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# Corrigendum for "Second-order reflected backward stochastic differential equations" and "Second-order BSDEs with general reflection and game options under uncertainty"\*

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## Abstract

The aim of this short note is to fill in a gap in our earlier paper [7] on 2BSDEs with reflections, and to explain how to correct the subsequent results in the second paper [6]. We also provide more insight on the properties of 2RBSDEs, in the light of the recent contributions [5, 13] in the so-called  $G$ -framework.

**Key words:** 2BSDEs, reflections, Skorokhod condition.

**AMS 2000 subject classifications:** 60H10, 60H30.

## 1 Introduction

In this short note, we fill in a gap in our earlier wellposedness result on so-called second-order reflected 2BSDEs (2RBSDEs for short). The issue stemmed from a wrongly defined minimality condition which ensures uniqueness of the solution, which we correct here. We also use this occasion to prove that an alternative minimality condition, taking the form of a Skorokhod-like condition, leads to the exact same solution, provided that an additional assumption on the finiteness of the  $p$ -variation of the lower obstacle is added (see Assumption 2.1 below). This new condition appeared recently in the two contributions [5, 13] on reflected  $G$ -BSDEs, and our result proves that the two notions do coincide, and that our formulation produces more general results.

Since some of the results of [7] were used in our subsequent paper [6] considering doubly reflected 2BSDEs, we also explain how to change the minimality condition there, as well as which results are impacted by this change. Roughly speaking, all our previous results still hold true, except for the *a priori* estimates, where we no longer control the total variation of the bounded variation process appearing in the solution, but only its  $\mathbb{D}_H^{2,\kappa}$ -norm.

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## 2 The lower reflected case

The notations in this section are the ones in [7]. Our only change is that to remain coherent with the notations in the next section, we will denote the lower obstacle in the 2RBSDE by  $L$  instead of  $S$ .

### 2.1 The gap

The mistake in the paper [7] can be found right after Equation (3.3), when we try to prove that the process  $K^{\mathbb{P}'} - k^{\mathbb{P}'}$  is non-decreasing. Indeed, appealing to the minimality condition (2.6) in [7] does not imply the sub-martingality of this process, since it only gives the required inequality for any  $t \in [0, T]$  and the *fixed* time  $T$ . The end of Step (ii) of the proof of Theorem 3.1 in [7] therefore does not go through. The issue here is that the minimality condition (2.6) that we wrote is only adapted to the case where the generator  $F$  is 0, described in Remark 3.1 in [7], and it has to be modified. One could argue there that  $K^{\mathbb{P}'} - k^{\mathbb{P}'}$  might still be non-decreasing, and that an appropriate minimality condition should reflect this fact. However, the following counter-example, communicated to us by Jianfeng Zhang, proves that this cannot be the case in general.

**Example 2.1** (Jianfeng Zhang). *Fix  $T = 2$  and take as a lower obstacle a process  $L$  satisfying the required assumptions in [7] as well as*

$$L_t := 2(1 - t), \quad 0 \leq t \leq 1, \quad \text{and } L_t \leq 2, \quad 1 \leq t \leq 2.$$

*Furthermore, take the generator  $F$  of the 2RBSDE to be 0, and the terminal condition to be  $L_2$ . In this case, the solution to the 2RBSDE being necessarily the supremum of the solutions to the associated RBSDEs, we will have automatically the representations*

$$Y_t = \operatorname{essup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} \operatorname{essup}_{\tau \in \mathcal{T}_{t,T}}^{\mathbb{P}'} \mathbb{E}^{\mathbb{P}'} [L_\tau | \mathcal{F}_T], \quad y_t^{\mathbb{P}'} = \operatorname{essup}_{\tau \in \mathcal{T}_{t,T}}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'} [L_\tau | \mathcal{F}_T].$$

*Furthermore, in this case since  $F = 0$ ,  $K^{\mathbb{P}'} - k^{\mathbb{P}'}$  being a  $\mathbb{P}'$ -submartingale is equivalent to  $Y - y^{\mathbb{P}'}$  being a  $\mathbb{P}'$ -supermartingale, which would imply in particular that*

$$Y_0 - y_0^{\mathbb{P}'} \geq \mathbb{E}^{\mathbb{P}'} [Y_1 - y_1^{\mathbb{P}'}]. \quad (2.1)$$

*However, it is clear by definition of  $L$  that  $Y_0 = y_0^{\mathbb{P}'} = 2$ . However, there is absolutely no reason why in general one could not have, for some  $\mathbb{P}'$ , and for an appropriate choice of  $L$ ,  $Y_1 > y_1^{\mathbb{P}'}$  (recall that we always have  $Y_1 \geq y_1^{\mathbb{P}'}$ ), at least with strictly positive  $\mathbb{P}'$ -probability, which then contradicts (2.1).*

The issue here was actually partially pointed out in Remark 3.6 of [6]. It is explained there (and the proof of this result is independent of the mistake in the minimality condition) that on the event  $\{Y_{t-} = L_{t-}\}$ , one has  $K^{\mathbb{P}} = k^{\mathbb{P}}$ ,  $\mathbb{P}$ -a.s., for any  $\mathbb{P} \in \mathcal{P}_H^\kappa$ , meaning that  $K^{\mathbb{P}} - k^{\mathbb{P}}$  is constant (and thus non-decreasing) as long as  $Y_{t-} = L_{t-}$ . Similarly, the Skorokhod condition satisfied by  $k^{\mathbb{P}}$  implies that  $K^{\mathbb{P}} - k^{\mathbb{P}}$  is still non-decreasing on the event  $\{y_{t-}^{\mathbb{P}} > L_{t-}\}$ . However, there is nothing we can say, as pointed out by the above counter-example, on the event  $\{Y_t > y_{t-}^{\mathbb{P}} = L_{t-}\}$ .

