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ON MORI CONE OF BOTT TOWERS

B. NARASIMHA CHARY

ABSTRACT. A Bott tower of height r is a sequence of projective bundles

$$X_r \xrightarrow{\pi_r} X_{r-1} \xrightarrow{\pi_{r-1}} \dots \xrightarrow{\pi_2} X_1 = \mathbb{P}^1 \xrightarrow{\pi_1} X_0 = \{pt\},$$

where $X_i = \mathbb{P}(\mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1})$ for a line bundle \mathcal{L}_{i-1} over X_{i-1} for all $1 \leq i \leq r$ and $\mathbb{P}(-)$ denotes the projectivization. These are smooth projective toric varieties and we refer to the top object X_r also as a Bott tower. In this article, we study the Mori cone and numerically effective (nef) cone of Bott towers, and we classify Fano, weak Fano and log Fano Bott towers. We prove some vanishing theorems for the cohomology of tangent bundle of Bott towers.

Keywords: Bott towers, Mori cone, primitive relations and toric varieties.

1. INTRODUCTION

In [BS58], R. Bott and H. Samelson introduced a family of (smooth differentiable) manifolds which may be viewed as the total spaces of iterated \mathbb{P}^1 -bundles over a point $\{pt\}$, where each \mathbb{P}^1 -bundle is the projectivization of a rank 2 decomposable vector bundle. In [GK94], M. Grossberg and Y. Karshon proved (in complex geometry setting) that these manifolds have a natural action of a compact torus and also obtained some applications to representation theory and symplectic geometry. In [Civ05], Y. Civan proved that these are smooth projective toric varieties. These are called Bott towers, we denote them by $\{(X_i, \pi_i) : 1 \leq i \leq r\}$, where

$$X_r \xrightarrow{\pi_r} X_{r-1} \xrightarrow{\pi_{r-1}} \dots \xrightarrow{\pi_2} X_1 = \mathbb{P}^1 \xrightarrow{\pi_1} \{pt\},$$

$X_i = \mathbb{P}(\mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1})$ for a line bundle \mathcal{L}_{i-1} over X_{i-1} for all $1 \leq i \leq r$ and r is the dimension of X_r . In [CS11], [CMS10] and [Ish12], the authors studied ‘‘cohomological rigidity’’ properties of Bott towers. These also play an important role in algebraic topology and K-theory (see [CR05], [DJ91] and references therein). In this article we refer to X_r also as a Bott tower (it is also called Bott manifold).

In this paper we study the geometry of Bott towers in more detail by methods of toric geometry. We work over the field \mathbb{C} of complex numbers. We study the **Mori cone** of X_r and prove that the class of curves corresponding to ‘primitive relations $r(P_i)$ ’ forms a basis of the real vector space of numerical classes of one-cycles in X_r (see Theorem 4.7 and Corollary 4.8). An extremal ray R in the Mori cone is called **Mori ray** if $R \cdot K_{X_r} < 0$, where K_{X_r} is the canonical divisor in X_r . We describe extremal rays and Mori rays of the Mori cone of X_r (see Theorem 8.1). We characterize the ampleness and numerical effectiveness of line bundles on X_r (see Lemma 5.1) and describe the generators of the *nef* cone of X_r (see Theorem 5.7).

Recall that a smooth projective variety X is called **Fano** (respectively, **weak Fano**) if its anti-canonical divisor $-K_X$ is ample (respectively, *nef* and *big*). Following [AS14], we say that

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a pair (X, D) of a normal projective variety X and an effective \mathbb{Q} -divisor D is **log Fano** if it is Kawamata log terminal and $-(K_X + D)$ is ample (see Section 7 for more details). We study the Fano, weak Fano and the log Fano (of the pair (X_r, D) for a suitably chosen divisor D in X_r) properties of the Bott tower X_r . To describe these results we need some notation. It is known that a Bott tower $\{(X_i, \pi_i) : 1 \leq i \leq r\}$ is uniquely determined by an upper triangular matrix M_r with integer entries, defined via the first Chern class of the line bundle \mathcal{L}_{i-1} on X_{i-1} , where $X_i = \mathbb{P}(\mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1})$ for $1 \leq i \leq r$ (see [GK94, Section 2.3], [Civ05] and [VT15, Section 7.8]). For more details see Section 2. Let

$$M_r := \begin{bmatrix} 1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} \\ 0 & 1 & \beta_{23} & \cdots & \beta_{2r} \\ 0 & 0 & 1 & \cdots & \beta_{3r} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}_{r \times r},$$

where β_{ij} 's are integers. Define for $1 \leq i \leq r$,

$$\eta_i^+ := \{r \geq j > i : \beta_{ij} > 0\}$$

and

$$\eta_i^- := \{r \geq j > i : \beta_{ij} < 0\}.$$

If $|\eta_i^+| = 1$ (respectively, $|\eta_i^+| = 2$), then let $\eta_i^+ = \{m\}$ (respectively, $\eta_i^+ = \{m_1, m_2\}$). If $|\eta_i^-| = 1$ (respectively, $|\eta_i^-| = 2$), then set $\eta_i^- = \{l\}$ (respectively, $\eta_i^- = \{l_1, l_2\}$). The following can be viewed as a condition on i^{th} row of the matrix M_r :

- N_i^1 is the condition that
 - (i) $|\eta_i^+| = 0$, $|\eta_i^-| \leq 1$, and if $|\eta_i^-| = 1$ then $\beta_{il} = -1$; or
 - (ii) $|\eta_i^-| = 0$, $|\eta_i^+| \leq 1$, and if $|\eta_i^+| = 1$ then $\beta_{im} = 1$ and $\beta_{mk} = 0$ for all $k > m$.
- N_i^2 is the condition that
 - Case 1: Assume that $|\eta_i^+| = 0$. Then $|\eta_i^-| \leq 2$, and if $|\eta_i^-| = 1$ (respectively, $|\eta_i^-| = 2$) then $\beta_{li} = -1$ or -2 (respectively, $\beta_{il_1} = -1 = \beta_{il_2}$).
 - Case 2: If $|\eta_i^-| = 1 = |\eta_i^+|$ and $l < m$, then $\beta_{il} = -1$, $\beta_{im} = 1$ and $\beta_{mk} = 0$ for all $k > m$.
 - Case 3: Assume that $|\eta_i^+| = 1$. Then $\beta_{im} = 1$ and either it satisfies
 - (i) Case 2; or
 - (ii) $|\eta_i^-| = 0$ and $\beta_{mk} = 0$ for all $k > m$; or
 - (iii) there exists a unique $r \geq s > m$ such that
 - $\beta_{ms} - \beta_{is} = 1$ and $\beta_{mk} - \beta_{ik} = 0$ for all $k > s$, or
 - $\beta_{ms} - \beta_{is} = -1$ and $\beta_{is} - \beta_{ms} - \beta_{sk} = 0$ for all $k > s$.

Definition 1.1. We say X_r satisfies condition I (respectively, condition II) if N_i^1 (respectively, N_i^2) holds for all $1 \leq i \leq r$.

Note that $N_i^1 \implies N_i^2$ for all $1 \leq i \leq r$. If X_r satisfies condition I, then it also satisfies conditions II. We prove,

Theorem (see Theorem 6.3).

- (1) X_r is Fano if and only if it satisfies I.
- (2) X_r is weak Fano if and only if it satisfies II.

As a consequence we get some vanishing results for the cohomology of tangent bundle of Bott towers and hence local rigidity results. Let T_{X_r} denote the tangent bundle of X_r .

Corollary (see Corollary 6.4 and Corollary 6.5). *If X_r satisfies I, then $H^i(X_r, T_{X_r}) = 0$ for all $i \geq 1$. In particular, X_r is locally rigid.*

For $1 \leq i \leq r$, we define some constants k_i which again depend on the given matrix M_r corresponding to the Bott tower X_r (for more details see Section 7). We prove,

Theorem (see Theorem 7.1). *The pair (X_r, D) is log Fano if and only if $k_i < 0$ for all $1 \leq i \leq r$.*

Remark 1.2. *By using the results of this article, in [Cha17b] we give some applications to Bott-Samelson-Demazure-Hansen (BSDH) variety, which can be described also as a iterated projective line bundle, by degeneration of this variety to a Bott tower. Precisely, we study Fano, weak Fano, log Fano properties for BSDH varieties (see also [Cha17a]). We obtain some vanishing theorems for the cohomology of tangent bundle (and line bundles) on BSDH varieties (see also [CKP15], [CKP] and [CK17]). We also recover the results in [PK16].*

The paper is organized as follows: In Section 2, we discuss preliminaries on Bott towers and toric varieties. In Section 3, we discuss the Picard group of the Bott tower and compute the relative tangent bundle. Section 4 contains detailed study of primitive collections and primitive relations of the Bott tower and we also describe the Mori cone. In Section 5 we describe ample and *nef* line bundles on the Bott tower, and we find the generators of the *nef* cone. In Section 6 and 7, we study Fano, weak Fano and log Fano properties for Bott towers. We also see some vanishing results. In Section 8, we describe extremal rays and Mori rays for the Bott tower.

2. PRELIMINARIES

In this section we recall toric varieties (see [CLS11]) and Bott towers (see [Civ05] and [VT15]). We work throughout the article over the field \mathbb{C} of complex numbers. We expect that the proofs work for algebraically closed fields of arbitrary characteristic, but did not find appropriate references in that generality.

2.1. Toric varieties. We briefly recall the structure of toric varieties from [CLS11] (see also [Ful93] and [Oda88]).

Definition 2.1. *A normal variety X is called a toric variety (of dimension n) if it contains an n -dimensional torus T (i.e. $T = (\mathbb{C}^*)^n$) as a Zariski open subset such that the action of the torus on itself by multiplication extends to an action of the torus on X .*

Toric varieties are completely described by the combinatorics of the corresponding fans. We briefly recall here, let N be the lattice of one-parameter subgroups of T and let M be the lattice of characters of T . Let $M_{\mathbb{R}} := M \otimes \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes \mathbb{R}$. Then we have a natural bilinear pairing

$$\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}.$$

A fan Σ in $N_{\mathbb{R}}$ is a collection of convex polyhedral cones that is closed under intersections and cone faces. Let $\check{\sigma}$ be the dual cone of $\sigma \in \Sigma$ in $M_{\mathbb{R}}$. For $\sigma \in \Sigma$, the semigroup algebra $\mathbb{C}[\check{\sigma} \cap M]$ is a normal domain and finitely generated \mathbb{C} -algebra. Then the scheme $\text{Spec}(\mathbb{C}[\check{\sigma} \cap M])$ is called

the affine toric variety corresponding to σ . For a given fan Σ , we can define a toric variety X_Σ by gluing the affine toric varieties $\text{Spec}(\mathbb{C}[\check{\sigma} \cap M])$ as σ varies in Σ . For all $1 \leq s \leq n$,

$$\Sigma(s) := \{\sigma \in \Sigma : \dim(\sigma) = s\}.$$

For each $\rho \in \Sigma(1)$, we denote u_ρ , the generator of $\rho \cap N$. For $\sigma \in \Sigma$,

$$\sigma(1) := \Sigma(1) \cap \sigma.$$

There is a bijective correspondence between the cones in Σ and the T -orbits in X_Σ . For each $\sigma \in \Sigma$, the dimension $\dim(O(\sigma))$ of the T -orbit $O(\sigma)$ corresponding to σ is $n - \dim(\sigma)$. Let $\tau, \sigma \in \Sigma$, then τ is a face of σ if and only if $O(\sigma) \subset \overline{O(\tau)}$, where $\overline{O(\sigma)}$ is the closure of T -orbit $O(\sigma)$. We denote $V(\sigma) = \overline{O(\sigma)}$ and it is a toric variety with the corresponding fan being $\text{Star}(\sigma)$, the star of σ which is the set of cones in Σ which have σ as a face. Let $D_\rho = \overline{O(\rho)}$ be the torus-invariant prime divisor in X_Σ corresponding to $\rho \in \Sigma(1)$. The group $T\text{Div}(X_\Sigma)$ of T -invariant divisors in X_Σ is given by

$$T\text{Div}(X_\Sigma) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho.$$

