



**HAL**  
open science

# SOME REMARKS ON THE NONLINEAR SCHRODINGER EQUATION WITH FRACTIONAL DISSIPATION.

Mohamad Darwich, Luc Molinet

► **To cite this version:**

Mohamad Darwich, Luc Molinet. SOME REMARKS ON THE NONLINEAR SCHRODINGER EQUATION WITH FRACTIONAL DISSIPATION.. Journal of Mathematical Physics, 2016, 57 (10), pp.101502. 10.1063/1.4965225 . hal-01545104

**HAL Id: hal-01545104**

**<https://hal.science/hal-01545104>**

Submitted on 22 Jun 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SOME REMARKS ON THE NONLINEAR SCHRÖDINGER EQUATION WITH FRACTIONAL DISSIPATION.

MOHAMAD DARWICH AND LUC MOLINET.

ABSTRACT. We consider the Cauchy problem for the  $L^2$ -critical focussing nonlinear Schrödinger equation with a fractional dissipation. According to the order of the fractional dissipation, we prove the global existence or the existence of finite time blowup dynamics with the log-log blow-up speed for  $\|\nabla u(t)\|_{L^2}$ .

## 1. INTRODUCTION

In this paper, we study the blowup and the global existence of solutions for the  $L^2$ -critical nonlinear Schrödinger equation (NLS) with a fractional dissipation term:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u + ia(-\Delta)^s u = 0, (t, x) \in [0, \infty[ \times \mathbb{R}^d, d \in \mathbb{N}^*. \\ u(0) = u_0 \in H^r(\mathbb{R}^d) \end{cases} \quad (1.1)$$

where  $a > 0$  is the coefficient of friction,  $s > 0$  and  $r \in \mathbb{R}$ .

The NLS equation ( $a = 0$ ) arises in various areas of nonlinear optics, plasma physics and fluid mechanics to describe propagation phenomena in dispersive media. To take into account weak dissipation effects, one usually add a linear damping term as in the linear damped NLS equation (see for instance Fibich [12]):

$$iu_t + \Delta u + iau + |u|^p u = 0, a > 0.$$

However, in a wide range of situations a frequency-dependent attenuation has been observed (cf [7]). This motivates to rather complete the NLS equation with a laplacian term as in the following complex Ginzburg-Landau equation studied in Passota-Sulem-Sulem [32]:

$$iu_t + \Delta u - ia\Delta u + |u|^p u = 0, a > 0,$$

Now, in many cases of practical importance the damping cannot be described by a local term even in the long-wavelength limit. In media with dispersion the weak dissipation is, in general, non local (see for instance Ott-Sudan [30]). It is thus quite natural to complete the NLS equation by a non local dissipative term in order to take into account some dissipation phenomena.

In this this paper we complete the  $L^2$ -critical NLS equation (1.2) with a fractional laplacian of order  $2s$ ,  $s > 0$ , and study the influence of this term on the blow-up phenomena for this equation. The fractional laplacian is commonly used to model fractal (anomalous) diffusion related to the Lévy

---

*Key words and phrases.* Damped Nonlinear Schrödinger Equation, Blow-up, Global existence.

flights (see e.g. Stroock [35], Bardos and all [3], Hanyga [18]). It also appears in the physical literature to model attenuation phenomena of acoustic waves in irregular porous random media (cf. Blackstock [4], Gaul [17], Chen-Holm [7]).

Finally, note that the case of a nonlinear damping of the type  $ia|u|^p u$ , has been studied by Antonelli-Sparber and Antonelli-Carles-Sparber (cf. [1] and [2]). In this case the origin of the nonlinear damping term is multiphoton absorption.

Recall that the Cauchy problem for the  $L^2$ -critical focussing nonlinear Schrödinger equation ( $a = 0$ ):

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}} u = 0 \\ u(0) = u_0 \in H^r(\mathbb{R}^d) \end{cases} \quad (1.2)$$

has been studied by a lot of authors (see for instance [19], [8], [6]) and it is known that the problem is locally well-posed in  $H^r(\mathbb{R}^d)$  for  $r \geq 0$ : For any  $u_0 \in H^r(\mathbb{R}^d)$ , with  $r \geq 0$ , there exist  $T > 0$  and a unique solution  $u$  of (1.2) with  $u(0) = u_0$  such that  $u \in C([0, T]; H^r(\mathbb{R}^d))$ . Moreover, if the maximal existence time  $T^*$  of the solution  $u$  in  $H^r(\mathbb{R}^d)$  is finite then  $\|u\|_{L^{\frac{4}{d}+2}([0, T^*] \times \mathbb{R}^d)} = \infty$ .

Let us mention that in the case  $a > 0$  the same results on the local Cauchy problem for (1.1) can be established in exactly the same way as in the case  $a = 0$ , since the same Strichartz estimates hold (see for instance [29]).

For  $u_0 \in H^1(\mathbb{R}^d)$ , a sharp criterion for global existence for (1.2) has been exhibited by Weinstein [37]: Let  $Q$  be the unique radial positive solution (see [5], [21]) to

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \quad (1.3)$$

If  $\|u_0\|_{L^2} < \|Q\|_{L^2}$  then the solution of (1.2) is global in  $H^1(\mathbb{R}^d)$ . This follows from the conservation of the energy and the  $L^2$  norm and the sharp Gagliardo-Nirenberg inequality which ensures that

$$\forall u \in H^1, E(u) \geq \frac{1}{2} \left( \int |\nabla u|^2 \right) \left( 1 - \left( \frac{\int |u|^2}{\int |Q|^2} \right)^{\frac{2}{d}} \right). \quad (1.4)$$

Actually, it was recently proven that any solution of (1.2) emanating from an initial datum  $u_0 \in L^2(\mathbb{R}^d)$  with  $\|u_0\|_{L^2} < \|Q\|_{L^2}$  is global and scatters, i.e.  $u$  behaves like the linear evolution of an  $L^2$  function for large time,  $u(t) \sim e^{it\Delta} u_+$  (cf. [11]).

On the other hand, NLS has unique minimal mass blow-up solution in  $H^1$  up to the symmetries of the equation (see [23]) that blow up at some time  $T > 0$  with a  $H^1$  norm that grows as  $\frac{1}{T-t}$ .

In the series of papers [24], [25], [26], [27], Merle and Raphael studied the blowup for (1.2) with  $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \delta$ ,  $\delta$  small and proved the existence of the blowup regime corresponding to the log-log law:

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \left( \frac{\log |\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \quad (1.5)$$

Recall that the evolution of (1.2) admits the following conservation laws in the energy space  $H^1$ :

$$L^2\text{-norm} : m(u) = \|u\|_{L^2} = \left( \int |u(x)|^2 dx \right)^{1/2}.$$

$$\text{Energy} : E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{d}{4+2d} \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

$$\text{Kinetic momentum} : P(u) = \text{Im} \left( \int \nabla u(x) \bar{u}(x) dx \right).$$

Now, for (1.1) with  $a > 0$ , there does not exist conserved quantities anymore. However, it is easy to prove that if  $u$  is a smooth solution of (1.1) on  $[0, T[$ , then for all  $t \in [0, T[$  it holds

$$\|u(t)\|_{L^2}^2 + a \int_0^t \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 = \|u_0\|_{L^2}^2. \quad (1.6)$$

$$\frac{d}{dt} E(u(t)) = -a \int |(-\Delta)^{\frac{s+1}{2}} u(t)|^2 + a \text{Im} \int ((-\Delta)^s u(t)) |u(t)|^{\frac{4}{d}} \bar{u}(t). \quad (1.7)$$

$$\frac{1}{2} \frac{d}{dt} P(u(t)) = a \text{Im} \int ((-\Delta)^s u(t)) \nabla \bar{u}(t). \quad (1.8)$$

In [10], the first author studied the case  $s = 0$ . He proved the global existence in  $H^1$  for  $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ , and showed that the log-log regime is stable by such perturbations (i.e. there exist solutions that blowup in finite time with the log-log law).