### 2.2 The new minimality condition and the proof of uniqueness

This being clarified, let us now explain what should be the appropriate minimality condition replacing (2.6) in [7]. Using the Lipschitz property of  $F$  (see Assumption 2.3(iii) in [7]), we can define bounded functions  $\lambda : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times D_H \rightarrow \mathbb{R}$  and  $\eta : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times D_H \rightarrow \mathbb{R}^d$  such that for any  $(t, \omega, y, y', z, z', a)$

$$F_t(\omega, y, z, a) - F_t(\omega, y', z', a) = \lambda_t(\omega, y, y', z, z', a)(y - y') + \eta_t(\omega, y, y', z, z', a) \cdot a^{1/2}(z - z').$$

Define then for any  $\mathbb{P} \in \mathcal{P}_H^\kappa$ , and for any  $t \in [0, T]$  the process

$$M_s^{t, \mathbb{P}} := \exp \left( \int_t^s \left( \lambda_u - \frac{1}{2} |\eta_u|^2 \right) (Y_u, y_u^\mathbb{P}, Z_u, z_u^\mathbb{P}, \hat{a}_u) du - \int_t^s \eta_u(Y_u, y_u^\mathbb{P}, Z_u, z_u^\mathbb{P}, \hat{a}_u) \cdot \hat{a}_u^{-1/2} dB_u \right).$$

Following the arguments in the beginning of the proof of Theorem 3.1 in [7], we have then for any  $t \in [0, T]$ , any  $\mathbb{P} \in \mathcal{P}_H^\kappa$  and any  $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$

$$Y_t - y_t^{\mathbb{P}'} = \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^T M_s^{t, \mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \right], \mathbb{P} - a.s. \quad (2.2)$$

Therefore, the representation formula

$$Y_t = \operatorname{essup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} y_t^{\mathbb{P}'}, \mathbb{P} - a.s.,$$

is equivalent to the new minimality condition

$$\operatorname{essinf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^T M_s^{t, \mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \right] = 0, \mathbb{P} - a.s. \quad (2.3)$$

If one replaces the minimality condition (2.6) in [7] by (2.3) above, as well as in the statement of Theorem 3.1 in [7] the representation (3.1) by simply

$$Y_t = \operatorname{essup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} y_t^{\mathbb{P}'}, \mathbb{P} - a.s., \quad (2.4)$$

then the proof of (2.4) is immediate as soon as one has proved (2.2). This allows us to recover uniqueness of the solution.

**Remark 2.1.** *The representation formula (3.1) in [7] does not only involve  $t$  and  $T$ , but any pair  $0 \leq t \leq s \leq T$ . If one only assumes the new minimality condition (2.3), then it cannot be proved immediately that*

$$Y_t = \operatorname{essup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} y_t^{\mathbb{P}'}(s, Y_s), \mathbb{P} - a.s.$$

*However, once we have proved that the solution  $Y$  of the 2RBSDE satisfies the dynamic programming principle, then the above is immediate. Furthermore, the case  $s = T$  is enough to obtain uniqueness, which is the purpose of Theorem 3.1 in [7].*

**Remark 2.2.** *Since in general  $K^{\mathbb{P}'} - k^{\mathbb{P}'}$  is not non-decreasing, we cannot reduce (2.3) to a statement involving only  $K^{\mathbb{P}'}$  and  $k^{\mathbb{P}'}$ , as is the case for non-reflected 2BSDEs, see for instance [12]. However, when  $L = -\infty$  and there is no reflection,  $k^{\mathbb{P}'}$  becomes identically 0, and (2.3) is indeed equivalent to*

$$\operatorname{essinf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'} \left[ K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'} \right] = 0, \mathbb{P} - a.s.,$$

*see the arguments in Steps (ii) of the proof of [12, Theorem 4.3, Theorem 4.6].*

*The need to depart from the "standard" minimality condition has also been pointed out by Popier and Zhou in a paper in preparation [9], when dealing with 2BSDEs under a monotonicity condition, with hypotheses relaxing the earlier work [10].*

**Remark 2.3.** *Let us check here that the new minimality condition (2.3) indeed allows to recover the classical RBSDE theory when  $\mathcal{P}_H^\kappa$  is reduced to a singleton  $\{\mathbb{P}\}$ . In this case, (2.3) says exactly that the bounded variation process  $\int_0^\cdot M_s^{0, \mathbb{P}} d(K_s^\mathbb{P} - k_s^\mathbb{P})$  is a  $\mathbb{P}$ -martingale. Since the filtration  $\overline{\mathbb{F}}^\mathbb{P}$  satisfies the predictable martingale representation property, it means that this process is identically 0. Now since  $M^{0, \mathbb{P}}$  is  $\mathbb{P} - a.s.$  positive, this implies that  $K^\mathbb{P} = k^\mathbb{P}$ , which is the desired property.*

### 2.3 Recovering existence

The second instance of the use of the wrong conclusion that  $K^\mathbb{P} - k^\mathbb{P}$  was non-decreasing in [7] is in the existence proof, during the discussion after Equation (4.6). At this point, the last thing to prove is that  $K$  satisfies the new minimality condition (2.3). However, we already have the result of Proposition 4.2 in [7] which shows that the process  $V^+$  satisfies the representation formula (2.4). Therefore, the fact that (2.3) is indeed satisfied is immediate, since both statements are equivalent.

To summarize, one should replace Definition 2.3 in [7] by the following, and use the corrections explained above in the proofs.

**Definition 2.1.** For  $\xi \in \mathbb{L}_H^{2,\kappa}$ , we say  $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$  is a solution to the 2RBSDE if

- $Y_T = \xi$ , and  $Y_t \geq L_t$ ,  $t \in [0, T]$ ,  $\mathcal{P}_H^\kappa - q.s.$
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$ , the process  $K^\mathbb{P}$  defined below has non-decreasing paths  $\mathbb{P} - a.s.$

$$K_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s) ds + \int_0^t Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s. \quad (2.5)$$

- We have the following minimality condition

$$\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^T M_s^{t, \mathbb{P}'} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \right] = 0, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s., \quad \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

### 2.4 An alternative: Skorokhod minimality condition

Readers familiar with the theory of standard reflected BSDEs should be wondering whether there is an equivalent, in the second-order setting, of the so-called Skorokhod condition. The latter states that the non-decreasing process appearing in the definition of a RBSDE acts in a minimal way, only when the solution actually reaches the obstacle, and implies uniqueness of the solution (see the seminal paper [4] for more details). There are actually two recent papers which treat the very related problem of reflected  $G$ -BSDEs, namely [5, 13], and which use a generalization of this condition. The aim of this section is to show that this condition also implies wellposedness in our framework, under an additional assumption on the obstacle  $L$ , and that the two definitions are actually equivalent. We also provide a more detailed comparison between [5, 13] and our work at the end of the section.

#### 2.4.1 Wellposedness under Skorokhod condition

Using the same notations as before, the Skorokhod condition for 2RBSDEs reads

$$\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^T (Y_{s-} - L_{s-}) dK_s^{\mathbb{P}'} \right] = 0, \quad t \in [0, T], \quad \mathbb{P} - a.s., \quad \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \quad (2.6)$$

For ease of reference, we provide the corresponding alternative definition of a solution to the 2RBSDE.