For each $m \in M$, the character χ^m of T is a rational function on X_Σ and the corresponding divisor is given by

$$\text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

2.2. Bott towers. In this section we recall some basic definitions and results on Bott towers. Let \mathcal{L}_0 be a trivial line bundle over a single point $X_0 := \{pt\}$, and let $X_1 := \mathbb{P}(\mathcal{O}_{X_0} \oplus \mathcal{L}_0)$, where $\mathbb{P}(-)$ denotes the projectivization. Let \mathcal{L}_1 be a line bundle on X_1 , then define $X_2 := \mathbb{P}(\mathcal{O}_{X_1} \oplus \mathcal{L}_1)$, which is a \mathbb{P}^1 -bundle over X_1 . Repeat this process r -times, so that each X_i is a \mathbb{P}^1 -bundle over X_{i-1} for $1 \leq i \leq r$. We get the following:

$$\begin{array}{c} X_r = \mathbb{P}(\mathcal{O}_{X_{r-1}} \oplus \mathcal{L}_{r-1}) \\ \downarrow \pi_r \\ X_{r-1} = \mathbb{P}(\mathcal{O}_{X_{r-2}} \oplus \mathcal{L}_{r-2}) \\ \downarrow \pi_{r-1} \\ \vdots \\ \downarrow \pi_2 \\ X_1 = \mathbb{P}(\mathcal{O}_{X_0} \oplus \mathcal{L}_0) \\ \downarrow \pi_1 \\ X_0 = \{pt\} \end{array}$$

For each $1 \leq i \leq r$, X_i is a smooth projective toric variety (see [Civ05, Theorem 22]). Consider the points $[1 : 0]$ and $[0 : 1]$ in \mathbb{P}^1 , we call them the south pole and the north pole respectively. The zero section of \mathcal{L}_{i-1} gives a section $s_i^0 : X_{i-1} \rightarrow X_i$, the south pole section; similarly, the north pole section $s_i^1 : X_{i-1} \rightarrow X_i$ by letting the first coordinate in $\mathbb{P}(\mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1})$ to vanish.

Let $1 \leq i \leq r$. Since $\pi_i : X_i \rightarrow X_{i-1}$ is a projective bundle, by a standard result on the cohomology ring of projective bundles we have the following (see [Har77, Page 429] for instance, and also [Mil16, Proposition 10.1]):

Theorem 2.2. *The cohomology ring $H^*(X_i, \mathbb{Z})$ of X_i is a free module over $H^*(X_{i-1}, \mathbb{Z})$ on generators 1 and u_i , which have degree 0 and 2 respectively, that is*

$$H^*(X_i, \mathbb{Z}) = H^*(X_{i-1}, \mathbb{Z})1 \oplus H^*(X_{i-1}, \mathbb{Z})u_i.$$

The ring structure is determined by the single relation

$$u_i^2 = c_1(\mathcal{L}_{i-1})u_i,$$

where $c_1(-)$ denotes the first Chern class and the restriction of u_i to the fiber $\mathbb{P}^1 \subset X_i$ is the first Chern class of the canonical line bundle over \mathbb{P}^1 . Hence we have

$$H^*(X_i, \mathbb{Z}) = H^*(X_{i-1}, \mathbb{Z})[u_i]/J_i,$$

where J_i is the ideal generated by $u_i^2 - c_1(\mathcal{L}_{i-1})u_i$.

Consider the exponential sequence (see [Har77, Page 446]):

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_{X_{i-1}} \rightarrow \mathcal{O}_{X_{i-1}}^* \rightarrow 0.$$

Then we get the following exact sequence:

$$0 \rightarrow H^1(X_{i-1}, \mathbb{Z}) \rightarrow H^1(X_{i-1}, \mathcal{O}_{X_{i-1}}) \rightarrow H^1(X_{i-1}, \mathcal{O}_{X_{i-1}}^*) \xrightarrow{c_1(-)} H^2(X_{i-1}, \mathbb{Z}) \rightarrow H^2(X_{i-1}, \mathcal{O}_{X_{i-1}}) \rightarrow \dots$$

Since X_{i-1} is toric, we have $H^j(X_{i-1}, \mathcal{O}_{X_{i-1}}) = 0$ for all $j > 0$ (see [Oda88, Corollary 2.8]). As $H^1(X_{i-1}, \mathcal{O}_{X_{i-1}}^*) = \text{Pic}(X_{i-1})$, we get $c_1(-) : \text{Pic}(X_{i-1}) \xrightarrow{\sim} H^2(X_{i-1}, \mathbb{Z})$. Then we have the following:

Theorem 2.3. *Each line bundle \mathcal{L}_{i-1} on X_{i-1} is determined (up to an algebraic isomorphism) by its first Chern class, which can be written as a linear combination*

$$c_1(\mathcal{L}_{i-1}) = - \sum_{k=1}^{i-1} \beta_{ki} u_k \in H^2(X_{i-1}, \mathbb{Z}),$$

where β_{ik} 's are integers for $1 \leq k \leq i-1$.

Then by Theorem 2.2 and 2.3, by iteration, we get the following:

Corollary 2.4. *We have*

$$H^*(X_r, \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_r]/J,$$

where J is the ideal generated by $\{u_j^2 + \sum_{i < j} \beta_{ij} u_i u_j : 1 \leq j \leq r\}$ and the integers β_{ij} 's are as in Theorem 2.3.

Write $\{\beta_{ij} : 1 \leq i < j \leq r\}$, the collection of $r(r-1)/2$ integers, as an upper triangular $r \times r$ matrix

$$M_r := \begin{bmatrix} 1 & \beta_{12} & \beta_{13} & \dots & \beta_{1r} \\ 0 & 1 & \beta_{23} & \dots & \beta_{2r} \\ 0 & 0 & 1 & \dots & \beta_{3r} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & & 1 \end{bmatrix}_{r \times r} \quad (2.1)$$

Then we get the following result (see for instance [GK94, Lemma 2.15] and also [Civ05, Section 3]).

Corollary 2.5. *There is a bijective correspondence between $\{\text{Bott towers of height } r\}$ and $\{r \times r \text{ upper triangular matrices with integer entries as in (2.1)}\}$.*

Two Bott towers $\{(X_i, \pi_i) : 1 \leq i \leq r\}$ and $\{(X'_i, \pi'_i) : 1 \leq i \leq r\}$ are isomorphic if there exists a collection of isomorphisms $\{\phi_i : X_i \rightarrow X'_i : 1 \leq i \leq r\}$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} X_r & \xrightarrow{\pi_r} & X_{r-1} & \xrightarrow{\pi_{r-1}} & \cdots & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & X_0 \\ \downarrow \phi_r & & \downarrow \phi_{r-1} & & & & \downarrow \phi_1 & & \downarrow \phi_0 \\ X'_r & \xrightarrow{\pi'_r} & X'_{r-1} & \xrightarrow{\pi'_{r-1}} & \cdots & \xrightarrow{\pi'_2} & X'_1 & \xrightarrow{\pi'_1} & X'_0 \end{array}$$

2.2.1. *Toric structure on Bott tower.* Let $\{e_1^+, \dots, e_r^+\}$ be the standard basis of the lattice \mathbb{Z}^r . Define, for all $i \in \{1, \dots, r\}$,

$$e_i^- := -e_i^+ - \sum_{j>i} \beta_{ij} e_j^+, \quad (2.2)$$

where β_{ij} 's are integers as above. Then we have the following theorem (see [Civ05, Section 3 and Theorem 22] and for algebraic topology setting see [VT15, Theorem 7.8.7]):

Theorem 2.6. *The Bott tower $\{(X_i, \pi_i) : 1 \leq i \leq r\}$ corresponding to a matrix M_r as in (2.1) is isomorphic to $\{(X_{\Sigma_i}, \pi_{\Sigma_i}) : 1 \leq i \leq r\}$, the collection of smooth projective toric varieties corresponding to the fan Σ_i with the 2^i maximal cones generated by the set of vectors*

$$\{e_j^\epsilon : 1 \leq j \leq i \text{ and } \epsilon \in \{+, -\}\},$$

and where $\pi_{\Sigma_i} : X_{\Sigma_i} \rightarrow X_{\Sigma_{i-1}}$ is the toric morphism induced by the projection $\overline{\pi_{\Sigma_i}} : \mathbb{Z}^i \rightarrow \mathbb{Z}^{i-1}$ for all $1 \leq i \leq r$.

Note that by Theorem 2.6, Σ_i has $2i$ one-dimensional cones generated by the vectors

$$\{e_j^+, e_j^- : 1 \leq j \leq i\},$$

and by (2.2), we can see that the divisors $D_{\rho_j^+}$ corresponding to e_j^+ for $1 \leq j \leq i$ form a basis of the Picard group of X_i (see Section 3 for more details).

3. ON PICARD GROUP OF A BOTT TOWER

Now we describe a basis of the Picard group $\text{Pic}(X_r)$ of X_r . Let $\epsilon \in \{+, -\}$ and for $1 \leq i \leq r$, let ρ_i^ϵ be the one-dimensional cone generated by e_i^ϵ . For all $1 \leq i \leq r$, we define $D_{\rho_i^\epsilon}$ to be the toric divisor corresponding to the one-dimensional cone ρ_i^ϵ . We prove,

Lemma 3.1. *The set $\{D_{\rho_i^\epsilon} : 1 \leq i \leq r \text{ and } \epsilon \in \{+, -\}\}$ forms a basis of $\text{Pic}(X_r)$.*

Proof. By Theorem 2.6, using the description of the one-dimensional cones we have the following decomposition of $\Sigma(1)$:

$$\Sigma(1) = \{\rho_i^+ : 1 \leq i \leq r\} \cup \{\rho_i^- : 1 \leq i \leq r\}. \quad (3.1)$$

Again by Theorem 2.6, $\{D_{\rho_i^+} : 1 \leq i \leq r\}$ forms a basis of the Picard group $Pic(X_r)$ of X_r . Since

$$0 \sim \text{div}(\chi^{e_i^+}) = \sum_{\rho \in \Sigma(1)} \langle u_\rho, e_i^+ \rangle D_\rho,$$

by (2.2) we can see that $\{D_{\rho_i^-} : 1 \leq i \leq r\}$ also forms a basis of $Pic(X_r)$. In general, let $\sigma \in \Sigma$ be the maximal cone generated by $\{e_i^\epsilon : 1 \leq i \leq r\}$. Take the torus-fixed point x^ϵ in X_r corresponding to the maximal cone σ . Let U be the torus-invariant open affine neighbourhood of x^ϵ in X_r . Then U is an affine space of dimension r ; in particular, $Pic(U) = 0$. Therefore, we get

$$X_r \setminus U = \cup_{i=1}^r D_{\rho_i^\epsilon}$$

and $Pic(X_r)$ is generated by $\{D_{\rho_i^\epsilon} : 1 \leq i \leq r\}$ (see [Har70, Chapter II, Proposition 3.1, page 66]). Since $\{D_{\rho_i^\epsilon} : 1 \leq i \leq r\}$ is linearly independent and the rank of $Pic(X_r)$ is r , this set $\{D_{\rho_i^\epsilon} : 1 \leq i \leq r\}$ forms a basis of $Pic(X_r)$. \square

By Lemma 3.1, the set $\{D_{\rho_i^+} : 1 \leq i \leq r\}$ forms a basis of $Pic(X_r)$. Now we express for each $1 \leq i \leq r$, $D_{\rho_i^-}$ in terms of $D_{\rho_j^+}$'s ($1 \leq j \leq r$). Let $1 \leq i \leq r$, define $h_i^{i-1} := -\beta_{i(i-1)}$ and

$$h_i^j := \begin{cases} 0 & \text{for } j > i. \\ 1 & \text{for } j = i. \\ -\sum_{k=j}^{i-1} \beta_{ik}(h_k^j) & \text{for } j < i. \end{cases}$$

Then we prove,

Lemma 3.2. *Let $1 \leq i \leq r$. The coefficient of $D_{\rho_j^+}$ in $D_{\rho_i^-}$ is h_i^j .*