In [32], Passot, Sulem and Sulem proved that the solutions are global in  $H^1(\mathbb{R}^2)$  for  $s = 1$ . However, their method does not seem to apply for any other values of  $d$ .

Our aim in this paper is to establish some results, for  $s > 0$ , on the global existence or the existence of finite time blowup dynamics with the log-log blow-up speed for  $\|\nabla u\|_{L^2}$ .

Let us now state our results:

**Theorem 1.1.** *Let  $d = 1, 2, 3, 4$  and  $0 < s < 1$  then there exists  $\delta_0 > 0$  such that  $\forall a > 0$  and  $\forall \delta \in ]0, \delta_0[$ , there exists  $u_0 \in H^1$  with  $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$ , such that the solution of (1.1) blows up in finite time in the log-log regime.*

**Theorem 1.2.** *Let  $d \in \mathbb{N}^*$ ,  $s \geq 1$  and  $r \geq 0$ . Then the Cauchy problem (1.1) is globally well-posed in  $H^r(\mathbb{R}^d)$ .*

**Theorem 1.3.** *Let  $d \in \mathbb{N}^*$ ,  $0 < s < 1$  and  $a > 0$ .*

- (1) *There exists a real number  $0 < \gamma = \gamma(d) \leq \|Q\|_{L^2}$  such that for any initial datum  $u_0 \in H^1(\mathbb{R}^d)$  with  $\|u_0\|_{L^2} < \gamma$ , the emanating solution  $u$  is global in  $H^1$  with an energy that is non increasing.*
- (2) *There does not exist any initial datum  $u_0$ , with  $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ , such that the solution  $u$  of (1.1) blows up at finite time  $T^*$  and satisfies*

$$\frac{1}{(T^* - t)^\alpha} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T^* - t)^\beta}, \quad \forall 0 < T - t \ll 1,$$

*for some pair  $(\alpha, \beta)$  satisfying  $0 < \beta < \frac{1}{2s}$  and  $\beta(1+s) - 1/2 < \alpha \leq \beta$ .*

**Remark 1.1.** *It is natural to expect that  $\gamma = \|Q\|_{L^2}$ . Indeed, for  $u_0 \in H^1(\mathbb{R}^d)$  satisfying  $\|u_0\|_{L^2} \leq \|Q\|_{L^2(\mathbb{R}^d)}$ , (1.6) ensures that  $\|u(t)\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^d)}$  as soon as  $t > 0$  and the solutions of the critical NLS equation with such initial data are global.*

*However, in contrary to the case  $s = 0$ , we do not succeed to prove this fact here since the constant  $C_d$  appearing in the estimate on the energy (see subsection 2.3) seems not to be directly related to  $Q$ . Assertion (2) of Theorem 1.3 is a partial result in this direction since it ensures that we do not have any blowup in the log-log regime, for any  $0 < s < 1$ , and in the regime  $\frac{1}{l}$ , for any  $0 < s < \frac{1}{2}$ , for initial data with critical or subcritical mass.*

**Acknowledgments :** The first author thanks the L.M.P.T. for his kind hospitality during the development of this work. Moreover, he would like to thank AUF for supporting this project. The authors would like to thank the anonymous Referee for valuable remarks and comments.

## 2. LOCAL AND GLOBAL EXISTENCE RESULTS

In this section, we prove Theorem 1.2 and part (1) of Theorem 1.3. Theorem 1.2 will follow from an a priori estimate on the critical Strichartz norm whereas part (1) of Theorem 1.3 follows from a monotonicity of the energy.

**2.1. Local existence result.** Recall that the main tools to prove the local existence results for (1.2) are the Strichartz estimates for the associated linear propagator  $e^{i\Delta t}$ . These Strichartz estimates read

$$\|e^{i\Delta t}\phi\|_{L_t^q L^r(\mathbb{R}^d)} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)}$$

for any pair  $(q, r)$  satisfying  $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$  and  $2 < q \leq \infty$ . Such ordered pair is called an admissible pair .

For  $a \geq 0$  and  $s \geq 0$  we denote by  $S_{a,s}$  the linear semi-group associated with (1.1), i.e.  $S_{a,s}(t) = e^{i\Delta t - a(-\Delta)^s t}$ . It is worth noticing that  $S_{a,s}$  is irreversible. The following lemma ensures that the linear semi-group  $S_{a,s}$  enjoys the same Strichartz estimates as  $e^{i\Delta t}$ .

**Lemma 2.1.** *Let  $\phi \in L^2(\mathbb{R}^d)$  and  $s > 0$ . Then for every admissible pair  $(q, r)$  it holds*

$$\|S_{a,s}(\cdot)\phi\|_{L_{t>0}^q L^r(\mathbb{R}^d)} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)}.$$

*Proof.* Setting, for any  $t \geq 0$ ,  $G_{a,s}(t, x) = \int e^{-ix\xi} e^{-at|\xi|^{2s}} d\xi$ , it holds

$$S_{a,s}(t)\varphi = G_{a,s}(t, \cdot) * e^{it\Delta}\varphi, \quad \forall t \geq 0.$$

Noticing that for  $s > 0$ ,  $\|G_{a,s}(t, \cdot)\|_{L^1} = \|G_{1,s}(1, \cdot)\|_{L^1}$  and that, according to Lemma 2.1 in [29],  $G_{1,s}(1, \cdot) \in L^1(\mathbb{R}^d)$  for  $s > 0$ , we get

$$\|S_{a,s}(t)\phi\|_{L_{t>0}^p L^q(\mathbb{R}^d)} = \|G_{a,s}(t, \cdot) * e^{it\Delta}(\phi)\|_{L_{t>0}^p L^q(\mathbb{R}^d)} \lesssim \|e^{it\Delta}\phi\|_{L_t^p L^q(\mathbb{R}^d)} \lesssim \|\phi\|_{L^2}.$$

□

With Lemma 2.1 in hand, it is not too hard to check that the local existence results for equation (1.2) (see for instance [8] and [6] for the local existence and [20] for the continuity with respect to initial data) also holds

for (1.1) with  $a \geq 0$  and  $s > 0$ . More precisely, we have the following statement:

**Proposition 2.1.** *Let  $a \geq 0$ ,  $s > 0$ ,  $r \geq 0$  and  $u_0 \in H^r(\mathbb{R}^d)$  with  $d \in \mathbb{N}^*$ . There exists  $T > 0$  and a unique solution  $u \in C([0, T]; H^r) \cap L_T^{\frac{4}{d}+2} L^{\frac{4}{d}+2}$  to (1.1) emanating from  $u_0$ . In addition, there exists a neighborhood  $\mathcal{V}_{u_0}$ <sup>(1)</sup> of  $u_0$  in  $H^r(\mathbb{R}^d)$  such that the associated solution map is continuous from  $\mathcal{V}_{u_0}$  into  $C([0, T]; H^r) \cap L_T^{\frac{4}{d}+2}([0, T] \times \mathbb{R}^d)$ . Finally, let  $T^*$  be the maximal time of existence of the solution  $u$  in  $H^r(\mathbb{R}^d)$ , then*

$$T^* < \infty \implies \|u\|_{L_{T^*}^{\frac{4}{d}+2} L^{\frac{4}{d}+2}} = +\infty. \quad (2.1)$$

**2.2. Proof of Theorem 1.2.** Let  $u \in C([0, T]; L^2(\mathbb{R}^d))$  be the solution emanating from some initial datum  $u_0 \in L^2(\mathbb{R}^d)$ . We have the following a priori estimates:

**Lemma 2.2.** *Let  $u \in C([0, T]; L^2(\mathbb{R}^d))$  be the solution of (1.1) emanating from  $u_0 \in L^2(\mathbb{R}^d)$ . Then*

$$\|u\|_{L_T^\infty L^2} \leq \|u_0\|_{L^2} \text{ and } \|(-\Delta)^{\frac{s}{2}} u\|_{L_T^2 L^2} \leq \frac{1}{\sqrt{2a}} \|u_0\|_{L^2}. \quad (2.2)$$

*Proof.* Assume first that  $u_0 \in H^\infty(\mathbb{R}^d)$ . Then (1.6) ensures that the mass is decreasing as soon as  $u$  is not the null solution and (1.6) leads to

$$\int_0^T \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 dt = -\frac{1}{2a} (\|u(T)\|_{L^2}^2 - \|u_0\|_{L^2}^2) \leq \frac{1}{2a} \|u_0\|_{L^2}^2.$$

This proves (2.2) for smooth solutions. The result for  $u_0 \in L^2(\mathbb{R}^d)$  follows by approximating  $u_0$  in  $L^2$  by a smooth sequence  $(u_{0,n}) \subset H^\infty(\mathbb{R}^d)$ .  $\square$

Note that the first estimate in (2.2) implies that  $\|u\|_{L_T^2 L^2} \leq T^{1/2} \|u_0\|_{L^2}$  and thus by interpolation:

$$\|\nabla u\|_{L_T^2 L^2} \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{L_T^2 L^2}^{\frac{1}{s}} \|u\|_{L_T^2 L^2}^{1-\frac{1}{s}} \lesssim T^{\frac{1}{2}(1-\frac{1}{s})} \quad (2.3)$$

Interpolating now between (2.3) and the first estimate of (2.2) we get

$$\|u\|_{L_T^{\frac{4}{d}+2} H^{\frac{2d}{4+2d}}} \lesssim T^{\frac{1}{2}(1-\frac{1}{s})}$$

and the embedding  $H^{\frac{2d}{4+2d}}(\mathbb{R}^d) \hookrightarrow L^{\frac{4}{d}+2}(\mathbb{R}^d)$  ensures that

$$\|u\|_{L_T^{\frac{4}{d}+2} L^{\frac{4}{d}+2}} \lesssim \|u\|_{L_T^{\frac{4}{d}+2} H^{\frac{2d}{4+2d}}} \lesssim T^{\frac{1}{2}(1-\frac{1}{s})}.$$

Denoting by  $T^*$  the maximal time of existence of  $u$  in  $L^2(\mathbb{R}^d)$  and letting  $T$  tends to  $T^*$ , this contradicts (2.1) whenever  $T^*$  is finite. This proves that the solutions are global in  $H^r(\mathbb{R}^d)$ .

Moreover, for  $s = 1$  we have  $\|u\|_{L_T^{\frac{4}{d}+2} L^{\frac{4}{d}+2}} \lesssim 1$  for any  $T > 0$  which ensures that

$$\|u\|_{L^{\frac{4}{d}+2}(\mathbb{R}_+^* \times \mathbb{R}^d)} \lesssim 1.$$

---

<sup>(1)</sup>It is worth noticing that for  $r = 0$ , the neighborhood  $\mathcal{V}_{u_0}$  does not depend only on  $\|u_0\|_{H^r}$  but on the Fourier profile of  $u_0$  (see for instance [20])

**2.3. Proof of Assertion 1 of Theorem 1.3.** Note that the global existence for any  $u_0 \in L^2(\mathbb{R}^d)$  with  $\|u_0\|_{L^2}$  small enough can be proven, as for the critical NLS equation, directly by a fixed point argument thanks to Lemma 2.1. This ensures the global existence in  $H^r(\mathbb{R}^d)$ ,  $r \geq 0$ , under the same smallness condition on  $\|u_0\|_{L^2}$ . We will not invoke this fact here and we will directly prove Assertion 1 of Theorem 1.3 by combining (1.4) and a monotony result on  $t \mapsto E(u(t))$ . To do this, we will work with smooth solutions and then get the result for  $H^1$ -solutions by continuity with respect initial data.

So, let  $u \in C([0, T]; H^\infty(\mathbb{R}^d))$  be a solution to (1.1) emanating from  $u_0 \in H^\infty(\mathbb{R}^d)$ . Then it holds

$$\frac{d}{dt}E(u(t)) = -a \int |(-\Delta)^{\frac{s+1}{2}} u(t)|^2 + a \operatorname{Im} \int ((-\Delta)^s u(t)) |u(t)|^{\frac{4}{d}} \overline{u(t)}$$

and Hölder inequalities in physical space and in Fourier space lead to

$$\left| \int ((-\Delta)^s u) |u|^{\frac{4}{d}} \overline{u} \right| \leq \|(-\Delta)^s u\|_{L^2} \|u\|_{L^{\frac{8}{\frac{d}{s}+2}}}^{\frac{4}{d}+1}$$

with

$$\|(-\Delta)^s u\|_{L^2} \leq \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2}^{\frac{2s}{s+1}} \|u\|_{L^2}^{\frac{1-s}{1+s}}.$$

Let us recall the following Gagliardo-Nirenberg inequality <sup>(2)</sup>

$$\|u\|_{L^{\frac{8}{\frac{d}{s}+2}}}^{\frac{8}{d}+2} \leq C_d^{\frac{8}{d}+2} \|\nabla u\|_{L^2}^4 \|u\|_{L^2}^{\frac{8}{d}-2}.$$

This estimate together with Cauchy-Schwarz inequality (in Fourier space)

$$\|\nabla u\|_{L^2}^2 \leq \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2}^{\frac{2}{s+1}} \|u\|_{L^2}^{\frac{2s}{s+1}}$$

lead to

$$\|u\|_{L^{\frac{8}{\frac{d}{s}+2}}}^{\frac{4}{d}+1} \leq C_d^{\frac{4}{d}+1} \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2}^{\frac{2}{s+1}} \|u\|_{L^2}^{\frac{2s}{s+1}} \|u\|_{L^2}^{\frac{4-d}{d}}.$$

Combining the above estimates we eventually obtain

$$\frac{d}{dt}E(u(t)) \leq a \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2}^2 (C_d^{\frac{4}{d}+1} \|u\|_{L^2}^{4/d} - 1)$$

which together with (2.2) implies that  $E(u(t))$  is not increasing for  $\|u_0\|_{L^2} \leq C_d^{1+\frac{d}{4}}$ .

