \tau) \mapsto \exp_{\Sigma'_i, s g_i}(p) \in M$ for small τ . The map F_i defines a smooth coordinate system in a neighborhood N of Σ . Let us write $F_i^{-1} = (P, T) : N \rightarrow \Sigma \times (-\tau, \tau)$. Let us remark that at a point $p \in \Sigma$, $DF_i^{-1}|_p : X \in T_p M \mapsto (\pi_p(X), (X, \nu)/f_i(p)) \in T_p \Sigma \times \mathbb{R}$ where π_p is the normal projection $T_p M \rightarrow T_p \Sigma$.

Let V be a neighborhood of Γ inside Σ'_i contained in N . There is τ' such that $G_i(V \times (-\tau', \tau')) \subset N$. Let η_i be the conormal to Γ in Σ'_i pointing to Σ_i . So neighboring points to Γ in Σ'_i can be parametrized by $(p, t) \in \Gamma \times (-\varepsilon, \varepsilon) \mapsto \exp_p^i(t\eta_i(p))$ where \exp^i is the exponential map in Σ'_i . Thus such a point has image by $G_i(\cdot, s)$ in the intersection $F_i(\Sigma, s) \cap G_i(\Sigma'_i, s)$ (s small) if

$$L_i(p, s, t) := T(G_i(\exp_p^i(t\eta_i(p)), s)) - s = 0$$

Solving t as a function of $(p, s) \in \Gamma \times \mathbb{R}$ can be done near $\Gamma \times \{0\}$ using the implicit function theorem since $L_i(p, 0, 0) = 0$. Indeed we have $\partial_t L_i(p, 0, 0) = (\nu(p), \eta_i(p))/f_i(p) > 0$ since both ν and η_i point to Ω_i . So

$t_i(p, s)$ can be defined near $\Gamma \times \{0\}$. At $(p, 0)$ we also have

$$0 = \partial_s(L(p, s, t_i(p, s))) = \frac{(\nu, \eta_i)}{f_i} \partial_s t_i + \frac{g_i(\nu, \nu_i)}{f_i} - 1$$

Thus $\partial_s t_i = \frac{f_i}{(\nu, \eta_i)} (1 - \frac{g_i(\nu, \nu_i)}{f_i}) > 0$ since $g_i/f_i < 1$. The curve $\gamma_{i,s}(p) = \exp_p^i(t_i(p, s)\eta(p))$ is sent by $G_i(\cdot, s)$ on the intersection $F_i(\Sigma, s) \cap G_i(\Sigma'_i, s)$: $\beta_i(\cdot, s) = G_i(\gamma_{i,s}(\cdot), s)$ is a parametrization of the intersection. Moreover $\gamma_{i,s}$ bounds a subdomain $B_{i,s}$ in Σ'_i whose image by $G_i(\cdot, s)$ is the piece of $\partial U_{i,s}$ contained in $G_i(\Sigma'_i, s)$. Since $\partial_s t_i > 0$, we have $B_{i,s} \subset \Sigma_i$. At $(p, 0)$, we also have

$$\partial_s \beta_i = g_i \nu_i + \frac{f_i}{(\nu, \eta)} (1 - \frac{g_i(\nu, \nu_i)}{f_i}) \eta_i$$

The curve $\gamma_s(\cdot) = P(\beta_i(\cdot, s))$ on Σ is such that $F_i(\gamma_s(\cdot), s)$ is also a parametrization of the intersection. γ_s bounds a subdomain $A_{i,s}$ in Σ such that $\partial U_{i,s}$ is $F_i(A_{i,s}, s) \cup G_i(B_{i,s}, s)$. We notice that, at $s = 0$,

$$\partial_s \gamma_s(p) = \pi_p(\partial_s \beta_i) = \pi_p(g_i \nu_i + \frac{f_i}{(\nu, \eta)} (1 - \frac{g_i(\nu, \nu_i)}{f_i}) \eta_i)$$