Proof. Proof is by induction on i and by using

$$0 \sim \text{div}(\chi^{e_i^+}) = \sum_{\rho \in \Sigma(1)} \langle u_\rho, e_i^+ \rangle D_\rho. \quad (3.2)$$

Recall the equation (2.2),

$$e_i^- = -e_i^+ - \sum_{j>i} \beta_{ij} e_j^+ \quad \text{for all } 1 \leq i \leq r.$$

If $i = 1$, by (3.2), we see

$$0 \sim \text{div}(\chi^{e_1^+}) = D_{\rho_1^+} - D_{\rho_1^-}.$$

Then we have

$$D_{\rho_1^-} \sim D_{\rho_1^+}. \quad (3.3)$$

If $i = 2$, by (3.2) and (2.2), we see

$$0 \sim \text{div}(\chi^{e_2^+}) = D_{\rho_2^+} - D_{\rho_2^-} - \beta_{21} D_{\rho_1^-}.$$

By (3.3), we get

$$D_{\rho_2^-} \sim D_{\rho_2^+} - \beta_{21} D_{\rho_1^+} = h_2^2 D_{\rho_2^+} + h_2^1 D_{\rho_1^+}.$$

By induction assume that

$$D_{\rho_k^-} \sim \sum_{j=1}^r h_k^j D_{\rho_j^+} \quad \text{for all } k < i.$$

Again by (3.2) and (2.2), we see

$$0 \sim \operatorname{div}(\chi^{e_i^+}) = D_{\rho_i^+} - D_{\rho_i^-} - \sum_{k < i} \beta_{ik} D_{\rho_k^-}.$$

Then

$$D_{\rho_i^-} \sim D_{\rho_i^+} - \sum_{k < i} \beta_{ik} D_{\rho_k^-}.$$

Hence

$$D_{\rho_i^-} \sim D_{\rho_i^+} - \sum_{k < i} \beta_{ik} \left(\sum_{j=1}^r h_k^j D_{\rho_j^+} \right).$$

Since $h_k^j = 0$ for $k < j$, we get

$$D_{\rho_i^-} \sim D_{\rho_i^+} - \sum_{k < i} \beta_{ik} \left(\sum_{j=1}^{i-1} h_k^j D_{\rho_j^+} \right).$$

Then

$$D_{\rho_i^-} \sim D_{\rho_i^+} + \sum_{j=1}^{i-1} \left(- \sum_{k=j}^{i-1} \beta_{ik} h_k^j \right) D_{\rho_j^+}.$$

Therefore, we conclude that $D_{\rho_i^-} \sim D_{\rho_i^+} + \sum_{j=1}^{i-1} h_i^j D_{\rho_j^+}$. This completes the proof of the lemma. \square

Let $\epsilon \in \{+, -\}$. Define $\Sigma(1)^\epsilon := \{\rho_i^\epsilon : 1 \leq i \leq r\}$. Then

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho = \sum_{\rho \in \Sigma(1)^+} a_\rho D_\rho + \sum_{\rho \in \Sigma(1)^-} a_\rho D_\rho.$$

For $1 \leq i \leq r$, let $g_i := a_{\rho_i^+} + \sum_{j=i}^r a_{\rho_j^-} h_j^i$. Then we have

Corollary 3.3. $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \sim \sum_{i=1}^r g_i D_{\rho_i^+}$.

Proof. We have $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho = \sum_{i=1}^r a_{\rho_i^+} D_{\rho_i^+} + \sum_{i=1}^r a_{\rho_i^-} D_{\rho_i^-}$. By Lemma 3.2, we can see that $\sum_{i=1}^r a_{\rho_i^-} D_{\rho_i^-} \sim \sum_{i=1}^r a_{\rho_i^-} \left(\sum_{j=1}^i h_i^j D_{\rho_j^+} \right)$. Then $\sum_{i=1}^r a_{\rho_i^-} D_{\rho_i^-} \sim \sum_{i=1}^r \left(\sum_{j=i}^r a_{\rho_j^-} h_j^i \right) D_{\rho_i^+}$. Hence we have $D \sim \sum_{i=1}^r \left(a_{\rho_i^+} + \sum_{j=i}^r a_{\rho_j^-} h_j^i \right) D_{\rho_i^+}$. Thus, $D \sim \sum_{i=1}^r g_i D_{\rho_i^+}$ and this completes the proof. \square

Remark 3.4. *By Corollary 3.3, we see some vanishing results of the cohomology of line bundles on BSDH varieties in [Cha17b].*

Let $1 \leq i \leq r$. We prove the following.

Lemma 3.5. *The relative tangent bundle T_{π_i} of $\pi_i : X_i \rightarrow X_{i-1}$ is given by*

$$T_{\pi_i} \simeq \mathcal{O}_{X_i}(D_{\rho_i^+} + D_{\rho_i^-}) \simeq \mathcal{O}_{X_i} \left(\sum_{j=1}^{i-1} \beta_{ij} D_{\rho_j^-} + 2D_{\rho_i^-} \right).$$

Proof. By definition of Bott tower, π_i is a \mathbb{P}^1 -fibration. Then the relative canonical bundle K_{π_i} is given by

$$K_{\pi_i} = \mathcal{O}_{X_i}(K_{X_i}) \otimes \pi_i^*(\mathcal{O}_{X_{i-1}}(-K_{X_{i-1}}))$$

(see [Kle80, Corollary 24, page 56]). By [CLS11, Theorem 8.2.3] (see also [Ful93, Page 74]), we have

$$K_{X_\Sigma} = - \sum_{\rho \in \Sigma(1)} D_\rho.$$

Then

$$K_{\pi_i} = \mathcal{O}_{X_i}(- \sum_{\rho \in \Sigma(1)} D_\rho) \otimes \pi_i^*(\mathcal{O}_{X_{i-1}}(\sum_{\rho' \in \Sigma'(1)} D_{\rho'})) ,$$

where Σ' is the fan of X_{i-1} . Since X_{i-1} smooth, any divisor of the form $D = \sum_{\rho' \in \Sigma'(1)} a_{\rho'} D_{\rho'}$ with $a_{\rho'} \in \mathbb{Z}$, in X_{i-1} is Cartier. Hence the pullback $\pi_i^*(D)$ is defined and given by

$$\pi_i^*(D) = \pi_i^*(\sum_{\rho' \in \Sigma'(1)} a_{\rho'} D_{\rho'}) = \sum_{\rho \in \Sigma(1)} -\varphi_D(\bar{\pi}_i(u_\rho)) D_\rho,$$

where φ_D is the support function corresponding to the divisor D (see [CLS11, Theorem 4.2.12] for the correspondence between support functions and Cartier divisors). Since the lattice map $\bar{\pi}_i : \mathbb{Z}^i \rightarrow \mathbb{Z}^{i-1}$ is the projection onto the first $i-1$ factors (see page 6), by definition of u_ρ and e_j^- (see (2.2)), for $\epsilon \in \{+, -\}$ we have

$$\bar{\pi}_i(u_{\rho_j^\epsilon}) = \begin{cases} u_{\rho_j'^\epsilon} & \text{if } 1 \leq j \leq i-1. \\ 0 & \text{if } j = i. \end{cases}$$

Hence

$$-\varphi_D(\bar{\pi}_i(u_{\rho_j^\epsilon})) = \begin{cases} a_{\rho_j'^\epsilon} & \text{if } 1 \leq j \leq i-1. \\ 0 & \text{if } j = i. \end{cases}$$

Thus we have,

$$\pi_i^*(\sum_{\rho' \in \Sigma'(1)} D_{\rho'}) = \sum_{\rho \in \Sigma(1) \setminus \{\rho_i^+, \rho_i^-\}} D_\rho.$$

Therefore, we see that

$$K_{\pi_i} = \mathcal{O}_{X_i}(-D_{\rho_i^+} - D_{\rho_i^-}). \quad (3.4)$$

By (2.2), we note that

$$0 \sim \text{div}(\chi^{e_i^+}) = D_{\rho_i^+} - D_{\rho_i^-} - \sum_{j=1}^{i-1} \beta_{ij} D_{\rho_j^-}. \quad (3.5)$$

Since $\check{K}_{\pi_i} = \det T_{\pi_i}$, we get $\check{K}_{\pi_i} = T_{\pi_i}$ as π_i is a \mathbb{P}^1 -fibration. Therefore, the result follows from (3.4) and (3.5). \square

Remark 3.6. By Lemma 3.2, the relative tangent bundle T_{π_i} can be expressed in terms of $D_{\rho_i^+}$ ($1 \leq i \leq r$).

The following is well known and proved here for completeness.

Lemma 3.7. *Let X and Y be smooth varieties. Let $f : X \rightarrow Y$ be a fibration with a section σ and denote by $\sigma(Y)$ its image in X . Then the restriction of the relative tangent bundle T_f to $\sigma(Y)$ is isomorphic to the normal bundle $\mathcal{N}_{\sigma(Y)/X}$ of $\sigma(Y)$ in X .*

Proof. Consider the normal bundle short exact sequence

$$0 \rightarrow T_{\sigma(Y)} \rightarrow T_X|_{\sigma(Y)} \rightarrow \mathcal{N}_{\sigma(Y)/X} \rightarrow 0, \quad (3.6)$$

where $T_{\sigma(Y)}$ and T_X are the tangent bundles of $\sigma(Y)$ and X respectively. Also consider the following short exact sequence

$$0 \rightarrow T_f \rightarrow T_X \rightarrow f^*T_Y \rightarrow 0. \quad (3.7)$$

By restricting (3.7) to $\sigma(Y)$, since σ is a section of f , we get the following short exact sequence

$$0 \rightarrow T_f|_{\sigma(Y)} \rightarrow T_X|_{\sigma(Y)} \rightarrow T_{\sigma(Y)} \rightarrow 0. \quad (3.8)$$

By using (3.6) and (3.8), we see $T_f|_{\sigma(Y)}$ is isomorphic to $\mathcal{N}_{\sigma(Y)/X}$. This completes the proof. \square

We prove,

Lemma 3.8. *Let $1 \leq i \leq r$. The normal bundle $\mathcal{N}_{X_{i-1}/X_i}$ of X_{i-1} in X_i is \mathcal{L}_{i-1}^\vee , where \mathcal{L}_{i-1} is as in the definition of Bott tower and \mathcal{L}_{i-1}^\vee is denotes the dual of \mathcal{L}_{i-1} .*

Proof. Fix $1 \leq i \leq r$ and let $\mathcal{L} := \mathcal{L}_{i-1}$. Recall that $\mathbb{P}(\mathcal{E})$ is by definition $\text{Proj}(S(\mathcal{E}))$, $S(\mathcal{E})$ is symmetric algebra of $\mathcal{E} = \mathcal{O}_{X_{i-1}} \oplus \mathcal{L}$ (see [Har77, Page 162]). Let $V(\mathcal{L}) = \text{Spec}(S(\mathcal{L}))$, the geometric vector bundle associated to the locally free sheaf (line bundle) \mathcal{L} (see [Har77, Exercise 5.18, Page 128]). Then, $V(\mathcal{L})$ is an open subvariety in $\mathbb{P}(\mathcal{E})$ and we have the following commutative diagram

$$\begin{array}{ccc} V(\mathcal{L}) & \hookrightarrow & \mathbb{P}(\mathcal{E}) = X_i \\ & \searrow \pi & \swarrow \pi_r \\ & X_{i-1} & \end{array}$$

Also note that the section $s_i^0(X_{i-1})$ of π_i corresponding to the projection $\mathcal{E} \rightarrow \mathcal{O}_{X_i}$ is same as the zero section $\sigma_\pi(X_{i-1})$ of π . Now consider the following short exact sequence

$$0 \rightarrow T_\pi \rightarrow T_{V(\mathcal{L})} \rightarrow \pi^*T_{X_{i-1}} \rightarrow 0. \quad (3.9)$$

Since the restriction $T_{\pi|_{\sigma_\pi(X_{i-1})}}$ of T_π to $\sigma_\pi(X_{i-1})$ is \mathcal{L}^\vee , by Lemma 3.7 and by above short exact sequence (3.9) we see that $\mathcal{N}_{\sigma_\pi(X_{i-1})/V(\mathcal{L})} \simeq \mathcal{L}^\vee$. Hence we conclude that $\mathcal{N}_{X_{i-1}/X_i} \simeq \mathcal{L}^\vee$ (here we are identifying X_{i-1} with the section corresponding to the projection $\mathcal{E} = \mathcal{O}_{X_{i-1}} \oplus \mathcal{L} \rightarrow \mathcal{O}_{X_{i-1}}$). This completes the proof of the lemma. \square

Let $1 \leq i \leq r$. We prove,

Lemma 3.9.