### 3. PROOF OF ASSERTION 2 OF THEOREM 1.3

Special solutions play a fundamental role for the description of the dynamics of (NLS). They are the solitary waves of the form  $u(t, x) = \exp(it)Q(x)$ , where  $Q$  the unique positive radial solution to

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \quad (3.1)$$

The pseudo-conformal transformation applied to the “stationary” solution  $e^{it}Q(x)$  yields an explicit solution for (NLS)

$$S(t, x) = \frac{1}{|t|^{\frac{d}{2}}} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}$$

<sup>(2)</sup>It is proven in [37] that the constant  $C_d$  is related for  $d = 1, 2, 3$  to the  $L^2$ -norm of the ground state solution of  $2\Delta\psi - (\frac{4}{d} - 1)\psi + \psi^{\frac{8}{d}+1} = 0$ .

which blows up at  $T^* = 0$ .

Note that

$$\|S(t)\|_{L^2} = \|Q\|_{L^2} \text{ and } \|\nabla S(t)\|_{L^2} \sim \frac{1}{t} \quad (3.2)$$

It turns out that  $S(t)$  is the unique minimal mass blow-up solution in  $H^1$  up to the symmetries of the equation ( see [23]).

A known lower bound ( see [8] and [6]) on the blow-up rate for (NLS) is

$$\|\nabla u(t)\|_{L^2} \geq \frac{C(u_0)}{\sqrt{T-t}}. \quad (3.3)$$

Note that this blow-up rate is strictly lower than the one of  $S(t)$  given by (3.2) and of the log-log law given by (1.5).

To prove assertion 2 of Theorem 1.3, we will need the following result ( see [16]) :

**Theorem 3.1.** *Let  $(v_n)_n$  be a bounded family of  $H^1(\mathbb{R}^d)$ , such that:*

$$\limsup_{n \rightarrow +\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^d)} \leq M \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \|v_n\|_{L^{\frac{4}{d}+2}} \geq m. \quad (3.4)$$

*Then, there exists  $(x_n)_n \subset \mathbb{R}^d$  such that:*

$$v_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly,}$$

*with  $\|V\|_{L^2(\mathbb{R}^d)} \geq \left(\frac{d}{d+4}\right)^{\frac{d}{4}} \frac{m^{\frac{d}{2}+1} + 1}{M^{\frac{d}{2}}} \|Q\|_{L^2(\mathbb{R}^d)}$ .*

Suppose that there exist an initial data  $u_0$  with  $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ , such that the corresponding solution  $u(t)$  blows up at time  $T > 0$  with the following behavior:

$$\frac{1}{(T-t)^\alpha} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T-t)^\beta}, \quad \forall t \in [0, T[, \quad (3.5)$$

where  $\beta > 0$  and  $\alpha \geq \beta$  satisfies  $\alpha > \beta(1+s) - 1/2$ .

Recalling that

$$E(u(t)) = E(u_0) - a \int_0^t K(u(\tau)) d\tau, \quad t \in [0, T[, \quad (3.6)$$

with  $K(u(t)) = \int |(-\Delta)^{\frac{s+1}{2}} u|^2 - \text{Im} \int ((-\Delta)^s u) |u|^{\frac{4}{d}} \bar{u}$ , we obtain that

$$E(u(t)) \lesssim E(u_0) + \left| \int_0^t ((-\Delta)^s u) |u|^{\frac{4}{d}} \bar{u} dx \right| \lesssim E(u_0) + \int_0^t \|(-\Delta)^s u\|_{L^2} \|u\|_{L^{\frac{8}{d}+2}}^{\frac{4}{d}+1}$$

This last estimate together with

$$\|u\|_{L^{\frac{8}{d}+2}}^{\frac{4}{d}+1} \lesssim \|\nabla u\|_{L^2}^2 \|u_0\|_{L^2}^{\frac{4-d}{d}} \quad \text{and} \quad \|(-\Delta)^s(u)\|_{L^2} \leq \|\nabla u\|_{L^2}^{2s} \|u\|_{L^2}^{1-2s} \lesssim \|\nabla u\|_{L^2}^{2s}$$

yield

$$E(u(t)) \lesssim E(u_0) + \int_0^t \|\nabla u\|_{L^2}^{2+2s}(\tau) d\tau. \quad (3.7)$$



Note that assumption (3.5) ensures that

$$0 \leq \frac{\int_0^t \|\nabla u(\tau)\|_{L^2}^{2+2s} d\tau}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2} \lesssim (T-t)^{-2\beta(1+s)+1+2\alpha} \rightarrow 0 \text{ as } t \nearrow T, \quad (3.8)$$

Now, let

$$\rho(t) = \frac{\|\nabla Q\|_{L^2(\mathbb{R}^d)}}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}} \quad \text{and} \quad v(t, x) = \rho^{\frac{d}{2}} u(t, \rho x)$$

and let  $(t_k)_k$  be a sequence of positive times such that  $t_k \nearrow T$ . We set  $\rho_k = \rho(t_k)$  and  $v_k = v(t_k, \cdot)$ . The family  $(v_k)_k$  satisfies

$$\|v_k\|_{L^2(\mathbb{R}^d)} = \|u(t_k, \cdot)\|_{L^2(\mathbb{R}^d)} < \|u_0\|_{L^2(\mathbb{R}^d)} \leq \|Q\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla v_k\|_{L^2(\mathbb{R}^d)} = \|\nabla Q\|_{L^2(\mathbb{R}^d)}.$$

The above estimate on  $\|v_k\|_{L^2}$  and (3.7) lead to

$$0 < \frac{1}{2} \left( \int |\nabla v_k|^2 \right) \left( 1 - \left( \frac{\int |v_k|^2}{\int |Q|^2} \right)^2 \right) \leq E(v_k) = \rho_k^2 E(u(t_k)) \lesssim \rho_k^2 E(u_0) + \rho_k^2 \int_0^{t_k} \|\nabla u(\tau)\|_{L^2}^{2+2s} d\tau$$

which, together with (3.8), ensures that  $\lim_{k \rightarrow +\infty} E(v_k) = 0$ . This forces

$$\|v_k\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \rightarrow \frac{d+2}{d} \|\nabla v_k\|_{L^2(\mathbb{R}^d)}^2 = \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 \quad (3.9)$$

and thus the family  $(v_k)_k$  satisfies the hypotheses of Theorem 3.1 with

$$m^{\frac{4}{d}+2} = \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 \quad \text{and} \quad M = \|\nabla Q\|_{L^2(\mathbb{R}^d)}.$$

Hence, there exists a family  $(x_k)_k \subset \mathbb{R}$  and a profile  $V \in H^1(\mathbb{R})$  with

$\|V\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}$ , such that,

$$\rho_k^{\frac{d}{2}} u(t_k, \rho_k \cdot + x_k) \rightharpoonup V \in H^1 \quad \text{weakly.} \quad (3.10)$$

Using (3.10),  $\forall A \geq 0$

$$\liminf_{n \rightarrow +\infty} \int_{B(0,A)} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{B(0,A)} |V|^2 dx.$$

But, since  $\lim_{n \rightarrow +\infty} \rho_n = 0$ ,  $\rho_n A < 1$  for  $n$  large enough and thus

$$\liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{|x-y| \leq 1} |u(t_n, x)|^2 dx \geq \liminf_{n \rightarrow +\infty} \int_{|x-x_n| \leq \rho_n A} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx.$$

Since this it is true for all  $A > 0$  we obtain that

$$\|u_0\|_{L^2}^2 > \liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{|x-y| \leq 1} |u(t_n, x)|^2 dx \geq \|Q\|_{L^2}^2$$

which contradicts the assumption  $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$  and the desired result is proven.

