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SOME REMARKS ON THE NONLINEAR SCHRÖDINGER EQUATION WITH FRACTIONAL DISSIPATION.

MOHAMAD DARWICH AND LUC MOLINET.

ABSTRACT. We consider the Cauchy problem for the L^2 -critical focussing nonlinear Schrödinger equation with a fractional dissipation. According to the order of the fractional dissipation, we prove the global existence or the existence of finite time blowup dynamics with the log-log blow-up speed for $\|\nabla u(t)\|_{L^2}$.

1. INTRODUCTION

In this paper, we study the blowup and the global existence of solutions for the L^2 -critical nonlinear Schrödinger equation (NLS) with a fractional dissipation term:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u + ia(-\Delta)^s u = 0, (t, x) \in [0, \infty[\times \mathbb{R}^d, d \in \mathbb{N}^*. \\ u(0) = u_0 \in H^r(\mathbb{R}^d) \end{cases} \quad (1.1)$$

where $a > 0$ is the coefficient of friction, $s > 0$ and $r \in \mathbb{R}$.

The NLS equation ($a = 0$) arises in various areas of nonlinear optics, plasma physics and fluid mechanics to describe propagation phenomena in dispersive media. To take into account weak dissipation effects, one usually add a linear damping term as in the linear damped NLS equation (see for instance Fibich [12]):

$$iu_t + \Delta u + iau + |u|^p u = 0, a > 0.$$

However, in a wide range of situations a frequency-dependent attenuation has been observed (cf [7]). This motivates to rather complete the NLS equation with a laplacian term as in the following complex Ginzburg-Landau equation studied in Passota-Sulem-Sulem [32]:

$$iu_t + \Delta u - ia\Delta u + |u|^p u = 0, a > 0,$$

Now, in many cases of practical importance the damping cannot be described by a local term even in the long-wavelength limit. In media with dispersion the weak dissipation is, in general, non local (see for instance Ott-Sudan [30]). It is thus quite natural to complete the NLS equation by a non local dissipative term in order to take into account some dissipation phenomena.

In this this paper we complete the L^2 -critical NLS equation (1.2) with a fractional laplacian of order $2s$, $s > 0$, and study the influence of this term on the blow-up phenomena for this equation. The fractional laplacian is commonly used to model fractal (anomalous) diffusion related to the Lévy

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flights (see e.g. Stroock [35], Bardos and all [3], Hanyga [18]). It also appears in the physical literature to model attenuation phenomena of acoustic waves in irregular porous random media (cf. Blackstock [4], Gaul [17], Chen-Holm [7]).

Finally, note that the case of a nonlinear damping of the type $ia|u|^p u$, has been studied by Antonelli-Sparber and Antonelli-Carles-Sparber (cf. [1] and [2]). In this case the origin of the nonlinear damping term is multiphoton absorption.

Recall that the Cauchy problem for the L^2 -critical focussing nonlinear Schrödinger equation ($a = 0$):

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}} u = 0 \\ u(0) = u_0 \in H^r(\mathbb{R}^d) \end{cases} \quad (1.2)$$

has been studied by a lot of authors (see for instance [19], [8], [6]) and it is known that the problem is locally well-posed in $H^r(\mathbb{R}^d)$ for $r \geq 0$: For any $u_0 \in H^r(\mathbb{R}^d)$, with $r \geq 0$, there exist $T > 0$ and a unique solution u of (1.2) with $u(0) = u_0$ such that $u \in C([0, T]; H^r(\mathbb{R}^d))$. Moreover, if the maximal existence time T^* of the solution u in $H^r(\mathbb{R}^d)$ is finite then $\|u\|_{L^{\frac{4}{d}+2}([0, T^*] \times \mathbb{R}^d)} = \infty$.

Let us mention that in the case $a > 0$ the same results on the local Cauchy problem for (1.1) can be established in exactly the same way as in the case $a = 0$, since the same Strichartz estimates hold (see for instance [29]).

For $u_0 \in H^1(\mathbb{R}^d)$, a sharp criterion for global existence for (1.2) has been exhibited by Weinstein [37]: Let Q be the unique radial positive solution (see [5], [21]) to

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \quad (1.3)$$

If $\|u_0\|_{L^2} < \|Q\|_{L^2}$ then the solution of (1.2) is global in $H^1(\mathbb{R}^d)$. This follows from the conservation of the energy and the L^2 norm and the sharp Gagliardo-Nirenberg inequality which ensures that

$$\forall u \in H^1, E(u) \geq \frac{1}{2} \left(\int |\nabla u|^2 \right) \left(1 - \left(\frac{\int |u|^2}{\int |Q|^2} \right)^{\frac{2}{d}} \right). \quad (1.4)$$

Actually, it was recently proven that any solution of (1.2) emanating from an initial datum $u_0 \in L^2(\mathbb{R}^d)$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ is global and scatters, i.e. u behaves like the linear evolution of an L^2 function for large time, $u(t) \sim e^{it\Delta} u_+$ (cf. [11]).

On the other hand, NLS has unique minimal mass blow-up solution in H^1 up to the symmetries of the equation (see [23]) that blow up at some time $T > 0$ with a H^1 norm that grows as $\frac{1}{T-t}$.

In the series of papers [24], [25], [26], [27], Merle and Raphael studied the blowup for (1.2) with $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \delta$, δ small and proved the existence of the blowup regime corresponding to the log-log law:

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \left(\frac{\log |\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \quad (1.5)$$

Recall that the evolution of (1.2) admits the following conservation laws in the energy space H^1 :

$$L^2\text{-norm} : m(u) = \|u\|_{L^2} = \left(\int |u(x)|^2 dx \right)^{1/2}.$$

$$\text{Energy} : E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{d}{4+2d} \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

$$\text{Kinetic momentum} : P(u) = \text{Im} \left(\int \nabla u(x) \bar{u}(x) dx \right).$$

Now, for (1.1) with $a > 0$, there does not exist conserved quantities anymore. However, it is easy to prove that if u is a smooth solution of (1.1) on $[0, T[$, then for all $t \in [0, T[$ it holds

$$\|u(t)\|_{L^2}^2 + a \int_0^t \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 = \|u_0\|_{L^2}^2. \quad (1.6)$$

$$\frac{d}{dt} E(u(t)) = -a \int |(-\Delta)^{\frac{s+1}{2}} u(t)|^2 + a \text{Im} \int ((-\Delta)^s u(t)) |u(t)|^{\frac{4}{d}} \bar{u}(t). \quad (1.7)$$

$$\frac{1}{2} \frac{d}{dt} P(u(t)) = a \text{Im} \int ((-\Delta)^s u(t)) \nabla \bar{u}(t). \quad (1.8)$$

In [10], the first author studied the case $s = 0$. He proved the global existence in H^1 for $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, and showed that the log-log regime is stable by such perturbations (i.e. there exist solutions that blowup in finite time with the log-log law).