- (1) *The toric sections of π_i are given by $D_{\rho_i^\epsilon}$, $\epsilon \in \{+, -\}$.*

(2) The normal bundle $\mathcal{N}_{X_{i-1}/X_i}$ of X_{i-1} in X_i is given by

$$\mathcal{N}_{X_{i-1}/X_i} = \check{\mathcal{L}}_{i-1} = \mathcal{O}_{X_i}(D_{\rho_i^+}),$$

where the line bundle \mathcal{L}_{i-1} is as in the definition of the Bott tower X_i .

Proof. Proof of (1): Recall that π_i is a \mathbb{P}^1 -fibration induced by the projection $\bar{\pi}_i : \mathbb{Z}^i \rightarrow \mathbb{Z}^{i-1}$. For each cone $\sigma \in \Sigma_F$ of dimension 1 (which is a maximal cone in Σ_F , where Σ_F denote the fan of the fiber \mathbb{P}^1), the subvariety $V(\sigma)$ is an invariant section of π_i , which is an invariant divisor in X_i . Hence we get two invariant divisors $V(\rho_i^+) = D_{\rho_i^+}$ and $V(\rho_i^-) = D_{\rho_i^-}$.

Proof of (2): By Lemma 3.8, we have $\mathcal{N}_{X_{i-1}/X_i} = \check{\mathcal{L}}_{i-1}$ and the section X_{i-1} is given by the projection $\mathcal{E} = \mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1} \rightarrow \mathcal{O}_{X_{i-1}}$. Hence (2) follows from (1). \square

4. PRIMITIVE RELATIONS OF THE BOTT TOWER

4.1. Primitive collections and primitive relations. First recall the notion of primitive collections and primitive relations of a fan Σ , which are basic tools for the classification of Fano toric varieties due to Batyrev (see [Bat91]).

Definition 4.1. We say $P \subset \Sigma(1)$ is a **primitive collection** if P is not contained in $\sigma(1)$ for some $\sigma \in \Sigma$ but any proper subset is. Note that if Σ is simplicial, primitive collection means that P does not generate a cone in Σ but every proper subset does.

Definition 4.2. Let $P = \{\rho_1, \dots, \rho_k\}$ be a primitive collection in a complete simplicial fan Σ . Recall u_ρ is the primitive vector of the ray $\rho \in \Sigma$. Then $\sum_{i=1}^k u_{\rho_i}$ is in the relative interior of a cone γ_P in Σ with a unique expression

$$\sum_{i=1}^k u_{\rho_i} = \sum_{\rho \in \gamma_P(1)} c_\rho u_\rho, \quad c_\rho \in \mathbb{Q}_{>0}. \quad \text{Hence we have } \sum_{i=1}^k u_{\rho_i} - \left(\sum_{\rho \in \gamma_P(1)} c_\rho u_\rho \right) = 0. \quad (4.1)$$

Then we call (4.1) the **primitive relation** of X_Σ corresponding to P .

Recall that $TDiv(X_\Sigma)$ denote the group of torus-invariant divisors in X_Σ (see Page 4). Since the fan Σ of X_r is full dimensional, we have the following short exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi_1} TDiv(X_r) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \xrightarrow{\varphi_2} Pic(X_r) \longrightarrow 0, \quad (4.2)$$

where the maps are given by $\varphi_1 : m \mapsto div(\chi^m)$ and $\varphi_2 : D \mapsto \mathcal{O}_{X_r}(D)$ (see [CLS11, Theorem 4.2.1]).

Now we recall some standard notations: Let X be a smooth projective variety, we define

$$N_1(X)_\mathbb{Z} := \left\{ \sum_{\text{finite}} a_i C_i : a_i \in \mathbb{Z}, C_i \text{ irreducible curve in } X \right\} / \equiv$$

where \equiv is the numerical equivalence, i.e. $Z \equiv Z'$ if and only if $D \cdot Z = D \cdot Z'$ for all divisors D in X . We denote by $[C]$ the class of C in $N_1(X)_\mathbb{Z}$. Let $N_1(X) := N_1(X)_\mathbb{Z} \otimes \mathbb{R}$. It is a well known fact that $N_1(X)$ is a finite dimensional real vector space (see [Kle66, Proposition 4, §1, Chapter IV]). In the case where X is a (smooth projective) toric variety, $N_1(X)_\mathbb{Z}$ is dual to

$Pic(X)$ via the natural pairing (see [CLS11, Proposition 6.3.15]). In our case $X = X_r$, there are dual exact sequences:

$$0 \longrightarrow M \xrightarrow{\varphi_1} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\varphi_2} Pic(X_r) \longrightarrow 0$$

and

$$0 \longrightarrow N_1(X_r)_{\mathbb{Z}} \xrightarrow{\varphi_2^*} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\varphi_1^*} N \longrightarrow 0, \quad (4.3)$$

where

$$\varphi_2^*([C]) = (D_{\rho} \cdot C)_{\rho \in \Sigma(1)}, \quad C \text{ is an irreducible complete curve in } X_r$$

and

$$\varphi_1^*(e_{\rho}) = u_{\rho}, \quad e_{\rho} \text{ is a standard basis vector of } \mathbb{R}^{\Sigma(1)}$$

(see [CLS11, Proposition 6.4.1]). Let P be a primitive collection in Σ . Note that since X_r is smooth projective, $P \cap \gamma_P(1) = \emptyset$ and

$$c_{\rho} \in \mathbb{Z}_{>0} \text{ for all } \rho \in \gamma_P(1) \quad (4.4)$$

(see [CLS11, Proposition 7.3.6]). As an element in $\mathbb{Z}^{\Sigma(1)}$, we write $r(P) = (r_{\rho})_{\rho \in \Sigma(1)}$, where

$$r_{\rho} = \begin{cases} 1 & \text{if } \rho \in P \\ -c_{\rho} & \text{if } \rho \in \gamma_P(1) \\ 0 & \text{otherwise} \end{cases} \quad (4.5)$$

Then by (4.1) we see that

$$\sum_{\rho \in \Sigma(1)} r_{\rho} u_{\rho} = 0.$$

Hence by the exact sequence (4.3) and by (4.4), we observe that $r(P)$ gives an element in $N_1(X_r)_{\mathbb{Z}}$ (see [CLS11, Page 305]). We prove,

Lemma 4.3. *Let $P_i := \{\rho_i^+, \rho_i^-\}$, $1 \leq i \leq r$. Then $\{P_i : 1 \leq i \leq r\}$ is the set of all primitive collections of the fan Σ of X_r .*

Proof. By Theorem 2.6, the cones in the fan Σ of X_r are generated by subsets of $\{e_1^+, \dots, e_r^+, e_1^-, \dots, e_r^-\}$ and containing no subset of the form $\{e_i^+, e_i^-\}$. Then by Definition 4.1, it is clear that $P_i = \{\rho_i^+, \rho_i^-\}$ is a primitive collection for all i . Also note that again by description of the cones in Σ , any primitive collection must contain a P_i for some $1 \leq i \leq r$.

Fix $1 \leq i \leq r$. Let Q be a collection of one-dimensional cones such that it properly contains P_i , i.e. there exists $1 \leq j \leq r$ and $j \neq i$ such that $\rho_j^{\epsilon} \in Q \supset P_i$, $\epsilon \in \{+, -\}$. Assume that Q is a primitive collection. Then by Definition 4.1, $\{\rho_i^+, \rho_i^-\} \subset Q$ generates a cone in Σ . This is a contradiction to the description of the cones in Σ . Therefore, we conclude that $\{P_i : 1 \leq i \leq r\}$ is the set of all primitive collections. \square

Now we define the **Contractible classes** from [Cas03]: Let X be a smooth projective toric variety. We define $NE(X)_{\mathbb{Z}}$ in $N_1(X)$ by

$$NE(X)_{\mathbb{Z}} := \left\{ \sum_{finite} a_i C_i : a_i \in \mathbb{Z}_{\geq 0} \text{ and } C_i \text{ irreducible curve in } X \right\}.$$

Let $\gamma \in NE(X)_{\mathbb{Z}}$ be primitive (i.e. the generator of $\mathbb{Z}_{\geq 0}\gamma$) and such that there exists some irreducible curve in X having numerical class in $\mathbb{Q}_{\geq 0}\gamma$. Then

Definition 4.4. (see [Cas03, Definition 2.3]) *The above class γ is called **contractible** if there exists a toric variety X_γ and an equivariant morphism $\phi_\gamma : X \rightarrow X_\gamma$, surjective with connected fibers, such that for every irreducible curve C in X ,*

$$\phi_\gamma(C) = \{pt\} \text{ if and only if } [C] \in \mathbb{Q}_{\geq 0}\gamma.$$

Remark 4.5. *Note that any contractible class is always a class of some invariant curve and also a primitive relation (see [Cas03, Theorem 2.2] and [Sca09, Page 74]).*

Recall the following result from [Cas03, Proposition 3.4].

Proposition 4.6. *Let $P = \{\rho_1, \dots, \rho_k\}$ be a primitive collection in Σ , with the primitive relation $r(P)$:*

$$\sum_{i=1}^k u_{\rho_i} - \sum_{\rho \in \gamma_P(1)} c_\rho u_\rho = 0.$$

Then $r(P)$ is contractible if and only if for every primitive collection Q of Σ such that $P \cap Q \neq \emptyset$ and $P \neq Q$, the set $(Q \setminus P) \cup \gamma_P(1)$ contains a primitive collection.

4.2. Mori cone. We use the notation as above. Let X be a smooth projective variety. We define $NE(X)$ the real convex cone in $N_1(X)$ generated by classes of irreducible curves. The **Mori cone** $\overline{NE}(X)$ is the closure of $NE(X)$ in $N_1(X)$ and it is a strongly convex cone of maximal dimension.

If X is a (smooth projective) toric variety, it is known that $NE(X)_\mathbb{Z}$ is generated by the finitely many torus-invariant irreducible curves in X and hence $NE(X)_\mathbb{Z}$ is a finitely generated monoid. Hence the cone $NE(X) = \overline{NE}(X)$ is a rational polyhedral cone and we have

$$\overline{NE}(X) = \sum_{\tau \in \Sigma(r-1)} \mathbb{R}_{\geq 0}[V(\tau)],$$

where $r = \dim(X)$ and $[V(\tau)] \in N_1(X)_\mathbb{Z}$ is the class of the toric curve $V(\tau)$. This is called the Toric Cone Theorem (see [CLS11, Theorem 6.3.20]). Let $\tau \in \Sigma(r-1)$ be a wall, that is $\tau = \sigma \cap \sigma'$ for some $\sigma, \sigma' \in \Sigma(r)$. Let σ (respectively, σ') is generated by $\{u_{\rho_1}, u_{\rho_2}, \dots, u_{\rho_r}\}$ (respectively, by $\{u_{\rho_2}, \dots, u_{\rho_{r+1}}\}$) and let τ be generated by $\{u_{\rho_2}, \dots, u_{\rho_r}\}$. Then we get a linear relation,

$$u_{\rho_1} + \sum_{i=2}^r b_i u_{\rho_i} + u_{\rho_{r+1}} = 0 \tag{4.6}$$

The relation (4.6) called **wall relation** and we have

$$D_\rho \cdot V(\tau) = \begin{cases} b_i & \text{if } \rho = \rho_i \text{ and } i \in \{2, 3, \dots, r\} \\ 1 & \text{if } \rho = \rho_i \text{ and } i \in \{1, r+1\} \\ 0 & \text{otherwise} \end{cases}$$

(see [CLS11, Proposition 6.4.4 and eq. (6.4.6) page 303]). Now we describe the Mori cone $\overline{NE}(X_r)$ of X_r in terms of the primitive relations of X_r .