We notice that since $B_{i,s} \subset \Sigma_i$, the mean curvature of $G_i(B_{i,s}, s)$ is positive ($s > 0$). The same is true for the mean curvature of $F_i(A_{i,s}, s)$. Moreover both surfaces intersect at an angle less than π . Finally using the first variation formula and Σ and Σ_i are minimal, we have at $s = 0$

$$\begin{aligned} \partial_s(|F_i(A_{i,s}, s)| + |G_i(B_{i,s}, s)|) &= - \int_{\Gamma} (\partial_s \gamma_s(p), \eta) + (\partial_s \gamma_{i,s}(p), \eta_i) \\ &= - \int_{\Gamma} (g_i \nu_i + \frac{f_i}{(\nu, \eta_i)} (1 - \frac{g_i(\nu, \nu_i)}{f_i}) \eta_i, \eta) \\ &\quad + \frac{f_i}{(\nu, \eta_i)} (1 - \frac{g_i(\nu, \nu_i)}{f_i}) \\ &= - \int_{\Gamma} (g_i \nu_i + \frac{f_i}{(\nu, \eta_i)} (1 - \frac{g_i(\nu, \nu_i)}{f_i}) \eta_i, \eta + \eta_i) \\ &= - \int_{\Gamma} (\partial_s \beta_i, \eta + \eta_i) \end{aligned}$$

where η is the unit conormal to Γ in Σ that points outside $N(R)$. We notice that for $s > 0$, β_s is inside $U_{i,0}$ so $\partial_s \beta_s$ points to $U_{i,0}$ and is orthogonal to the tangent space to Γ . $\eta + \eta_i$ is a vector that bisects the wedge corresponding to $U_{i,0}$ and contained in the orthogonal to the tangent space to Γ . So $(\partial_s \beta_i, \eta + \eta_i) > 0$ along Γ and $\partial_s(|F_i(A_{i,s}, s)| + |G_i(B_{i,s}, s)|) < 0$. This implies that $|\partial U_{i,s}| < |\partial U_{i,0}|$ for $s > 0$ small. So all the stated properties are satisfied. \square

8.2. The lower semi-continuity. We have the following result.

Proposition 25. *Let $M_i \rightarrow \overline{M}$ be a converging sequence of cusp manifolds. Then $\liminf \mathcal{A}_1(M_i) \geq \mathcal{A}_1(\overline{M})$.*

Proof. Let us consider a minimal surface Σ_i in M_i such that $a(\Sigma_i) = \mathcal{A}(M_i)$. By Proposition 21, we know that there is A_0 such that $|\Sigma_i| < A_0$. Moreover, by Corollary 4, there is ℓ_0, δ_0 and \mathfrak{s}_0 such that if M_i contains a geodesic loop of length $\ell \leq \ell_0$ then its tubular neighborhood N_{R_ℓ} satisfies $S_{R_\ell} = \partial N_{R_\ell}$ has diameter less than δ_0 and systole larger than \mathfrak{s}_0 . Thus there is \bar{R} such that $\Sigma_i \cap N_{R_\ell - \bar{R}}$ is empty by Proposition 24.

This implies that there is $\varepsilon > 0$ such that $\Sigma_i \subset M_{i[\varepsilon, \infty)}$ for any i . Since $M_i \rightarrow \bar{M}$ there is $\varphi_i : \bar{M}_{[\varepsilon/2, \infty)} \rightarrow M_i$ which is a κ_i quasi-isometry where $\kappa_i \rightarrow 1$. Moreover we have $M_{i[\varepsilon, \infty)} \subset \varphi_i(\bar{M}_{[\varepsilon/2, \infty)})$. Let $\tilde{g}_i = \varphi_i^* g_i$ and $\tilde{\Sigma}_i = \varphi_i^{-1}(\Sigma_i) \subset \bar{M}_{[\varepsilon/2, \infty)}$ which is a minimal surface for the metric \tilde{g}_i . Since $M_i \rightarrow \bar{M}$ we have $\tilde{g}_i \rightarrow \bar{g}$ in the C^∞ topology. Moreover $a_{\tilde{g}_i}(\tilde{\Sigma}_i) \leq A_0$ and the index of Σ_i is 0 or 1.

Thus we can apply the compactness result of Sharp (Theorem A.6 in [16]). It implies that there is a closed connected embedded minimal surface $\bar{\Sigma}$ in (\bar{M}, \bar{g}) such that $(\tilde{\Sigma}_i)$ converges in the varifold sense to $\bar{\Sigma}$ with some multiplicity. Moreover, the convergence is smooth outside a finite number of points. If $\bar{\Sigma}$ is orientable, then

$$\mathcal{A}_1(\bar{M}) \leq a_{\bar{g}}(\bar{\Sigma}) = |\bar{\Sigma}|_{\bar{g}} \leq \lim |\tilde{\Sigma}_i|_{\tilde{g}_i} \leq \liminf \mathcal{A}_1(M_i)$$

If $\bar{\Sigma}$ is non-orientable, then either $\tilde{\Sigma}_i$ is non orientable or $\tilde{\Sigma}_i$ is orientable and the convergence must be with multiplicity at least 2. In both cases, we have

$$\mathcal{A}_1(\bar{M}) \leq a_{\bar{g}}(\bar{\Sigma}) = 2|\bar{\Sigma}|_{\bar{g}} \leq \lim a_{\tilde{g}_i}(\tilde{\Sigma}_i) = \liminf \mathcal{A}_1(M_i)$$

So the proposition is proved. \square

APPENDIX A.