## 4. BLOW UP SOLUTION.

In this section, we prove the existence of the explosive solutions in the case  $0 < s < 1$ .

**Theorem 4.1.** *Let  $0 < s < 1$  and  $1 \leq d \leq 4$ . There exist a set of initial data  $\Omega$  in  $H^1$ , such that for any  $0 < a < a_0$  with  $a_0 = a_0(s)$  small enough, the emanating solution  $u(t)$  to (1.1) blows up in finite time in the log-log regime.*

The set of initial data  $\Omega$  is the set described in [24], in order to initialize the log-log regime. It is open in  $H^1$ . Using the continuity with regard to the initial data and the parameters, we easily obtain the following corollary:

**Corollary 4.1.** *Let  $0 < s < 1$ ,  $1 \leq d \leq 4$  and  $u_0 \in H^1$  be an initial data such that the corresponding solution  $u(t)$  of (1.2) blows up in the loglog regime. There exist  $\beta_0 > 0$  and  $a_0 > 0$  such that if  $v_0 = u_0 + h_0$ ,  $\|h_0\|_{H^1} \leq \beta_0$  and  $a \leq a_0$ , the solution  $v(t)$  for (1.1) with the initial data  $v_0$  blows up in finite time.*

Now to prove Theorem 4.1, we look for a solution of (1.1) such that for  $t$  close enough to blowup time, we shall have the following decomposition:

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} (Q_{b(t)} + \epsilon)(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}, \quad (4.1)$$

for some geometrical parameters  $(b(t), \lambda(t), x(t), \gamma(t)) \in (0, \infty) \times (0, \infty) \times \mathbb{R}^d \times \mathbb{R}$ ,  $\lambda(t) \sim \frac{1}{\|\nabla u(t)\|_{L^2}}$ ,  $b \sim -\frac{\lambda_s}{\lambda}$  where  $\frac{ds}{dt} = \frac{1}{\lambda^2(t)}$ . The profiles  $Q_b$  are suitable deformations of  $Q$  related to some extra degeneracy of the problem, in fact these profile  $Q_b$  are a regularization of the exact self similar solutions to (1.2) which satisfy the nonlinear elliptic equation:

$$\Delta Q_b - Q_b + ib(\frac{d}{2}Q_b + y \cdot \nabla Q_b) + Q_b|Q_b|^{\frac{4}{d}} = 0. \quad (4.2)$$

Note that our proof is very close to the case of  $s = 0$  (see Darwich [10]). Actually, as noticed in [33], we only need to prove that in the log-log regime the  $L^2$  norm does not grow, and the growth of the energy( resp the momentum) is below  $\frac{1}{\lambda(t)^2}$  (resp  $\frac{1}{\lambda(t)}$ ). In this paper, we will prove that in the log-log regime, the growths of the energy and the momentum are bounded by:

$$E(u(t)) \lesssim \log(\lambda(t))\lambda(t)^{-2s}, \quad P(u(t)) \lesssim \log(\lambda(t))\lambda(t)^{\frac{-2s}{s+1}}.$$

Let us recall that a fonction  $u : [0, T] \mapsto H^1$  follows the log-log regime if the following uniform controls on the decomposition (4.1) hold on  $[0, T]$ :

- Control of  $b(t)$

$$b(t) > 0, \quad b(t) < 10b(0). \quad (4.3)$$

- Control of  $\lambda$ :

$$\lambda(t) \leq e^{-e^{\frac{\pi}{100b(t)}}} \quad (4.4)$$

and the monotonicity of  $\lambda$ :

$$\lambda(t_2) \leq \frac{3}{2}\lambda(t_1), \quad \forall 0 \leq t_1 \leq t_2 \leq T. \quad (4.5)$$

Let  $k_0 \leq k_+$  be integers and  $T^+ \in [0, T]$  such that

$$\frac{1}{2^{k_0}} \leq \lambda(0) \leq \frac{1}{2^{k_0-1}}, \frac{1}{2^{k_+}} \leq \lambda(T^+) \leq \frac{1}{2^{k_+-1}} \quad (4.6)$$

and for  $k_0 \leq k \leq k_+$ , let  $t_k$  be a time such that

$$\lambda(t_k) = \frac{1}{2^k}, \quad (4.7)$$

then we assume the control of the doubling time interval:

$$t_{k+1} - t_k \leq k\lambda^2(t_k). \quad (4.8)$$

- control of the excess of mass:

$$\int_{\mathbb{R}^d} |\nabla \epsilon(t)|^2 + \int_{\mathbb{R}^d} |\epsilon(t)|^2 e^{-|y|} \leq \Gamma_{b(t)}^{\frac{1}{4}}, \text{ where } \Gamma_b \sim e^{-\frac{\pi}{b}}. \quad (4.9)$$

The main point is to establish that (4.3)-(4.9) determine a trapping region for the flow. Actually, after the decomposition (4.1) of  $u$ , the log-log regime corresponds to the following asymptotic controls

$$b_s \sim C e^{-\frac{c}{b}}, \quad -\frac{\lambda_s}{\lambda} \sim b \quad (4.10)$$

and

$$\int_{\mathbb{R}^d} |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}}, \quad (4.11)$$

where we have introduced the rescaled time  $\frac{ds}{dt} = \frac{1}{\lambda^2}$ .

In fact, (4.11) is partly a consequence of the preliminary estimate:

$$\int_{\mathbb{R}^d} |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}} + \lambda^2(t) E(t). \quad (4.12)$$

One then observes that in the log-log regime, the integration of the laws (4.10) yields

$$\lambda \sim e^{-e^{\frac{c}{b}}} \ll e^{-\frac{c}{b}}, \quad b(t) \rightarrow 0, \quad t \rightarrow T. \quad (4.13)$$

Hence, the term involving the conserved Hamiltonian is asymptotically negligible with respect to the leading order term  $e^{-\frac{c}{b}}$  which drives the decay (4.12) of  $b$ . This was a central observation made by Planchon and Raphael in [33]. In fact, any growth of the Hamiltonian algebraically below  $\frac{1}{\lambda^2(t)}$  would be enough. In this paper, we will prove that in the log-log regime, the growth of the energy is estimated by:

$$E(u(t)) \lesssim (\log(\lambda(t))) \lambda^{-2s}(t). \quad (4.14)$$

It then follows from (4.12) that:

$$\int_{\mathbb{R}^d} |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}}. \quad (4.15)$$

An important feature of this estimate of  $H^1$  flavor is that it relies on a flux computation in  $L^2$ . This allows one to recover the asymptotic laws for the geometrical parameters (4.10) and to close the bootstrap estimates of the log-log regime.