Theorem 4.7. $\overline{NE}(X_r)_\mathbb{Z} = \sum_{i=1}^r \mathbb{Z}_{\geq 0}r(P_i)$.

Proof. We have

$$\overline{NE}(X_r) = \sum_{P \in \mathcal{P}} \mathbb{R}_{\geq 0} r(P),$$

where \mathcal{P} is the set of all primitive collections in X_r (see [CLS11, Theorem 6.4.11]). By Lemma 4.3, $\{P_i : 1 \leq i \leq r\}$ is the set of all primitive collections of X_r . Therefore, we get

$$\overline{NE}(X_r) = \sum_{i=1}^r \mathbb{R}_{\geq 0} r(P_i).$$

By [Cas03, Theorem 4.1], we have

$$\overline{NE}(X_r)_{\mathbb{Z}} = \sum_{\gamma \in \mathcal{C}} \mathbb{Z}_{\geq 0} \gamma,$$

where \mathcal{C} is the set of all contractible classes in X_r .

By Proposition 4.6, we can see that the primitive relations $r(P_i)$ are contractible classes for $1 \leq i \leq r$. Since any contractible class is a primitive relation, we get

$$\mathcal{C} = \{r(P_i) : 1 \leq i \leq r\}.$$

Hence we conclude that

$$\overline{NE}(X_r)_{\mathbb{Z}} = \sum_{i=1}^r \mathbb{Z}_{\geq 0} r(P_i).$$

This completes the proof of the theorem. \square

We have

Corollary 4.8. *The set $\{r(P_i) : 1 \leq i \leq r\}$ forms a basis of $N_1(X_r)_{\mathbb{Z}}$.*

Proof. By Theorem 4.7, $\{r(P_i) : 1 \leq i \leq r\}$ generates the monoid $\overline{NE}(X_r)_{\mathbb{Z}}$ and the cone $\overline{NE}(X_r)$ is of dimension r . So $r(P_i)$ for $1 \leq i \leq r$ are linearly independent. Also the group $N_1(X_r)_{\mathbb{Z}}$ is generated by $\overline{NE}(X_r)_{\mathbb{Z}}$, hence by $r(P_i)$ for $1 \leq i \leq r$. Hence these form a basis of $N_1(X_r)_{\mathbb{Z}}$. \square

Next we describe the primitive relation $r(P_i)$ explicitly by finding the cone γ_{P_i} in (4.1) for $1 \leq i \leq r$. We also observe that these cones depend on the given matrix corresponding to the Bott tower. We need some notation to state the result. Recall the matrix M_r corresponding to the Bott tower X_r is

$$M_r = \begin{bmatrix} 1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} \\ 0 & 1 & \beta_{23} & \cdots & \beta_{2r} \\ 0 & 0 & 1 & \cdots & \beta_{3r} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}_{r \times r}$$

(see Section 2). Fix $1 \leq i \leq r$. Define:

- (1) Let $r \geq j > j_1 = i \geq 1$ and define $a_{1,j} := \beta_{j_1 j}$.
- (2) Let $r \geq j_2 > j_1$ be the least integer such that $a_{1,j} > 0$, then define for $j > j_2$

$$a_{2,j} := \beta_{i j_2} \beta_{j_2 j} - \beta_{i j}.$$

- (3) Let $k > 2$ and let $r \geq j_k > j_{k-1}$ be the least integer such that $a_{k-1,j} < 0$, then inductively, define for $j > j_k$

$$a_{k,j} := -a_{k-1,j_k} \beta_{j_k j} + a_{k-1,j}.$$

- (4) For $j \leq i$, $b_j := 0$, and for $j > i$ define

$$b_j := a_{l,j} \text{ if } j_{l+1} \geq j > j_l, l \geq 1. \quad (4.7)$$

Note that we have

$$b_j = \begin{cases} 0 & \text{for } j \leq i \\ < 0 & \text{for } j \in \{j_3, \dots, j_m\} \\ \geq 0 & \text{otherwise.} \end{cases}$$

- (5) Let $I_i := \{j_1, \dots, j_m\}$.

Example 4.9. Let

$$M_7 = \begin{bmatrix} 1 & -1 & -1 & -1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 2 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{7 \times 7}$$

Let $i = 1$, then $j_1 = 1$ and (1) $a_{1,2} = \beta_{12} = -1$; (2) $a_{1,3} = \beta_{13} = -1$; (3) $a_{1,4} = \beta_{14} = -1$; (4) $a_{1,5} = \beta_{15} = 2$; (5) $a_{1,6} = \beta_{16} = -1$; (6) $a_{1,7} = \beta_{17} = 2$.

Then $j_2 = 5$ and (1) $a_{2,6} = \beta_{15} \beta_{56} - \beta_{16} = -1$; (2) $a_{2,7} = \beta_{15} \beta_{57} - \beta_{17} = 2$.

Then $j_3 = 6$ and $a_{3,7} = -a_{2,6} \beta_{67} + a_{2,7} = -(-1)(-1) + (2) = 1$.

Therefore, $I_1 = \{1, 5, 6\}$.

Let $1 \leq i \leq r$. Let $\mathcal{A}_i := \{e_j^{\epsilon_j} : 1 \leq j \leq r, b_j \neq 0 \text{ and}$

$$\epsilon_j = \begin{cases} + & \text{for } j \notin I_i \\ - & \text{for } j \in I_i \end{cases} \}.$$

Remark 4.10. Note that as $b_j = 0$ for $j \leq i$, we can take $i < j \leq r$ in the definition of \mathcal{A}_i .

Now we have,

Proposition 4.11. Let $1 \leq i \leq r$. The cone γ_{P_i} in the primitive relation of X_r corresponding to P_i is generated by \mathcal{A}_i .

Before going to the proof we see an example.

Example 4.12. We use same setting as in Example 4.9. By Lemma 4.3, we have $P_i = \{\rho_i^+, \rho_i^-\}$ for all $1 \leq i \leq 7$. By definition of e_i^- (see (2.2)), we have

- (i) $e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ - 2e_5^+ + e_6^+ - 2e_7^+$; (ii) $e_2^- + e_2^+ = -2e_4^+ + e_5^+ - 2e_6^+ + e_7^+$; (iii) $e_3^- + e_3^+ = e_5^+ + e_7^+$; (iv) $e_4^- + e_4^+ = e_5^+ - 2e_6^+ + e_7^+$; (v) $e_5^- + e_5^+ = e_6^+ - 2e_7^+$; (vi) $e_6^- + e_6^+ = e_7^+$; (vii) $e_7^- + e_7^+ = 0$.

Now we describe the cone γ_{P_1} . Observe that in (i) coefficient of e_5^+ is negative. By (v), we can see

$$e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ + 2(e_5^- - e_6^+ + 2e_7^+) + e_6^+ - 2e_7^+.$$

Then $e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ + 2e_5^- - e_6^+ + 2e_7^+$. By (vi),

$$e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ + 2e_5^- + e_6^- + e_7^+. \quad (4.8)$$

In this case, $I_1 = \{1, 5, 6\}$ (see Example 4.9) and the cone γ_{P_1} is generated by

$$\{e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}.$$

Now we prove Proposition 4.11:

Proof. By (2.2), for all $1 \leq i \leq r$, we have

$$e_i^- + e_i^+ = - \sum_{j>i} \beta_{ij} e_j^+. \quad (4.9)$$

If for all $j > i$, $\beta_{ij} \leq 0$, then the cone γ_{P_i} is generated by $\{e_j^+ : j > i, \beta_{ij} < 0\}$. If not, choose the least integer $j_2 > i$ such that $\beta_{ij_2} > 0$. Now write

$$e_i^- + e_i^+ = - \left(\sum_{j_2 > j > i} \beta_{ij} e_j^+ \right) + \beta_{ij_2} (-e_{j_2}^+) - \left(\sum_{j > j_2} \beta_{ij} e_j^+ \right).$$

Again by using (4.9), we have

$$e_i^- + e_i^+ = - \left(\sum_{j_2 > j > i} \beta_{ij} e_j^+ \right) + \beta_{ij_2} (e_{j_2}^- + \sum_{j > j_2} \beta_{j_2 j} e_j^+) - \left(\sum_{j > j_2} \beta_{ij} e_j^+ \right).$$

Then

$$e_i^- + e_i^+ = - \left(\sum_{j_2 > j > i} \beta_{ij} e_j^+ \right) + \beta_{ij_2} e_{j_2}^- + \sum_{j > j_2} (\beta_{ij_2} \beta_{j_2 j} - \beta_{ij}) e_j^+.$$

By definition $a_{2,j} = \beta_{ij_2} \beta_{j_2 j} - \beta_{ij}$, then we have

$$e_i^- + e_i^+ = - \left(\sum_{j_2 > j > i} \beta_{ij} e_j^+ \right) + \beta_{ij_2} e_{j_2}^- + \left(\sum_{j > j_2} a_{2,j} e_j^+ \right).$$

If $a_{2,j} \geq 0$ for all $j > j_2$, then γ_{P_i} is generated by

$$\{e_j^{\epsilon_j} : j > i, \epsilon_j = + \forall j \neq j_2, \text{ and } \epsilon_j = - \text{ for } j = j_2\}.$$

Otherwise, choose the least integer $j_3 > j_2$ such that $a_{2,j_3} < 0$. By substituting $-e_{j_3}^+$ from (4.9), we get

$$e_i^- + e_i^+ = - \left(\sum_{j_2 > j > i} a_{1,j} e_j^+ \right) + \beta_{ij_2} e_{j_2}^- + \left(\sum_{j_3 > j > j_2} a_{2,j} e_j^+ \right) - a_{2,j_3} (e_{j_3}^- + \sum_{j > j_3} \beta_{j_3 j} e_j^+) + \left(\sum_{j > j_3} a_{2,j} e_j^+ \right).$$

Then,

$$e_i^- + e_i^+ = - \left(\sum_{j_2 > j > i} a_{1,j} e_j^+ \right) + \beta_{ij_2} e_{j_2}^- + \left(\sum_{j_3 > j > j_2} a_{2,j} e_j^+ \right) - a_{2,j_3} e_{j_3}^- + \sum_{j > j_3} (-a_{2,j_3} \beta_{j_3 j} + a_{2,j}) e_j^+.$$

By definition $a_{3,j} = -a_{2,j_3} \beta_{j_3 j} + a_{2,j}$, then we have

$$e_i^- + e_i^+ = - \left(\sum_{j_2 > j > i} a_{1,j} e_j^+ \right) + 2e_{j_2}^- + \left(\sum_{j_3 > j > j_2} a_{2,j} e_j^+ \right) - a_{2,j_3} e_{j_3}^- + \left(\sum_{j > j_3} a_{3,j} e_j^+ \right).$$

By repeating this process, we get the cone γ_{P_i} as we required. \square

Let $1 \leq i \leq r$. Recall $I_i = \{i = j_1, \dots, j_m\}$ as in page 13. Define for $1 \leq j \leq r$,

$$c_j := \begin{cases} -b_j & \text{if } j \in I_i \setminus \{j_1, j_2\} \\ b_j & \text{otherwise} \end{cases}$$

Set $\gamma_{P_i}(1) := \{\gamma_1, \dots, \gamma_i\}$. Then we have

Corollary 4.13. *For $1 \leq i \leq r$, the primitive relation $r(P_i)(= (r_\rho)_{\rho \in \Sigma(1)})$ of X_r given by*

$$r_\rho = \begin{cases} 1 & \text{for } \rho = \rho_i^+ \text{ or } \rho_i^- \\ -c_j & \text{for } \rho = \gamma_j \in \gamma_{P_i}(1) \\ 0 & \text{otherwise} \end{cases}$$

Example 4.14. *We use Example 4.12. The following can be seen easily from (4.8).*

$$(1) \quad \gamma_{P_1}(1) = \{\rho_2^+, \rho_3^+, \rho_4^+, \rho_5^-, \rho_6^-, \rho_7^+\}.$$

(2) *The primitive relation $r(P_1) = (r_\rho)_{\rho \in \Sigma(1)}$ is given by*

$$r_\rho = \begin{cases} 1 & \text{for } \rho = \rho_1^+ \text{ or } \rho_1^- \\ -1 & \text{for } \rho = \rho_k^+, k \in \{2, 3, 4, 7\} \text{ and } \rho = \rho_6^- \\ -2 & \text{for } \rho = \rho_5^- \\ 0 & \text{otherwise} \end{cases}$$

Now we describe the primitive relations $r(P_i)$ in terms of intersection of two maximal cones in the fan of X_r . Let $1 \leq i \leq r$. Let $\mathcal{C}'_i := \{e_j^{\epsilon_j} : 1 \leq j \leq r \text{ and}$

$$\epsilon_j = \begin{cases} + & \text{if } j \notin I_i \setminus \{j_1\} \\ - & \text{if } j \in I_i \end{cases} \}.$$

Let $\mathcal{C}''_i := \{e_j^{\epsilon_j} : 1 \leq j \leq r \text{ and}$

$$\epsilon_j = \begin{cases} + & \text{if } j \notin I_i \\ - & \text{if } j \in I_i \end{cases} \}.$$

Example 4.15. *We use Example 4.12, for $i = 1$, we have $I_1 = \{1, 5, 6\}$. Then*

$$\mathcal{C}'_1 = \{e_1^+, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\} \text{ and } \mathcal{C}''_1 = \{e_1^-, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}.$$

We prove the following by using *wall relation* (see page 12).