In [32], Passot, Sulem and Sulem proved that the solutions are global in $H^1(\mathbb{R}^2)$ for $s = 1$. However, their method does not seem to apply for any other values of d .

Our aim in this paper is to establish some results, for $s > 0$, on the global existence or the existence of finite time blowup dynamics with the log-log blow-up speed for $\|\nabla u\|_{L^2}$.

Let us now state our results:

Theorem 1.1. *Let $d = 1, 2, 3, 4$ and $0 < s < 1$ then there exists $\delta_0 > 0$ such that $\forall a > 0$ and $\forall \delta \in]0, \delta_0[$, there exists $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$, such that the solution of (1.1) blows up in finite time in the log-log regime.*

Theorem 1.2. *Let $d \in \mathbb{N}^*$, $s \geq 1$ and $r \geq 0$. Then the Cauchy problem (1.1) is globally well-posed in $H^r(\mathbb{R}^d)$.*

Theorem 1.3. *Let $d \in \mathbb{N}^*$, $0 < s < 1$ and $a > 0$.*

- (1) *There exists a real number $0 < \gamma = \gamma(d) \leq \|Q\|_{L^2}$ such that for any initial datum $u_0 \in H^1(\mathbb{R}^d)$ with $\|u_0\|_{L^2} < \gamma$, the emanating solution u is global in H^1 with an energy that is non increasing.*
- (2) *There does not exist any initial datum u_0 , with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, such that the solution u of (1.1) blows up at finite time T^* and satisfies*

$$\frac{1}{(T^* - t)^\alpha} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T^* - t)^\beta}, \quad \forall 0 < T - t \ll 1,$$

for some pair (α, β) satisfying $0 < \beta < \frac{1}{2s}$ and $\beta(1+s) - 1/2 < \alpha \leq \beta$.

Remark 1.1. *It is natural to expect that $\gamma = \|Q\|_{L^2}$. Indeed, for $u_0 \in H^1(\mathbb{R}^d)$ satisfying $\|u_0\|_{L^2} \leq \|Q\|_{L^2(\mathbb{R}^d)}$, (1.6) ensures that $\|u(t)\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^d)}$ as soon as $t > 0$ and the solutions of the critical NLS equation with such initial data are global.*

However, in contrary to the case $s = 0$, we do not succeed to prove this fact here since the constant C_d appearing in the estimate on the energy (see subsection 2.3) seems not to be directly related to Q . Assertion (2) of Theorem 1.3 is a partial result in this direction since it ensures that we do not have any blowup in the log-log regime, for any $0 < s < 1$, and in the regime $\frac{1}{l}$, for any $0 < s < \frac{1}{2}$, for initial data with critical or subcritical mass.

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2. LOCAL AND GLOBAL EXISTENCE RESULTS

In this section, we prove Theorem 1.2 and part (1) of Theorem 1.3. Theorem 1.2 will follow from an a priori estimate on the critical Strichartz norm whereas part (1) of Theorem 1.3 follows from a monotonicity of the energy.

2.1. Local existence result. Recall that the main tools to prove the local existence results for (1.2) are the Strichartz estimates for the associated linear propagator $e^{i\Delta t}$. These Strichartz estimates read

$$\|e^{i\Delta t}\phi\|_{L_t^q L^r(\mathbb{R}^d)} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)}$$

for any pair (q, r) satisfying $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $2 < q \leq \infty$. Such ordered pair is called an admissible pair .

For $a \geq 0$ and $s \geq 0$ we denote by $S_{a,s}$ the linear semi-group associated with (1.1), i.e. $S_{a,s}(t) = e^{i\Delta t - a(-\Delta)^s t}$. It is worth noticing that $S_{a,s}$ is irreversible. The following lemma ensures that the linear semi-group $S_{a,s}$ enjoys the same Strichartz estimates as $e^{i\Delta t}$.

Lemma 2.1. *Let $\phi \in L^2(\mathbb{R}^d)$ and $s > 0$. Then for every admissible pair (q, r) it holds*

$$\|S_{a,s}(\cdot)\phi\|_{L_{t>0}^q L^r(\mathbb{R}^d)} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)}.$$

Proof. Setting, for any $t \geq 0$, $G_{a,s}(t, x) = \int e^{-ix\xi} e^{-at|\xi|^{2s}} d\xi$, it holds

$$S_{a,s}(t)\varphi = G_{a,s}(t, \cdot) * e^{it\Delta}\varphi, \quad \forall t \geq 0.$$

Noticing that for $s > 0$, $\|G_{a,s}(t, \cdot)\|_{L^1} = \|G_{1,s}(1, \cdot)\|_{L^1}$ and that, according to Lemma 2.1 in [29], $G_{1,s}(1, \cdot) \in L^1(\mathbb{R}^d)$ for $s > 0$, we get

$$\|S_{a,s}(t)\phi\|_{L_{t>0}^p L^q(\mathbb{R}^d)} = \|G_{a,s}(t, \cdot) * e^{it\Delta}(\phi)\|_{L_{t>0}^p L^q(\mathbb{R}^d)} \lesssim \|e^{it\Delta}\phi\|_{L_t^p L^q(\mathbb{R}^d)} \lesssim \|\phi\|_{L^2}.$$