**Definition 2.2.** For  $\xi \in \mathbb{L}_H^{2,\kappa}$ , we say  $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$  is a Skorokhod-solution to the 2RBSDE if

- $Y_T = \xi$ , and  $Y_t \geq L_t$ ,  $t \in [0, T]$ ,  $\mathcal{P}_H^\kappa - q.s.$
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$ , the process  $K^\mathbb{P}$  defined below has non-decreasing paths  $\mathbb{P} - a.s.$

$$K_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s) ds + \int_0^t Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s. \quad (2.7)$$

- We have the following minimality condition

$$\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'} \left[ \int_t^T (Y_{s^-} - L_{s^-}) dK_s^{\mathbb{P}'} \right] = 0, \quad t \in [0, T], \quad \mathbb{P} - a.s., \quad \forall \mathbb{P}' \in \mathcal{P}_H^\kappa.$$

In more mundane terms, this condition is saying that if there is a probability measure  $\mathbb{P}$  such that the supremum in the representation formula (2.4) is attained, then on the support of  $\mathbb{P}$ , the classical Skorokhod condition is satisfied by the solution of the 2RBSDE.

Let us now argue how (2.6) can be used in stead of (2.3) to recover wellposedness, and that both conditions actually lead to the exact same solution. Notice however that the method of proof here requires the following condition on  $L$ , which basically asks that it has a finite  $p$ -variation, for some  $p \geq 1$ .

**Assumption 2.1.** Let  $\Pi_{[0, T]}$  be the set of all partitions of  $[0, T]$  (which allow for stopping times). We have for some  $p \geq 1$

$$\ell := \sup_{(\rho_i)_i \in \Pi_{[0, T]}} \sup_{\mathbb{P}' \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}'} \left[ \sum_{i=0}^{n-1} |L_{\rho_{i+1}^-} - L_{\rho_i^-}|^p \right] < +\infty.$$

The result is as follows, and its proof borrows a lot from the seminal paper of Ekren, Touzi and Zhang [3].

**Theorem 2.1.** Let Assumption 2.1 hold, as well as the necessary assumptions for wellposedness in [7]. Then there is a unique Skorokhod-solution to the 2RBSDE which coincides with the unique solution to the 2RBSDE.

*Proof.* We now argue in two steps.

### Step 1: uniqueness

This is the easiest part. Assume that there exists a Skorokhod-solution  $(\tilde{Y}, \tilde{Z})$ . We will argue that  $\tilde{Y} = Y$ , which implies immediately that  $\tilde{Z} = Z$ , since  $Z$  is uniquely defined by the quadratic co-variation between  $Y$  and  $B$ .

Fix first some  $\mathbb{P} \in \mathcal{P}_H^\kappa$ . By definition,  $\tilde{Y}$  is a super-solution under  $\mathbb{P}$  to the standard BSDE with terminal condition  $\xi$ , generator  $\hat{F}$ . Since it is also always above  $L$ , and since solutions to reflected BSDEs are also the minimal super-solutions of the associated BSDEs, we deduce that necessarily we have  $\tilde{Y} \geq y^\mathbb{P}$ ,  $\mathbb{P}$ -a.s., which implies by arbitrariness of  $\mathbb{P}$  and by (2.4) that

$$\tilde{Y} \geq Y.$$

For the converse inequality, fix some  $\varepsilon > 0$  and some  $\mathbb{P} \in \mathcal{P}_H^\kappa$ , and define the following stopping time

$$\tilde{\tau}_\varepsilon^\mathbb{P} := \inf\{t \geq 0, \tilde{Y}_{t^-} - L_{t^-} \leq \varepsilon\} \wedge T.$$

The Skorokhod condition implies that

$$0 = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'} \left[ \int_t^T (\tilde{Y}_{s^-} - L_{s^-}) dK_s^{\mathbb{P}'} \right] \geq \varepsilon \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'} [K_{\tilde{\tau}_\varepsilon}^{\mathbb{P}'} - K_t^{\mathbb{P}'}].$$

Next, let  $(\mathcal{Y}^{\mathbb{P}'}(\tilde{\tau}_\varepsilon, L_{\tilde{\tau}_\varepsilon}), \mathcal{Z}^{\mathbb{P}'}(\tilde{\tau}_\varepsilon, L_{\tilde{\tau}_\varepsilon}))$  be the solution, under  $\mathbb{P}$  and on  $[0, \tilde{\tau}_\varepsilon]$ , of the BSDE with generator  $\hat{F}$  and terminal condition  $L_{\tilde{\tau}_\varepsilon}$ . We have  $\mathbb{P}$ -a.s.

$$\begin{aligned} \tilde{Y}_t &= \tilde{Y}_{\tilde{\tau}_\varepsilon} + \int_t^{\tilde{\tau}_\varepsilon} \hat{F}_s(\tilde{Y}_s, \tilde{Z}_s) ds - \int_t^{\tilde{\tau}_\varepsilon} \mathcal{Z}_s dB_s + \int_t^{\tilde{\tau}_\varepsilon} dK_s^{\mathbb{P}'}, \quad 0 \leq t \leq \tilde{\tau}_\varepsilon, \\ \mathcal{Y}_t(\tilde{\tau}_\varepsilon, L_{\tilde{\tau}_\varepsilon}) &= L_{\tilde{\tau}_\varepsilon} + \int_t^{\tilde{\tau}_\varepsilon} \hat{F}_s(\mathcal{Y}_s(\tilde{\tau}_\varepsilon, L_{\tilde{\tau}_\varepsilon}), \mathcal{Z}_s(\tilde{\tau}_\varepsilon, L_{\tilde{\tau}_\varepsilon})) ds - \int_t^{\tilde{\tau}_\varepsilon} \mathcal{Z}_s(\tilde{\tau}_\varepsilon, L_{\tilde{\tau}_\varepsilon}) dB_s, \quad 0 \leq t \leq \tilde{\tau}_\varepsilon. \end{aligned}$$

Classical estimates using the same linearization arguments as above show that there is some constant  $C > 0$  such that

$$\tilde{Y}_t - \mathcal{Y}_t(\tilde{\tau}_\varepsilon, L_{\tilde{\tau}_\varepsilon}) \leq C \mathbb{E}_t^{\mathbb{P}'} \left[ \tilde{Y}_{\tilde{\tau}_\varepsilon} - L_{\tilde{\tau}_\varepsilon} + K_{\tilde{\tau}_\varepsilon}^{\mathbb{P}'} - K_t^{\mathbb{P}'} \right] \leq C\varepsilon + C \mathbb{E}_t^{\mathbb{P}'} \left[ K_{\tilde{\tau}_\varepsilon}^{\mathbb{P}'} - K_t^{\mathbb{P}'} \right].$$

Using the Skorokhod condition, this implies immediately that

$$\tilde{Y}_t \leq \operatorname{essup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} \mathcal{Y}_t(\tilde{\tau}_\varepsilon, L_{\tilde{\tau}_\varepsilon}) + C\varepsilon.$$

By the classical comparison theorem, we have that  $\mathcal{Y}_t(\tilde{\tau}_\varepsilon, L_{\tilde{\tau}_\varepsilon}) \leq y_t$ , so that we deduce

$$\tilde{Y}_t \leq \operatorname{essup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} y_t + C\varepsilon \leq Y_t + C\varepsilon,$$

which implies the required result by arbitrariness of  $\varepsilon$ .