**Remark 4.1.** *Actually, one also needs the bound on the momentum to control the geometrical parameters (see Lemma 7.2 in [10]).*

**4.1. Control of the energy and the kinetic momentum.** Let us recall that we say that an ordered pair  $(q, r)$  is admissible whenever  $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$  and  $2 < q \leq \infty$ . We define the Strichartz norm of functions  $u : [0, T] \times \mathbb{R}^d \mapsto \mathbb{C}$  by:

$$\|u\|_{S^0([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (4.16)$$

and

$$\|u\|_{S^1([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|\nabla u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (4.17)$$

We will sometimes abbreviate  $S^i([0, T] \times \mathbb{R}^d)$  with  $S_T^i$  or  $S^i[0, T]$ ,  $i = 1, 2$ . Now we will derive an estimate on the energy, to check that it remains small with respect to  $\lambda^{-2}$ :

**Proposition 4.1.** *Assuming that (4.4)-(4.9) hold, then the energy and kinetic momentum are controlled on  $[0, T^+]$  by:*

$$|E(u(t))| \lesssim (\log(\lambda(t))) \lambda^{-2s}(t), \quad (4.18)$$

$$|P(u(t))| \lesssim (\log(\lambda(t))) \lambda^{\frac{-2s}{s+1}}(t). \quad (4.19)$$

To prove Proposition 4.1, we will need the two following lemmas.

**Lemma 4.1.** *Let  $u \in C([0, T]; H^1(\mathbb{R}^d))$  be a solution of (1.1). Then we have the following estimation:*

$$\|\nabla u\|_{L_T^\infty L_x^2} + \|(\Delta)^{\frac{s+1}{2}} u\|_{L_T^2 L_x^2} \lesssim \|\nabla u_0\|_{L_x^2} + \| |u|^{\frac{4}{d}} \nabla u \|_{L_T^1 L_x^2}$$

*Proof.* Multiply Equation 1.1 by  $\overline{\Delta u}$ , integrate and take the imaginary part, to obtain :

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + a \int |(-\Delta)^{\frac{s+1}{2}} u|^2 \leq \left| \int |u|^{\frac{4}{d}} u \Delta u \right| = \left| \int \nabla(|u|^{\frac{4}{d}} u) \nabla u \right|$$

By integrating in time, we get

$$\frac{1}{2} \|\nabla u\|_{L_T^\infty L^2}^2 + a \|(-\Delta)^{\frac{s+1}{2}} u\|_{L_T^2 L_x^2}^2 \leq \frac{1}{2} \|\nabla u_0\|_{L^2}^2 + \|\nabla(|u|^{\frac{4}{d}} u)\|_{L_T^1 L_x^2} \|\nabla u\|_{L_T^\infty L^2}$$

Dividing by  $\sqrt{\frac{1}{2} \|\nabla u\|_{L_T^\infty L^2}^2 + a \|(-\Delta)^{\frac{s+1}{2}} u\|_{L_T^2 L_x^2}^2}$  we obtain:

$$\|\nabla u\|_{L_T^\infty L_x^2} + \|(\Delta)^{\frac{s+1}{2}} u\|_{L_T^2 L_x^2} \lesssim \|\nabla u_0\|_{L^2} + \| |u|^{\frac{4}{d}} \nabla u \|_{L_T^1 L_x^2}$$

□

**Lemma 4.2.** *There exists a real number  $0 < \alpha \ll 1$  such that the following holds: Let  $u \in C([0, T]; H^1(\mathbb{R}^d))$  be the solution of (1.1) emanating from  $u_0 \in H^1$ . For a fixed  $t \in ]0, T[$  we set  $\Delta t = \alpha \|u_0\|_{L^2}^{\frac{d-4}{d}} \|u(t)\|_{H^1}^{-2}$ . Then  $u \in C([t, t + \Delta t]; H^1(\mathbb{R}^d))$  and we have the following controls*

$$\|u\|_{S^0[t, t+\Delta t]} \leq 2 \|u_0\|_{L^2}$$

and

$$\|u\|_{S^1[t, t+\Delta t]} + \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2([t, t+\Delta t]) L_x^2} \leq 2 \|u(t)\|_{H^1(\mathbb{R}^d)}$$

*Proof.* We first assume that  $\Delta t > 0$  is such that  $t + \Delta t < T$ . Then, according to Lemma 2.1, it holds

$$\left\| \int_0^t S_{a,s}(t-\tau) |u(\tau)|^{\frac{4}{d}} u(\tau) d\tau \right\|_{S^1[t, t+\Delta t]} \lesssim \left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t, t+\Delta t]) L_x^2}.$$

Using the Hölder inequality we obtain:

$$\left( \int |u|^{\frac{8}{d}} |\nabla u|^2 \right)^{\frac{1}{2}} \leq \left( \int |u|^{\frac{2(4+d)}{d}} \right)^{\frac{2}{4+d}} \left( \int |\nabla u|^{\frac{2(4+d)}{d}} \right)^{\frac{d}{2(4+d)}}.$$

Integrating in time and applying again Hölder inequality we get:

$$\begin{aligned} \left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t, t+\Delta t]) L^2(\mathbb{R}^d)} &\leq \left( \int \left( \int |u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{2}{4+d} \frac{4+d}{4}} dt \right)^{\frac{4}{4+d}} \\ &\quad \times \left( \int \left( \int |\nabla u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{d}{2(4+d)} \frac{4+d}{d}} dt \right)^{\frac{d}{4+d}}. \end{aligned}$$

Thus:

$$\left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t, t+\Delta t]) L^2(\mathbb{R}^d)} \leq \|u\|_{L^{\frac{4+d}{d}}([t, t+\Delta t]) L^{\frac{8+2d}{d}}(\mathbb{R}^d)}^{\frac{4}{d}} \|\nabla u\|_{L^{\frac{4+d}{d}}([t, t+\Delta t]) L^{\frac{8+2d}{d}}(\mathbb{R}^d)}.$$

But  $(\frac{4+d}{d}, \frac{8+2d}{d})$  is admissible, thus we have:

$$\left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t, t+\Delta t]) L^2(\mathbb{R}^d)} \leq \|u\|_{L^{\frac{4+d}{d}}([t, t+\Delta t]) L^{\frac{8+2d}{d}}(\mathbb{R}^d)}^{\frac{4}{d}} \|u\|_{S^1[t, t+\Delta t]}.$$

By Sobolev inequalities we have:

$$\begin{aligned} \|u\|_{L^{\frac{4+d}{d}}([t, t+\Delta t]) L^{\frac{8+2d}{d}}(\mathbb{R}^d)} &\lesssim \|u\|_{L^{\frac{4+d}{d}}([t, t+\Delta t]) H^{\frac{2d}{d+4}}(\mathbb{R}^d)} \\ &\leq (\Delta t)^{\frac{d}{d+4}} \|u\|_{L^\infty([t, t+\Delta t]) H^{\frac{2d}{d+4}}(\mathbb{R}^d)}. \end{aligned}$$

Now by interpolation we obtain for  $d = 1, 2, 3, 4$ :

$$\|u\|_{L^{\frac{4+d}{d}}([t, t+\Delta t]) L^{\frac{8+2d}{d}}(\mathbb{R}^d)} \leq (\Delta t)^{\frac{d}{d+4}} \|u\|_{L^\infty([t, t+\Delta t]) L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{L^\infty([t, t+\Delta t]) H^1(\mathbb{R}^d)}^{\frac{2d}{d+4}},$$

which, according to (1.6), leads to

$$\|u^{\frac{4}{d}} \nabla u\|_{L_t^1 L_x^2} \leq (\Delta t)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[t, t+\Delta t]}^{\frac{2d}{d+4}} \|u\|_{S^1[t, t+\Delta t]}. \quad (4.20)$$