□

With Lemma 2.1 in hand, it is not too hard to check that the local existence results for equation (1.2) (see for instance [8] and [6] for the local existence and [20] for the continuity with respect to initial data) also holds

for (1.1) with $a \geq 0$ and $s > 0$. More precisely, we have the following statement:

Proposition 2.1. *Let $a \geq 0$, $s > 0$, $r \geq 0$ and $u_0 \in H^r(\mathbb{R}^d)$ with $d \in \mathbb{N}^*$. There exists $T > 0$ and a unique solution $u \in C([0, T]; H^r) \cap L_T^{\frac{4}{d}+2} L^{\frac{4}{d}+2}$ to (1.1) emanating from u_0 . In addition, there exists a neighborhood \mathcal{V}_{u_0} ⁽¹⁾ of u_0 in $H^r(\mathbb{R}^d)$ such that the associated solution map is continuous from \mathcal{V}_{u_0} into $C([0, T]; H^r) \cap L_T^{\frac{4}{d}+2}([0, T] \times \mathbb{R}^d)$. Finally, let T^* be the maximal time of existence of the solution u in $H^r(\mathbb{R}^d)$, then*

$$T^* < \infty \implies \|u\|_{L_{T^*}^{\frac{4}{d}+2} L^{\frac{4}{d}+2}} = +\infty. \quad (2.1)$$

2.2. Proof of Theorem 1.2. Let $u \in C([0, T]; L^2(\mathbb{R}^d))$ be the solution emanating from some initial datum $u_0 \in L^2(\mathbb{R}^d)$. We have the following a priori estimates:

Lemma 2.2. *Let $u \in C([0, T]; L^2(\mathbb{R}^d))$ be the solution of (1.1) emanating from $u_0 \in L^2(\mathbb{R}^d)$. Then*

$$\|u\|_{L_T^\infty L^2} \leq \|u_0\|_{L^2} \text{ and } \|(-\Delta)^{\frac{s}{2}} u\|_{L_T^2 L^2} \leq \frac{1}{\sqrt{2a}} \|u_0\|_{L^2}. \quad (2.2)$$

Proof. Assume first that $u_0 \in H^\infty(\mathbb{R}^d)$. Then (1.6) ensures that the mass is decreasing as soon as u is not the null solution and (1.6) leads to

$$\int_0^T \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 dt = -\frac{1}{2a} (\|u(T)\|_{L^2}^2 - \|u_0\|_{L^2}^2) \leq \frac{1}{2a} \|u_0\|_{L^2}^2.$$

This proves (2.2) for smooth solutions. The result for $u_0 \in L^2(\mathbb{R}^d)$ follows by approximating u_0 in L^2 by a smooth sequence $(u_{0,n}) \subset H^\infty(\mathbb{R}^d)$. \square

Note that the first estimate in (2.2) implies that $\|u\|_{L_T^2 L^2} \leq T^{1/2} \|u_0\|_{L^2}$ and thus by interpolation:

$$\|\nabla u\|_{L_T^2 L^2} \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{L_T^2 L^2}^{\frac{1}{s}} \|u\|_{L_T^2 L^2}^{1-\frac{1}{s}} \lesssim T^{\frac{1}{2}(1-\frac{1}{s})} \quad (2.3)$$

Interpolating now between (2.3) and the first estimate of (2.2) we get

$$\|u\|_{L_T^{\frac{4}{d}+2} H^{\frac{2d}{4+2d}}} \lesssim T^{\frac{1}{2}(1-\frac{1}{s})}$$

and the embedding $H^{\frac{2d}{4+2d}}(\mathbb{R}^d) \hookrightarrow L^{\frac{4}{d}+2}(\mathbb{R}^d)$ ensures that

$$\|u\|_{L_T^{\frac{4}{d}+2} L^{\frac{4}{d}+2}} \lesssim \|u\|_{L_T^{\frac{4}{d}+2} H^{\frac{2d}{4+2d}}} \lesssim T^{\frac{1}{2}(1-\frac{1}{s})}.$$

Denoting by T^* the maximal time of existence of u in $L^2(\mathbb{R}^d)$ and letting T tends to T^* , this contradicts (2.1) whenever T^* is finite. This proves that the solutions are global in $H^r(\mathbb{R}^d)$.

Moreover, for $s = 1$ we have $\|u\|_{L_T^{\frac{4}{d}+2} L^{\frac{4}{d}+2}} \lesssim 1$ for any $T > 0$ which ensures that

$$\|u\|_{L^{\frac{4}{d}+2}(\mathbb{R}_+^* \times \mathbb{R}^d)} \lesssim 1.$$

⁽¹⁾It is worth noticing that for $r = 0$, the neighborhood \mathcal{V}_{u_0} does not depend only on $\|u_0\|_{H^r}$ but on the Fourier profile of u_0 (see for instance [20])

2.3. Proof of Assertion 1 of Theorem 1.3. Note that the global existence for any $u_0 \in L^2(\mathbb{R}^d)$ with $\|u_0\|_{L^2}$ small enough can be proven, as for the critical NLS equation, directly by a fixed point argument thanks to Lemma 2.1. This ensures the global existence in $H^r(\mathbb{R}^d)$, $r \geq 0$, under the same smallness condition on $\|u_0\|_{L^2}$. We will not invoke this fact here and we will directly prove Assertion 1 of Theorem 1.3 by combining (1.4) and a monotony result on $t \mapsto E(u(t))$. To do this, we will work with smooth solutions and then get the result for H^1 -solutions by continuity with respect initial data.