A.1. A uniform graph lemma. Let us consider \mathbb{R}^3 endowed with the metric $\bar{g} = h^2(x_3)(dx_1^2 + dx_2^2) + dx_3^2$. For $k_1, k_2, k_3, k_4 \in \{1, 2, 3\}$ and $p \leq 4$, we recall that

$$n_p(k_1, \dots, k_p) = \#\{i \in \{1, \dots, p\} | k_i \in \{1, 2\}\}.$$

We consider a second metric $g = a_{kl}(x_1, x_2, x_3)dx_k dx_l$. We assume that there is some A such that the following hypotheses occurs

- H1 $\frac{1}{A^2}\bar{g} \leq g \leq A^2\bar{g}$
- H2 $\frac{|h'|}{h} \leq A$ and $\frac{|h''|}{h} \leq A$.
- H3 $|a_{kl}| \leq Ah^{n_2(k,l)}(x_3)$, $|\partial_i a_{kl}| \leq Ah^{n_3(k,l,i)}(x_3)$ and $|\partial_i \partial_j a_{kl}| \leq Ah^{n_4(k,l,i,j)}(x_3)$.

We notice that the metric \bar{g} satisfies also the hypotheses of the last item.

Lemma 26. *Let \bar{g} and g as above and consider ε_0, k_0 , then there is $C > 0$ such that the following is true. Let Σ be a surface in $\mathbb{R}^2 \times [a, b]$ endowed with the metric g which is tangent to $\mathbb{R}^2 \times \{\bar{t}\}$ at $\bar{p} = (0, 0, \bar{t})$ such that $d_\Sigma(\bar{p}, \partial\Sigma) \geq \varepsilon_0$ and $|A_\Sigma| \leq k_0$.*

Then there is a function u defined on the disk $\{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 2C^2/h^2(\bar{t})\}$ such that $(x_1, x_2) \mapsto (x_1, x_2, \bar{t} + u(x_1, x_2))$ is a parametrization of a neighborhood of \bar{p} in Σ . Moreover u satisfies

$$|u| \leq A\varepsilon_0, \quad \|\nabla u\| \leq h(\bar{t}) \quad \text{and} \quad \|\text{Hess } u\| \leq \frac{1}{C}h^2(\bar{t})$$

Proof. First we replace Σ by the geodesic disk of center \bar{p} and ε_0 . Since $a_{33} \geq \frac{1}{A^2}$, the distance between $\{x_3 = \bar{t}\}$ and $\{x_3 = \bar{t} \pm t\}$ is at least t/A . So Σ is contained in $\mathbb{R}^2 \times [\bar{t} - A\varepsilon_0, \bar{t} + A\varepsilon_0]$. Let us also remark that since $\frac{|h'|}{h} \leq A$ we have $e^{-A|x_3 - \bar{t}|}h(\bar{t}) \leq h(x_3) \leq e^{A|x_3 - \bar{t}|}h(\bar{t})$. Then $e^{-A^2\varepsilon_0}h(\bar{t}) \leq h(x_3) \leq e^{A^2\varepsilon_0}h(\bar{t})$ on $[\bar{t} - A\varepsilon_0, \bar{t} + A\varepsilon_0]$.

Let us consider $\Psi : \mathbb{R}^2 \times [\bar{t} - A\varepsilon_0, \bar{t} + A\varepsilon_0] \rightarrow \mathbb{R}^2 \times [\bar{t} - A\varepsilon_0, \bar{t} + A\varepsilon_0] : (y_1, y_2, y_3) \mapsto (\frac{1}{h(\bar{t})}y_1, \frac{1}{h(\bar{t})}y_2, y_3)$. Then the metric $g^* = \Psi^*g$ can be written $b_{kl}(y_1, y_2, y_3)dy_kdy_l$ where $b_{kl} = h(\bar{t})^{-n_2(k,l)}a_{kl} \circ \Psi$. Thus

$$|b_{kl}| = h(\bar{t})^{-n_2(k,l)}|a_{kl}| \circ \Psi \leq A \frac{h^{n_2(k,l)}(y_3)}{h^{n_2(k,l)}(\bar{t})} \leq Ae^{n_2(k,l)A^2\varepsilon_0} \leq Ae^{2A^2\varepsilon_0}$$