### Step 2: existence

The only thing that needs to be done here is to prove that the solution we constructed in the sense of Definition 2.1 is also a Skorokhod-solution. In other words, we simply have to prove that  $Y$  satisfies the Skorokhod minimality condition

$$\operatorname{essinf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^T (Y_{s-} - L_{s-}) dK_s^{\mathbb{P}'} \right] = 0, \quad t \in [0, T], \quad \mathbb{P} - a.s., \quad \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Without loss of generality, we prove that this holds for  $t = 0$ , which is equivalent to proving that

$$\inf_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T (Y_{s-} - L_{s-}) dK_s^{\mathbb{P}} \right] = 0.$$

Let us start by fixing some  $\varepsilon > 0$ , and define the following sequence of stopping times  $(\tau_n)_{n \geq 1}$  by

$$\tau_0 := 0, \quad \tau_1 := \inf\{t \geq 0, Y_{t-} - L_{t-} \leq \varepsilon\} \wedge T,$$

$$\tau_{2n} := \inf\{t > \tau_{2n-1}, Y_{t-} - L_{t-} \geq 2\varepsilon\} \wedge T, \quad \tau_{2n+1} := \inf\{t > \tau_{2n}, Y_{t-} - L_{t-} \leq \varepsilon\} \wedge T, \quad n \geq 1.$$

We start by proving that for any  $n \geq 1$  and any  $\mathbb{P} \in \mathcal{P}_H^\kappa$

$$\inf_{\mathbb{P}' \in \mathcal{P}_H^\kappa(\tau_{2n}^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'} [K_{\tau_1}^{\mathbb{P}'}] = \operatorname{essinf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(\tau_{2n}^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'} [K_{\tau_{2n+1}}^{\mathbb{P}'} - K_{\tau_{2n}}^{\mathbb{P}'}] = 0. \quad (2.8)$$

By the dynamic programming principle (see [7, Proposition 4.1]), we know that for any  $n \geq 0$ , and using the link between reflected BSDEs and optimal stopping problems, where for any  $0 \leq s \leq t \leq T$ ,  $\mathcal{T}_{s,t}$  denotes the set of stopping times taking values in  $[s, t]$

$$Y_{\tau_{2n}} = \operatorname{essup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(\tau_{2n}^+, \mathbb{P})}^{\mathbb{P}} \operatorname{essup}_{\tau \in \mathcal{T}_{\tau_{2n}, \tau_{2n+1}}}^{\mathbb{P}'} \mathcal{Y}_{\tau_{2n}}^{\mathbb{P}'}(\tau, L_\tau \mathbf{1}_{\{\tau < \tau_{2n+1}\}} + Y_{\tau_{2n+1}} \mathbf{1}_{\{\tau = \tau_{2n+1}\}}).$$

Now for any  $\delta > 0$ , there exists some  $\mathbb{P}'_\delta \in \mathcal{P}_H^\kappa(\tau_{2n}^+, \mathbb{P})$  and some  $\tau^\delta \in \mathcal{T}_{\tau_{2n}, \tau_{2n+1}}$  such that

$$Y_{\tau_{2n}} \leq \mathcal{Y}_{\tau_{2n}}^{\mathbb{P}'_\delta}(\tau^\delta, L_{\tau^\delta} \mathbf{1}_{\{\tau^\delta < \tau_{2n+1}\}} + Y_{\tau_{2n+1}} \mathbf{1}_{\{\tau^\delta = \tau_{2n+1}\}}) + \delta.$$

Define then

$$\mathcal{M}_s^{t, \mathbb{P}'_\delta} := \exp \left( \int_t^s (\lambda_u - \frac{1}{2} |\eta_u|^2) (Y_u, \mathcal{Y}_u^{\mathbb{P}'_\delta}, Z_u, \mathcal{Z}_u^{\mathbb{P}'_\delta}, \hat{a}_u) du - \int_t^s \eta_u (Y_u, \mathcal{Y}_u^{\mathbb{P}'_\delta}, Z_u, \mathcal{Z}_u^{\mathbb{P}'_\delta}, \hat{a}_u) \cdot \hat{a}_u^{-1/2} dB_u \right),$$

where we denoted for simplicity

$$\mathcal{Y}^{\mathbb{P}'_\delta} := \mathcal{Y}^{\mathbb{P}'_\delta}(\tau^\delta, L_{\tau^\delta} \mathbf{1}_{\{\tau^\delta < \tau_{2n+1}\}} + Y_{\tau_{2n+1}} \mathbf{1}_{\{\tau^\delta = \tau_{2n+1}\}}), \quad \mathcal{Z}^{\mathbb{P}'_\delta} := \mathcal{Z}^{\mathbb{P}'_\delta}(\tau^\delta, L_{\tau^\delta} \mathbf{1}_{\{\tau^\delta < \tau_{2n+1}\}} + Y_{\tau_{2n+1}} \mathbf{1}_{\{\tau^\delta = \tau_{2n+1}\}}).$$

Notice that for any  $p \in \mathbb{R}$ , the boundedness of  $\lambda$  and  $\nu$  imply that for some constant  $C_p > 0$

$$\sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}} \left[ \left( \sup_{t \leq s \leq T} \mathcal{M}_s^{t, \mathbb{P}} \right)^p + \left( \inf_{t \leq s \leq T} \mathcal{M}_s^{t, \mathbb{P}} \right)^p \right] \leq C_p.$$

Then, linearization arguments similar to the ones used before in this note imply that

$$Y_t - \mathcal{Y}_t^{\mathbb{P}'_\delta} = \mathbb{E}_t^{\mathbb{P}'_\delta} \left[ \mathcal{M}_{\tau^\delta}^{t, \mathbb{P}'_\delta} (Y_{\tau^\delta} - L_{\tau^\delta}) \mathbf{1}_{\{\tau^\delta < \tau_{2n+1}\}} + \int_t^{\tau^\delta} \mathcal{M}_s^{t, \mathbb{P}'_\delta} dK_s^{\mathbb{P}'_\delta} \right], \quad \tau_{2n} \leq t \leq \tau_{2n+1}, \quad \mathbb{P} - a.s.$$