So, let $u \in C([0, T]; H^\infty(\mathbb{R}^d))$ be a solution to (1.1) emanating from $u_0 \in H^\infty(\mathbb{R}^d)$. Then it holds

$$\frac{d}{dt}E(u(t)) = -a \int |(-\Delta)^{\frac{s+1}{2}} u(t)|^2 + a \operatorname{Im} \int ((-\Delta)^s u(t)) |u(t)|^{\frac{4}{d}} \overline{u(t)}$$

and Hölder inequalities in physical space and in Fourier space lead to

$$\left| \int ((-\Delta)^s u) |u|^{\frac{4}{d}} \overline{u} \right| \leq \|(-\Delta)^s u\|_{L^2} \|u\|_{L^{\frac{8}{\frac{d}{d}+2}}}^{\frac{4}{d}+1}$$

with

$$\|(-\Delta)^s u\|_{L^2} \leq \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2}^{\frac{2s}{s+1}} \|u\|_{L^2}^{\frac{1-s}{1+s}}.$$

Let us recall the following Gagliardo-Nirenberg inequality ⁽²⁾

$$\|u\|_{L^{\frac{8}{\frac{d}{d}+2}}}^{\frac{8}{d}+2} \leq C_d^{\frac{8}{d}+2} \|\nabla u\|_{L^2}^4 \|u\|_{L^2}^{\frac{8}{d}-2}.$$

This estimate together with Cauchy-Schwarz inequality (in Fourier space)

$$\|\nabla u\|_{L^2}^2 \leq \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2}^{\frac{2}{s+1}} \|u\|_{L^2}^{\frac{2s}{s+1}}$$

lead to

$$\|u\|_{L^{\frac{8}{\frac{d}{d}+2}}}^{\frac{4}{d}+1} \leq C_d^{\frac{4}{d}+1} \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2}^{\frac{2}{s+1}} \|u\|_{L^2}^{\frac{2s}{s+1}} \|u\|_{L^2}^{\frac{4-d}{d}}.$$

Combining the above estimates we eventually obtain

$$\frac{d}{dt}E(u(t)) \leq a \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2}^2 (C_d^{\frac{4}{d}+1} \|u\|_{L^2}^{4/d} - 1)$$

which together with (2.2) implies that $E(u(t))$ is not increasing for $\|u_0\|_{L^2} \leq C_d^{1+\frac{d}{4}}$.

3. PROOF OF ASSERTION 2 OF THEOREM 1.3

Special solutions play a fundamental role for the description of the dynamics of (NLS). They are the solitary waves of the form $u(t, x) = \exp(it)Q(x)$, where Q the unique positive radial solution to

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \quad (3.1)$$

The pseudo-conformal transformation applied to the “stationary” solution $e^{it}Q(x)$ yields an explicit solution for (NLS)

$$S(t, x) = \frac{1}{|t|^{\frac{d}{2}}} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}$$

⁽²⁾It is proven in [37] that the constant C_d is related for $d = 1, 2, 3$ to the L^2 -norm of the ground state solution of $2\Delta\psi - (\frac{4}{d} - 1)\psi + \psi^{\frac{8}{d}+1} = 0$.

which blows up at $T^* = 0$.

Note that

$$\|S(t)\|_{L^2} = \|Q\|_{L^2} \text{ and } \|\nabla S(t)\|_{L^2} \sim \frac{1}{t} \quad (3.2)$$

It turns out that $S(t)$ is the unique minimal mass blow-up solution in H^1 up to the symmetries of the equation (see [23]).

A known lower bound (see [8] and [6]) on the blow-up rate for (NLS) is

$$\|\nabla u(t)\|_{L^2} \geq \frac{C(u_0)}{\sqrt{T-t}}. \quad (3.3)$$

Note that this blow-up rate is strictly lower than the one of $S(t)$ given by (3.2) and of the log-log law given by (1.5).

To prove assertion 2 of Theorem 1.3, we will need the following result (see [16]) :

Theorem 3.1. *Let $(v_n)_n$ be a bounded family of $H^1(\mathbb{R}^d)$, such that:*

$$\limsup_{n \rightarrow +\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^d)} \leq M \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \|v_n\|_{L^{\frac{4}{d}+2}} \geq m. \quad (3.4)$$

Then, there exists $(x_n)_n \subset \mathbb{R}^d$ such that:

$$v_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly,}$$

$$\text{with } \|V\|_{L^2(\mathbb{R}^d)} \geq \left(\frac{d}{d+4}\right)^{\frac{d}{4}} \frac{m^{\frac{d}{2}+1} + 1}{M^{\frac{d}{2}}} \|Q\|_{L^2(\mathbb{R}^d)}.$$

Suppose that there exist an initial data u_0 with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, such that the corresponding solution $u(t)$ blows up at time $T > 0$ with the following behavior:

$$\frac{1}{(T-t)^\alpha} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T-t)^\beta}, \quad \forall t \in [0, T[, \quad (3.5)$$

where $\beta > 0$ and $\alpha \geq \beta$ satisfies $\alpha > \beta(1+s) - 1/2$.

Recalling that

$$E(u(t)) = E(u_0) - a \int_0^t K(u(\tau)) d\tau, \quad t \in [0, T[, \quad (3.6)$$

with $K(u(t)) = \int |(-\Delta)^{\frac{s+1}{2}} u|^2 - \text{Im} \int ((-\Delta)^s u) |u|^{\frac{4}{d}} \bar{u}$, we obtain that

$$E(u(t)) \lesssim E(u_0) + \left| \int_0^t ((-\Delta)^s u) |u|^{\frac{4}{d}} \bar{u} dx \right| \lesssim E(u_0) + \int_0^t \|(-\Delta)^s u\|_{L^2} \|u\|_{L^{\frac{8}{d}+2}}^{\frac{4}{d}+1}$$

This last estimate together with

$$\|u\|_{L^{\frac{8}{d}+2}}^{\frac{4}{d}+1} \lesssim \|\nabla u\|_{L^2}^2 \|u_0\|_{L^2}^{\frac{4-d}{d}} \quad \text{and} \quad \|(-\Delta)^s(u)\|_{L^2} \leq \|\nabla u\|_{L^2}^{2s} \|u\|_{L^2}^{1-2s} \lesssim \|\nabla u\|_{L^2}^{2s}$$

yield

$$E(u(t)) \lesssim E(u_0) + \int_0^t \|\nabla u\|_{L^2}^{2+2s}(\tau) d\tau. \quad (3.7)$$

Note that assumption (3.5) ensures that

$$0 \leq \frac{\int_0^t \|\nabla u(\tau)\|_{L^2}^{2+2s} d\tau}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2} \lesssim (T-t)^{-2\beta(1+s)+1+2\alpha} \rightarrow 0 \text{ as } t \nearrow T, \quad (3.8)$$