So there is a constant B such that $|b_{kl}| \leq B$. A similar computation proves that $|\nabla b_{kl}| \leq B$ and $|\text{Hess } b_{kl}| \leq B$. Using Hypothesis H1, we also have $\frac{1}{A^2}\Psi^*\bar{g} \leq g^* \leq A^2\Psi^*\bar{g}$ where $\Psi^*\bar{g} = \frac{h^2(y_3)}{h^2(\bar{t})}(dy_1^2 + dy_2^2) + dy_3^2$. This implies that $\det g^*$ is far from 0 and ∞ . So the coefficients b^{kl} of the inverse of g^* satisfy $|b^{kl}| \leq B$ and, for any $k \in \{1, 2, 3\}$, $\frac{1}{B} \leq b_{kk} \leq B$ and $\frac{1}{B} \leq b^{kk} \leq B$.

Let us define $\Sigma^* = \Psi^{-1}(\Sigma)$, $\Sigma^* \subset (\mathbb{R}^3, g^*)$ is a geodesic disk of radius ε_0 and curvature bounded by k_0 . Let us consider $g_e = dy_1^2 + dy_2^2 + dy_3^2$ the Euclidean metric. Because of the the control we have on g^* , there is ε_1 that depends only on ε_0 , A and B and k_1 that depends only on k_0 , A and B such that (Σ^*, g_e) has curvature bounded by k_1 and $d_{\Sigma^*, g_e}(\bar{p}, \partial\Sigma^*) \geq \varepsilon_1$ (the proof of this result can be found in the Appendix of [14] more precisely see the proof of Propositions 4.1 and 4.3).

So we have a surface in the Euclidean space \mathbb{R}^3 with curvature bounded by k_1 , $d_{\Sigma^*, g_e}(\bar{p}, \partial\Sigma^*) \geq \varepsilon_1$ and that is tangent to $\mathbb{R}^2 \times \{t\}$ at \bar{p} . Then a classical uniform graph lemma (see Proposition 2.3 in [14]) implies that there is C that depends only on k_1 and ε_1 such the following is true. There is a function u defined on the Euclidean disk of radius $\sqrt{2}C$ centered at the origin such that $(y_1, y_2) \mapsto (y_1, y_2, \bar{t} + u(y_1, y_2))$ is a parametrization of a neighborhood of \bar{p} in Σ^* . Moreover

$$|u| \leq 2C, \quad |\nabla u| \leq 1 \quad \text{and} \quad \|\text{Hess } u\| \leq \frac{1}{C}$$

In order to come back to the original coordinate system we define the function $v(x_1, x_2) = u(h(\bar{t})x_1, h(\bar{t})x_2)$ which is defined on $\{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 2\delta^2/h^2(\bar{t})\}$ and satisfies

$$|v| \leq 2C, \quad |\nabla v| \leq h(\bar{t}) \quad \text{and} \quad \|\text{Hess } v\| \leq \frac{1}{C}h^2(\bar{t})$$

We notice that, since $\Sigma \subset \mathbb{R}^2 \times [\bar{t} - A\varepsilon_0, \bar{t} + A\varepsilon_0]$, we have $|v| \leq A\varepsilon_0$. \square

A.2. The minimal surface equation. Several times we consider graphs that are minimal surfaces; let us write the equation solved by these graphs.

On \mathbb{R}^3 we consider the metric $g = a_1^2(x_3)dx_1^2 + a_2^2(x_3)dx_2^2 + dx_3^2$ which is a model for the metric in cusp or tubular ends. Let u be a function in a domain of \mathbb{R}^2 and consider the graph parametrized by $X(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$. The induced metric is

$$(a_1^2(u) + u_{x_1}^2)dx_1^2 + 2u_{x_1}u_{x_2}dx_1dx_2 + (a_2^2(u) + u_{x_2}^2)dx_2^2$$

So the area element is $Wdx_1dx_2 = (a_1^2(u)a_2^2(u) + a_2^2(u)u_{x_1}^2 + a_1^2(u)u_{x_2}^2)dx_1dx_2$.