Since by Lemma 4.1 it holds

$$\|u\|_{L^\infty([t, t+\Delta t]) H^1} + \|(\Delta)^{\frac{s+1}{2}} u\|_{L^2([t, t+\Delta t]) L_x^2} \lesssim \|u(t)\|_{H^1} + \||u|^{\frac{4}{d}} \nabla u\|_{L^1([t, t+\Delta t]) L_x^2},$$

we finally get

$$\|u\|_{S^1[t, t+\Delta t]} + \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2([t, t+\Delta t]) L^2} \lesssim \|u(t)\|_{H^1} + (\Delta t)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[t, t+\Delta t]}^{\frac{2d}{d+4} + 1}.$$

In view of (2.1) and a continuity argument, it follows that  $u \in C([t, t + \Delta t]; H^1(\mathbb{R}^d))$  for some  $\Delta t \sim \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{d-4}{d}} \|u(t)\|_{H^1(\mathbb{R}^d)}^{-2}$  and

$$\|u\|_{S^1[t, t+\Delta t]} + \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2([t, t+\Delta t]) L^2} \leq 2\|u_0\|_{H^1}.$$

In the same way

$$\|u\|_{S^0[t, t+\Delta t]} \lesssim \|u(t)\|_{L^2(\mathbb{R}^d)} + (\Delta t)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[t, t+\Delta t]}^{2\frac{d}{d+4}} \|u\|_{S^0[t, t+\Delta t]}$$

which ensures that

$$\|u\|_{S^0[t, t+\Delta t]} \leq 2 \|u_0\|_{L^2(\mathbb{R}^d)} .$$

□

*Proof of Proposition 4.1* : According to (4.8), each interval  $[t_k, t_{k+1}]$ , can be divided into  $k$  intervals,  $[\tau_k^j, \tau_k^{j+1}]$  of length less than  $\lambda(t_k)$ . From (1.7), we have

$$|E(u(\tau_k^{j+1})) - E(u(\tau_k^j))| \lesssim \left| \int_{\tau_k^j}^{\tau_k^{j+1}} \int_{\mathbb{R}^d} ((-\Delta)^s u) |u|^{\frac{4}{d}} \bar{u} dx \right| .$$

For notation convenience we set  $\Theta = ]\tau_k^j, \tau_k^{j+1}[ \times \mathbb{R}^d$ . By Plancherel formula and Hölder inequality, it holds

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \int_{\mathbb{R}^d} ((-\Delta)^s u) |u|^{\frac{4}{d}} \bar{u} dx dt \leq \|(-\Delta)^{\frac{s}{2}}(u|u|^{\frac{4}{d}})\|_{L^{\frac{4+2d}{4+d}}(\Theta)} \|(-\Delta)^{\frac{s}{2}}u\|_{L^{\frac{4+2d}{d}}(\Theta)}$$

and, by interpolation, we have

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^{\frac{4+2d}{d}}(\Theta)}^2 \leq \|\nabla u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{2s} \|u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{2-2s} .$$

Noticing that the fractional Leibniz rule (see [20]) leads to

$$\|(-\Delta)^{\frac{s}{2}}(|u|^{\frac{4}{d}}u)\|_{L^{\frac{4+2d}{4+d}}(\Theta)} \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{L^{\frac{4+2d}{d}}(\Theta)} \|u|^{\frac{4}{d}}\|_{L^{\frac{4+2d}{4}}(\Theta)} \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{L^{\frac{4+2d}{d}}(\Theta)} \|u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{\frac{4}{d}} ,$$

we finally obtain

$$|E(u(\tau_k^{j+1})) - E(u(\tau_k^j))| \lesssim \|u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{\frac{4}{d}} \|\nabla u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{2s} \|u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{2-2s} .$$

Since  $(\frac{4}{d} + 2, \frac{4}{d} + 2)$  is an admissible pair, Lemma 4.2 yields

$$|E(u(\tau_k^{j+1})) - E(u(\tau_k^j))| \lesssim \lambda(t_k)^{-2s}$$

and summing over  $j$  we get

$$|E(u(t_{k+1})) - E(u(t_k))| \lesssim k \lambda(t_k)^{-2s} .$$

Finally, taking  $T^+ = T$  and summing from  $k_0$  to  $k^+$ , we obtain:

$$|E(u(T^+)) - E(u_0)| \lesssim k^+ \lambda^{-2s}(T^+) \lesssim \log(\lambda(T)) \lambda^{-2s}(T) .$$

Note that the growth of the energy is small with respect to  $\frac{1}{\lambda^2}$ , because  $s < 1$ .

Let us now proceed with the momentum. According to (1.8) we have :

$$|P(u(\tau_k^{j+1})) - P(u(\tau_k^j))| \lesssim \int_{\tau_k^j}^{\tau_k^{j+1}} \left| \int_{\mathbb{R}^d} \bar{\nabla} u (-\Delta)^s u \right| .$$

But

$$\left| \int_{\mathbb{R}^d} \bar{\nabla} u (-\Delta)^s u \right| = \left| \int_{\mathbb{R}^d} ((-\Delta)^{\frac{s}{2} + \frac{1}{4}} u) (-\Delta)^{\frac{s}{2} - \frac{1}{4}} \bar{\nabla} u \right| \leq \|(-\Delta)^{\frac{s}{2} + \frac{1}{4}} u\|_{L^2(\mathbb{R}^d)} \|(-\Delta)^{\frac{s}{2} - \frac{1}{4}} \bar{\nabla} u\|_{L^2(\mathbb{R}^d)}$$

with

$$\|(-\Delta)^{\frac{s}{2} - \frac{1}{4}} \bar{\nabla} u\|_{L^2(\mathbb{R}^d)} = \|(-\Delta)^{\frac{s}{2} + \frac{1}{4}} u\|_{L^2(\mathbb{R}^d)}$$

and, by interpolation,

$$\|(-\Delta)^{\frac{s}{2}+\frac{1}{4}}u\|_{L^2(\mathbb{R}^d)}^2 \leq \|(-\Delta)^{\frac{s}{2}+\frac{1}{2}}u\|_{L^2(\mathbb{R}^d)}^{2\theta} \|u\|_{L^2(\mathbb{R}^d)}^{2-2\theta},$$

where  $0 < \theta = \frac{2s+1}{2s+2} < 1$ . Therefore we get

$$|P(u(\tau_k^{j+1})) - P(u(\tau_k^j))| \lesssim (\tau_{k+1} - \tau_k)^{1-\theta} \|u_0\|_{L^2(\mathbb{R}^d)}^{2-2\theta} \|(-\Delta)^{\frac{s}{2}+\frac{1}{2}}u\|_{L^2(\Theta)}^{2\theta}$$

and Lemma 4.2 ensures that

$$|P(u(\tau_k^{j+1})) - P(u(\tau_k^j))| \lesssim \lambda^{2-2\theta} \lambda^{-2\theta} = \lambda^{2-4\theta} = \lambda^{\frac{-2s}{s+1}}.$$

Summing over  $j$  we obtain that:

$$|P(u(\tau_k)) - P(u(\tau_k))| \lesssim k \lambda^{\frac{-2s}{s+1}}$$

and summing from  $k_0$  to  $k^+$ , we finally get

$$|P(u(T^+)) - P(u_0)| \lesssim \log(\lambda(T)) \lambda(T)^{\frac{-2s}{s+1}}.$$

Note that the growth of the momentum is small with respect  $\frac{1}{\lambda}$  since  $1 - \frac{2s}{s+1} > 0$ .