Proposition 4.16. *Fix $1 \leq i \leq r$. The class of curve $r(P_i)$ is given by*

$$r(P_i) = [V(\tau_i)],$$

where $\tau_i = \sigma \cap \sigma'$ and σ (respectively, σ') is the cone generated by \mathcal{C}'_i (respectively, by \mathcal{C}''_i).

Proof. From Corollary 4.13, we have the following.

$$e_i^+ + e_i^- - \sum_{j>i} c_j e_j^{\epsilon_j} = 0, \quad (4.10)$$

where ϵ_j is as in Proposition 4.11. First we show that the set $Q := \{\rho \in \Sigma(1) : D_\rho \cdot V(\tau_i) > 0\}$ is not contained in $\sigma(1)$ for any $\sigma \in \Sigma$ (we adapt the arguments of [CLS11, Proof of Theorem 6.4.11, page 306], here we are not assuming the curve $V(\tau_i)$ is extremal). Indeed, suppose $Q \subseteq \sigma(1)$ for some $\sigma \in \Sigma$. Let D be an ample divisor in X_r (such exists as X_r is projective). Then, we can assume that D is of the form

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho, \quad a_\rho = 0 \text{ for all } \rho \in \sigma(1) \text{ and } a_\rho \geq 0 \text{ for all } \rho \notin \sigma(1)$$

(see [CLS11, (6.4.10), page 306]). Then we can see

$$D \cdot V(\tau_i) = \sum_{\rho \notin \sigma(1)} a_\rho D_\rho \cdot V(\tau_i).$$

As $Q \subseteq \sigma(1)$, by definition of Q , $D_\rho \cdot V(\tau_i) \leq 0$ for $\rho \notin \sigma(1)$. Since $a_\rho \geq 0$ for $\rho \notin \sigma(1)$, we get $D \cdot V(\tau_i) \leq 0$, which is a contradiction as D is ample. Therefore, Q is not contained in $\sigma(1)$ for any $\sigma \in \Sigma$. Hence to prove the proposition it is enough to prove

$$P_i = Q(= \{\rho \in \Sigma(1) : D_\rho \cdot V(\tau_i) > 0\})$$

(see again [CLS11, Proof of Theorem 6.4.11, page 306]). From (4.10) and by using *wall relation*, we can see that

$$D_\rho \cdot V(\tau_i) = \begin{cases} 1 & \text{if } \rho = \rho_i^+ \text{ or } \rho_i^- \\ -c_j & \text{if } \rho = \rho_j^{\epsilon_j} \text{ and } j \in I_i \setminus \{j_1\} \\ 0 & \text{otherwise.} \end{cases}$$

Since c_j 's are all positive integers (see (4.4)), by Lemma 4.3 we conclude that

$$P_i = \{\rho \in \Sigma(1) : D_\rho \cdot V(\tau_i) > 0\}$$

and hence $r(P_i) = [V(\tau_i)]$. This completes the proof of the proposition. \square

Example 4.17. In Example 4.12, the curve $r(P_1) = [V(\tau_1)]$ with $\tau_1 = \sigma \cap \sigma'$ where σ is the cone generated by

$$\mathcal{C}'_1 = \{e_1^+, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}$$

and σ' is the cone generated by

$$\mathcal{C}''_1 = \{e_1^-, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}.$$

Corollary 4.18. $\overline{NE}(X_r)_\mathbb{Z} = \sum_{i=1}^r \mathbb{Z}_{\geq 0} [V(\tau_i)]$, where τ_i is as in Proposition 4.16.

Proof. This follows from Theorem 4.7 and Proposition 4.16 \square

5. AMPLE AND *nef* LINE BUNDLES ON THE BOTT TOWER

Let X be a smooth projective variety. Recall $N^1(X)$ is the real finite dimensional vector space of numerical classes of real divisors in X (see [Kle66, §1, Chapter IV]). In $N^1(X)$, we define the *nef* cone $Nef(X)$ to be the cone generated by classes of numerically effective divisors and it is a strongly convex closed cone in $N^1(X)$. The ample cone $Amp(X)$ of X is the cone in $N^1(X)$ generated by classes of ample divisors. Note that the ample cone $Amp(X)$ is interior of the *nef* cone $Nef(X)$ (see [Kle66, Theorem 1, §2, Chapter IV]). Recall that the *nef* cone $Nef(X)$ and the Mori cone $\overline{NE}(X)$ are closed convex cones and are dual to each other (see [Kle66, §2, Chapter IV]).

In our case, we have $Pic(X_r)_{\mathbb{R}} = N^1(X_r)$, as the numerical equivalence and linear equivalence coincide (see [CLS11, Proposition 6.3.15]).

In this section, we characterize the ampleness and numerical effectiveness of line bundles on X_r and we study the generators of the *nef* cone of X_r . We use the notation as in Section 4. Let $D = \sum a_{\rho} D_{\rho}$ be a toric divisor in X_r and for $1 \leq i \leq r$, define

$$d_i := (a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}).$$

Then we prove,

Lemma 5.1.

- (1) *The divisor D is ample if and only if $d_i > 0$ for all $1 \leq i \leq r$.*
- (2) *The divisor D is numerically effective (*nef*) if and only if $d_i \geq 0$ for all $1 \leq i \leq r$.*

Proof. Proof of (2): Recall that the primitive relation $r(P_i)$ is given by

$$r(P_i) = (r_{\rho})_{\rho \in \Sigma(1)}$$

(see page 11). First observe that we have the following

$$D \cdot r(P_i) = \sum_{\rho \in \Sigma(1)} a_{\rho} (D_{\rho} \cdot r(P_i)) = \sum_{\rho \in \Sigma(1)} a_{\rho} r_{\rho}$$

(see [CLS11, Proposition 6.4.1, page 299]). Then by (4.5), we get

$$D \cdot r(P_i) = \sum_{\rho \in P_i} a_{\rho} - \sum_{\rho \in \gamma_{P_i}(1)} r_{\rho} a_{\rho}.$$

By Lemma 4.3, we have $P_i = \{\rho_i^+, \rho_i^-\}$. Then by Corollary 4.13, we get

$$D \cdot r(P_i) = (a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}) =: d_i. \tag{5.1}$$

Since the *nef* cone $Nef(X_r)$ and the Mori cone $\overline{NE}(X_r)$ are dual to each other, the divisor D is *nef* if and only if $D \cdot C \geq 0$ for all torus-invariant irreducible curves C in X_r . By Theorem 4.7, we have

$$\overline{NE}(X_r) = \sum_{i=1}^r \mathbb{R}_{\geq 0} r(P_i).$$

Hence D is *nef* if and only if $D \cdot r(P_i) \geq 0$ for all $1 \leq i \leq r$. Therefore, by (5.1), we conclude that the divisor D is *nef* if and only if $d_i \geq 0$ for all $1 \leq i \leq r$. This completes the proof of (2).

Proof of (1): Recall that the divisor D is ample if and only if its class in $\text{Pic}(X_r)_{\mathbb{R}}$ lies in the interior of the *nef* cone $\text{Nef}(X_r)$. Hence by using similar arguments as in the proof of (2) and the toric Kleiman criterion for ampleness [CLS11, Theorem 6.3.13], we can see that D is ample if and only if $d_i > 0$ for all $1 \leq i \leq r$. \square

Next we describe the generators of the *nef* cone $\text{Nef}(X_r)$ of X_r .

Example 5.2. Let $M_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$. Then $X_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, the Hirzebruch surface \mathcal{H}_1 and the rays $\rho_1^+, \rho_1^-, \rho_2^+$ and ρ_2^- of the fan (shown below) of X_2 are generated by $e_1^+, e_1^- = -e_1^+ + e_2^+, e_2^+$ and $-e_2^+$ respectively.

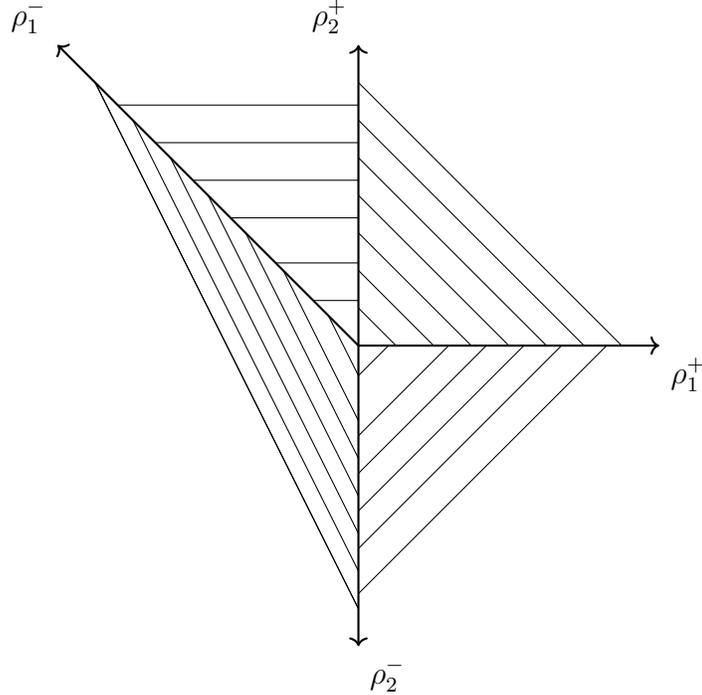


Figure. Fan of Hirzebruch surface \mathcal{H}_1 .

The primitive relations $r(P_1)$ and $r(P_2)$ are given by

$$r(P_1) : e_1^+ + e_1^- = e_2 \text{ and } r(P_2) : e_2^+ + e_2^- = 0.$$

By wall relation, we observe that

$$(1) D_{\rho_1^+} \cdot r(P_1) = 1 \text{ and } D_{\rho_1^+} \cdot r(P_2) = 0.$$

$$(2) D_{\rho_2^-} \cdot r(P_1) = 0 \text{ and } D_{\rho_2^-} \cdot r(P_2) = 1.$$

Then the dual basis of $\{r(P_1), r(P_2)\}$ is $\{D_{\rho_1^+}, D_{\rho_2^-}\}$. Hence the generators of the *nef* cone $\text{Nef}(\mathcal{H}_1)$ are $D_{\rho_1^+}$ and $D_{\rho_2^-}$. Note that by Lemma 3.1, $\text{Pic}(\mathcal{H}_1)$ is generated by $\{D_{\rho_1^+}, D_{\rho_2^-}\}$.

Let $D = aD_{\rho_1^+} + bD_{\rho_2^-} \in \text{Pic}(\mathcal{H}_1)$. Then

D is ample if and only if $a > 0$ and $b > 0$

(this gives back [CLS11, Example (6.1.16), page 273]).