Now, let

$$\rho(t) = \frac{\|\nabla Q\|_{L^2(\mathbb{R}^d)}}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}} \quad \text{and} \quad v(t, x) = \rho^{\frac{d}{2}} u(t, \rho x)$$

and let $(t_k)_k$ be a sequence of positive times such that $t_k \nearrow T$. We set $\rho_k = \rho(t_k)$ and $v_k = v(t_k, \cdot)$. The family $(v_k)_k$ satisfies

$$\|v_k\|_{L^2(\mathbb{R}^d)} = \|u(t_k, \cdot)\|_{L^2(\mathbb{R}^d)} < \|u_0\|_{L^2(\mathbb{R}^d)} \leq \|Q\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla v_k\|_{L^2(\mathbb{R}^d)} = \|\nabla Q\|_{L^2(\mathbb{R}^d)}.$$

The above estimate on $\|v_k\|_{L^2}$ and (3.7) lead to

$$0 < \frac{1}{2} \left(\int |\nabla v_k|^2 \right) \left(1 - \left(\frac{\int |v_k|^2}{\int |Q|^2} \right)^2 \right) \leq E(v_k) = \rho_k^2 E(u(t_k)) \lesssim \rho_k^2 E(u_0) + \rho_k^2 \int_0^{t_k} \|\nabla u(\tau)\|_{L^2}^{2+2s} d\tau$$

which, together with (3.8), ensures that $\lim_{k \rightarrow +\infty} E(v_k) = 0$. This forces

$$\|v_k\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \rightarrow \frac{d+2}{d} \|\nabla v_k\|_{L^2(\mathbb{R}^d)}^2 = \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 \quad (3.9)$$

and thus the family $(v_k)_k$ satisfies the hypotheses of Theorem 3.1 with

$$m^{\frac{4}{d}+2} = \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 \quad \text{and} \quad M = \|\nabla Q\|_{L^2(\mathbb{R}^d)}.$$

Hence, there exists a family $(x_k)_k \subset \mathbb{R}$ and a profile $V \in H^1(\mathbb{R})$ with

$\|V\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}$, such that,

$$\rho_k^{\frac{d}{2}} u(t_k, \rho_k \cdot + x_k) \rightharpoonup V \in H^1 \quad \text{weakly.} \quad (3.10)$$

Using (3.10), $\forall A \geq 0$

$$\liminf_{n \rightarrow +\infty} \int_{B(0,A)} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{B(0,A)} |V|^2 dx.$$

But, since $\lim_{n \rightarrow +\infty} \rho_n = 0$, $\rho_n A < 1$ for n large enough and thus

$$\liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{|x-y| \leq 1} |u(t_n, x)|^2 dx \geq \liminf_{n \rightarrow +\infty} \int_{|x-x_n| \leq \rho_n A} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx.$$

Since this it is true for all $A > 0$ we obtain that

$$\|u_0\|_{L^2}^2 > \liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{|x-y| \leq 1} |u(t_n, x)|^2 dx \geq \|Q\|_{L^2}^2$$

which contradicts the assumption $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ and the desired result is proven.

4. BLOW UP SOLUTION.

In this section, we prove the existence of the explosive solutions in the case $0 < s < 1$.

Theorem 4.1. *Let $0 < s < 1$ and $1 \leq d \leq 4$. There exist a set of initial data Ω in H^1 , such that for any $0 < a < a_0$ with $a_0 = a_0(s)$ small enough, the emanating solution $u(t)$ to (1.1) blows up in finite time in the log-log regime.*

The set of initial data Ω is the set described in [24], in order to initialize the log-log regime. It is open in H^1 . Using the continuity with regard to the initial data and the parameters, we easily obtain the following corollary:

Corollary 4.1. *Let $0 < s < 1$, $1 \leq d \leq 4$ and $u_0 \in H^1$ be an initial data such that the corresponding solution $u(t)$ of (1.2) blows up in the loglog regime. There exist $\beta_0 > 0$ and $a_0 > 0$ such that if $v_0 = u_0 + h_0$, $\|h_0\|_{H^1} \leq \beta_0$ and $a \leq a_0$, the solution $v(t)$ for (1.1) with the initial data v_0 blows up in finite time.*

Now to prove Theorem 4.1, we look for a solution of (1.1) such that for t close enough to blowup time, we shall have the following decomposition:

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} (Q_{b(t)} + \epsilon)(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}, \quad (4.1)$$

for some geometrical parameters $(b(t), \lambda(t), x(t), \gamma(t)) \in (0, \infty) \times (0, \infty) \times \mathbb{R}^d \times \mathbb{R}$, $\lambda(t) \sim \frac{1}{\|\nabla u(t)\|_{L^2}}$, $b \sim -\frac{\lambda_s}{\lambda}$ where $\frac{ds}{dt} = \frac{1}{\lambda^2(t)}$. The profiles Q_b are suitable deformations of Q related to some extra degeneracy of the problem, in fact these profile Q_b are a regularization of the exact self similar solutions to (1.2) which satisfy the nonlinear elliptic equation:

$$\Delta Q_b - Q_b + ib(\frac{d}{2}Q_b + y \cdot \nabla Q_b) + Q_b|Q_b|^{\frac{4}{d}} = 0. \quad (4.2)$$

Note that our proof is very close to the case of $s = 0$ (see Darwich [10]). Actually, as noticed in [33], we only need to prove that in the log-log regime the L^2 norm does not grow, and the growth of the energy(resp the momentum) is below $\frac{1}{\lambda(t)^2}$ (resp $\frac{1}{\lambda(t)}$). In this paper, we will prove that in the log-log regime, the growths of the energy and the momentum are bounded by:

$$E(u(t)) \lesssim \log(\lambda(t))\lambda(t)^{-2s}, \quad P(u(t)) \lesssim \log(\lambda(t))\lambda(t)^{\frac{-2s}{s+1}}.$$

Let us recall that a fonction $u : [0, T] \mapsto H^1$ follows the log-log regime if the following uniform controls on the decomposition (4.1) hold on $[0, T]$:

- Control of $b(t)$

$$b(t) > 0, \quad b(t) < 10b(0). \quad (4.3)$$

- Control of λ :

$$\lambda(t) \leq e^{-e^{\frac{\pi}{100b(t)}}} \quad (4.4)$$

and the monotonicity of λ :

$$\lambda(t_2) \leq \frac{3}{2}\lambda(t_1), \quad \forall 0 \leq t_1 \leq t_2 \leq T. \quad (4.5)$$

Let $k_0 \leq k_+$ be integers and $T^+ \in [0, T]$ such that

$$\frac{1}{2^{k_0}} \leq \lambda(0) \leq \frac{1}{2^{k_0-1}}, \frac{1}{2^{k_+}} \leq \lambda(T^+) \leq \frac{1}{2^{k_+-1}} \quad (4.6)$$

and for $k_0 \leq k \leq k_+$, let t_k be a time such that

$$\lambda(t_k) = \frac{1}{2^k}, \quad (4.7)$$

then we assume the control of the doubling time interval:

$$t_{k+1} - t_k \leq k\lambda^2(t_k). \quad (4.8)$$

- control of the excess of mass:

$$\int_{\mathbb{R}^d} |\nabla \epsilon(t)|^2 + \int_{\mathbb{R}^d} |\epsilon(t)|^2 e^{-|y|} \leq \Gamma_{b(t)}^{\frac{1}{4}}, \text{ where } \Gamma_b \sim e^{-\frac{\pi}{b}}. \quad (4.9)$$

The main point is to establish that (4.3)-(4.9) determine a trapping region for the flow. Actually, after the decomposition (4.1) of u , the log-log regime corresponds to the following asymptotic controls

$$b_s \sim C e^{-\frac{c}{b}}, \quad -\frac{\lambda_s}{\lambda} \sim b \quad (4.10)$$

and

$$\int_{\mathbb{R}^d} |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}}, \quad (4.11)$$

where we have introduced the rescaled time $\frac{ds}{dt} = \frac{1}{\lambda^2}$.

In fact, (4.11) is partly a consequence of the preliminary estimate:

$$\int_{\mathbb{R}^d} |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}} + \lambda^2(t) E(t). \quad (4.12)$$

One then observes that in the log-log regime, the integration of the laws (4.10) yields

$$\lambda \sim e^{-e^{\frac{c}{b}}} \ll e^{-\frac{c}{b}}, \quad b(t) \rightarrow 0, \quad t \rightarrow T. \quad (4.13)$$

Hence, the term involving the conserved Hamiltonian is asymptotically negligible with respect to the leading order term $e^{-\frac{c}{b}}$ which drives the decay (4.12) of b . This was a central observation made by Planchon and Raphael in [33]. In fact, any growth of the Hamiltonian algebraically below $\frac{1}{\lambda^2(t)}$ would be enough. In this paper, we will prove that in the log-log regime, the growth of the energy is estimated by:

$$E(u(t)) \lesssim (\log(\lambda(t))) \lambda^{-2s}(t). \quad (4.14)$$

It then follows from (4.12) that:

$$\int_{\mathbb{R}^d} |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}}. \quad (4.15)$$

An important feature of this estimate of H^1 flavor is that it relies on a flux computation in L^2 . This allows one to recover the asymptotic laws for the geometrical parameters (4.10) and to close the bootstrap estimates of the log-log regime.

Remark 4.1. *Actually, one also needs the bound on the momentum to control the geometrical parameters (see Lemma 7.2 in [10]).*

4.1. Control of the energy and the kinetic momentum. Let us recall that we say that an ordered pair (q, r) is admissible whenever $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $2 < q \leq \infty$. We define the Strichartz norm of functions $u : [0, T] \times \mathbb{R}^d \mapsto \mathbb{C}$ by:

$$\|u\|_{S^0([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (4.16)$$

and

$$\|u\|_{S^1([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|\nabla u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (4.17)$$

We will sometimes abbreviate $S^i([0, T] \times \mathbb{R}^d)$ with S_T^i or $S^i[0, T]$, $i = 1, 2$. Now we will derive an estimate on the energy, to check that it remains small with respect to λ^{-2} :

Proposition 4.1. *Assuming that (4.4)-(4.9) hold, then the energy and kinetic momentum are controlled on $[0, T^+]$ by:*

$$|E(u(t))| \lesssim (\log(\lambda(t))) \lambda^{-2s}(t), \quad (4.18)$$

$$|P(u(t))| \lesssim (\log(\lambda(t))) \lambda^{\frac{-2s}{s+1}}(t). \quad (4.19)$$

To prove Proposition 4.1, we will need the two following lemmas.

Lemma 4.1. *Let $u \in C([0, T]; H^1(\mathbb{R}^d))$ be a solution of (1.1). Then we have the following estimation:*

$$\|\nabla u\|_{L_T^\infty L_x^2} + \|(\Delta)^{\frac{s+1}{2}} u\|_{L_T^2 L_x^2} \lesssim \|\nabla u_0\|_{L_x^2} + \| |u|^{\frac{4}{d}} \nabla u \|_{L_T^1 L_x^2}$$

Proof. Multiply Equation 1.1 by $\overline{\Delta u}$, integrate and take the imaginary part, to obtain :

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + a \int |(-\Delta)^{\frac{s+1}{2}} u|^2 \leq \left| \int |u|^{\frac{4}{d}} u \Delta u \right| = \left| \int \nabla(|u|^{\frac{4}{d}} u) \nabla u \right|$$

By integrating in time, we get

$$\frac{1}{2} \|\nabla u\|_{L_T^\infty L^2}^2 + a \|(-\Delta)^{\frac{s+1}{2}} u\|_{L_T^2 L_x^2}^2 \leq \frac{1}{2} \|\nabla u_0\|_{L^2}^2 + \|\nabla(|u|^{\frac{4}{d}} u)\|_{L_T^1 L_x^2} \|\nabla u\|_{L_T^\infty L^2}$$