By definition of the  $(\tau_n)_{n \geq 0}$ , we deduce that

$$\delta \geq Y_{\tau_{2n}} - \mathcal{Y}_{\tau_{2n}}^{\mathbb{P}'_\delta} \geq \varepsilon \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ \mathcal{M}_{\tau_{2n}}^{\tau_{2n}, \mathbb{P}'_\delta} \mathbf{1}_{\{\tau^\delta < \tau_{2n+1}\}} \right] + \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ \inf_{\tau_{2n} \leq s \leq \tau^\delta} \mathcal{M}_s^{\tau_{2n}, \mathbb{P}'_\delta} (K_{\tau^\delta}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta}) \right].$$

We then estimate that

$$\begin{aligned} \mathbb{P}'_\delta[\tau^\delta \leq \tau_{2n+1}] &= \mathbb{E}^{\mathbb{P}'_\delta} \left[ \left( \mathcal{M}_{\tau^\delta}^{\tau_{2n}, \mathbb{P}'_\delta} \right)^{-\frac{1}{2}} \left( \mathcal{M}_{\tau^\delta}^{\tau_{2n}, \mathbb{P}'_\delta} \right)^{\frac{1}{2}} \mathbf{1}_{\{\tau^\delta \leq \tau_{2n+1}\}} \right] \\ &\leq \left( \mathbb{E}^{\mathbb{P}'_\delta} \left[ \left( \mathcal{M}_{\tau^\delta}^{\tau_{2n}, \mathbb{P}'_\delta} \right)^{-1} \right] \mathbb{E}^{\mathbb{P}'_\delta} \left[ \left( \mathcal{M}_{\tau^\delta}^{\tau_{2n}, \mathbb{P}'_\delta} \right)^{\frac{1}{2}} \mathbf{1}_{\{\tau^\delta \leq \tau_{2n+1}\}} \right] \right)^{\frac{1}{2}} \\ &\leq C_{-1}^{\frac{1}{2}} \sqrt{\frac{\delta}{\varepsilon}}. \end{aligned}$$

Recall as well that by Step (iii) of the proof of [7, Theorem 3.1] that for some  $\bar{C} > 0$

$$\operatorname{esssup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(\tau_{2n}^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}} \left[ (K_{\tau_{2n+1}}^{\mathbb{P}'} - K_{\tau_{2n}}^{\mathbb{P}'})^2 \right] \leq \bar{C}.$$

Therefore, we have

$$\begin{aligned} &\mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ K_{\tau_{2n+1}}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta} \right] \\ &\leq \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ K_{\tau^\delta}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta} \right] + \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ (K_{\tau_{2n+1}}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta}) \mathbf{1}_{\{\tau^\delta < \tau_{2n+1}\}} \right] \\ &= \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ \left( \inf_{\tau_{2n} \leq s \leq \tau^\delta} \mathcal{M}_s^{t, \mathbb{P}'_\delta} (K_{\tau^\delta}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta}) \right)^{\frac{1}{3}} (K_{\tau^\delta}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta})^{\frac{2}{3}} \left( \inf_{\tau_{2n} \leq s \leq \tau^\delta} \mathcal{M}_s^{t, \mathbb{P}} \right)^{-\frac{1}{3}} \right] \\ &\quad + \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ (K_{\tau_{2n+1}}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta}) \mathbf{1}_{\{\tau^\delta < \tau_{2n+1}\}} \right] \\ &\leq \left( \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ \inf_{\tau_{2n} \leq s \leq \tau^\delta} \mathcal{M}_s^{t, \mathbb{P}} (K_{\tau^\delta}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta}) \right] \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ (K_{\tau^\delta}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta})^2 \right] \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ \left( \inf_{\tau_{2n} \leq s \leq \tau^\delta} \mathcal{M}_s^{t, \mathbb{P}} \right)^{-1} \right] \right)^{\frac{1}{3}} \\ &\quad + \left( \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'_\delta} \left[ (K_{\tau^\delta}^{\mathbb{P}'_\delta} - K_{\tau_{2n}}^{\mathbb{P}'_\delta})^2 \right] \mathbb{P}'_\delta[\tau^\delta < \tau_{2n+1}] \right)^{\frac{1}{2}} \\ &\leq (\bar{C} C_{-1})^{\frac{1}{3}} \delta^{\frac{1}{3}} + \bar{C}^{\frac{1}{2}} C_{-1}^{\frac{1}{4}} \left( \frac{\delta}{\varepsilon} \right)^{\frac{1}{4}}. \end{aligned}$$



This implies immediately that

$$\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(\tau_{2n}^+, \mathbb{P})} \mathbb{E}_{\tau_{2n}}^{\mathbb{P}'} [K_{\tau_{2n+1}}^{\mathbb{P}'} - K_{\tau_{2n}}^{\mathbb{P}'}] \leq (\bar{C}C_{-1})^{\frac{1}{3}} \delta^{\frac{1}{3}} + \bar{C}^{\frac{1}{2}} C_{-1}^{\frac{1}{4}} \left( \frac{\delta}{\varepsilon} \right)^{\frac{1}{4}},$$

which proves (2.8) by letting  $\delta$  go to 0.

Therefore, for any  $n \geq 0$ , we can find some  $\mathbb{P}_1 \in \mathcal{P}_H^\kappa$ , and some  $\mathbb{P}_{n+1} \in \mathcal{P}_H^\kappa(\tau_{2n}^+, \mathbb{P}_n)$  such that for some  $\tilde{C} > 0$

$$\mathbb{E}_{\tau_{2n}}^{\mathbb{P}_{n+1}} [K_{\tau_{2n+1}}^{\mathbb{P}_{n+1}} - K_{\tau_{2n}}^{\mathbb{P}_{n+1}}] \leq \frac{\varepsilon}{2^n}.$$

By definition, we have  $Y_t - L_t \leq 2\varepsilon$  for  $t \in [\tau_{2n-1}, \tau_{2n}]$ , so that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_n} \left[ \int_0^{\tau_{2n}} (Y_{s^-} - L_{s^-}) dK_s^{\mathbb{P}_n} \right] &= \sum_{i=0}^{n-1} \mathbb{E}^{\mathbb{P}_n} \left[ \int_{\tau_{2i+1}}^{\tau_{2(i+1)}} (Y_{s^-} - L_{s^-}) dK_s^{\mathbb{P}_n} + \int_{\tau_{2i}}^{\tau_{2i+1}} (Y_{s^-} - L_{s^-}) dK_s^{\mathbb{P}_n} \right] \\ &= \sum_{i=0}^{n-1} \mathbb{E}^{\mathbb{P}_{i+1}} \left[ \int_{\tau_{2i+1}}^{\tau_{2(i+1)}} (Y_{s^-} - L_{s^-}) dK_s^{\mathbb{P}_{i+1}} + \int_{\tau_{2i}}^{\tau_{2i+1}} (Y_{s^-} - L_{s^-}) dK_s^{\mathbb{P}_{i+1}} \right] \\ &\leq \sum_{i=0}^{n-1} 2\varepsilon \mathbb{E}^{\mathbb{P}_n} [K_{\tau_{2(i+1)}}^{\mathbb{P}_{i+1}} - K_{\tau_{2i+1}}^{\mathbb{P}_{i+1}}] + \frac{\varepsilon}{2^i} \mathbb{E}^{\mathbb{P}_n} \left[ \sup_{\tau_{2i} \leq s \leq \tau_{2i+1}} (Y_s - L_s) \right] \\ &\leq \tilde{C}\varepsilon, \end{aligned}$$

where we used the *a priori* estimates satisfied by the solution of the 2RBSDE, see [7, Theorem 3.3] and the definition of the  $\mathbb{P}_n$ .