Now we prove the similar results for X_r . For $1 \leq m \leq r$, define

$$J_m := \{1 \leq i < m : \{\rho_m^+\} \cap \gamma_{P_i}(1) \neq \emptyset\}.$$

Remark 5.3. Note that the set J_m is the collection of indices $i < m$ for which $u_{\rho_m^+}$ appear in the γ_{P_i} part of the expression (4.1) for the primitive relation $r(P_i)$.

We set $D_1 := D_{\rho_1^+}$, and for $m > 1$ define inductively

$$D_m := \begin{cases} D_{\rho_m^+} & \text{if } J_m = \emptyset \\ (\sum_{k \in J_m} c_{\rho_m^+}^{\gamma_{P_k}} D_k) + D_{\rho_m^+} & \text{if } J_m \neq \emptyset, \end{cases}$$

where $-c_{\rho_m^+}^{\gamma_{P_k}}$ is the coefficient of e_m^+ in the primitive relation $r(P_k)$.

Example 5.4. In Example 5.2, $D_1 = D_{\rho_1^+}$, $J_2 = \{1\}$ and $D_2 = D_1 + D_{\rho_2^+}$. By using (2.2), we see that $0 \sim \text{div}(\chi^{e_1^+}) \sim D_{\rho_1^+} - D_{\rho_1^-}$ and $0 \sim \text{div}(\chi^{e_2^+}) \sim D_{\rho_2^+} - D_{\rho_2^-} + D_{\rho_1^-}$. Hence $D_2 = D_1 + D_{\rho_2^+} = D_{\rho_2^-}$.

Example 5.5. In Example 4.12,

- (1) Recall by (4.8), we have $e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ + 2e_5^- + e_6^- + e_7^+$. Then, $\gamma_{P_1}(1) = \{\rho_2^+, \rho_3^+, \rho_4^+, \rho_5^-, \rho_6^-, \rho_7^+\}$.
- (2) $\gamma_{P_2}(1) = \{\rho_4^-, \rho_5^-, \rho_6^+, \rho_7^+\}$ (since $e_2^+ + e_2^- = 2e_4^- + e_5^- + e_6^+ + e_7^+$).
- (3) $\gamma_{P_3}(1) = \{\rho_5^+, \rho_7^+\}$ (since $e_3^+ + e_3^- = e_5^+ + e_7^+$).
- (4) $\gamma_{P_4}(1) = \{\rho_5^+, \rho_6^-, \rho_7^-\}$ (since $e_4^+ + e_4^- = e_5^+ + 2e_6^- + e_7^-$).
- (5) $\gamma_{P_5}(1) = \{\rho_6^+, \rho_7^-\}$ (since $e_5^+ + e_5^- = e_6^+ + 2e_7^-$).
- (6) $\gamma_{P_6}(1) = \{\rho_7^+\}$ (since $e_6^+ + e_6^- = e_7^+$).
- (7) $\gamma_{P_7}(1) = \emptyset$. (since $e_7^+ + e_7^- = 0$).

Then ,

- (1) If $m = 1$, then $D_1 = D_{\rho_1^+}$.
- (2) If $m = 2$, then $J_2 = \{1\}$ and $c_{\rho_2^+}^{\gamma_{P_1}} = 1$. Hence $D_2 = D_1 + D_{\rho_2^+}$.
- (3) If $m = 3$, then $J_3 = \{1\}$ and $c_{\rho_3^+}^{\gamma_{P_1}} = 1$. Hence $D_3 = D_1 + D_{\rho_3^+}$.
- (4) If $m = 4$, then $J_4 = \{1\}$ and $c_{\rho_4^+}^{\gamma_{P_1}} = 1$. Hence $D_4 = D_1 + D_{\rho_4^+}$.

(5) If $m = 5$, then $J_5 = \{3, 4\}$ and $c_{\rho_5^+}^{\gamma_{P_3}} = 1$; $c_{\rho_5^+}^{\gamma_{P_4}} = 1$. Hence

$$D_5 = D_3 + D_4 + D_{\rho_5^+}.$$

(6) If $m = 6$, then $J_6 = \{2, 5\}$ and $c_{\rho_6^+}^{\gamma_{P_2}} = 1$; $c_{\rho_6^+}^{\gamma_{P_5}} = 1$. Hence

$$D_6 = D_2 + D_5 + D_{\rho_6^+}.$$

(7) If $m = 7$, then $J_7 = \{1, 2, 3, 6\}$ and

$$c_{\rho_7^+}^{\gamma_{P_1}} = 1 ; c_{\rho_7^+}^{\gamma_{P_2}} = 1 ; c_{\rho_7^+}^{\gamma_{P_3}} = 1 ; \text{ and } c_{\rho_7^+}^{\gamma_{P_6}} = 1 . \text{ Hence}$$

$$D_7 = D_1 + D_2 + D_3 + D_6 + D_{\rho_7^+}.$$

We prove,

Proposition 5.6. *The set $\{D_i : 1 \leq i \leq r\}$ is dual basis of $\{r(P_i) : 1 \leq i \leq r\}$.*

Proof. Fix $1 \leq i \leq r$. By Proposition 4.16, the class of curve corresponding to the primitive relation $r(P_i)$ is given by

$$r(P_i) = [V(\tau_i)]$$

(where τ_i is described as in Proposition 4.16). From Corollary 4.13, the primitive relation $r(P_i) (= [V(\tau_i)])$ is

$$e_i^+ + e_i^- - \sum_{j>i} c_j e_j^{\epsilon_j} = 0, \quad (5.2)$$

where ϵ_j is as in Proposition 4.16. Note that this is the *wall relation* for the torus-invariant curve $V(\tau_i)$. We prove

$$D_m \cdot r(P_i) = D_m \cdot V(\tau_i) = \begin{cases} 1 & \text{if } i = m. \\ 0 & \text{if } i \neq m. \end{cases} \quad (5.3)$$

By (5.2) and by *wall relation*, we have

$$D_{\rho_m^+} \cdot V(\tau_i) = \begin{cases} 1 & \text{for } m = i \\ 0 & \text{for } m < i \\ -c_{\rho_m^+}^{\gamma_{P_i}} & \text{for } m > i \text{ and } i \in J_m \\ 0 & \text{for } m > i \text{ and } i \notin J_m \end{cases} \quad (5.4)$$

Hence by definition of D_m , it is clear that

$$D_m \cdot V(\tau_i) = \begin{cases} 1 & \text{for } m = i \\ 0 & \text{for } m < i \end{cases} \quad (5.5)$$

Now we claim $D_m \cdot V(\tau_i) = 0$ for all $m > i$. Assume that $m > i$ and write $m = i + j$, where $1 \leq j \leq r - i$. We prove the claim by induction on j . If $j = 1$, then $D_m = D_{i+1}$.

Case 1: If $J_{i+1} = \emptyset$, then $D_{i+1} = D_{\rho_{i+1}^+}$. By (5.4), we see that $D_{i+1} \cdot V(\tau_i) = 0$.

Case 2: Assume that $J_{i+1} \neq \emptyset$.

Subcase 1: If $i \notin J_{i+1}$, then by (5.4) and (5.5), we can see that $D_{i+1} \cdot V(\tau_i) = 0$.

Subcase 2: If $i \in J_{i+1}$, then by (5.5), we have $D_{i+1} \cdot V(\tau_i) = c_{\rho_{i+1}^+}^{\gamma P_i} + (D_{\rho_{i+1}^+} \cdot V(\tau_i))$.

By (5.4), $D_{\rho_{i+1}^+} \cdot V(\tau_i) = -c_{\rho_{i+1}^+}^{\gamma P_i}$ and hence $D_{i+1} \cdot V(\tau_i) = 0$. This proves the claim for $j = 1$.

Now assume that $j > 1$.

Case 1: If $J_m = \emptyset$, then by (5.4) and (5.5), we see that $D_m \cdot V(\tau_i) = 0$.

Case 2: Assume that $J_m \neq \emptyset$.

Subcase 1: If $i \notin J_m$, then by (5.4) and (5.5), we can see that

$$D_m \cdot V(\tau_i) = \left(\sum_{k \in J_m, k > i} c_{\rho_m^+}^{\gamma P_k} D_k \right) \cdot V(\tau_i) + (D_{\rho_m^+} \cdot V(\tau_i)).$$

By induction on j , $D_k \cdot V(\tau_i) = 0$ for all $i < k < m$. By (5.4), as $m > i$ and $m \notin J_m$, we have $D_{\rho_m^+} \cdot V(\tau_i) = 0$. Hence we conclude that $D_m \cdot V(\tau_i) = 0$. This completes the proof of the proposition. \square

We have,

Theorem 5.7.

- (1) The nef cone $Nef(X_r)$ of X_r is generated by $\{D_i : 1 \leq i \leq r\}$.
- (2) The divisor $D = \sum_i a_i D_i$ is ample if and only if $a_i > 0$ for all $1 \leq i \leq r$.

Proof. Since the nef cone $Nef(X_r)$ is dual of the Mori cone $\overline{NE}(X_r)$, (1) follows from Proposition 5.6.

Proof of (2): This follows from (1) as the ample cone $Amp(X_r)$ is interior of the nef cone $Nef(X_r)$. \square

6. FANONESS AND WEAK FANONESS OF BOTT TOWERS

In this section we describe the matrices M_r such that the corresponding Bott tower X_r is Fano or weak Fano. First recall the *Itaka dimension* of a Cartier divisor D in a normal projective variety X . Let

$$N(D) := \{m \geq 0 : H^0(X, \mathcal{L}(mD)) \neq 0\},$$

where $\mathcal{L}(mD)$ is the line bundle associated to mD . For $m \in N(D)$, we have a rational map

$$\phi_m : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{L}(mD))^*).$$

If $N(D)$ is empty we define the *Itaka dimension* $\kappa(D)$ of D as $-\infty$. Otherwise we define

$$\kappa(D) := \max_{m \in N(D)} \{\dim(\phi_m(X))\}.$$

Observe that $\kappa(D) \in \{-\infty, 0, 1, \dots, \dim(X)\}$. We say D is *big* if $\kappa(D) = \dim(X)$ (see [Laz04, Section 2.2, page 139]). Note that an ample divisor is *big*.

Lemma 6.1. *Let X be a smooth projective variety, let U be an open affine subset of X . Let D be an effective divisor with support $X \setminus U$. Then D is big.*

Proof. It suffices to show that there exists an effective divisor E with support $X \setminus U$ such that E is *big*. Indeed, we then have $mD = E + F$ for some $m \geq 0$ and for some effective divisor F . Then $E + F$ is big and hence so is D .

There exists $f_1, \dots, f_n \in \mathcal{O}_X(U)$ algebraically independent over \mathbb{C} , where $n = \dim(X)$. View f_1, \dots, f_n as rational functions on X , then $f_1, \dots, f_n \in H^0(X, \mathcal{O}_X(E))$ for some effective divisor E with support $X \setminus U$ (since $\text{div}(f_i)$ is an effective divisor with support in $X \setminus U$ for $1 \leq i \leq r$). Thus, the monomials in f_1, \dots, f_n of any degree m are linearly independent elements of $H^0(X, \mathcal{O}_X(mE))$. So $\dim(H^0(X, \mathcal{O}_X(mE)))$ grows like m^n as $m \rightarrow \infty$. Hence E is *big* (see [Laz04, Corollary 2.1.38 and Lemma 2.2.3]) and this completes the proof. \square

We get the following as a variant of Lemma 6.1.

Corollary 6.2. *Let X be a smooth projective variety and D be an effective divisor. Let $\text{supp}(D)$ denotes the support of D . If $X \setminus \text{supp}(D)$ is affine, then D is big.*

A smooth projective variety X is called Fano (respectively, weak Fano) if its anti-canonical line bundle $-K_X$ is ample (respectively, *nef* and *big*). To describe our results we use the notation and terminology from Section 1 (see page 2). We prove,

Theorem 6.3.

- (1) X_r is Fano if and only if it satisfies I.
- (2) X_r is weak Fano if and only if it satisfies II.

Proof. Proof of (2): We have

$$K_{X_r} = - \sum_{\rho \in \Sigma(1)} D_\rho \tag{6.1}$$

(see [CLS11, Theorem 8.2.3] or [Ful93, Page 74]). The anti-canonical line bundle of any projective toric variety is *big*, since we have

$$\text{supp}(-K_{X_r}) = X_r \setminus (\mathbb{C}^*)^r,$$

$(\mathbb{C}^*)^r$ is an affine open subset of X_r , by Corollary 6.2, $-K_{X_r}$ is *big*.