#### REFERENCES

- [1] P. Antonelli and C. Sparber. *Global well-posedness for cubic NLS with nonlinear damping*. Comm. Partial Differential Equations, 35 (2010) 48324845.
- [2] P. Antonelli, R. Carles and C. Sparber. *On nonlinear Schrödinger-type equations with nonlinear damping*. Int. Math. Res. Not. (2015), no. 3, 740–762.
- [3] C. Bardos, P. Penel, P. Frisch and P.L. Sulem. *Modified dissipativity for a nonlinear evolution equation arising in turbulence*. Arch. Rat. Mech. Anal, 71 (1979) 4237–256.
- [4] D.T. Blackstock. *Generalized Burgers equation for plane waves*. J. Acoust. Soc. Am. 77 (1985), no. 3, 2050–2053.
- [5] H. Berestycki and P.-L. Lions. *Nonlinear scalar field equations. II. Existence of infinitely many solutions*. Arch. Rational Mech. Anal., 82(1983):347–375.
- [6] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.
- [7] W. Chen and S. Holm *Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law dependency*. J. Acoust. Soc. Am. 115 (2004), no. 4, 1424–1430.
- [8] T. Cazenave and F. Weissler. *The Cauchy problem for the critical nonlinear Schrödinger equation*. Nonlinear Anal. 14 (1990), 807–836.
- [9] J. Colliander and P. Raphael. *Rough blowup solutions to the  $L^2$  critical NLS*. Math. Ann., 345(2009):307–366.
- [10] M. Darwich. *Blowup for the Damped  $L^2$  critical nonlinear Schrödinger equations*. Advances in Differential Equations. volume 17, Numbers 3-4 (2012),337-367.
- [11] B. Dodson. *Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state*. Advances in Mathematics 285 (2015), 1589-1618.
- [12] G. Fibich. *Self-focusing in the damped nonlinear Schrödinger equation*. SIAM J. Appl. Math, 61 (2001), no. 5, 16801705.
- [13] G. Fibich and F. Merle. *Self-focusing on bounded domains*. Phys. D, 155(2001), 132–158.
- [14] G. Fibich and M. Klein. *Nonlinear-damping continuation of the nonlinear Schrödinger equation—a numerical study*. Physica D, 241 (2012), 519-527.
- [15] A. Friedman *Partial Differential Equations*.
- [16] T. Hmidi and S. Keraani. *Blowup theory for the critical nonlinear Schrödinger equations revisited*. Int. Math. Res. Not., 46 (2005), 2815–2828.

- [17] L. Gaul. *The influence of damping on waves and vibrations*. Mech. Syst. Signal Process. 13 (1999), no. 1, 1–30.
- [18] A. Hanyga. *Multi-dimensional solutions of space-fractional diffusion equations*. Proc. R. Soc. London, Ser. A 457 (2001), 2993–3005.
- [19] T. Kato. *On nonlinear Schrödinger equations* Ann. Inst. H. Poincaré Phys. Théor., 46(1987), 113–129.
- [20] C. E. Kenig, G. Ponce, and L. Vega. *Well-posedness and scattering results for the generalized Korteweg-de-Vries equation via the contraction principle*. Comm. Pure App. Math, 46 (1993), no. 4, 527–620.
- [21] M.K Kwong. *Uniqueness of positive solutions of  $\Delta u - u + u^p = 0$  in  $\mathbf{R}^n$* . Arch. Rational Mech. Anal., 105(1989):243–266.
- [22] P.-L. Lions. *The concentration-compactness principle in the calculus of variations. The locally compact case. II*. Ann. Inst. H. Poincaré Anal. Non Linéaire, 1(1984):223–283.
- [23] F. Merle. *Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power*, Duke Math. J. 69:2, (1993), 427–454.
- [24] F. Merle and P. Raphael. *Blow up dynamic and upper bound on the blow up rate for critical nonlinear Schrödinger equation*. In Journées “Équations aux Dérivées Partielles” (Forges-les-Eaux, 2002), pages Exp. No. XII, 5. Univ. Nantes, Nantes, 2002.
- [25] F. Merle and P. Raphael. *Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation*. Geom. Funct. Anal., 13(2003):591–642.
- [26] F. Merle and P. Raphael. *On universality of blow-up profile for  $L^2$  critical nonlinear Schrödinger equation*. Invent. Math., 156(2004):565–672.
- [27] F. Merle and P. Raphael. *Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation*. Comm. Math. Phys., 253(2005):675–704.
- [28] F. Merle and P. Raphael. *On a sharp lower bound on the blow-up rate for the  $L^2$  critical nonlinear Schrödinger equation*. J. Amer. Math. Soc., 19(2006):37–90 (electronic).
- [29] C. Miao, B. Yuan and B. Zhang. *Well-posedness of the Cauchy problem for the fractional power dissipative equations*. Nonlinear Analysis, 68 (2008) 461–484
- [30] E. Ott and R.N. Sudan. *Damping of Solitary Waves*. Phys. Fluids, 13 (1970) , 1432–1434.
- [31] M. Ohta and G. Todorova. *Remarks on global existence and blowup for damped nonlinear Schrödinger equations*. Discrete Contin. Dyn. Syst., 23(2009):1313–1325.
- [32] T. Passot, C. Sulem and P.L. Sulem. *Linear versus nonlinear dissipation for critical NLS equation*. Physica D, 203 (2005) 167184
- [33] F. Planchon and P. Raphaël. *Existence and stability of the log-log blow-up dynamics for the  $L^2$ -critical nonlinear Schrödinger equation in a domain*. Ann. Henri Poincaré, 8(2007):1177–1219.
- [34] P. Raphael. *Stability of the log-log bound for blow up solutions to the critical nonlinear Schrödinger equation*. Math. Ann., 331(2005):577–609.
- [35] D. Stroock. *Diffusion processes associated with Lévy generators*. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32 (1975), no. 3, 209–244.
- [36] M. Tsutsumi. *Nonexistence of global solutions to the Cauchy problem for the damped nonlinear Schrödinger equations*. SIAM J. Math. Anal., 15(1984):357–366.
- [37] M.I. Weinstein. *Nonlinear Schrödinger equations and sharp interpolation estimates*. Comm. Math. Phys., 87(1982/83):567–576.

MOHAMAD DARWICH: FACULTÉ DES SCIENCES DE L’UNIVERSITÉ LIBANAISE, DÉPARTEMENT DE MATHÉMATIQUES PURES, HADATH, LIBAN

*E-mail address:* Mohamad.Darwich@lmpt.univ-tours.fr

LUC MOLINET: LABORATOIRE DE MATHÉMATIQUES ET PHYSIQUE THÉORIQUE, UNIVERSITÉ FRANCOIS RABELAIS TOURS, CNRS UMR 7350- FÉDÉRATION DENIS POISSON, PARC GRANDMONT, 37200 TOURS, FRANCE

*E-mail address:* Luc.Molinet@lmpt.univ-tours.fr