Next, notice that  $Y - L$  is right-continuous, and therefore uniformly continuous from the right. Besides, by definition, we have for any  $n \geq 0$ , that on  $\{\tau_{n+1} < T\}$

$$\left| (Y_{\tau_{n+1}^-} - L_{\tau_{n+1}^-}) - (Y_{\tau_n^-} - L_{\tau_n^-}) \right| \geq \varepsilon.$$

Therefore the  $\tau_n$  cannot accumulate and for  $n$  large enough we necessarily have  $\tau_n = T$ . We now assume that the  $n$  we have chosen satisfies this property.

Finally, fix some  $m \leq n$ . We have the following estimate for any  $\mathbb{P} \in \mathcal{P}_H^\kappa$

$$\begin{aligned} \mathbb{P}[\tau_{2n} < T] &\leq \mathbb{P} \left[ \bigcap_{i=0}^{n-1} \left\{ |Y_{\tau_{2(i+1)}^-} - Y_{\tau_{2i+1}^-}| + |L_{\tau_{2(i+1)}^-} - L_{\tau_{2i+1}^-}| \geq \varepsilon \right\} \right] \\ &\leq \mathbb{P} \left[ \bigcap_{i=0}^{n-1} \left( \left\{ |Y_{\tau_{2(i+1)}^-} - Y_{\tau_{2i+1}^-}| \geq \frac{\varepsilon}{2} \right\} \cup \left\{ |L_{\tau_{2(i+1)}^-} - L_{\tau_{2i+1}^-}| \geq \frac{\varepsilon}{2} \right\} \right) \right] \\ &\leq \mathbb{P} \left[ \left\{ \sum_{i=0}^{n-1} |Y_{\tau_{2(i+1)}^-} - Y_{\tau_{2i+1}^-}|^2 \geq \frac{m\varepsilon^2}{4} \right\} \cup \left\{ \sum_{i=0}^{n-1} |L_{\tau_{2(i+1)}^-} - L_{\tau_{2i+1}^-}|^p \geq \frac{(n-m)\varepsilon^p}{2^p} \right\} \right] \\ &\leq \frac{4}{m\varepsilon^2} \mathbb{E}^{\mathbb{P}} \left[ \sum_{i=0}^{n-1} |Y_{\tau_{2(i+1)}^-} - Y_{\tau_{2i+1}^-}|^2 \right] + \frac{2^p}{(n-m)\varepsilon^p} \mathbb{E}^{\mathbb{P}} \left[ \sum_{i=0}^{n-1} |L_{\tau_{2(i+1)}^-} - L_{\tau_{2i+1}^-}|^p \right]. \end{aligned}$$

Now notice that we have for some constant  $C$  which may change value from line to line, by definition and using Doob's inequality as well as the elementary inequality  $\sum_i a_i^2 \leq (\sum_i |a_i|)^2$  and the estimates of [7, Theorem 3.3]

$$\mathbb{E}^{\mathbb{P}} \left[ \sum_{i=0}^{n-1} |Y_{\tau_{i+1}^-} - Y_{\tau_i^-}|^2 \right] \leq C \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |\hat{F}_s(Y_s, Z_s)|^2 ds + \int_0^T |\hat{a}_s^{1/2} Z_s|^2 ds + (K_T^{\mathbb{P}})^2 \right] \leq C.$$

Consequently, using Assumption 2.1

$$\begin{aligned}
\inf_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \int_0^T (Y_{s^-} - L_{s^-}) dK_s^\mathbb{P} \right] &\leq \mathbb{E}^{\mathbb{P}^n} \left[ \int_0^T (Y_{s^-} - L_{s^-}) dK_s^{\mathbb{P}^n} \right] \\
&\leq \tilde{C}\varepsilon + \left( \mathbb{E}^{\mathbb{P}^n} \left[ \sup_{0 \leq s \leq T} (Y_s - L_s)^2 \right] \mathbb{E}^{\mathbb{P}^n} \left[ (K_T^{\mathbb{P}^n})^2 \right] \right)^{\frac{1}{2}} \mathbb{P}^n[\tau_{2n} < T] \\
&\leq \tilde{C}\varepsilon + \frac{4C}{m\varepsilon^2} + \frac{2^p \ell}{(n-m)\varepsilon^p}.
\end{aligned}$$

It thus suffices to let  $n$  go to  $+\infty$  first, then  $m$  to  $+\infty$  and finally  $\varepsilon$  to 0.  $\square$

## 2.4.2 Comparison with the literature

In the recent months, two independent studies of the so-called reflected  $G$ -BSDEs have appeared, the first by Li and Peng [5], and the second in the PhD thesis of Soumana Hima [13]. Both these papers obtain wellposedness, in the  $G$ -framework of Peng of solutions to reflected  $G$ -BSDEs with a lower obstacle. Unlike our first paper [7], they ensure uniqueness by using the Skorokhod minimality condition (2.6). However, as shown by the result of the previous section, under Assumption 2.1, both minimality conditions actually lead to the exact same solution. Let us now detail a bit more the other differences between the two different approaches.