Dividing by $\sqrt{\frac{1}{2} \|\nabla u\|_{L_T^\infty L^2}^2 + a \|(-\Delta)^{\frac{s+1}{2}} u\|_{L_T^2 L_x^2}^2}$ we obtain:

$$\|\nabla u\|_{L_T^\infty L_x^2} + \|(\Delta)^{\frac{s+1}{2}} u\|_{L_T^2 L_x^2} \lesssim \|\nabla u_0\|_{L^2} + \| |u|^{\frac{4}{d}} \nabla u \|_{L_T^1 L_x^2}$$

□

Lemma 4.2. *There exists a real number $0 < \alpha \ll 1$ such that the following holds: Let $u \in C([0, T]; H^1(\mathbb{R}^d))$ be the solution of (1.1) emanating from $u_0 \in H^1$. For a fixed $t \in]0, T[$ we set $\Delta t = \alpha \|u_0\|_{L^2}^{\frac{d-4}{d}} \|u(t)\|_{H^1}^{-2}$. Then $u \in C([t, t + \Delta t]; H^1(\mathbb{R}^d))$ and we have the following controls*

$$\|u\|_{S^0[t, t+\Delta t]} \leq 2 \|u_0\|_{L^2}$$

and

$$\|u\|_{S^1[t, t+\Delta t]} + \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2([t, t+\Delta t]) L_x^2} \leq 2 \|u(t)\|_{H^1(\mathbb{R}^d)}$$

Proof. We first assume that $\Delta t > 0$ is such that $t + \Delta t < T$. Then, according to Lemma 2.1, it holds

$$\left\| \int_0^t S_{a,s}(t-\tau) |u(\tau)|^{\frac{4}{d}} u(\tau) d\tau \right\|_{S^1[t,t+\Delta t]} \lesssim \left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t,t+\Delta t])L_x^2}.$$

Using the Hölder inequality we obtain:

$$\left(\int |u|^{\frac{8}{d}} |\nabla u|^2 \right)^{\frac{1}{2}} \leq \left(\int |u|^{\frac{2(4+d)}{d}} \right)^{\frac{2}{4+d}} \left(\int |\nabla u|^{\frac{2(4+d)}{d}} \right)^{\frac{d}{2(4+d)}}.$$

Integrating in time and applying again Hölder inequality we get:

$$\begin{aligned} \left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t,t+\Delta t])L^2(\mathbb{R}^d)} &\leq \left(\int \left(\int |u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{2}{4+d} \frac{4+d}{4}} dt \right)^{\frac{4}{4+d}} \\ &\quad \times \left(\int \left(\int |\nabla u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{d}{2(4+d)} \frac{4+d}{d}} dt \right)^{\frac{d}{4+d}}. \end{aligned}$$

Thus:

$$\left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t,t+\Delta t])L^2(\mathbb{R}^d)} \leq \|u\|_{L^{\frac{4+d}{d}}([t,t+\Delta t])L^{\frac{8+2d}{d}}(\mathbb{R}^d)} \|\nabla u\|_{L^{\frac{4+d}{d}}([t,t+\Delta t])L^{\frac{8+2d}{d}}(\mathbb{R}^d)}.$$

But $(\frac{4+d}{d}, \frac{8+2d}{d})$ is admissible, thus we have:

$$\left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t,t+\Delta t])L^2(\mathbb{R}^d)} \leq \|u\|_{L^{\frac{4+d}{d}}([t,t+\Delta t])L^{\frac{8+2d}{d}}(\mathbb{R}^d)} \|u\|_{S^1[t,t+\Delta t]}.$$

By Sobolev inequalities we have:

$$\begin{aligned} \|u\|_{L^{\frac{4+d}{d}}([t,t+\Delta t])L^{\frac{8+2d}{d}}(\mathbb{R}^d)} &\lesssim \|u\|_{L^{\frac{4+d}{d}}([t,t+\Delta t])H^{\frac{2d}{d+4}}(\mathbb{R}^d)} \\ &\leq (\Delta t)^{\frac{d}{d+4}} \|u\|_{L^\infty([t,t+\Delta t])H^{\frac{2d}{d+4}}(\mathbb{R}^d)}. \end{aligned}$$

Now by interpolation we obtain for $d = 1, 2, 3, 4$:

$$\|u\|_{L^{\frac{4+d}{d}}([t,t+\Delta t])L^{\frac{8+2d}{d}}(\mathbb{R}^d)} \leq (\Delta t)^{\frac{d}{d+4}} \|u\|_{L^\infty([t,t+\Delta t])L^2(\mathbb{R}^d)} \|u\|_{L^\infty([t,t+\Delta t])H^1(\mathbb{R}^d)},$$

which, according to (1.6), leads to

$$\|u^{\frac{4}{d}} \nabla u\|_{L_t^1 L_x^2} \leq (\Delta t)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[t,t+\Delta t]}^{\frac{2d}{d+4}} \|u\|_{S^1[t,t+\Delta t]}. \quad (4.20)$$

Since by Lemma 4.1 it holds

$$\|u\|_{L^\infty([t,t+\Delta t])H^1} + \|(\Delta)^{\frac{s+1}{2}} u\|_{L^2([t,t+\Delta t])L_x^2} \lesssim \|u(t)\|_{H^1} + \||u|^{\frac{4}{d}} \nabla u\|_{L^1([t,t+\Delta t])L_x^2},$$

we finally get

$$\|u\|_{S^1[t,t+\Delta t]} + \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2([t,t+\Delta t])L^2} \lesssim \|u(t)\|_{H^1} + (\Delta t)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[t,t+\Delta t]}^{\frac{2d}{d+4}+1}.$$

In view of (2.1) and a continuity argument, it follows that $u \in C([t, t + \Delta t]; H^1(\mathbb{R}^d))$ for some $\Delta t \sim \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{d-4}{d}} \|u(t)\|_{H^1(\mathbb{R}^d)}^{-2}$ and

$$\|u\|_{S^1[t,t+\Delta t]} + \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^2([t,t+\Delta t])L^2} \leq 2\|u_0\|_{H^1}.$$