So if v is an other function with zero boundary values and $A(t)$ is the area of the graph of $u + tv$, the derivative of A at $t = 0$ is

$$\begin{aligned} A'(0) &= \int \frac{1}{W} \left(a_1(u)a_1'(u)a_2^2(u)v + a_1^2(u)a_2(u)a_2'(u)v + a_2(u)a_2'(u)u_{x_1}^2v \right. \\ &\quad \left. + a_2^2(u)u_{x_1}v_{x_1} + a_1(u)a_1'(u)u_{x_2}^2v + a_1^2(u)u_{x_2}v_{x_2} \right) \\ &= \int \frac{a_1(u)a_2(u)}{W} \left(a_1'(u)a_2(u) + a_1(u)a_2'(u) + \frac{a_2'(u)}{a_1(u)}u_{x_1}^2 + \frac{a_1'(u)}{a_2(u)}u_{x_2}^2 \right) v \\ &\quad - v \operatorname{div} \frac{(a_2^2(u)u_{x_1}, a_1^2(u)u_{x_2})}{W} \end{aligned}$$

Thus the graph is minimal if u satisfies

$$(2) \quad 0 = \operatorname{div} \frac{(a_2^2(u)u_{x_1}, a_1^2(u)u_{x_2})}{W} - \frac{a_1(u)a_2(u)}{W} \left(a_1'(u)a_2(u) + a_1(u)a_2'(u) + \frac{a_2'(u)}{a_1(u)}u_{x_1}^2 + \frac{a_1'(u)}{a_2(u)}u_{x_2}^2 \right)$$

REFERENCES

- [1] Frederick Justin Almgren, Jr. The homotopy groups of the integral cycle groups. *Topology*, 1:257–299, 1962.
- [2] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.
- [3] Tobias H. Colding and Camillo De Lellis. The min-max construction of minimal surfaces. In *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*, Surv. Differ. Geom., VIII, pages 75–107. Int. Press, Somerville, MA, 2003.
- [4] Pascal Collin, Laurent Hauswirth, Laurent Mazet, and Harold Rosenberg. Minimal surfaces in finite volume non compact hyperbolic 3-manifolds. *Trans. Amer. Math. Soc.*, 369:4293–4309, 2017.
- [5] Pascal Collin, Laurent Hauswirth, Laurent Mazet, and Harold Rosenberg. Corrigendum to "minimal surfaces in finite volume non compact hyperbolic 3-manifolds". *Trans. Amer. Math. Soc.*, 372(10):7521–7524, 2019.
- [6] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.
- [7] Robert Hardt and Leon Simon. Boundary regularity and embedded solutions for the oriented Plateau problem. *Ann. of Math. (2)*, 110:439–486, 1979.
- [8] Joel Hass. Minimal fibrations of hyperbolic 3-manifold. preprint, arXiv:1512.04145.

- [9] Zheng Huang and Biao Wang. Complex length of short curves and minimal fibrations of hyperbolic three-manifolds fibering over the circle. *Proc. Lond. Math. Soc. (3)*, 118(6):1305–1327, 2019.
- [10] Daniel Ketover, Fernando Marques, and Neves André. The catenoid estimate and its geometric applications. *J. Differential Geom.*, 115:1–26, 2020.
- [11] Laurent Mazet and Harold Rosenberg. Minimal hypersurfaces of least area. *J. Differential Geom.*, 106:283–316, 2017.
- [12] Robert Meyerhoff. A lower bound for the volume of hyperbolic 3-manifolds. *Canad. J. Math.*, 39:1038–1056, 1987.
- [13] Jon T. Pitts. Existence and regularity of minimal surfaces on Riemannian manifolds. *Bull. Amer. Math. Soc.*, 82:503–504, 1976.
- [14] Harold Rosenberg, Rabah Souam, and Eric Toubiana. General curvature estimates for stable H -surfaces in 3-manifolds and applications. *J. Differential Geom.*, 84:623–648, 2010.
- [15] Richard Schoen. Estimates for stable minimal surfaces in three-dimensional manifolds. In *Seminar on minimal submanifolds*, volume 103 of *Ann. of Math. Stud.*, pages 111–126. Princeton Univ. Press, 1983.
- [16] Ben Sharp. Compactness of minimal hypersurfaces with bounded index. *J. Differential Geom.*, 106:317–339, 2017.
- [17] Leon Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University Centre for Mathematical Analysis, 1983.
- [18] Antoine Song. Embeddedness of least area minimal hypersurfaces. *J. Differential Geom.*, 110(2):345–377, 2018.
- [19] Xin Zhou. Min-max minimal surface in (M^{n+1}, g) with $Ric_g > 0$ and $2 \leq n \leq 6$. *J. Differential Geom.*, 100:129–160, 2015.
- [20] Xin Zhou. Min-max minimal hypersurface in manifold with positive Ricci curvature. *J. Differential Geom.*, 105:291–343, 2017.

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