By using Lemma 5.1, we prove that $-K_{X_r}$ is *nef* if and only if X_r satisfies III.

Let $D = -K_{X_r}$. By (6.1) and by definition of d_i for D (see Lemma 5.1), we have

$$d_i = 2 - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j.$$

Then by Lemma 5.1(2), $-K_{X_r}$ is *nef* if and only if $\sum_{\gamma_j \in \gamma_{P_i}(1)} c_j \leq 2$ for all $1 \leq i \leq r$.

First assume that $-K_{X_r}$ is *nef*. Fix $1 \leq i \leq r$. By above discussion, we have

$$\sum_{\gamma_j \in \gamma_{P_i}(1)} c_j \leq 2. \tag{6.2}$$

Since c_j 's are positive integers (see (4.4)), we get the following situation:

$$|\gamma_{P_i}(1)| = 0 \text{ or } |\gamma_{P_i}(1)| = 1, \text{ or } |\gamma_{P_i}(1)| = 2.$$

Case 1: If $|\gamma_{P_i}(1)| = 0$, then by definition of γ_{P_i} (see Definition 4.2), we have

$$r(P_i) : e_i^+ + e_i^- = 0.$$

Then $|\eta_i^+| = 0 = |\eta_i^-|$. Hence we see X_r satisfies the condition N_i^1 .

Case 2: If $|\gamma_{P_i}(1)| = 1$, then there exists a unique $r \geq j > i$, such that $\gamma_j \in \gamma_{P_i}(1)$ and the primitive relation is either

$$r(P_i) : e_i^+ + e_i^- = c_j e_j^+ \quad (6.3)$$

or

$$r(P_i) : e_i^+ + e_i^- = c_j e_j^- \quad (6.4)$$

By (6.2), we get $c_j = 1$ or 2 .

Subcase (i): Assume that $c_j = 1$. If the primitive relation is (6.3), then we can see that $|\eta_i^+| = 0$ and $c_j = -\beta_{ij} = 1$. Then $\beta_{ij} = -1$ and hence X_r satisfies the condition N_i^1 .

If the primitive relation is (6.4), then by (2.2) $|\eta_i^-| = 0$ and $|\eta_i^+| = 1$. Hence $c_j = \beta_{ij} = 1$ and $\beta_{jk} = 0$ for all $k > j$.

Subcase (ii): Assume that $c_j = 2$. If the primitive relation $r(P_i)$ is (6.3), then $|\eta_i^+| = 0$ and $|\eta_i^-| = 1$. So by (2.2), we have $c_j = -\beta_{ij}$. If the primitive relation $r(P_i)$ is (6.4), then $|\eta_i^+| = 1$, $|\eta_i^-| = 0$ and $\beta_{jk} = 0$ for all $k > j$. Again by (2.2), we have $c_j = \beta_{ij}$. Thus,

$$\text{either } \beta_{ij} = -2 \text{ or } \beta_{ij} = 2.$$

Hence X_r satisfies the condition N_i^2 .

Case 3: If $|\gamma_{P_i}(1)| = 2$, then there exists $r \geq s_1 > s_2 > i$ with $\gamma_{s_1}, \gamma_{s_2} \in \gamma_{P_i}(1)$ such that the primitive relation $r(P_i)$ is

$$r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^\pm + c_{s_2} e_{s_2}^\pm \quad (6.5)$$

Subcase (i): If the primitive relation is $r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^+ + c_{s_2} e_{s_2}^+$, by (2.2) we see $|\eta_i^+| = 0$ and $|\eta_i^-| = 2$. By (6.2) and (4.4) (c_i 's are positive integers), we get

$$c_{s_1} = 1, c_{s_2} = 1 \text{ and } \beta_{is_1} = \beta_{is_2} = -1.$$

Hence X_r satisfies the condition N_i^2 .

Subcase (ii): If the primitive relation is $r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^+ + c_{s_2} e_{s_2}^-$, by (2.2) we see $|\eta_i^+| = 1 = |\eta_i^-|$. Then $\beta_{is_1} = -1$, $\beta_{is_2} = 1$ and $\beta_{s_2 k} = 0$ for all $k > s_2$.

Subcase (iii): If the primitive relation is $r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^- + c_{s_2} e_{s_2}^+$, by (2.2) we see $|\eta_i^+| = 1$ and $\beta_{is_1} = 1$. Then $\beta_{s_1 s_2} - \beta_{is_2} = 1$ and $\beta_{s_1 k} - \beta_{ik} = 0$ for all $k > s_2$.

Subcase (iv): If the primitive relation is $r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^- + c_{s_2} e_{s_2}^-$, by (2.2) we see $|\eta_i^+| = 1$ and $\beta_{is_1} = 1$. Then $\beta_{s_1 s_2} - \beta_{is_2} = -1$ and $\beta_{is_2} - \beta_{s_1 s_2} - \beta_{s_2 k} = 0$ for all $k > s_2$.

Hence X_r satisfies the condition N_i^2 . Therefore, we conclude that if X_r is weak Fano then X_r satisfies the condition II . Similarly, we can prove by using Lemma 5.1(2), if X_r satisfies II then X_r is weak Fano. This completes the proof of (2).

Proof of (1): This follows by using similar arguments as in the proof of (2) and Lemma 5.1(1). \square

6.1. Local rigidity of Bott towers. Now we prove some vanishing results for the cohomology of tangent bundle of the Bott tower X_r and we get some local rigidity results. Let T_{X_r} denotes the tangent bundle of X_r . Then we have

Corollary 6.4. *If $X_{\bar{w}}$ satisfies I , then $H^i(X_{\bar{w}}, T_{X_{\bar{w}}}) = 0$ for all $i \geq 1$.*

Proof. If X_r satisfies I , then by Theorem 6.3, X_r is Fano variety. By [BB96, Proposition 4.2], since X_r is a smooth Fano toric variety, we get $H^i(X_r, T_{X_r}) = 0$ for all $i \geq 1$. \square

It is well known that by Kodaira-Spencer theory, the vanishing of $H^1(X, T_X)$ implies that X is locally rigid, i.e. admits no local deformations (see [Huy06, Proposition 6.2.10, page 272]). Then by above result we have

Corollary 6.5. *The Bott tower X_r is locally rigid if it satisfies I .*

7. LOG FANONESS OF BOTT TOWERS

Recall that a pair (X, D) of a normal projective variety X and an effective \mathbb{Q} -divisor D is **Kawamata log terminal (klt)** if $K_X + D$ is \mathbb{Q} -Cartier, and for all proper birational maps $f : Y \rightarrow X$, the pull back $f^*(K_X + D) = K_Y + D'$ satisfies $f_*K_Y = K_X$ and $\lfloor D' \rfloor \leq 0$, where $\lfloor \sum_i a_i D_i \rfloor = \sum_i \lfloor a_i \rfloor D_i$, $\lfloor x \rfloor$ is the greatest integer $\leq x$. The pair (X, D) is called **log Fano** if it is klt and $-(K_X + D)$ is ample.

We recall here, a condition for the anti-canonical line bundle to be *big* (see [CG13]). Let X be a \mathbb{Q} -Gorenstein projective normal variety over \mathbb{C} . If X admits a divisor D with the pair (X, D) being log Fano then $-K_X$ is big (In [CG13] there is a necessary and sufficient condition that X is log Fano (or ‘‘Fano type’’) variety, see [CG13, Theorem 1.1] for more details on this).

If X is smooth and D is a normal crossing divisor, the pair (X, D) is log Fano if and only if $\lfloor D \rfloor = 0$ and $-(K_X + D)$ is ample (see [KM08, Lemma 2.30, Corollary 2.31 and Definition 2.34]). In case of toric variety X see also [CLS11, Definition 11.4.23 and Proposition 11.4.24, page 558]. We use notation as in Lemma 5.1. Let $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a toric divisor in X_r , with a'_ρ s in $\mathbb{Q}_{\geq 0}$ and $\lfloor D \rfloor = 0$. For $1 \leq i \leq r$, define

$$k_i := d_i - 2 + \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j.$$

Then we prove,

Theorem 7.1. *The pair (X_r, D) is log Fano if and only if $k_i < 0$ for all $1 \leq i \leq r$.*

Proof. From the above discussion by the condition on D , the pair (X_r, D) is log Fano if and only if $-(K_{X_r} + D)$ is ample. Note that as $-K_{X_r} = \sum_{\rho \in \Sigma(1)} D_\rho$, we get

$$-(K_{X_r} + D) = \sum_{\rho \in \Sigma(1)} (1 - a_\rho) D_\rho.$$

By Lemma 5.1, $-(K_{X_r} + D)$ is ample if and only if

$$((1 - a_{\rho_i^+}) + (1 - a_{\rho_i^-}) - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j(1 - a_{\gamma_j})) > 0 \text{ for all } 1 \leq i \leq r. \quad (7.1)$$

Recall the definition of d_i for D ,

$$d_i = a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}.$$

Then we have

$$((1 - a_{\rho_i^+}) + (1 - a_{\rho_i^-}) - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j (1 - a_{\gamma_j})) = -(d_i - 2 + \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j).$$

Hence in (7.1)

$$((1 - a_{\rho_i^+}) + (1 - a_{\rho_i^-}) - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j (1 - a_{\gamma_j})) = -k_i \text{ for all } 1 \leq i \leq r$$

and we conclude that $-(K_{X_r} + D)$ is ample if and only if $k_i < 0$ for all $1 \leq i \leq r$. This completes the proof of the theorem. \square

8. EXTREMAL RAYS AND MORI RAYS OF THE BOTT TOWER

In this section we study the extremal rays and Mori rays of Mori cone of X_r . First we recall some definitions. Let V be a finite dimensional vector space over \mathbb{R} and let K be a (closed) cone in V . A subcone Q in K is called extremal if $u, v \in K, u + v \in Q$ then $u, v \in Q$. A face of K is an extremal subcone. A one-dimensional face is called an extremal ray. Note that an extremal ray is contained in the boundary of K .

Let X be a smooth projective variety. An extremal ray R in $\overline{NE}(X) \subset N_1(X)$ is called Mori if $R \cdot K_X < 0$, where K_X is the canonical divisor in X . Recall that $\overline{NE}(X_r)$ is a strongly convex rational polyhedral cone of maximal dimension in $N_1(X_r)$. We prove,

Theorem 8.1.

- (1) *The class of curves $r(P_i)$ for $1 \leq i \leq r$ are all extremal rays in the Mori cone $\overline{NE}(X_r)$ of X_r .*
- (2) *Fix $1 \leq i \leq r$, the class of curve $r(P_i)$ is Mori ray if and only if either $|\gamma_{P_i}(1)| = 0$, or $|\gamma_{P_i}(1)| = 1$ with $c_j = 1$ for $\gamma_j \in \gamma_{P_i}(1)$.*

Proof. Proof of (1): This follows from Theorem 4.7 and Corollary 4.8.

Proof of (2): By (1), $r(P_i)$ $1 \leq i \leq r$ are all extremal rays in $\overline{NE}(X_r)$. Hence for $1 \leq i \leq r$, $r(P_i)$ is Mori if $K_{X_r} \cdot r(P_i) < 0$. Since $K_{X_r} = -\sum_{\rho \in \Sigma(1)} D_\rho$, we can see by Corollary 4.13 and by similar arguments as in the proof of Lemma 5.1,

$$K_{X_r} \cdot r(P_i) = -2 + \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j. \quad (8.1)$$

Thus if $K_{X_r} \cdot r(P_i) < 0$, then

$$\sum_{\gamma_j \in \gamma_{P_i}(1)} c_j < 2.$$

As c_j are all positive integers (see (4.4)), we get either $|\gamma_{P_i}(1)| = 0$, or $|\gamma_{P_i}(1)| = 1$ and $c_j = 1$ for $\gamma_j \in \gamma_{P_i}(1)$. Similarly, by using (8.1) we can prove the converse. This completes the proof of the theorem. \square

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