In the same way

$$\|u\|_{S^0[t,t+\Delta t]} \lesssim \|u(t)\|_{L^2(\mathbb{R}^d)} + (\Delta t)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[t,t+\Delta t]}^{2\frac{d}{d+4}} \|u\|_{S^0[t,t+\Delta t]}$$

which ensures that

$$\|u\|_{S^0[t, t+\Delta t]} \leq 2 \|u_0\|_{L^2(\mathbb{R}^d)} .$$

□

Proof of Proposition 4.1 : According to (4.8), each interval $[t_k, t_{k+1}]$, can be divided into k intervals, $[\tau_k^j, \tau_k^{j+1}]$ of length less than $\lambda(t_k)$. From (1.7), we have

$$|E(u(\tau_k^{j+1})) - E(u(\tau_k^j))| \lesssim \left| \int_{\tau_k^j}^{\tau_k^{j+1}} \int_{\mathbb{R}^d} ((-\Delta)^s u) |u|^{\frac{4}{d}} \bar{u} dx \right| .$$

For notation convenience we set $\Theta =]\tau_k^j, \tau_k^{j+1}[\times \mathbb{R}^d$. By Plancherel formula and Hölder inequality, it holds

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \int_{\mathbb{R}^d} ((-\Delta)^s u) |u|^{\frac{4}{d}} \bar{u} dx dt \leq \|(-\Delta)^{\frac{s}{2}}(u|u|^{\frac{4}{d}})\|_{L^{\frac{4+2d}{4+d}}(\Theta)} \|(-\Delta)^{\frac{s}{2}}u\|_{L^{\frac{4+2d}{d}}(\Theta)}$$

and, by interpolation, we have

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^{\frac{4+2d}{d}}(\Theta)}^2 \leq \|\nabla u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{2s} \|u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{2-2s} .$$

Noticing that the fractional Leibniz rule (see [20]) leads to

$$\|(-\Delta)^{\frac{s}{2}}(|u|^{\frac{4}{d}}u)\|_{L^{\frac{4+2d}{4+d}}(\Theta)} \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{L^{\frac{4+2d}{d}}(\Theta)} \|u|^{\frac{4}{d}}\|_{L^{\frac{4+2d}{4}}(\Theta)} \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{L^{\frac{4+2d}{d}}(\Theta)} \|u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{\frac{4}{d}} ,$$

we finally obtain

$$|E(u(\tau_k^{j+1})) - E(u(\tau_k^j))| \lesssim \|u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{\frac{4}{d}} \|\nabla u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{2s} \|u\|_{L^{\frac{4+2d}{d}}(\Theta)}^{2-2s} .$$

Since $(\frac{4}{d} + 2, \frac{4}{d} + 2)$ is an admissible pair, Lemma 4.2 yields

$$|E(u(\tau_k^{j+1})) - E(u(\tau_k^j))| \lesssim \lambda(t_k)^{-2s}$$

and summing over j we get

$$|E(u(t_{k+1})) - E(u(t_k))| \lesssim k\lambda(t_k)^{-2s} .$$

Finally, taking $T^+ = T$ and summing from k_0 to k^+ , we obtain:

$$|E(u(T^+)) - E(u_0)| \lesssim k^+ \lambda^{-2s}(T^+) \lesssim \log(\lambda(T)) \lambda^{-2s}(T) .$$

Note that the growth of the energy is small with respect $\frac{1}{\lambda^2}$, because $s < 1$.

Let us now proceed with the momentum. According to (1.8) we have :

$$|P(u(\tau_k^{j+1})) - P(u(\tau_k^j))| \lesssim \int_{\tau_k^j}^{\tau_k^{j+1}} \left| \int_{\mathbb{R}^d} \bar{\nabla} u (-\Delta)^s u \right| .$$

But

$$\left| \int_{\mathbb{R}^d} \bar{\nabla} u (-\Delta)^s u \right| = \left| \int_{\mathbb{R}^d} ((-\Delta)^{\frac{s}{2} + \frac{1}{4}} u) (-\Delta)^{\frac{s}{2} - \frac{1}{4}} \bar{\nabla} u \right| \leq \|(-\Delta)^{\frac{s}{2} + \frac{1}{4}} u\|_{L^2(\mathbb{R}^d)} \|(-\Delta)^{\frac{s}{2} - \frac{1}{4}} \bar{\nabla} u\|_{L^2(\mathbb{R}^d)}$$

with

$$\|(-\Delta)^{\frac{s}{2} - \frac{1}{4}} \bar{\nabla} u\|_{L^2(\mathbb{R}^d)} = \|(-\Delta)^{\frac{s}{2} + \frac{1}{4}} u\|_{L^2(\mathbb{R}^d)}$$

and, by interpolation,

$$\|(-\Delta)^{\frac{s}{2}+\frac{1}{4}}u\|_{L^2(\mathbb{R}^d)}^2 \leq \|(-\Delta)^{\frac{s}{2}+\frac{1}{2}}u\|_{L^2(\mathbb{R}^d)}^{2\theta} \|u\|_{L^2(\mathbb{R}^d)}^{2-2\theta},$$

where $0 < \theta = \frac{2s+1}{2s+2} < 1$. Therefore we get

$$|P(u(\tau_k^{j+1})) - P(u(\tau_k^j))| \lesssim (\tau_{k+1} - \tau_k)^{1-\theta} \|u_0\|_{L^2(\mathbb{R}^d)}^{2-2\theta} \|(-\Delta)^{\frac{s}{2}+\frac{1}{2}}u\|_{L^2(\Theta)}^{2\theta}$$

and Lemma 4.2 ensures that

$$|P(u(\tau_k^{j+1})) - P(u(\tau_k^j))| \lesssim \lambda^{2-2\theta} \lambda^{-2\theta} = \lambda^{2-4\theta} = \lambda^{\frac{-2s}{s+1}}.$$

Summing over j we obtain that:

$$|P(u(\tau_k)) - P(u(\tau_k))| \lesssim k \lambda^{\frac{-2s}{s+1}}$$

and summing from k_0 to k^+ , we finally get

$$|P(u(T^+)) - P(u_0)| \lesssim \log(\lambda(T)) \lambda(T)^{\frac{-2s}{s+1}}.$$

Note that the growth of the momentum is small with respect $\frac{1}{\lambda}$ since $1 - \frac{2s}{s+1} > 0$.

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