- (i) First of all, concerning the assumptions made, the main difference is on the obstacle. In [5, Theorems 5.1 and 5.2], in addition to our own assumptions, it is assumed to either be bounded from above or that it is a semimartingale under every measure considered (see their Assumptions (H4) and (H4')). Similarly, [13] requires the obstacle to be a semimartingale (see the equation just after (5.4) in [13]). In our framework, if one is satisfied with Definition 2.1, then we only require classical square integrability on  $L$ . If one also wants to recover the Skorokhod condition, then we need more in the form of Assumption 2.1. In any case, this does not imply that  $L$  has to be a semimartingale nor bounded from above, and merely asks for  $L$  to have finite  $p$ -variation for some  $p \geq 1$ . Notice also that our condition is strictly weaker than being a semimartingale, since the latter have finite quadratic variation.
- (ii) Concerning the method of proof, both [5] and [13] use the classical penalisation method introduced by [4] to prove existence, while uniqueness is obtained through *a priori* estimates. Our proof is more constructive and in the spirit of the original paper [12]. We expect that the penalisation approach should be applicable in our setting as well, but we leave this interesting question to future research.
- (ii) Maybe more important than the above point, one has to keep in mind that the very essence of the  $G$ -BSDE theory requires that the data of the equation, meaning here the generator  $\widehat{F}$ , the terminal condition  $\xi$  and the obstacle  $L$ , have to have some degree of regularity with respect to the  $\omega$  variable. More precisely, they have to be quasi-continuous in  $\omega$ , which loosely speaking means that they must be uniformly continuous (for the uniform convergence topology) outside a "small" set (see the references for more details). This is inherent to the construction itself, as soon as the set  $\mathcal{P}_H^\kappa$  is non-dominated, and cannot be avoided with this approach. Granted, it is also the case in our paper [7]. However, since then, many progresses have been achieved in the 2BSDE theory, and the recent paper [11] has proved that the (non-reflected) 2BSDE theory worked perfectly without *any* regularity assumption. Furthermore, a general *modus operandi* is given in [11, Proposition 2.1 and Remark 4.2] to extend those result to many type of 2BSDEs, including the reflected ones. This program has actually been carried out in the recent PhD thesis Noubiagain [8] (see also [1] and [2]). Combined with the results and discussions of the present note, the 2RBSDEs can therefore be defined in a much more general framework than the reflected  $G$ -BSDEs.

### 3 General reflections

#### 3.1 Uniqueness

Let us now consider our second paper [6]. First of all, the definition of a solution should be replaced by the following.

**Definition 3.1.** We say  $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$  is a solution to a 2DRBSDE if

- $Y_T = \xi, \mathcal{P}_H^\kappa - q.s.$
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$ , the process  $V^\mathbb{P}$  defined below has paths of bounded variation  $\mathbb{P} - a.s.$

$$V_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s) ds + \int_0^t Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s., \quad (3.1)$$

and admits the following decomposition

$$V_t^\mathbb{P} = K_t^\mathbb{P} - \mathcal{K}_t^{\mathbb{P},+}, \quad t \in [0, T], \quad \mathbb{P} - a.s., \quad (3.2)$$

where the two processes  $K^\mathbb{P}$  and  $\mathcal{K}^{\mathbb{P},+}$  are non-decreasing, and where  $\mathcal{K}^{\mathbb{P},+}$  satisfies the following Skorokhod condition

$$\int_0^T (S_{s-} - Y_{s-}) d\mathcal{K}_s^{\mathbb{P},+} = 0, \quad \mathbb{P} - a.s. \quad (3.3)$$

- We have the following minimality condition for  $0 \leq t \leq T$

$$\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^T M_s^{t, \mathbb{P}'} d(V_s^{\mathbb{P}'} + k_s^{\mathbb{P}',+} - k_s^{\mathbb{P}',-}) \right] = 0, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s., \quad \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \quad (3.4)$$

- $L_t \leq Y_t \leq S_t, \mathcal{P}_H^\kappa - q.s.$

There are two main differences with the earlier definition in our paper [6]. The first one is obviously the new minimality condition (3.4), which is simply the version with two obstacles of (2.3). The second main difference is the decomposition (3.2) of the bounded variation process  $V^\mathbb{P}$ . It is not really new, *per se*, as it was already implicit in the existence proof we provided in [6], see in particular the lignes between the statements of Lemma 4.3 and Proposition 4.4. In particular, it does not require any additional argument in the existence proof.

Under this new definition, the proof of uniqueness of a solution follows exactly the same lignes as in the lower obstacle case described above, it suffices to use the new minimality condition (3.4), which is equivalent to the representation formula of the solution to the 2DRBSDE as an essential supremum of solutions of the associated DRBSDEs.

#### 3.2 A priori estimates

The main change in [6] with the introduction of the new minimality condition (3.4) above concerns the *a priori* estimates for 2DRBSDEs. Let us start with Proposition 3.5 in [6], which has to be corrected as follows.

**Proposition 3.1.** Let Assumption 2.3 hold. Assume  $\xi \in \mathbb{L}_H^{2,\kappa}$  and  $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$  is a solution to the 2DRBSDE (2.5). Let  $\{(y^\mathbb{P}, z^\mathbb{P}, k^{\mathbb{P},+}, k^{\mathbb{P},-})\}_{\mathbb{P} \in \mathcal{P}_H^\kappa}$  be the solutions of the corresponding DRBSDEs (2.6). Then we have the following results for all  $t \in [0, T]$  and for all  $\mathbb{P} \in \mathcal{P}_H^\kappa$

(i)  $V_t^{\mathbb{P},+} := \int_0^t \mathbf{1}_{Y_{s-}=L_{s-}} dV_s^{\mathbb{P}} = \int_0^t \mathbf{1}_{Y_{s-}=L_{s-}} dk_s^{\mathbb{P},-}$ ,  $\mathbb{P}$ -a.s., and is therefore a non-decreasing process.

(ii)  $V_t^{\mathbb{P},-} := \int_0^t \mathbf{1}_{y_{s-}^{\mathbb{P}}=S_{s-}} dV_s^{\mathbb{P}} = - \int_0^t \mathbf{1}_{y_{s-}^{\mathbb{P}}=S_{s-}} dk_s^{\mathbb{P},+}$ ,  $\mathbb{P}$ -a.s., and is therefore a non-increasing process.

The proof of (ii) above is given in [6] and does not use the minimality condition and is thus correct. (i) can be proved similarly. The issue now is that we no longer have a nice Jordan decomposition of  $V^{\mathbb{P}}$ , which changes a lot how we can prove and obtain *a priori* estimates for the solution.

Actually, the main point here is to rely on the decomposition (3.2), which is almost a Jordan decomposition. In the proof of Theorem 3.7 in [6], the proof of the estimates for  $Y$ ,  $y^{\mathbb{P}}$ ,  $z^{\mathbb{P}}$ ,  $k^{\mathbb{P},+}$  and  $k^{\mathbb{P},-}$  does not change and is still correct. In the estimate for  $Z$ , corresponding to the calculations in (3.15) in [6], one has to use the decomposition (3.2) for  $V^{\mathbb{P}}$ , and the fact that we know that for some constant  $C$  independent of  $\mathbb{P}$

$$\mathbb{E}^{\mathbb{P}} [ |K_T^{\mathbb{P}}|^2 + |\mathcal{K}_T^{\mathbb{P}}|^2 ] \leq C.$$

Indeed, this is a consequence of [6, Lemma A.11] and the fact that the unique solution to the 2DRBSDE is constructed through the Doob-Meyer decomposition of a doubly reflected  $g$ -supermartingale. The rest of the proof is then the same, still using the decomposition (3.2). Thus Theorem 3.7 in [6] should be replaced by

**Theorem 3.1.** *Let Assumptions 2.3, 2.5 and 2.8 hold. Assume  $\xi \in \mathbb{L}_H^{2,\kappa}$  and  $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$  is a solution to the 2DRBSDE (2.5). Let  $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P},+}, k^{\mathbb{P},-})\}_{\mathbb{P} \in \mathcal{P}_H^{\kappa}}$  be the solutions of the corresponding DRBSDEs (2.6). Then, there exists a constant  $C_{\kappa}$  depending only on  $\kappa$ ,  $T$  and the Lipschitz constant of  $\widehat{F}$  such that*

$$\begin{aligned} & \|Y\|_{\mathbb{D}_H^{2,\kappa}}^2 + \|Z\|_{\mathbb{H}_H^{2,\kappa}}^2 + \sup_{\mathbb{P} \in \mathcal{P}_H^{\kappa}} \left\{ \|y^{\mathbb{P}}\|_{\mathbb{D}^2(\mathbb{P})}^2 + \|z^{\mathbb{P}}\|_{\mathbb{H}^2(\mathbb{P})}^2 \right\} \\ & + \sup_{\mathbb{P} \in \mathcal{P}_H^{\kappa}} \mathbb{E}^{\mathbb{P}} \left[ \text{Var}_{0,T}(V^{\mathbb{P}})^2 + (K_T^{\mathbb{P}})^2 + (\mathcal{K}_T^{\mathbb{P},+})^2 + (k_T^{\mathbb{P},+})^2 + (k_T^{\mathbb{P},-})^2 \right] \\ & \leq C_{\kappa} \left( \|\xi\|_{\mathbb{L}_H^{2,\kappa}}^2 + \phi_H^{2,\kappa} + \psi_H^{2,\kappa} + \varphi_H^{2,\kappa} + \zeta_H^{2,\kappa} \right). \end{aligned}$$

Next, concerning the estimates for the difference between two solutions, the proof of Theorem 3.8 in [6] also has to be modified. More precisely, the three lines after (3.19) should be erased. Then the proof of the estimate for  $\delta Z$  is still correct. However, we only have control over the difference between  $V^{\mathbb{P},1}$  and  $V^{\mathbb{P},2}$ , not individually for  $K^{\mathbb{P},1}$  and  $K^{\mathbb{P},2}$  on the one hand, and  $\mathcal{K}^{\mathbb{P},1}$  and  $\mathcal{K}^{\mathbb{P},2}$  on the other hand. Theorem 3.8 of [6] should therefore be replaced by

**Theorem 3.2.** *Let Assumptions 2.3, 2.5 and 2.8 hold. For  $i = 1, 2$ , let  $(Y^i, Z^i)$  be the solutions to the 2DRBSDE (2.5) with terminal condition  $\xi^i$ , upper obstacle  $S$  and lower obstacle  $L$ . Then, there exists a constant  $C_{\kappa}$  depending only on  $\kappa$ ,  $T$  and the Lipschitz constant of  $F$  such that*

$$\begin{aligned} & \|Y^1 - Y^2\|_{\mathbb{D}_H^{2,\kappa}} \leq C \|\xi^1 - \xi^2\|_{\mathbb{L}_H^{2,\kappa}} \\ & \|Z^1 - Z^2\|_{\mathbb{H}_H^{2,\kappa}}^2 + \sup_{\mathbb{P} \in \mathcal{P}_H^{\kappa}} \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |V_t^{\mathbb{P},1} - V_t^{\mathbb{P},2}|^2 \right] \\ & \leq C \|\xi^1 - \xi^2\|_{\mathbb{L}_H^{2,\kappa}} \left( \|\xi^1\|_{\mathbb{L}_H^{2,\kappa}} + \|\xi^1\|_{\mathbb{L}_H^{2,\kappa}} + (\phi_H^{2,\kappa})^{1/2} + (\psi_H^{2,\kappa})^{1/2} + (\varphi_H^{2,\kappa})^{1/2} + (\zeta_H^{2,\kappa})^{1/2} \right). \end{aligned}$$

Notice also that Remark 3.9 in [6] no longer holds. Similarly, Remark 3.12 should be deleted. As a consequence, in Proposition 3.10 in [6], the constant  $\gamma$  should always be taken as equal to 0. Finally, direct computations using the decomposition (3.2) prove that Proposition 3.14 in [6] should be replaced by

**Proposition 3.2.** *Let Assumptions 2.3, 2.5, 2.8 and 3.13 hold. Let  $(Y, Z)$  be the solution to the 2DRBSDE, then for all  $\mathbb{P} \in \mathcal{P}_H^\kappa$*

$$Z_t = P_t, \quad dt \times \mathbb{P} - \text{a.s. on the set } \{Y_{t-} = S_{t-}\}, \quad (3.5)$$

*and there exists a progressively measurable process  $(\alpha_t^\mathbb{P})_{0 \leq t \leq T}$  such that  $0 \leq \alpha \leq 1$  and*

$$d\mathcal{K}_t^{\mathbb{P},+} = \alpha_t^\mathbb{P} \mathbf{1}_{Y_{t-} = S_{t-}} \left( \left[ \widehat{F}_t(S_t, P_t) + U_t \right]^+ dt + dC_t^+ + dK_t^\mathbb{P} \right).$$

### 3.3 Existence

Because we no longer control the total variation of  $V^\mathbb{P}$  in Theorem 3.2, the proof of existence we gave in [6] only holds for  $\xi \in \text{UC}_b(\Omega)$ . However, this is not an issue at all, since the only reason we had to restrict to uniformly continuous terminal condition was to obtain the measurability result in [6, Lemma 4.1] and the dynamic programming principle of [6, Proposition]. Using the results of [11], in particular Proposition 2.1, these two results were obtained in [8] (see also [2]) for doubly reflected BSDEs, and allow to extend the construction carried out in [6] to any  $\xi \in \mathbb{L}_H^{2,\kappa}$ .

Finally, notice that similar arguments as in the lower reflected case allow to prove that wellposedness can be recovered for 2DRBSDEs when the minimality condition (3.4) is replaced by asking that the process  $K^\mathbb{P}$  in the decomposition (3.2) satisfies the Skorokhod condition (2.6), provided that Assumption 2.1 holds. In such a situation, both processes  $K^\mathbb{P}$  and  $\mathcal{K}^{\mathbb{P},+}$  thus satisfy Skorokhod type conditions.

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